Diagram spaces and symmetric spectra

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Outline

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Spectra and cohomology theories

In algebraic topology, homology and cohomology theories are represented by spectra.

Definition

A spectrum E is a sequence of based spaces $E(n)$ for $n \geq 0$, together with structure maps $S^1 \wedge E(n) \to E(n+1)$.

Here $\wedge$ is the smash product of based spaces:

$$X \wedge Y = X \times Y \big/ \{*\} \times Y \cup X \times \{*\}.$$

A basic example is the *sphere spectrum* $\mathbb{S}$ with $\mathbb{S}(n) = S^n$.

Some constructions:

The shift suspension spectrum $\Sigma^k E$ has $\Sigma^k E(n) = E(n+k)$.
For a based space X, the spectrum $E \wedge X$ has nth space $E(n) \wedge X$.

The associated homology and cohomology theories on based spaces are defined by

$$E_n(X) = [\Sigma^n \mathbb{S}, E \wedge X] \quad \text{and} \quad E^n(X) = [\mathbb{S} \wedge X, \Sigma^n E]$$

where $[-,-]$ denotes the abelian group of morphism in the homotopy category of spectra (aka the stable homotopy category).

In particular, the homotopy groups are defined by

$$\pi_k E = [\Sigma^k \mathbb{S}, E].$$

A multiplicative structure on a cohomology theory corresponds to a multiplicative structure on the representing spectrum.

In order to make this precise it is convenient to work in a category of spectra with a strictly associative "smash product". There are several good choices for such a category, for instance

- the category of symmetric spectra (Hovey-Shipley-Smith),
- the category of orthogonal spectra (Mandell-May-Schwede-Shipley),
- the category of simplicial functors (Lydakis),
- the category of S-modules (Elmendorf-Kriz-Mandell-May).

In this talk we shall concentrate on the category of symmetric spectra.

Symmetric spectra

Definition (Hovey-Shipley-Smith)

A symmetric spectrum E is a spectrum in which each space $E(n)$ is equipped with a based Σ_n-action such that the iterated structure maps

$$S^m \wedge E(n) \to E(m+n)$$

are $\Sigma_m \times \Sigma_n$-equivariant.

The sphere spectrum $\mathbb{S}$ is an example of a symmetric spectrum.

Let Sp^Σ be the category of symmetric spectra.

Theorem (Hovey-Shipley-Smith)

The category Sp^Σ has an associative smash product $\wedge$ such that $(Sp^\Sigma, \wedge, \mathbb{S})$ is a symmetric monoidal category.

$\mathbb{S}$-algebras

The universal property of the smash product is such that a map of symmetric spectra $E \wedge E' \to E''$ amounts to a "compatible" family of $\Sigma_m \times \Sigma_n$-equivariant maps

$$E(n) \wedge E'(m) \to E''(m+n).$$

A *symmetric ring spectrum* R is a monoid in $(Sp^\Sigma, \wedge, \mathbb{S})$. Thus, there are maps of symmetric spectra $\mathbb{S} \to R$ and $R \wedge R \to R$, which amount to maps

$$S^n \to R(n) \quad \text{and} \quad R(m) \wedge R(n) \to R(m+n).$$

Since $\mathbb{S}$ is the unit for the smash product, we have that

$\mathbb{S}$-**modules** $= Sp^\Sigma$,

$\mathbb{S}$-**algebras** $=$ **Symmetric ring spectra**,

Comm $\mathbb{S}$-**algebras** $=$ **Comm symmetric ring spectra**.

Using the smash product $\wedge$ in analogy with the tensor product $\otimes$ of abelian groups, one may define topological versions of constructions from algebra.

The basic idea is to replace the ground ring $\mathbb{Z}$ by $\mathbb{S}$.

Forgetting the symmetric monoidal structure, the homotopy theory associated to Sp^{Σ} is equivalent to the homotopy theory for ordinary (non-symmetric) spectra:

Theorem (Hovey-Shipley-Smith)

There is a model structure on Sp^{Σ} that makes it Quillen equivalent to the category of ordinary spectra. In particular, the homotopy category of symmetric spectra is equivalent to the stable homotopy category.

