

TWISTED CYCLIC THEORY, EQUIVARIANT KK THEORY AND KMS STATES

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Dedicated to the memory of Gerard Murphy

Abstract

Given a C^* -algebra A with a KMS weight for a circle action, we construct and compute a secondary invariant on the equivariant K -theory of the mapping cone of $A^{\mathbb{T}} \hookrightarrow A$, both in terms of equivariant KK -theory and in terms of a semifinite spectral flow. This in particular puts the previously considered examples of Cuntz algebras [9] and $SU_q(2)$ [13] in a general framework. As a new example we consider the Araki-Woods III_λ representations of the Fermion algebra.

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1. INTRODUCTION

The main subject of this paper is a study of the cohomological information contained in a non-tracial weight ϕ on a C^* -algebra A . As the replacement to the tracial property we will assume that ϕ has the KMS_β property with respect to a one-parameter group of automorphisms σ of A and, for reasons to be apparent later, that $\sigma : \mathbb{R} \rightarrow \text{Aut}(A)$ factorizes through a circle action $\sigma : \mathbb{T} \rightarrow \text{Aut}(A)$. Since ϕ is not a trace, it only has well-defined pairing with the K -theory of the fixed point algebra A^σ . On the other hand,

$$\psi = (b + B)\phi$$

is a cocycle on the quotient complex $(CC_{ev}(A)/CC_{ev}(A^\sigma), b + B)$, hence can be expected to give a pairing with the K -theory of the mapping cone M of the inclusion $A^\sigma \hookrightarrow A$.

This is made precise in this paper. In fact, we get two pictures of this pairing.

The KK-theoretic picture

The action of σ induces a natural element $[\hat{\mathcal{D}}]$ of $KK_0^\mathbb{T}(M, A^\sigma)$ and the Kasparov product gives us the homomorphism

$$[\hat{\mathcal{D}}] \cap : K_0^\mathbb{T}(M) \rightarrow K_0^\mathbb{T}(A^\sigma) = K_0(A^\sigma)[\chi, \chi^{-1}],$$

where χ is the fundamental character of \mathbb{T} .

The analytic spectral flow picture

This is based on identifying a natural II_∞ -algebra \mathcal{N} with a semifinite normal trace Tr_ϕ , and studying the spectral flow of a path of operators of the form

$$[0, 1] \ni t \rightarrow \mathcal{D} + a_t$$

where \mathcal{D} is the generator of the action of \mathbb{T} and a_t is a path of bounded perturbations associated to a class in $K_0^\mathbb{T}(M)$. The KMS-property of ϕ on A suggests a natural choice for \mathcal{N} and the associated trace Tr_ϕ . However, as the examples show, this leads to a non-summable (even non- θ -summable) situation, hence the existing technology for computing semifinite spectral flow does not work. This forces on us a choice of a *renormalisation procedure* - once performed, we get an explicit functional on the equivariant K -theory of the mapping cone M with values in \mathbb{R} .

The constructions given in this paper are inspired by the computations done before (compare [9, 13]), which produced examples of pairings of K -theoretic type with non-tracial states.

For the convenience of the reader, we will summarize the main results.

1.1. The KK-theoretic results. Our basic data consists of a C^* -algebra A , together with a strongly continuous action of the circle \mathbb{T} by $*$ -automorphisms $\sigma: \mathbb{T} \rightarrow \text{Aut}(A)$. Throughout the paper we will make an additional assumption on σ (see Definition 2.2), which is slightly weaker than assuming that the spectral subspaces of σ are full, but is general enough to cover all interesting examples we are aware of (in fact, most of our analytic results hold in full generality, but the proofs become technically more complicated and not very illuminating).

We let F denote the fixed point subalgebra A^σ of A , and Φ the canonical conditional expectation $A \rightarrow F$. Consider the left Hilbert C^* -module $X = L^2(A, \Phi)$ over F . Let π denote the representation of A on X given by the left multiplication and let \mathcal{D} denote the generator of the action of \mathbb{T} on X induced by σ . We prove that (π, X, \mathcal{D}) defines an element $[\mathcal{D}]$ of $KK_1^\mathbb{T}(A, F)$ (cf. Proposition 2.9).

Let $M = M(F, A)$ denote the mapping cone of the inclusion $F \subset A$. The class $[\mathcal{D}]$ has a canonical lifting to a class $[\hat{\mathcal{D}}] \in KK_1^\mathbb{T}(M, F)$ which is the main KK -theoretic object of this paper (cf. Section 2.2), and the pairing with the equivariant theory of the mapping cone is given by the Kasparov product

$$K_0^\mathbb{T}(M) \times KK_0^\mathbb{T}(M, F) \rightarrow K_0^\mathbb{T}(F).$$

The group $K_0^\mathbb{T}(M)$ can be described as follows. Every class in $K_0^\mathbb{T}(M)$ has a representative given by an isometry v in $(A^\sim \otimes \mathcal{B}(\mathcal{H}))^\mathbb{T}$, where \mathcal{H} is the space of a finite dimensional unitary representation of \mathbb{T} , such that vv^* and v^*v are in $F^\sim \otimes \mathcal{B}(\mathcal{H})$, and $vv^* = v^*v$ modulo $F \otimes \mathcal{B}(\mathcal{H})$. As usual, the superscript \sim denotes unitization. We will denote by $[v]$ the corresponding class in $K_0^\mathbb{T}(M)$. Let $P = \chi_{[0, \infty)}(\mathcal{D})$.

In this notation, the Kasparov product

$$[\hat{\mathcal{D}}] \cap : K_0^{\mathbb{T}}(M) \rightarrow K_0^{\mathbb{T}}(F),$$

which we denote by $\text{Index}_{\hat{\mathcal{D}}}$, is described by the following result.

Theorem 1.1 (cf. Theorem 2.10). *Let $[v]$ be a class in $K_0^{\mathbb{T}}(M)$ represented by a partial isometry v in $(A^{\sim} \otimes B(\mathcal{H}))^{\mathbb{T}}$ as above. Then*

$$\text{Index}_{\hat{\mathcal{D}}}(v) = -\text{Index}((P \otimes 1)v(P \otimes 1) : v^*v(P\mathcal{H}_F \otimes \mathcal{H}) \rightarrow vv^*(P\mathcal{H}_F \otimes \mathcal{H})) \in K_0^{\mathbb{T}}(F).$$

1.2. The analytic results - semifinite spectral flow. We assume that ϕ is a faithful positive, semifinite weight on A satisfying the KMS_{β} condition with respect to the action of \mathbb{R} induced by σ .

Let $\mathcal{H}_{\phi} = L^2(A, \phi)$, $\pi_{\phi} : A \rightarrow \mathcal{B}(\mathcal{H}_{\phi})$ denote the GNS-representation and \mathcal{N} be the commutant of $J_{\phi}\pi_{\phi}(F)J_{\phi}$ in $\mathcal{B}(\mathcal{H}_{\phi})$. Then \mathcal{N} is a semifinite von Neumann algebra and $A \simeq \pi_{\phi}(A) \subset \mathcal{N}$ with a positive, faithful, semifinite trace Tr_{ϕ} (see Section 3.2).

