Brave new rings from K3 surfaces

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Brave new rings...

...in moduli contexts

The brave new algebra program

Generalize from commutative rings to E_∞ ring spectra whenever possible.

This is possible in many different contexts.

The context for this talk is: structure sheaves (of commutative rings) on moduli spaces of objects related to formal groups.

Examples

Lubin-Tate moduli of formal groups (Hopkins, Miller, Goerss, Rezk)
Moduli of elliptic curves (Hopkins, Miller, Goerss, Rezk, Lurie)
Moduli of abelian varieties (Ravenel, Behrens, Lawson)

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Observation (Artin, Mazur)

K3 surfaces have an associated formal Brauer group.

Question

Can the structure sheaf of some moduli stack of K3 surfaces be refined to a sheaf of brave new rings?

Answer (for today)

Yes (locally and generically)

Mindset

K3 surfaces are to be studied in the light of their crystals.

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“Unfortunately, it appears that there is now in your world a race of vampires called referees, who clamp down mercilessly upon mathematicians unless they know the right passwords. I shall do my best to modernize my language and notations, but I am well aware of my short-comings in that respect; I can assure you, at any rate, that my intentions are honourable and my results invariant, probably canonical, perhaps even functorial. But please allow me to assume that the characteristic is not 2.”

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A. Weil

K3 surfaces

Definitions

Let k be an algebraically closed field of characteristic $p > 2$.

Definition

A K3 surface over k is a smooth projective surface X such that the canonical bundle Ω^2_X is trivial, and the surface X is not abelian.

Definition

A polarization is an ample line bundle L on X which is not a p-th power.

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The Fermat quartic defined by

$$T_1^4 + T_2^4 + T_3^4 + T_4^4$$

is a K3 surface, and so is more generally any smooth quartic inside $\mathbb{P}_k^3$.

Example

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Local moduli: deformations

Let W be the ring of p-typical Witt vectors of k.

Theorem (Deligne, Illusie)

The formal deformation space S of a K3 surface X is formally smooth over W of dimension 20, so that there is a (non-canonical) isomorphism

$$S \cong \hat{\mathbb{A}}^{20}_{W},$$

and there is a universal formal deformation $\mathcal{X}$ over S.

Theorem (Deligne, Illusie)

Let L be a polarization on X. The formal deformation space of (X, L) is representable by a closed formal subscheme $S_L \subset S$, defined by a single equation. It is flat over W of relative dimension 19.

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K3 spectra

Definition

A K3 spectrum is a triple (E, X, ι), where
E is an even periodic ring spectrum with associated formal group Γ_E,
X is a K3 surface over $\pi_0 E$ with associated formal Brauer group $\hat{\mathrm{Br}}_X$, and
$\iota \colon \Gamma_E \cong \hat{\mathrm{Br}}_X$ is an isomorphism.

In this definition the term 'ring spectrum' is understood in the weak 'up to homotopy' sense.

Example

If (X, L) is a polarized K3 surface of finite height, then there is an even periodic ring spectrum E such that $\pi_0 E \cong \mathcal{O}(S_L)$ and $\Gamma_E \cong \hat{\mathrm{Br}}_{\mathcal{X}}$.

This follows from Landweber exactness and properties of the height stratification of the moduli space (van der Geer, Katsura, Ogus).

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The crystalline perspective

Crystals associated with K3 surfaces

Let $\mathcal{X}/S$ be a universal formal deformation of a K3 surface X.

The crystal (H, ∇)

The de Rham cohomology $H = \mathrm{H}^2_{\mathrm{dR}}(\mathcal{X}/S)$ supports the Gauss-Manin connection $\nabla = \nabla_{\mathrm{GM}}$.

The F-cyrstal $(H, \nabla, F_\bullet)$

If ϕ is a lift of Frobenius to S which is compatible with the canonical lift of Frobenius to W, there is an induced ϕ-linear map $F_\phi \colon H \to H$.

The Hodge F-crystal $(H, \nabla, F_\bullet, F^\bullet)$

The Hodge filtration $H = F^0 \supset F^1 \supset F^2 \supset F^3 = 0$ lifts the Hodge filtration on the reduction.

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Ordinary Hodge F-crystals

A Hodge F-crystal has two polygons associated with it:

Geometry

The Hodge polygon encodes the Hodge numbers derived from the filtration.

Arithmetic

The Newton polygon encodes the multiplicities and valuations of the eigenvalues of Frobenius.

Definition

A Hodge F-crystal is ordinary if its Hodge and Newton polygons agree.

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Ordinary K3 surfaces

Definition

A K3 surface X is ordinary if its Hodge F-crystal is ordinary.

Example (the Newton/Hodge polygon of an ordinary K3 surface)

Characterization

A K3 surface X is ordinary if its formal Brauer group is multiplicative.

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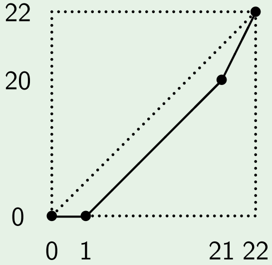

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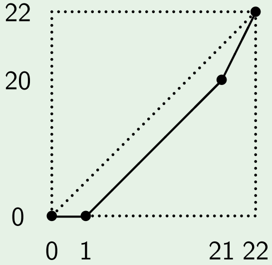

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Canonical coordinates

Let $\mathcal{X}/S$ be a universal formal deformation of an ordinary K3 surface X.

Theorem (Deligne, Illusie)

There is a basis $(a, b_1, \ldots, b_{20}, c)$ for the associated crystal H as well as coordinates $t_1, \ldots, t_{20}$ on S such that there are explicit formulas for the Gauss-Manin connection and the action of Frobenius.