Examples of $\mathbb{S}$-algebras

Example (Eilenberg-Mac Lane spectra)

An ordinary ring R gives rise to the Eilenberg-Mac Lane spectrum HR defined by

$$HR(n) = |R(S^n_\bullet)|,$$

where

- $S^n_\bullet$ is the n-fold smash product of the simplicial circle $S^1_\bullet$,
- $R(S^n_\bullet)$ is the simplicial set $[k] \mapsto R[S^n_k]/R\{*\}$.

This is an $\mathbb{S}$-algebra which is commutative if R is. The multiplication is induced by the simplicial maps

$$R(S^m_\bullet) \wedge R(S^n_\bullet) \to R[S^m_\bullet \wedge S^n_\bullet] = R[S^{m+n}_\bullet].$$

The $\mathbb{S}$-algebra HR represents ordinary cohomology with coefficients in R.

Example (Thom spectra)

The cobordism spectrum MO can naturally be defined as a commutative $\mathbb{S}$-algebra:

$$MO(n) = B(*, O(n), S^n)/BO(n),$$

where S^n is the one-point compactification of $\mathbb{R}^n$ with the usual action of the orthogonal group $O(n)$. The multiplication is induced by

$$\begin{aligned} B(*, O(m), S^m) \times B(*, O(n), S^n) &\to B(*, O(m) \times O(n), S^m \wedge S^n) \\ &\to B(*, O(m+n), S^{m+n}). \end{aligned}$$

A similar construction applies to MSO. These $\mathbb{S}$-algebras represent the unoriented and the oriented cobordism theories.

We shall later see more examples of Thom spectra that are $\mathbb{S}$-algebras (for instance MU).

Example (Topological K-theory)

Commutative $\mathbb{S}$-algebra models of the topological K-theory spectra KO and KU have been constructed by M. Joachim and M. Mandell. These $\mathbb{S}$-algebras represent real and complex topological K-theory.

Example (Spherical monoid rings)

Given a (topological) monoid M, the spherical monoid ring $\mathbb{S}[M]$ is defined by

$$\mathbb{S}[M](n) = S^n \wedge M_+, \qquad (M_+ = M \cup \{+\}).$$

This is an $\mathbb{S}$-algebra which is commutative if M is. It is related to the ordinary monoid ring via $\pi_0 \mathbb{S}[M] = \mathbb{Z}[\pi_0 M]$.

E_∞ spaces and infinite loop spaces

A spectrum E has an associated infinite loop space $\Omega^\infty(E)$ defined by

$$\Omega^\infty(E) = (\mathrm{ho})\operatorname{colim}_n \Omega^n E(n), \quad \Omega^n E(n) = \mathrm{Map}_*(S^n, E(n)).$$

The recognition theorem of Boardman-Vogt and May gives a criterion for when a space has the form $\Omega^\infty(E)$.

Recall that an *operad* $\mathcal{E}$ is a sequence of spaces $\mathcal{E}(n)$, for $n \geq 0$, such that Σ_n acts on $\mathcal{E}(n)$ and there are structure maps

$$\mathcal{E}(k) \times \mathcal{E}(j_1) \times \cdots \times \mathcal{E}(j_k) \to \mathcal{E}(j_1 + \cdots + j_k)$$

which satisfy certain associativity and equivariance conditions.
An action of an operad $\mathcal{E}$ on a space X is given by a compatible sequence of maps

$$\mathcal{E}(k) \times X^k \to X.$$

In this way $\mathcal{E}(k)$ parametrizes k-fold multiplications in X.

An operad $\mathcal{E}$ is an E_∞ operad if the space $\mathcal{E}(n)$ is Σ_n-free and contactible for all n.

Definition

An E_∞ space is a space with an action of an E_∞ operad.

The operad action on an E_∞ space X induces a homotopy associative and commutative multiplication. In fact, the term "E_∞" indicates that X is "homotopy everything" in the sense that it has all possible coherence homotopies for associativity and commutativity.

We say that X is *grouplike* if $\pi_0 X$ is a group.

Theorem (Boardman-Vogt, May)

If X is a grouplike E_∞ space, then there exists a spectrum E such that $X \simeq \Omega^\infty E$.