1.2.1. Renormalization. $(A, \pi_{\phi}, \mathcal{H}_{\phi}, \mathcal{D})$ is a semifinite spectral triple with respect to $(\mathcal{N}, \text{Tr}_{\phi})$, and hence it is tempting to apply the spectral flow formulas developed in this context to obtain some numerical invariants of the equivariant spectral flow. However, as the examples show, $(1 + \mathcal{D}^2)^{-\frac{1}{2}}$ is almost never finitely summable with respect to Tr_{ϕ} . To circumvent this problem, we will use the methods of [9]. We replace the pair (\mathcal{N}, ϕ) by the semifinite von Neumann algebra $\mathcal{M} = \mathcal{N}_{\phi}$ and the trace Tr_{ϕ} by $\phi_{\mathcal{D}} \equiv \text{Tr}_{\phi}(e^{-\beta\mathcal{D}/2} \cdot e^{-\beta\mathcal{D}/2})$, which is a trace on \mathcal{M} (see Section 3.2).

In fact, the natural construction turns out to work the other way round. The weight $\phi_{\mathcal{D}}$ is the restriction of the dual weight $\hat{\phi}$ of ϕ on the crossed product $A \rtimes_{\sigma} \mathbb{T}$, and Tr_{ϕ} can be easily constructed from $\hat{\phi}$.

1.2.2. Modular index pairing. The operator \mathcal{D} is $(1, \infty)$ -summable with respect to $\phi_{\mathcal{D}}$, and for a unitary operator u such that $u[\mathcal{D}, u^*]$ is bounded and belongs to \mathcal{M} we can compute the spectral flow along the linear path from \mathcal{D} to $u\mathcal{D}u^*$.

We formalize this situation in Section 4, where we construct the group $K_1(A, \sigma)$ (see Definition 4.2) of homotopy classes of partial isometries v in A^{\sim} (more precisely, in matrix algebras over A^{\sim}) which satisfy

$$\sigma_t(v)v^*, \sigma_t(v^*)v \in \mathcal{M} \quad \forall t \in \mathbb{R}.$$

We will call such v modular. A modular isometry has the form $v = \sum_k v_k$, where the v_k 's are homogeneous partial isometries and only finitely many of them are non-zero.

Given such a modular isometry, we can associate to it a class $\llbracket v \rrbracket \in K_0^{\mathbb{T}}(M)$, and the analytic formula computes the semifinite spectral flow of the pair $(\mathcal{D}, v\mathcal{D}v^*)$ with respect to $(\mathcal{M}, \phi_{\mathcal{D}})$.

Theorem 1.2 (The residue formula, cf. Theorem 5.6). *Let $v \in A$ be a modular partial isometry. Then $\text{sf}_{\phi_{\mathcal{D}}}(vv^*\mathcal{D}, v\mathcal{D}v^*)$ is given by*

$$\text{Res}_{r=1/2} \left(r \mapsto \phi_{\mathcal{D}}(v[\mathcal{D}, v^*](1 + \mathcal{D}^2)^{-r}) + \frac{1}{2} \int_1^{\infty} \phi_{\mathcal{D}}((\sigma_{-i\beta}(v^*)v - vv^*)\mathcal{D}(1 + s\mathcal{D}^2)^{-r})s^{-1/2} ds \right).$$

The spectral flow defines a map $K_1(A, \sigma) \rightarrow \mathbb{R}$. Using our KK -theoretic constructions, it can be described as follows.

Theorem 1.3 (cf. Theorem 4.10). *Let v be a modular partial isometry. The $\phi_{\mathcal{D}}$ -spectral flow $sf_{\phi_{\mathcal{D}}}(vv^*\mathcal{D}, v\mathcal{D}v^*)$ along the linear path joining $vv^*\mathcal{D}$ to $v\mathcal{D}v^*$ is given by the composition:*

$$K_1(A, \sigma) \xrightarrow{\ll \gg} K_0^{\mathbb{T}}(M) \xrightarrow{-\text{Index}_{\hat{\mathcal{D}}}} K_0(F)[\chi, \chi^{-1}] \xrightarrow{\phi_*} \mathbb{R}[\chi, \chi^{-1}] \xrightarrow{\text{Ev}(e^{-\beta})} \mathbb{R},$$

Here ϕ_* denotes the map defined by the trace $\phi|_F$ and $\text{Ev}(e^{-\beta})$ is the evaluation at $\chi = e^{-\beta}$.

1.3. Twisted cyclic cohomology. The above spectral flow formula turns out to be related to twisted cyclic cohomology as follows (see Section 5.3). Denote by \mathcal{A} the algebra spanned by the homogeneous elements in the domain of ϕ .

Theorem 1.4 (cf. Theorem 5.9). *The bilinear functional on \mathcal{A} given by*

$$\psi^r(a_0, a_1) = \phi_{\mathcal{D}}(a_0[\mathcal{D}, a_1](1 + \mathcal{D}^2)^{-r}) + \frac{1}{2} \int_1^{\infty} \phi_{\mathcal{D}}((\sigma_{-i\beta}(a_1)a_0 - a_0a_1)\mathcal{D}(1 + s\mathcal{D}^2)^{-r})s^{-1/2}ds$$

depends holomorphically on r for $\Re(r) > 1/2$ and modulo functions which are holomorphic for $\Re(r) > 0$ is a function valued $\sigma_{-i\beta}$ -twisted (b, B) -cocycle.

Therefore if the residue at $r = 1/2$ exists, we get a twisted cyclic cocycle. This is the case for example when the action has full spectral subspaces and the cocycle is just $(a_0, a_1) \mapsto \phi(a_0[\mathcal{D}, a_1])$.

The appearance of twisted cyclic cohomology can be explained as follows. We would like to compute the spectral flow from \mathcal{D} to $u\mathcal{D}u^*$ with respect to Tr_{ϕ} . To do this, we would like to define a \mathbb{T} -equivariant Chern character Ch_g , $g \in \mathbb{T}$. However, this is not possible because of summability issues, and what we are in fact able to define is a ‘‘continuation’’ of this hypothetical Chern character to $g = e^{-\beta} \in \mathbb{T}_{\mathbb{C}} = \mathbb{C}^*$. The evaluation of an equivariant cyclic cocycle at a group-like element different from 1 is exactly what produces twisted cyclic cocycles, compare with [27].

2. EQUIVARIANT KK CLASS ASSOCIATED TO A CIRCLE ACTION

2.1. A Kasparov module from a circle action. Let A be a C^* -algebra, $\sigma: \mathbb{T} \rightarrow \text{Aut}(A)$ a strongly continuous action of the circle. It will be convenient to consider σ as a 2π -periodic one-parameter group of automorphisms. We denote by F the fixed point algebra $\{a \in A \mid \sigma_t(a) = a \ \forall t \in \mathbb{R}\}$. Since \mathbb{T} is a compact group, the map

$$\Phi: A \rightarrow F, \quad \Phi(a) = \frac{1}{2\pi} \int_0^{2\pi} \sigma_t(a) dt$$

is a faithful conditional expectation. Next define an F -valued inner product on A by $(a|b)_R := \Phi(a^*b)$. The properties of Φ allow us to see that this is a (pre)- C^* -inner product on A , and so we may complete A in the topology determined by the norm $\|a\|_X^2 = \|(a|a)_R\|_F$ to obtain a C^* -module, for the right action of F .