The formulas use multiplicative notation $q_j = t_j + 1$, and $\omega_j = \mathrm{dlog}(q_j)$. Then (ω_j) is a W-basis of $\Omega_{S/W}$.

Definition

The Katz lift ψ_{can} of Frobenius on S is given by $\psi_{\mathrm{can}}(q_j) = q_j^p$.

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The Katz lift ψ_{can} on S maps S_L into itself.

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Trivialized K3 surfaces

Definition

A trivialised K3 surface is a K3 surface together with a chosen canonical coordinate a.

The group $\mathbb{Z}_p^\times$ acts on the situation by changing the choice of a. This gives rise to a Galois cover T_L of S_L.

$$\begin{array}{ccc} T_L & \longrightarrow & \mathcal{M}_{\mathrm{K3}}^{\mathrm{triv}} \\ \downarrow & & \downarrow \\ S_L & \longrightarrow & \mathcal{M}_{\mathrm{K3}}^{\mathrm{ord}} \end{array}$$

The Katz lift ψ_{can} extends to T_L.

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Brave new K3 spectra

K(1)-local E_∞ ring spectra

From now on: every spectrum is localized at K(1).

Observation

If E is an E_∞ ring spectrum, there are maps ψ^p, θ on $\pi_0 E$ such that

$$\psi^p(f) = f^p + p\theta(f),$$

so that ψ^p is a lift of Frobenius, and θ is the error term.

Definition

A ring A with operations ψ^p, θ as above is called a θ-algebra.

Characterization

A θ-algebra is a ring with a section $s = (\mathrm{id}, \theta) \colon A \to W_2(A)$ of $w_0 \colon W_2(A) \to A$: define $\psi^p = w_1 s$.

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An obstruction theory

Summary

If E is an E_∞ ring spectrum, $\mathrm{K}_0 E = \pi_0(\mathrm{K} \wedge E)$ is a θ-algebra with Adams operations.

Theorem (Goerss, Hopkins)

There is an obstruction theory for E_∞ structures and spaces of E_∞ maps between $\mathrm{K}(1)$-local ring spectra.

Thus, E_∞ ring spectra E may be shown to exist if one guesses the θ-algebra $\mathrm{K}_* E$ with its Adams operations, and E_∞ equivalences of such E may be shown to exist if one guesses their effect on $\mathrm{K}_* E$.

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Theorem

For each ordinary polarized K3 surface (X, L), there is an even periodic K(1)-local E_∞ ring spectrum $E(X, L)$ such that

$$\mathrm{K}_0 E(X, L) \cong \mathcal{O}(T_L), \quad \pi_0 E(X, L) \cong \mathcal{O}(S_L).$$

The E_∞ structure is unique up to equivalence.

In fact, $\mathrm{K}_{2n} E(X, L) \cong \mathrm{H}^0(T_L, \omega^{\otimes n})$ and $\pi_{2n} E(X, L) \cong \mathrm{H}^0(S_L, \omega^{\otimes n})$.

Proof

The relevant obstruction groups vanish.

Note: There is no claim that the equivalence is unique.

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$$\mathrm{K}_0 E(X, L) \cong \mathcal{O}(T_L), \quad \pi_0 E(X, L) \cong \mathcal{O}(S_L).$$

The E_∞ structure is unique up to equivalence.

In fact, $\mathrm{K}_{2n} E(X, L) \cong \mathrm{H}^0(T_L, \omega^{\otimes n})$ and $\pi_{2n} E(X, L) \cong \mathrm{H}^0(S_L, \omega^{\otimes n})$.

Proof

The relevant obstruction groups vanish.

Note: There is no claim that the equivalence is unique.

Question

The finite group $\mathrm{Aut}(X, L)$ acts on S_L. Does it act on $E(X, L)$?

Theorem

For each ordinary polarized K3 surface (X, L), there is a unique homotopy action of its automorphism group through E_∞ maps on $E(X, L)$.

Proof

The relevant obstruction groups vanish, so that the Hurewicz map

$$\pi_0\mathcal{E}_\infty(E(X, L), E(X, L)) \to \mathrm{Hom}_{\theta\,\mathrm{Alg}\,/\,\mathrm{K}_*}(\mathrm{K}_*E(X, L), \mathrm{K}_*E(X, L))$$

is bijective.

Note: There is no claim that the components are contractible.

Brave new K3 spectra

E_∞ maps

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Brave new K3 spectra
Rigidification

Question
Can the homotopy action of $\mathrm{Aut}(X, L)$ be rigidified to a topological action?

Definition
An automorphism in characteristic p is wild if p divides its order. If an automorphism group does not contain a wild automorphism it is tame.

Theorem
If the automorphism group of an ordinary polarized K3 surface (X, L) is tame, it acts through E_∞ maps on $E(X, L)$.

Proof
This uses another obstruction theory (Cooke, Dwyer, Kan).
It shows that the space of actions is contractible.

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Examples

Theorem (Dolgachev, Keum)

If $p > 11$, the automorphism group of (X, L) is tame.

Corollary

If $p > 11$, there is an E_∞ action of $\mathrm{Aut}(X, L)$ on $E(X, L)$.

Example

The Fermat quartic is ordinary if and only if $p \equiv 1$ modulo 4.
For $p \neq 3$, the automorphism group of (X, L) is isomorphic to

$$((\mathbb{Z}/4)^4/\Delta) \rtimes \Sigma_4.$$

Therefore, there is always an E_∞ action of $\mathrm{Aut}(X, L)$ on $E(X, L)$ if the Fermat surface is ordinary.

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