Example

The Barratt-Eccles operad $\mathcal{E}$ is the E_∞ operad with

$$\mathcal{E}(n) = E\Sigma_n \quad (= B(*, \Sigma_n, \Sigma_n)).$$

If $\mathcal{A}$ is a symmetric (strict) monoidal category, then $\mathcal{E}$ acts on the classifying space $B\mathcal{A}$ which is therefore an E_∞ space.

Example

The linear isometries operad $\mathcal{L}$ is the E_∞ operad with

$$\mathcal{L}(n) = \left\{ \text{Linear isometries } \underbrace{\mathbb{R}^\infty \oplus \cdots \oplus \mathbb{R}^\infty}_{n} \to \mathbb{R}^\infty \right\}$$

Let $O(n)$ be the orthogonal group and let $BO(\infty)$ be the stabilization of the classifying spaces $BO(n)$. Then $\mathcal{L}$ acts on $BO(\infty)$ which is therefore an E_∞ space.

Remark

We may think of an E_∞ space as the topological analogy of a commutative monoid in algebra.

It is important to realize that not every E_∞ space is homotopy equivalent to a strictly commutative topological monoid. In fact, for this to be the case the space must up to homotopy be a product of Eilenberg Mac Lane spaces.

The E_∞ space $BO(\infty)$ is an example of a space which is not homotopy equivalent to a product of Eilenberg Mac Lane spaces.

In the following we shall describe an approach to E_∞ spaces which overcomes this difficulty and which is convenient when working with symmetric spectra.

Remark

A. Blumberg has introduced an approach to "structured spaces" which is convenient for working with the Elmendorf-Kriz-Mandell-May S-modules.

The category of $\mathcal{I}$-spaces

Let $\mathcal{I}$ be the category with objects the finite sets $\mathbf{n} = \{1, \dots, n\}$ (including the empty set $\mathbf{0}$) and morphisms the injective maps.

The ordered concatenation $\mathbf{m} \sqcup \mathbf{n}$ of ordered sets makes this a symmetric monoidal category $(\mathcal{I}, \sqcup, \mathbf{0})$.

Let $\mathcal{S}$ be the category of spaces (or simplicial sets).

Definition

We define an *$\mathcal{I}$-space* to be a functor $X\colon \mathcal{I} \to \mathcal{S}$ and we write $\mathcal{S}^{\mathcal{I}}$ for the category of such functors.

The symmetric monoidal structure of $\mathcal{I}$ induces a symmetric monoidal structure on $\mathcal{S}^{\mathcal{I}}$:

$$X \boxtimes Y(n) = \operatorname*{colim}_{\mathbf{n}_1 \sqcup \mathbf{n}_2 \to \mathbf{n}} X(n_1) \times Y(n_2),$$

where the colimit is over the comma category $(\sqcup \downarrow \mathbf{n})$. The unit is the constant $\mathcal{I}$-diagram $*$.

The universal property of the $\boxtimes$-product is such that a map of $\mathcal{I}$-spaces $X \boxtimes Y \to Z$ amounts to a natural map of $\mathcal{I} \times \mathcal{I}$-spaces

$$X(m) \times Y(n) \to Z(m+n).$$

An *$\mathcal{I}$-space monoid* A is a monoid in $(\mathcal{S}^{\mathcal{I}}, \boxtimes, *)$. Thus, there are maps $* \to A$ and $A \boxtimes A \to A$ which amounts to maps

$$* \to A(n) \quad \text{and} \quad A(m) \times A(n) \to A(m+n).$$

The following notation is often convenient in order to keep track of the combinatorics involved with $\mathcal{I}$-spaces:

Given a morphism $\alpha\colon \mathbf{m} \to \mathbf{n}$ in $\mathcal{I}$, we write $\mathbf{n} - \alpha$ for the complement of $\alpha(\mathbf{m})$ in $\mathbf{n}$.