Definition 2.1. *We let $X = \overline{A}$ be the C^* -module completion of A with inner product $(\cdot|\cdot)_R$.*

We remark that the general theory of C^* -modules (or Hilbert modules) is discussed in many places and we will use [26, 33]. For a right C^* - B -module Y , we let $\mathcal{L}(Y)$ be the C^* -algebra (for the operator norm) of adjointable endomorphisms, $\mathcal{K}(Y)$ the closed ideal of compact endomorphisms, which is the completion of the ideal of finite rank endomorphisms. The latter is generated by the rank one endomorphisms $\Theta_{x,y}$, $x, y \in Y$, defined by $\Theta_{x,y}z = x(y|z)_R$, $z \in Y$.

The circle action is defined on the dense subspace $A \subset X$ and extends to a unitary action on X , and hence defines a circle action on $\mathcal{L}(X)$, which we continue to denote by σ . The F -module X is a full F -module for the right inner product. For $k \in \mathbb{Z}$, denote the eigenspaces of the action σ by

$$A_k = \{a \in A : \sigma_t(a) = e^{ikt}a \text{ for all } t \in \mathbb{R}\}.$$

Then $F = A_0$, which guarantees the fullness of X over F . Also, A is a \mathbb{Z} -graded algebra in an obvious way, $A_{-k} = (A_k)^*$ and, in particular, each A_k is an F -module. Furthermore, $A_0 A_k$ is dense in A_k , and hence $\overline{F A} = A$ and similarly $\overline{A F} = A$ (this also follows from the existence of a σ -invariant approximate unit in A). Note also that the norm on A_k defined by the above inner product coincides with the C^* -norm. We denote by X_k the space A_k considered as a closed submodule of X . For $k \in \mathbb{Z}$ we set $F_k = \overline{A_k A_k^*}$.

For each $k \in \mathbb{Z}$, the projection onto the k -th spectral subspace for the circle action is defined by an operator Φ_k on X via

$$\Phi_k(x) = \frac{1}{2\pi} \int_0^{2\pi} e^{-ikt} \sigma_t(x) dt, \quad x \in X.$$

The range of Φ_k is the submodule X_k . These ranges give us the natural \mathbb{Z} -grading of X . The operators Φ_k are adjointable endomorphisms of the F -module X such that $\Phi_k^* = \Phi_k = \Phi_k^2$ and $\Phi_k \Phi_l = \delta_{k,l} \Phi_k$. If $K \subset \mathbb{Z}$ then the sum $\sum_{k \in K} \Phi_k$ converges strictly to a projection in the endomorphism algebra, [28]. In particular, the sum $\sum_{k \in \mathbb{Z}} \Phi_k$ converges strictly to the identity operator on X .

Definition 2.2. *The action σ on A satisfies the **Spectral Subspace Assumption (SSA)** if F_k is a complemented ideal in F for every $k \in \mathbb{Z}$. Equivalently, the representation $\pi_k: F \rightarrow \text{End}_F(X_k)$ given by left multiplication satisfies $\pi_k(F) = \pi_k(F_k)$ (then $\ker \pi_k$ is the complementary ideal to F_k).*

There is a special case of this assumption which is well known, namely A is said to have **full spectral subspaces** if $F_k = F$ for all $k \in \mathbb{Z}$. The gauge action on the Cuntz algebras \mathcal{O}_n provides examples where fullness holds. The quantum group $SU_q(2)$ with its Haar state and associated circle action is an example of an algebra satisfying the SSA but not having full spectral subspaces [13].

Lemma 2.3. *If $\overline{A_1 A_1^*} = \overline{A_1^* A_1} = A_0$, the modules X_k and $\overline{X_k}$ are full for all $k \in \mathbb{Z}$.*

Proof. Observe that as $A_1 A_1^* \subset A_0$, we have $A_1 = \overline{A_0 A_1}$. So if $A_0 = \overline{A_1^* A_1}$, by induction we get $A_0 = \overline{(A_1^k)^* A_1^k}$ for $k \geq 1$. Since $A_1^k \subset A_k$, we conclude that X_k is full. Similarly, if $k \leq -1$ then $(A_1^*)^{-k} \subset A_k$, so $A_0 = \overline{A_1 A_1^*}$ implies that X_k is full. \square

The following Lemma is the key step in obtaining a Kasparov module.

Lemma 2.4. *For a circle action on A the following conditions are equivalent:*

- (i) *the action satisfies the SSA;*
- (ii) *for all $a \in A$ and $k \in \mathbb{Z}$, the endomorphism $a\Phi_k$ of the right F -module X is compact.*

Proof. Assume the action satisfies the SSA. If $x, y \in A_k$ and $z \in X$, then

$$\Theta_{x,y} z = x\Phi(y^* z) = x\Phi(y^* z_k) = xy^* z_k = xy^* \Phi_k z.$$

Thus $\Theta_{x,y} = xy^* \Phi_k$. It follows that $a\Phi_k$ is compact for any $a \in A_k A_k^*$. Since $A_k A_k^*$ is dense in F_k , we see that $f\Phi_k$ is compact for any $f \in F_k$, and hence $f\Phi_k$ is compact for any $f \in F$ by the SSA. But then $af\Phi_k$ is compact for any $f \in F$ and $a \in A$. Since AF is dense in A , we can approximate $b\Phi_k$ for any $b \in A$ by (compact) endomorphisms of the form $af\Phi_k$.

Conversely, assume $f\Phi_k$ is compact for some $f \in F$, so that $f\Phi_k$ can be approximated by finite sums of operators $\Theta_{x,y}$, $x, y \in X_k$. We have seen, however, that $\Theta_{x,y} = xy^*\Phi_k$, and so $\pi_k(f)$ is in $\pi_k(F_k)$. Therefore if $f\Phi_k$ is compact for all $f \in F$ and $k \in \mathbb{Z}$, the SSA is satisfied. \square

From now on we will assume that the Spectral Subspace Assumption is satisfied, even though some of our results hold in full generality.

Since we have the circle action defined on X , we may use the generator of this action to define an unbounded operator \mathcal{D} . We will not define or study \mathcal{D} from the generator point of view, instead taking a more bare-hands approach. It is easy to check that \mathcal{D} as defined below is the generator of the circle action. The theory of unbounded operators on C^* -modules that we require is all contained in Lance's book, [26, Chapters 9,10]. We quote the following definitions (adapted to our situation).

Definition 2.5. [26] *Let Y be a right C^* - B -module. A densely defined unbounded operator*

$$\mathcal{D}: \text{dom } \mathcal{D} \subset Y \rightarrow Y$$

is a B -linear operator defined on a dense B -submodule $\text{dom } \mathcal{D} \subset Y$. The operator \mathcal{D} is closed if the graph $G(\mathcal{D}) = \{(x, \mathcal{D}x) : x \in \text{dom } \mathcal{D}\}$ is a closed submodule of $Y \oplus Y$.

If $\mathcal{D}: \text{dom } \mathcal{D} \subset Y \rightarrow Y$ is densely defined and unbounded, define a submodule

$$\text{dom } \mathcal{D}^* := \{y \in Y : \exists z \in Y \text{ such that } \forall x \in \text{dom } \mathcal{D}, (\mathcal{D}x|y)_R = (x|z)_R\}.$$

Then for $y \in \text{dom } \mathcal{D}^*$ define $\mathcal{D}^*y = z$. Given $y \in \text{dom } \mathcal{D}^*$, the element z is unique, so $\mathcal{D}^*: \text{dom } \mathcal{D}^* \rightarrow Y$, $\mathcal{D}^*y = z$ is well-defined, and moreover is closed.