Example

Given an ordinary ring R, the correspondence $\mathbf{n} \mapsto \mathrm{GL}_n(R)$ defines a functor $\mathcal{I} \to \mathbf{Groups}$: a morphism $\alpha\colon \mathbf{m} \to \mathbf{n}$ in $\mathcal{I}$ gives rise to the group homomorphism $\alpha_*\colon \mathrm{GL}_m(R) \to \mathrm{GL}_n(R)$ defined by

$$\alpha_*(f)\colon R^n \simeq R^{(n-\alpha)} \oplus R^m \xrightarrow{id \oplus f} R^{(n-\alpha)} \oplus R^m \simeq R^n.$$

Applying the classifying space functor we get the commutative $\mathcal{I}$-space monoid $B\mathrm{GL}(R)$.

The multiplication is induced by the homomorphisms

$$\mathrm{GL}_m(R) \times \mathrm{GL}_n(R) \to \mathrm{GL}_{m+n}(R), \quad (A, B) \mapsto \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$$

For $R = \mathbb{R}$ we may remember the topology on $\mathrm{GL}(\mathbb{R})$ and get the commutative $\mathcal{I}$-space monoids $B\mathrm{GL}(\mathbb{R})$ with $\mathcal{I}$-space submonids BO and BSO.

The model category structure on $\mathcal{I}$-spaces

- A map of $\mathcal{I}$-spaces $X \to Y$ is an $\mathcal{I}$-*equivalence* if the induced map of homotopy colimits $X_{h\mathcal{I}} \to Y_{h\mathcal{I}}$ is a weak homotopy equivalence.
  Here we use the Bousfield-Kan simplicial replacement of the diagram in order to define the homotopy colimit:

$$X_{h\mathcal{I}} = \operatorname*{hocolim}_{\mathcal{I}} X = \left| \coprod_{\mathbf{n}_0 \leftarrow \ldots \leftarrow \mathbf{n}_k} X(n_k) \right|$$

- A map of $\mathcal{I}$-spaces $X \to Y$ is an $\mathcal{I}$-*fibration* if it is a level fibration and if for each morphism $\alpha \colon \mathbf{m} \to \mathbf{n}$ in $\mathcal{I}$

$$\begin{array}{ccc} X(m) & \longrightarrow & Y(m) \\ \downarrow{\scriptstyle \alpha_*} & & \downarrow{\scriptstyle \alpha_*} \\ X(n) & \longrightarrow & Y(n) \end{array}$$

  is a homotopy cartesian square.

- A map of $\mathcal{I}$-spaces $X \to Y$ is an *$\mathcal{I}$-cofibration* if for each $\mathbf{n}$ the map

$$X(n) \cup_{L_n(X)} L_n(Y) \to Y(n)$$

is a projective Σ_n-cofibration. Here $L_n(X)$ denotes the nth *latching space*,

$$L_n(X) = \operatorname*{colim}_{\substack{\mathbf{m}\to\mathbf{n} \\ m<n}} X(m).$$

That a map of spaces is a projective Σ_n-cofibration means that it is a retract of a relative free Σ_n-cell complex.

Theorem (Sagave-S)

The $\mathcal{I}$-equivalences, $\mathcal{I}$-fibrations, and $\mathcal{I}$-cofibrations define a cofibrantly generated proper model category structure on $\mathcal{S}^{\mathcal{I}}$.

The fibrant objects are the "homotopy constant" $\mathcal{I}$-spaces.

This is the $\mathcal{I}$-space analogue of the stable model structure on Sp^{Σ}. We shall sometimes refer to it as the *absolute* model structure.

There are adjoint functors

$$\text{colim}\colon \mathcal{S}^{\mathcal{I}} \rightleftarrows \mathcal{S} : \text{const}$$

These are strong symmetric monoidal functors, hence preserve multiplicative structures.

Theorem (Sagave-S)

The functors colim *and* const *define a Quillen equivalence between* $\mathcal{S}^{\mathcal{I}}$ *and* $\mathcal{S}$. *In particular, the homotopy category of* $\mathcal{I}$*-spaces is equivalent to the usual homotopy category of spaces.*

It follows that an $\mathcal{I}$-space X is equivalent to the constant $\mathcal{I}$-space $X_{h\mathcal{I}}$. We think of $X_{h\mathcal{I}}$ as the underlying space associated with X.

Remark

If X is a commutative $\mathcal{I}$-space monoid, then $X_{h\mathcal{I}}$ has a canonical action of the Barratt-Eccles operad.

Example

Consider the commutative $\mathcal{I}$-space monoid $BO\colon \mathbf{n} \mapsto BO(n)$. In this case the connectivity of the maps $BO(m) \to BO(n)$ tends to infinity with m and n which implies that

$$BO_{h\mathcal{I}} \simeq \operatorname{colim}\{BO(0) \to BO(1) \to BO(2) \to \dots\} = BO(\infty)$$

Thus, we have "represented" the E_∞ space $BO(\infty)$ as a strictly commutative monoid in $\mathcal{S}^{\mathcal{I}}$.

Example

Let R be an ordinary ring and consider the commutative $\mathcal{I}$-space monoid $\mathbf{n} \mapsto B\mathrm{GL}_n(R)$. In this case the underlying space can be identified in terms of Quillen's plus construction:

$$B\mathrm{GL}(R)_{h\mathcal{I}} \simeq B\mathrm{GL}_\infty(R)^+.$$

Thus, we have "represented" the higher algebraic K-theory of R by a commutative monoid in $\mathcal{S}^{\mathcal{I}}$.

Example

Let X be a based space and consider the commutative $\mathcal{I}$-space monoid $X^\bullet\colon \mathbf{n} \mapsto X^n$, where $\alpha\colon \mathbf{m} \to \mathbf{n}$ induces the map

$$\alpha_*\colon X^m \xrightarrow{\mathrm{id}\times\{*\}} X^m \times X^{(n-\alpha)} \xrightarrow{\simeq} X^n.$$

In this case the underlying space can be identified in terms of the Barratt-Eccles construction

$$X^\bullet_{h\mathcal{I}} \simeq \Gamma^+(X) = \coprod_{n=0}^{\infty} E\Sigma_n \times_{\Sigma_n} X^n \Big/ \sim .$$

It follows from the Barratt-Priddy-Quillen Theorem that if X is connected then $X^\bullet_{h\mathcal{I}} \simeq \Omega^\infty\Sigma^\infty(X)$, and by the Dold-Thom Theorem that the canonical map

$$X_{h\mathcal{I}} = \operatorname*{hocolim}_{\mathcal{I}} X^\bullet \to \operatorname*{colim}_{\mathcal{I}} X^\bullet = SP^\infty(X)$$

induces the Hurewicz homomorphism on homotopy groups.

Example (Symmetric Thom spectra from $\mathcal{I}$-spaces)

Let $\mathcal{S}^{\mathcal{I}}/BO$ be the category of $\mathcal{I}$-spaces over the commutative $\mathcal{I}$-space monoid $BO\colon \mathbf{n} \mapsto BO(n)$. The Thom space functor has a refinement to a strong symmetric monoidal Thom spectrum functor

$$T\colon \mathcal{S}^{\mathcal{I}}/BO \to Sp^{\Sigma}.$$

In particular, T takes commutative $\mathcal{I}$-space monoids over BO to commutative $\mathbb{S}$-algebras.

For instance, the Thom spectra MO and MSO are obtained from the identity $BO \to BO$ and the inclusion $BSO \to BO$.

In order to realize the complex cobordism spectrum MU, notice that the inclusions

$$U(n) \subseteq O(\underbrace{\mathbf{2} \sqcup \cdots \sqcup \mathbf{2}}_{n})$$

make $\mathbf{n} \mapsto BU(n)$ a diagram on a certain subcategory of $\mathcal{I}$. The left Kan extension defines a commutative $\mathcal{I}$-space monoid over BO and the associated Thom spectrum represents MU as a commutative $\mathbb{S}$-algebra.

The model category of commutative $\mathcal{I}$-space monoids

We wish to define a model category of commutative $\mathcal{I}$-space monoids such that the weak equivalences are the $\mathcal{I}$-equivalences of the underlying $\mathcal{I}$-spaces.

In such a model structure the fibrant objects will in general not be homotopy constant: if this was the case, then $A_{h\mathcal{I}}$ would be equivalent to a strictly commutative monoid for any commutative $\mathcal{I}$-space monoid A. The commutative $\mathcal{I}$-space monoid BO is a counter example to this.