Definition 2.6. [26] *Let Y be a right C^* - B -module. A densely defined unbounded operator \mathcal{D} is symmetric if for all $x, y \in \text{dom } \mathcal{D}$ we have $(\mathcal{D}x|y)_R = (x|\mathcal{D}y)_R$. A symmetric operator \mathcal{D} is self-adjoint if $\text{dom } \mathcal{D} = \text{dom } \mathcal{D}^*$ (and so \mathcal{D} is necessarily closed). A densely defined unbounded operator \mathcal{D} is regular if \mathcal{D} is closed, \mathcal{D}^* is densely defined, and $1 + \mathcal{D}^*\mathcal{D}$ has dense range.*

The extra requirement of regularity is necessary in the C^* -module context for the continuous functional calculus, and is not automatic, [26, Chapter 9]. With these definitions in hand, we return to our C^* -module X . The following can be proved just as in [28, Proposition 4.6], or equivalently by observing that the operator \mathcal{D} is presented in diagonal form.

Proposition 2.7. *Let X be the right C^* - F -module of Definition 2.1. Define $X_{\mathcal{D}} \subset X$ to be the linear space*

$$X_{\mathcal{D}} = \{x = \sum_{k \in \mathbb{Z}} x_k \in X : \|\sum_{k \in \mathbb{Z}} k^2(x_k|x_k)_R\| < \infty\}.$$

For $x = \sum_{k \in \mathbb{Z}} x_k \in X_{\mathcal{D}}$ define $\mathcal{D}x = \sum_{k \in \mathbb{Z}} kx_k$. Then $\mathcal{D}: X_{\mathcal{D}} \rightarrow X$ is a self-adjoint regular operator on X .

There is a continuous functional calculus for self-adjoint regular operators, [26, Theorem 10.9], and we use this to obtain spectral projections for \mathcal{D} at the C^* -module level. Let $f_k \in C_c(\mathbb{R})$ be 1 in a small neighbourhood of $k \in \mathbb{Z}$ and zero on $(-\infty, k - 1/2] \cup [k + 1/2, \infty)$. Then it is clear that $\Phi_k = f_k(\mathcal{D})$. That is the spectral projections of \mathcal{D} are the same as the projections onto the spectral subspaces of the circle action.

Lemma 2.8. *For all $a \in A$, the operator $a(1 + \mathcal{D}^2)^{-1/2}$ is a compact endomorphism of the F -module X .*

Proof. Since $a\Phi_k$ is a compact endomorphism for all $a \in A$, and $a\Phi_k, a\Phi_m$ have orthogonal initial spaces, the sum

$$a(1 + \mathcal{D}^2)^{-1/2} = \sum_{k \in \mathbb{Z}} (1 + k^2)^{-1/2} a\Phi_k$$

converges in norm to a compact endomorphism. \square

Proposition 2.9. *The pair (X, \mathcal{D}) is an unbounded Kasparov module defining a class in the equivariant KK-group $KK_1^{\mathbb{T}}(A, F)$.*

Proof. We will use the approach of [22, Section 4]. Let $V = \mathcal{D}(1 + \mathcal{D}^2)^{-1/2}$. We need to show that various operators belong to $\text{End}_F^0(X)$. First, $V - V^* = 0$, so $a(V - V^*)$ is compact for all $a \in A$. Also $a(1 - V^2) = a(1 + \mathcal{D}^2)^{-1}$ which is compact from Lemma 2.8 and the boundedness of $(1 + \mathcal{D}^2)^{-1/2}$. Finally, we need to show that $[V, a]$ is compact for all $a \in A$. First we suppose that $a = a_m$ is homogenous for the circle action. Then

$$\begin{aligned} [V, a] &= [\mathcal{D}, a](1 + \mathcal{D}^2)^{-1/2} - \mathcal{D}(1 + \mathcal{D}^2)^{-1/2}[(1 + \mathcal{D}^2)^{1/2}, a](1 + \mathcal{D}^2)^{-1/2} \\ &= b_1(1 + \mathcal{D}^2)^{-1/2} + Vb_2(1 + \mathcal{D}^2)^{-1/2}, \end{aligned}$$

where $b_1 = [\mathcal{D}, a] = ma$ and $b_2 = [(1 + \mathcal{D}^2)^{1/2}, a]$. Provided that $b_2(1 + \mathcal{D}^2)^{-1/2}$ is a compact endomorphism, Lemma 2.8 will show that $[V, a]$ is compact for all homogenous a . So consider the action of $[(1 + \mathcal{D}^2)^{1/2}, a](1 + \mathcal{D}^2)^{-1/2}$ on $x = \sum_{k \in \mathbb{Z}} x_k$. We find

$$\begin{aligned} \sum_{k \in \mathbb{Z}} [(1 + \mathcal{D}^2)^{1/2}, a](1 + \mathcal{D}^2)^{-1/2} x_k &= \sum_{k \in \mathbb{Z}} \left((1 + (m + k)^2)^{1/2} - (1 + k^2)^{1/2} \right) (1 + k^2)^{-1/2} a x_k \\ (1) \qquad \qquad \qquad &= \sum_{k \in \mathbb{Z}} f_m(k) a \Phi_k x. \end{aligned}$$

The function

$$f_m(k) = \left((1 + (m + k)^2)^{1/2} - (1 + k^2)^{1/2} \right) (1 + k^2)^{-1/2}$$

goes to 0 as $k \rightarrow \pm\infty$, and as the $a_m \Phi_k$ are compact with orthogonal ranges, the sum in (1) converges in the operator norm on endomorphisms and so converges to a compact endomorphism. For $a \in A$ a finite sum of homogenous terms, we apply the above reasoning to each term in the sum to find that $[(1 + \mathcal{D}^2)^{1/2}, a](1 + \mathcal{D}^2)^{-1/2}$ is a compact endomorphism.

Now let $a \in A$ be the norm limit of a Cauchy sequence $\{a_i\}_{i \geq 0}$ where each a_i is a finite sum of homogenous terms. Then

$$\|[V, a_i - a_j]\| \leq 2\|a_i - a_j\| \rightarrow 0,$$

so the sequence $[V, a_i]$ is also Cauchy in norm, and so the limit is compact. \square

2.2. Lifting KK-classes to the mapping cone. As remarked in the introduction, the class $[(X, \mathcal{D})]$ in $KK_1^{\mathbb{T}}(A, F)$ lifts to a class in $KK_0^{\mathbb{T}}(M, F)$ which we will describe in this subsection, where M is the mapping cone of the inclusion $F \subset A$. To begin with, we can just as well work in a somewhat more general context.

2.2.1. Construction of $\hat{\mathcal{D}}$. Let A and F be separable C^* -algebras equipped with an action α of a compact group G and suppose that we are given a class $[\mathcal{D}]$ in $KK_1^G(A, F)$. Such a class can be represented as by a G -equivariant extension of A by $\mathcal{K} \otimes F$ with the following structure.