- A map of $\mathcal{I}$-spaces $X \to Y$ is a *positive $\mathcal{I}$-fibration* if it is a level fibration in positive degrees and if for any morphism $\alpha\colon \mathbf{m} \to \mathbf{n}$ with $m > 0$ the square

$$\begin{array}{ccc} X(m) & \longrightarrow & Y(m) \\ \downarrow{\scriptstyle \alpha_*} & & \downarrow{\scriptstyle \alpha_*} \\ X(n) & \longrightarrow & Y(n) \end{array}$$

  is homotopy cartesian.

Theorem (Sagave-S)

There is a cofibrantly generated model structure on the category of commutative $\mathcal{I}$-space monoids in which the weak equivalences are the $\mathcal{I}$-equivalences and the fibrations are the positive $\mathcal{I}$-fibrations.

We shall refer to this as the *positive model structure*.

In this model structure the cofibrations are retracts of relative cell complexes obtained by attaching free commutative cells generated by $\mathcal{I}$-spaces of the form $\mathcal{I}(d, -) \times D^n$ for $d > 0$.

Remark

The above theorem is the $\mathcal{I}$-space version of the analogous result for symmetric spectra due to Mandell-May-Schwede-Shipley.

E_∞ spaces and commutative $\mathcal{I}$-space monoids

In the following we fix an E_∞ operad (e.g., the Barratt-Eccles operad) and consider the associated category of E_∞ spaces.

Theorem (Sagave-S)

The category of E_∞ spaces is Quillen equivalent to the category of commutative $\mathcal{I}$-space monoids. In particular, these categories have equivalent homotopy categories.

In order to prove this one first observes that there is a natural notion of an E_∞ $\mathcal{I}$-space, defined using the $\boxtimes$-product instead of the cartesian product of spaces.

Furthermore, the category of E_∞ $\mathcal{I}$-spaces both has an absolute and a positive model structure with weak equivalences the $\mathcal{I}$-equivalences of the underlying $\mathcal{I}$-spaces.

We then have the following chain of Quillen equivalences:

$$
\begin{array}{c}
(E_\infty \text{ spaces, “projective” model structure}) \\
\text{colim} \uparrow \quad \downarrow \text{const} \\
(E_\infty\ \mathcal{I}\text{-spaces, absolute model structure}) \\
\mathrm{id} \uparrow \quad \downarrow \mathrm{id} \\
(E_\infty\ \mathcal{I}\text{-spaces, positive model structure}) \\
\text{“induction”} \downarrow \quad \uparrow \text{“restriction”} \\
(\text{Commutative } \mathcal{I}\text{-space monoids, positive model structure})
\end{array}
$$

Remark

The (“induction”, “restriction”) adjunction is the $\mathcal{I}$-space version of the analogues Quillen equivalence for symmetric spectra established by Elmendorf-Mandell.

Example

Consider for each $d \geq 0$ the $\mathcal{I}$-space $F_d(*) = \mathcal{I}(\mathbf{d}, -)$. The "induction" functor takes the E_∞ $\mathcal{I}$-space

$$\mathbb{E}(F_d(*)) = \coprod_{k \geq 0} (E\Sigma_k \times F_d(*)^{\boxtimes k})/\Sigma_k$$

to the commutative $\mathcal{I}$-space monoid

$$\mathbb{C}(F_d(*)) = \coprod_{n \geq 0} F_d(*)^{\boxtimes k}/\Sigma_k.$$

There is a Σ_k-equivariant isomorphism $F_d(*)^{\boxtimes k} = \mathcal{I}(\overbrace{\mathbf{d} \sqcup \cdots \sqcup \mathbf{d}}^{k}, -)$ which shows that the Σ_k-action is level free except for $d = 0$. It follows that the canonical map of E_∞ spaces

$$\mathbb{E}(F_d(*)) \to \mathbb{C}(F_d(*))$$

is an $\mathcal{I}$-equivalence for $d > 0$ but not for $d = 0$. This corresponds to the fact that $\mathbb{E}(F_d(*))$ is positively cofibrant if and only if $d > 0$.