- (i) A right Hilbert C^* -module \mathcal{H}_F over F ;

(ii) a unitary representation λ of G on \mathcal{H}_F compatible with the Hilbert C^* -module structure:

$$\alpha(g)(\langle \xi, \eta \rangle) = \langle \lambda(g)\xi, \lambda(g)\eta \rangle;$$

(iii) an injective representation $\pi : A \rightarrow \mathcal{L}(\mathcal{H}_F)$ such that, for all $g \in G$ and $a \in A$,

$$\pi(\alpha(g)(a)) = \text{Ad } \lambda(g)(\pi(a));$$

(iv) a projection $P \in \mathcal{L}(\mathcal{H}_F)$ commuting with $\lambda(G)$ and satisfying

$$[\pi(a), P] \in \mathcal{K}(\mathcal{H}_F) \text{ for all } a \in A, \quad P\pi(a)P \notin \mathcal{K}(\mathcal{H}_F) \text{ for all } a \in A \setminus \{0\}.$$

The extension in question is

$$(2) \quad 0 \rightarrow P\mathcal{K}(\mathcal{H}_F)P \rightarrow P(\mathcal{K}(\mathcal{H}_F) + \pi(A))P \rightarrow A \rightarrow 0.$$

For an inclusion $B \subset A$ of C^* -algebras we denote by $M(B, A)$ the corresponding mapping cone,

$$M(B, A) = \{f \in C_0((0, 1]; A) \mid f(1) \in B\}.$$

Let B be a C^* -subalgebra of A consisting of elements commuting with P . Using the extension (2), we get a G -equivariant extension for the mapping cone

$$(3) \quad 0 \rightarrow C_0((0, 1), P\mathcal{K}(\mathcal{H}_F)P) \rightarrow M(PB, P(\mathcal{K}(\mathcal{H}_F) + \pi(A))P) \rightarrow M(B, A) \rightarrow 0.$$

In turn, this extension defines a class in $KK_1^G(M(B, A), C_0((0, 1), P\mathcal{K}(\mathcal{H}_F)P))$. We will use $[\hat{\mathcal{D}}]$ to denote the image of this class in $KK_0^G(M(B, A), F)$ under the Bott periodicity isomorphism:

$$KK_1^G(M(B, A), C_0((0, 1), P\mathcal{K}(\mathcal{H}_F)P)) \xrightarrow{\cong} KK_0^G(M(B, A), F).$$

2.2.2. Pairing of $[\hat{\mathcal{D}}]$ with $K_0^G(M(B, A))$. Recall that a K_0 -class of a mapping cone algebra $M(B, A)$ is given (after stabilisation) by a pair of paths of projections $p_t, q_t \in A^\sim$, $t \in [0, 1]$ such that $p_0 = q_0$, $p_1, q_1 \in B^\sim$ and $p_1 - q_1 \in B$. Since this implies that p_1 and q_1 are Murray - von Neumann equivalent in A^\sim , such a class $[p_t] - [q_t]$ can be equivalently described by a partial isometry $v \in A^\sim$ satisfying $v^*v = p_1$ and $vv^* = q_1$, with p_1 and q_1 as above, see [31] for details.

We can then give the following index-type description of the pairing of $[\hat{\mathcal{D}}]$ with $K_0^G(M(B, A))$.

Theorem 2.10. *Let $[v]$ be a class in $K_0^G(M(B, A))$ represented by a partial isometry in $(A^\sim \otimes B(\mathcal{H}))^G$, where \mathcal{H} carries a finite dimensional representation of G and v^*v, vv^* are in $(B^\sim \otimes B(\mathcal{H}))^G$. Then*

$$\langle [\hat{\mathcal{D}}], [v] \rangle = -\text{Index}((P \otimes 1)v(P \otimes 1): v^*v(P\mathcal{H}_F \otimes \mathcal{H}) \rightarrow vv^*(P\mathcal{H}_F \otimes \mathcal{H})) \in K_0^G(F).$$

Proof. This follows from the Lemma below and the fact that the pairing of $[\hat{\mathcal{D}}]$ with $K_*^G(M(B, A))$ coincides with the boundary map $\delta : K_*^G(M(B, A)) \rightarrow K_{*+1}^G(C_0((0, 1), F))$ in the six term exact sequence of the extension (3). \square

Lemma 2.11. *Let F be a unital separable C^* -algebra, X a countably generated right Hilbert F -module. Assume P, p and q are projections in $\mathcal{L}(X)$ such that $[P, p] = [P, q] = 0$. Assume also that v_t , $0 \leq t \leq 1$, is a continuous path of partial isometries in $\mathcal{L}(X)$ such that $p = v_0$, $p = v_t^*v_t$ for all t , $[P, v_t] \in \mathcal{K}(X)$ for all t , and $q = v_1v_1^*$. Therefore the unitaries $\exp(2\pi i P v_t v_t^* P)$ are equal to 1 modulo $\mathcal{K}(X)$, and equal to 1 for $t = 0, 1$, hence they define a unitary V in $C_0((0, 1), \mathcal{K}(X))^\sim$. Then the class of V in $K_0(F) = K_1(C_0((0, 1), F))$ is equal to $-\text{Index}(Pv_1P: pPX \rightarrow qPX)$.*

Proof. Let us see first that the class of V depends only on v_1 . So assume that u_t is another path with the same properties and $u_1 = v_1$. It defines a unitary U . Consider

$$w_t(s) = R_s \begin{pmatrix} v_t \sqrt{1-s} & 0 \\ u_t \sqrt{s} & 0 \end{pmatrix}, \quad \text{where } R_s = \begin{pmatrix} \cos \frac{\pi s}{2} & \sin \frac{\pi s}{2} \\ -\sin \frac{\pi s}{2} & \cos \frac{\pi s}{2} \end{pmatrix}.$$

Then $w_t(s)$ is a partial isometry for all s and t , $w_t(0) = \begin{pmatrix} v_t & 0 \\ 0 & 0 \end{pmatrix}$, $w_t(1) = \begin{pmatrix} u_t & 0 \\ 0 & 0 \end{pmatrix}$. Since $w_0(s)w_0(s)^*$ and $w_1(s)w_1(s)^*$ commute with $P_2 := P \oplus P$, the unitaries W_s defined by the paths

$$(\exp(2\pi i P_2 w_t(s) w_t(s)^* P_2))_t$$

belong to $C_0((0,1), \mathcal{K}(X \oplus X))^\sim$ and connect $\begin{pmatrix} V & 0 \\ 0 & 1 \end{pmatrix}$ to $\begin{pmatrix} U & 0 \\ 0 & 1 \end{pmatrix}$ as s varies from 0 to 1.