Units of $\mathbb{S}$-algebras

An ordinary commutative ring R has an underlying commutative monoid $(R, \cdot)$.
An ordinary commutative monoid M has an associated commutative monoid ring $\mathbb{Z}[M]$.
These are adjoint functors

$$\textbf{Comm monoids} \underset{(-,\cdot)}{\overset{\mathbb{Z}[-]}{\rightleftarrows}} \textbf{Comm rings}$$

There are adjoint functors

$$\textbf{Comm groups} \underset{(-)^*}{\overset{\text{forgetful}}{\rightleftarrows}} \textbf{Comm monoids}$$

where M^* is the submonoid of invertible elements. The units of the commutative ring R is the commutative group

$$\mathrm{GL}_1(R) = (R, 1)^*.$$

A commutative $\mathbb{S}$-algebra R has an "underlying" commutative $\mathcal{I}$-space monoid $\Omega^{\mathcal{I}}(R)$:

$$\Omega^{\mathcal{I}}(R)(n) = \Omega^n(R(n)) \quad \big(= \mathrm{Map}_*(S^n, R(n))\big)$$

where $\alpha\colon \mathbf{m} \to \mathbf{n}$ in $\mathcal{I}$ induces $\alpha_*\colon \Omega^{\mathcal{I}}(R)(m) \to \Omega^{\mathcal{I}}(R)(n)$,

$$\alpha_*(f)\colon S^n \simeq S^{n-\alpha} \wedge S^m \xrightarrow{\mathrm{id}\wedge f} S^{n-\alpha} \wedge R(m) \to R(n).$$

A commutative $\mathcal{I}$-space monoid M gives rise to a commutative $\mathbb{S}$-algebra $\mathbb{S}^{\mathcal{I}}[M]$:

$$\mathbb{S}^{\mathcal{I}}[M](n) = S^n \wedge M(n)_+$$

These are adjoint functors (in fact a Quillen adjunction)

$$\textbf{Comm } \mathcal{I}\textbf{-space monoids} \underset{\Omega^{\mathcal{I}}}{\overset{\mathbb{S}^{\mathcal{I}}[-]}{\rightleftarrows}} \textbf{Comm } \mathbb{S}\textbf{-algebras}$$

Definition

A commutative $\mathcal{I}$-space monoid M is *grouplike* if the topological monoid $M_{h\mathcal{I}}$ is grouplike (that is, $\pi_0 M_{h\mathcal{I}}$ is a group).

There are adjoint functors

$$\textbf{Grouplike comm } \mathcal{I}\textbf{-space monoids} \underset{(-)^*}{\overset{\text{forgetful}}{\rightleftarrows}} \textbf{Comm } \mathcal{I}\textbf{-space monoids}$$

where $M^*(n) \subseteq M(n)$ is the subspace of components that map into the homotopy invertible elements in $M_{h\mathcal{I}}$.

Definition

The units of a commutative $\mathbb{S}$-algebra R is the commutative $\mathcal{I}$-space monoid $\mathrm{GL}_1(R) = \Omega^{\mathcal{I}}(R)^*$.

Remark

If R is fibrant (or semistable), then $\mathrm{GL}_1(R)_{h\mathcal{I}} \simeq (\Omega^{\mathcal{I}}(R)_{h\mathcal{I}})^*$ and

$$\pi_0 \mathrm{GL}_1(R) = \pi_0 \mathrm{GL}_1(R)_{h\mathcal{I}} = \mathrm{GL}_1(\pi_0(R)).$$

Remark

There are adjoint functors

$$\mathcal{I}\textbf{-spaces} \underset{\Omega^{\mathcal{I}}}{\overset{\mathbb{S}^{\mathcal{I}}[-]}{\rightleftarrows}} \mathbb{S}\textbf{-modules}$$

which restrict to the above adjunctions. The main point of the above is that using the factorization

$$\Omega^\infty\colon\ \mathbb{S}\textbf{-modules} \xrightarrow{\Omega^{\mathcal{I}}} \mathcal{I}\textbf{-spaces} \to \textbf{Spaces}$$

we can "represent" $\Omega^\infty(R)$ as the strictly commutative object $\Omega^{\mathcal{I}}(R)$ if R is a commutative $\mathbb{S}$-algebra.