In particular, instead of the original path we can consider the path

$$u_t = R_t \begin{pmatrix} p \sqrt{1-t} & 0 \\ v_1 \sqrt{t} & 0 \end{pmatrix}.$$

Assume $x := P v_1 P: pPX \rightarrow qPX$ is a regular Fredholm operator. Let p_0X and q_0X be its kernel and cokernel, respectively, and put $p_1 = pP - p_0$, $q_1 = qP - q_0$. Then we have

$$(4) \quad P_2 u_t u_t^* P_2 = R_t \begin{pmatrix} pP(1-t) & x^* \sqrt{t(1-t)} \\ x \sqrt{t(1-t)} & qPt \end{pmatrix} R_t^*$$

$$(5) \quad = R_t \begin{pmatrix} p_0(1-t) & 0 \\ 0 & q_0t \end{pmatrix} R_t^* + R_t \begin{pmatrix} p_1(1-t) & x^* \sqrt{t(1-t)} \\ x \sqrt{t(1-t)} & q_1t \end{pmatrix} R_t^*.$$

Since x is an isomorphism between p_1X and q_1X and a partial isometry modulo compacts, the polar decomposition $x = w|x|$ is well-defined, $w^*w = p_1$, $ww^* = q_1$, and $|x| = p_1$ modulo compacts. Using the straight line homotopy between $|x|$ and p_1 , we can replace path (5) by the homotopic path with w instead of x . But then the second summand in (5) becomes a projection orthogonal to the first summand, hence it does not contribute to the exponent. Thus it remains to understand the class of the unitary defined by the path

$$\exp \left(2\pi i R_t \begin{pmatrix} p_0(1-t) & 0 \\ 0 & q_0t \end{pmatrix} R_t^* \right),$$

which is clearly homotopic to the path

$$\exp \left(2\pi i \begin{pmatrix} p_0(1-t) & 0 \\ 0 & q_0t \end{pmatrix} \right).$$

The class in $K_0(\mathcal{K}(X)) \cong K_1(C_0((0,1), \mathcal{K}(X)))$ defined by this path is $[q_0] - [p_0]$.

Turning to the case when $x = P v_1 P$ is not regular, observe that we can replace pP and qP by $pP + r$ and $qP + r$, where r is any compact projection orthogonal to pP and qP , since the class of the path $\exp \left(2\pi i \begin{pmatrix} -rt & 0 \\ 0 & rt \end{pmatrix} \right)$ in $K_0(\mathcal{K}(X)) \cong K_1(C_0((0,1), \mathcal{K}(X)))$ is zero. Furthermore, any compact perturbation of x on the right hand side of (4) leads to a homotopic path of unitaries. It follows that if $y: pPX \oplus F^n \rightarrow qPX \oplus F^n$ is a regular Fredholm operator with $qPy pP = P v_1 P$ then the path $\exp(2\pi i P_2 u_t u_t^* P_2)$ defines the same class as the path

$$\exp \left(2\pi i R_t \begin{pmatrix} (pP + r)(1-t) & y^* \sqrt{t(1-t)} \\ y \sqrt{t(1-t)} & (qP + r)t \end{pmatrix} R_t^* \right),$$

where r is the projection $X \oplus F^n \rightarrow F^n$. The same computation as in the regular case shows that this class equals $-\text{Index}(y)$, which is $-\text{Index}(P v_1 P: pPX \rightarrow qPX)$ by definition. \square

2.2.3. *The case of a circle action.* In the particular case of this paper we therefore obtain a class $[\hat{\mathcal{D}}] \in KK_0^{\mathbb{T}}(M, F)$, where $M = M(F, A)$. The pairing of $[\hat{\mathcal{D}}]$ with $K_0^{\mathbb{T}}(M)$ defines a map $K_0^{\mathbb{T}}(M) \rightarrow K_0^{\mathbb{T}}(F)$, which we denote by $\text{Index}_{\hat{\mathcal{D}}}$.

Remarks 2.12. There is a strong evidence that the class $[\hat{\mathcal{D}}]$ coincides with the one obtained from (X, \mathcal{D}) via a C^* -algebra analogue of the Atiyah-Patodi-Singer (APS) theory developed in [2].

Let us also make a few remarks about the equivariant K -theory. The group $K_0^{\mathbb{T}}(M)$ is a module over the representation ring of \mathbb{T} , which we identify with the ring $\mathcal{R}_{\mathbb{T}} = \mathbb{Z}[\chi, \chi^{-1}]$ of Laurent polynomials with integral coefficients; therefore χ^n denotes the one-dimensional representation $t \mapsto e^{int}$. For a \mathbb{T} -module \mathcal{H} we denote by $\mathcal{H}[n]$ the module with the same underlying space but with the action tensored with χ^n . Now in terms of partial isometries, the $\mathcal{R}_{\mathbb{T}}$ -module structure on $K_0^{\mathbb{T}}(M)$ is described as follows: if $v \in (A^{\sim} \otimes \mathcal{B}(\mathcal{H}))^{\mathbb{T}}$ then $\chi[v]$ is the class of the partial isometry v considered as an element of $(A^{\sim} \otimes \mathcal{B}(\mathcal{H}[1]))^{\mathbb{T}}$. Furthermore, since the action of \mathbb{T} on F is trivial, we have $K_0^{\mathbb{T}}(F) = K_0(F) \otimes_{\mathbb{Z}} \mathcal{R}_{\mathbb{T}} = K_0(F)[\chi, \chi^{-1}]$.

3. ANALYTIC INTERLUDE - DUAL WEIGHTS

A KMS weight provides some analytic tools that we now explain.

Definition 3.1. *A weight ϕ on a C^* -algebra A is (σ, β) -KMS weight (KMS_{β} weight for short) if ϕ is a semifinite, norm lower semicontinuous, σ -invariant weight such that $\phi(aa^*) = \phi(\sigma_{i\beta/2}(a)^* \sigma_{i\beta/2}(a))$ for all $a \in \text{dom}(\sigma_{i\beta/2})$.*

Here $\text{dom}(\sigma_{i\beta/2})$ consists of all elements $a \in A$ such that $t \mapsto \sigma_t(a)$ extends to a continuous function from $0 \leq \Im(t) \leq \beta/2$ which is analytic in the open strip. We will assume throughout the rest of the paper that ϕ is a faithful KMS_{β} weight on A . Introduce the notation

$$\text{dom}(\phi)_+ = \{a \in A_+ : \phi(a) < \infty\}, \quad \text{dom}(\phi)^{1/2} = \{a \in A : a^*a \in \text{dom}(\phi)_+\},$$

$$\text{dom}(\phi) = \text{span}\{\text{dom}(\phi)_+\} = (\text{dom}(\phi)^{1/2})^* \text{dom}(\phi)^{1/2},$$

and extend ϕ to a linear functional on $\text{dom}(\phi)$. Recall that we defined a conditional expectation $\Phi: A \rightarrow F$.

The weight $\phi|_F$ is a norm lower semicontinuous semifinite trace. We will sometimes denote it by τ . Then $\phi = \tau \circ \Phi$, as ϕ is assumed to be σ -invariant.

The GNS construction yields a Hilbert space $\mathcal{H} := \mathcal{H}_{\phi}$, and a map $\Lambda: \text{dom}(\phi)^{1/2} \rightarrow \mathcal{H}$ with dense image and $\langle \Lambda(a), \Lambda(b) \rangle = \phi(a^*b)$, where $\langle \cdot, \cdot \rangle$ is the inner product. In fact, $\Lambda(\text{dom}(\phi)^{1/2} \cap (\text{dom}(\phi)^{1/2})^*)$ is a left Hilbert algebra. The algebra A is represented on \mathcal{H} as left multiplication operators, $a\Lambda(b) = \Lambda(ab)$, and the weight ϕ extends to a normal semifinite faithful weight on the von Neumann algebra $\pi(A)''$. We have $\sigma_t^{\phi} = \sigma_{-\beta t}$ on A , [23].

Consider the space \mathcal{H}_{τ} of the GNS-representation of the trace τ on F ; then \mathcal{H} can be identified with $X \otimes_F \mathcal{H}_{\tau}$ via the map $a \otimes \Lambda_{\tau}(f) \mapsto \Lambda_{\phi}(af)$. It follows that the action of A on \mathcal{H} extends to a representation of $\mathcal{L}(X)$ on \mathcal{H} . Denote by \mathcal{N} the von Neumann algebra generated by $\mathcal{L}(X)$ in $B(\mathcal{H})$.

3.1. Dual weights. Assume a compact group G acts on a von Neumann algebra \mathcal{L} . Recall that the action is called saturated if the central support $z(p_0)$ of the projection $p_0 := \int_G \lambda_g dg \in \mathcal{L} \rtimes G$ is equal to 1. Since $\lambda_g p_0 = p_0 \lambda_g = p_0$, the weak operator closure of $\mathcal{L} p_0 \mathcal{L}$ in $\mathcal{L} \rtimes G$ is an ideal, and the action is saturated if and only if this ideal coincides with $\mathcal{L} \rtimes G$.

Fix a G -invariant semifinite normal faithful weight ϕ . We have a canonical representation of G on \mathcal{H}_ϕ , $g \mapsto u_g$, $u_g \Lambda(a) = \Lambda(\alpha_g(a))$ for $a \in \mathcal{L}$.

The following should be known (the first part definitely is).

Proposition 3.2. *The covariant representation of (\mathcal{L}, G) on \mathcal{H}_ϕ extends to a normal representation of $\mathcal{L} \rtimes G$ on \mathcal{H}_ϕ . The kernel of this representation is $(\mathcal{L} \rtimes G)(1 - z(p_0))$.*

In particular, the representation is faithful if and only if the action is saturated.

Proof. As usual we represent $\mathcal{L} \rtimes G$ on $L^2(G, \mathcal{H}_\phi)$ so that λ_g are the operators of left translations and

$$(\pi(a)\xi)(g) = \alpha_g^{-1}(a)\xi(g).$$

Define an isometry $v: \mathcal{H}_\phi \rightarrow L^2(G, \mathcal{H}_\phi)$ by

$$(v\xi)(g) = u_g^*\xi.$$

Then

$$vu_g\xi = \lambda_g v\xi \quad \text{and} \quad va\xi = \pi(a)v\xi.$$

It follows that $v\mathcal{H}_\phi$ is a $\mathcal{L} \rtimes G$ -invariant subspace of $L^2(G, \mathcal{H}_\phi)$, so that $x \mapsto v^*xv$ is the required normal representation Π of $\mathcal{L} \rtimes G$ on \mathcal{H}_ϕ .

To understand its kernel consider the right Hilbert \mathcal{L}^G -module X obtained by completing \mathcal{L} with respect to the norm

$$\|a\| = \|\Phi(a^*a)\|^{1/2},$$

where $\Phi: \mathcal{L} \rightarrow \mathcal{L}^G$ is the canonical conditional expectation. Denote by ψ the restriction of ϕ to \mathcal{L}^G . Identifying \mathcal{H}_ϕ with $X \otimes_{\mathcal{L}^G} \mathcal{H}_\psi$, we see that $\mathcal{L}(X)$ acts \mathcal{H}_ϕ . Since $u_g \in \mathcal{L}(X)$, we conclude that $\Pi(\mathcal{L} \rtimes G) \subset \mathcal{L}(X)''$. On the other hand, using that $\Pi(p_0) = \int_G u_g dg$ is the projection defined by $\Phi: \mathcal{L} \rightarrow \mathcal{L}^G$, we have

$$\Pi(ap_0b^*) = \Theta_{a,b} \quad \text{for } a, b \in \mathcal{L}.$$

Since $\mathcal{K}(X)$ is strictly dense in $\mathcal{L}(X)$, we see that $\Pi(\mathcal{L}p_0\mathcal{L})$ is weakly operator dense in $\Pi(\mathcal{L} \rtimes G)$. In particular, $\Pi(z(p_0)) = 1$. In other words, if z is the central projection such that $(\mathcal{L} \rtimes G)z$ is the kernel of Π , then $z \geq 1 - z(p_0)$. If $z \neq 1 - z(p_0)$ then $x := zp_0 \neq 0$. We have $p_0(\mathcal{L} \rtimes G)p_0 = \mathcal{L}^G p_0$. Thus $x \in \mathcal{L}^G p_0$, $x \neq 0$ and $\Pi(x) = 0$. But this is impossible since, $\Pi(ap_0) = a$ on $\mathcal{H}_\psi \subset \mathcal{H}_\phi$ for any $a \in \mathcal{L}^G$. \square

Remarks 3.3. Using the one-to-one correspondence between ideals of Morita equivalent algebras, one can modify the last part of the above argument to show that if we have a C^* -dynamical system (A, G) and a covariant representation on a Hilbert space \mathcal{H} such that the representation of A^G on \mathcal{H}^G is faithful, then the restriction of the representation of $A \rtimes G$ on \mathcal{H} to the ideal $\overline{Ap_0A}$ is faithful. In particular, if the action is saturated in the C^* -sense, then the representation of $A \rtimes G$ is faithful (assuming faithfulness of A^G on \mathcal{H}^G).

Consider now the dual weight $\hat{\phi}$ on $\mathcal{L} \rtimes G$,

$$\hat{\phi}\left(\int_G a_g \lambda_g dg\right) = \phi(a_e).$$

By restricting it to $\overline{\mathcal{L}p_0\mathcal{L}}^{wo} = (\mathcal{L} \rtimes G)z(p_0) \cong \Pi(\mathcal{L} \rtimes G) = \mathcal{L}(X)''$ we get a normal semifinite faithful weight $\phi_{\mathcal{D}}$ on $\mathcal{L}(X)''$ such that

$$\phi_{\mathcal{D}}(\Theta_{a,a}) = \hat{\phi}(ap_0a^*) = \phi(aa^*).$$

3.2. The modular spectral triple. Turning to our case $G = \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, $\mathcal{L} = \pi_\phi(A)''$, we get a weight $\phi_{\mathcal{D}}$ on $\mathcal{N} = \mathcal{L}(X)'' \subset B(\mathcal{H})$. Denote by \mathcal{A} the algebra consisting of finite sums of σ -homogeneous elements in the domain $\text{dom}(\phi)$ of ϕ .

Definition 3.4. *The data $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \mathcal{N}, \phi_{\mathcal{D}})$ will be called the **modular spectral triple** for (A, σ, ϕ) .*

It will provide us with a way to compute the spectral flow from $vv^*\mathcal{D}$ and $v\mathcal{D}v^*$ with respect to the trace $\phi_{\mathcal{D}}$ on $\mathcal{M} := \mathcal{N}^\sigma$ for appropriate partial isometries in \mathcal{A} .

By construction, for $f \in F$, $f \geq 0$, we have

$$(6) \quad \phi_{\mathcal{D}}(f\Phi_0) = \tau(f).$$

More generally, we get the following inequality.

Lemma 3.5. *For all $f \in F$, $f \geq 0$ and $k \in \mathbb{Z}$ we have*

$$\phi_{\mathcal{D}}(f\Phi_k) \leq \tau(f),$$

and equality holds if A has full spectral subspaces. In particular, we have $f(1+\mathcal{D}^2)^{-1/2} \in \mathcal{L}^{(1,\infty)}(\mathcal{M}, \phi_{\mathcal{D}})$ if $f \in \mathcal{F} := \text{dom}(\phi) \cap F$.

Proof. If we identify $\mathcal{L} \rtimes \mathbb{T}$ with its image in $B(\mathcal{H} \otimes L^2(\mathbb{T}))$, then the preimage of $f\Phi_k$ under the representation Π (from the proof of Proposition 3.2) contains the element $f \otimes p_k$, where p_k is the projection onto the one-dimensional space spanned by e^{-ikt} . Since $\hat{\phi}$ is the restriction of $\phi \otimes \text{Tr}$ to $\mathcal{L} \rtimes \mathbb{T}$, the inequality follows. Furthermore, if A has full spectral subspaces then the action of \mathbb{T} is saturated, so the representation Π is faithful and equality holds. \square

Since the modular group of $\hat{\phi}$ is inner by construction, $\hat{\phi}$ can be modified to a trace, called the dual trace. In addition to the weight $\phi_{\mathcal{D}}$ it will be convenient to consider the trace on \mathcal{N} obtained by restricting this dual trace. Explicitly, the modular group of $\phi_{\mathcal{D}}$ is $\text{Ad } e^{-i\beta t\mathcal{D}}$, so

$$\text{Tr}_\phi := \phi_{\mathcal{D}}(e^{\beta\mathcal{D}/2} \cdot e^{\beta\mathcal{D}/2})$$

is a trace, and

$$\text{Tr}_\phi(\Theta_{a,a}) = \phi_{\mathcal{D}}(e^{\beta\mathcal{D}/2}\Theta_{a,a}e^{\beta\mathcal{D}/2}) = \phi_{\mathcal{D}}(\Theta_{\sigma_{i/2}^\phi(a), \sigma_{i/2}^\phi(a)}) = \phi(\sigma_{i/2}^\phi(a)\sigma_{i/2}^\phi(a)^*) = \phi(a^*a).$$

Remarks 3.6. Therefore Tr_ϕ is the normal extension to $\mathcal{L}(X)''$ of the trace on $\mathcal{L}(X)$ induced from $\phi|_F$, see [25]. Yet another way of constructing Tr_ϕ is to say that \mathcal{N} is given by the basic construction associated with the conditional expectation Φ , while Tr_ϕ is the canonical trace on \mathcal{N} defined by the trace $\phi|_F$, [30].

4. MODULAR INDEX PAIRING

4.1. Modular \mathbf{K}_1 . Recall the notion of spectral flow, see e.g. [3] for details. According to Section 6 of that paper, if \mathcal{N} is a semifinite von Neumann algebra with faithful normal semifinite trace τ and $\mathcal{D}_1, \mathcal{D}_2$ are closed self-adjoint operators affiliated with \mathcal{N} which differ by a bounded operator and whose spectral projections $P_1 = \chi_{[0,+\infty)}(\mathcal{D}_1)$ and $P_2 = \chi_{[0,+\infty)}(\mathcal{D}_2)$ are such that the operator $P_1P_2 \in P_1\mathcal{N}P_2$ is Breuer-Fredholm, then the spectral flow is defined by

$$sf(\mathcal{D}_1, \mathcal{D}_2) = \text{Index}_\tau(P_1P_2).$$

In the case when P_1 and P_2 are finite we clearly have $sf(\mathcal{D}_1, \mathcal{D}_2) = \tau(P_2) - \tau(P_1)$.

Consider now the modular spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \mathcal{N}, \phi_{\mathcal{D}})$ defined in the previous Section. We would like to obtain an analytic formula for the spectral flow from \mathcal{D} to $u\mathcal{D}u^*$ with respect to $\phi_{\mathcal{D}} \otimes \text{Tr}$, where u is a unitary in A . However, just to define it we need conditions which guarantee that the operator and $u\mathcal{D}u^*$ belongs to \mathcal{M} . This motivates the following definition, which is essentially from [9], but is slightly modified and extended to adapt to our current considerations.

Definition 4.1. *A partial isometry in A^{\sim} is modular if $v\sigma_t(v^*)$ and $v^*\sigma_t(v)$ are in $(A^{\sim})^{\sigma}$ for all $t \in \mathbb{R}$. By a modular partial isometry over A we mean a modular partial isometry in $\text{Mat}_n(A^{\sim}) = A^{\sim} \otimes \text{Mat}_n(\mathbb{C})$ for some $n \in \mathbb{N}$ with respect to the action $\sigma \otimes \iota$.*

In [9] only modular unitaries were considered. Observe that every modular partial isometry v over A defines a modular unitary by

$$u_v = \begin{pmatrix} 1 - v^*v & v^* \\ v & 1 - vv^* \end{pmatrix}.$$

Define the modular K_1 group as follows.

Definition 4.2. *Let $K_1(A, \sigma)$ be the abelian group with one generator $[v]$ for each partial isometry v over A satisfying the modular condition and with the following relations:*

- 1) $[v] = 0$ if v is over F ,
- 2) $[v] + [w] = [v \oplus w]$,
- 3) if v_t , $t \in [0, 1]$, is a continuous path of modular partial isometries in $\text{Mat}_n(A^{\sim})$ then $[v_0] = [v_1]$.

Remarks 4.3. It is easy to show that $v \oplus w \sim w \oplus v$, see [9], however the inverse of $[v]$ is not $[v^*]$ in general. Equivalently, even though u_v is a self-adjoint unitary and hence is homotopic to the identity, such a homotopy cannot always be chosen to consist of modular unitaries.

Observe that σ -homogeneous partial isometries are modular. It turns out that they generate the whole group $K_1(A, \sigma)$. We need some preparation to prove this.

Lemma 4.4. *A unitary $u \in A^{\sim}$ is modular if and only if there exists a self-adjoint element $a \in F^{\sim}$ such that $uau^* \in F^{\sim}$ and $\sigma_t(u) = ue^{ita}$ for $t \in \mathbb{R}$.*

Proof. Put $u_t = u^*\sigma_t(u)$. Then

$$u_{t+s} = u^*\sigma_{t+s}(u) = u^*\sigma_t(u)\sigma_t(u^*\sigma_s(u)) = u_t u_s.$$

Thus $\{u_t\}_t$ is a norm-continuous one-parameter group of unitary operators in F^{\sim} . Hence there exists a self-adjoint element $a \in F^{\sim}$ such that $u_t = e^{ita}$. Therefore

$$\sigma_t(u) = ue^{ita} = e^{itau^*} u.$$

Since u is modular, the second equality implies that $uau^* \in F^{\sim}$. The converse is obvious. \square

For an element $x \in \text{Mat}_n(A^{\sim})$ we denote by x_k the spectral component of x with respect to $\sigma \otimes \iota$, so $(\sigma_t \otimes \iota)(x_k) = e^{ikt} x_k$.

Lemma 4.5. *A partial isometry $v \in \text{Mat}_n(A^{\sim})$ is modular if and only if the elements v_k are partial isometries which are zero for all but a finite number of k 's and the source projections $v_k^* v_k$, $k \in \mathbb{Z}$, as well as the range projections $v_k v_k^*$, $k \in \mathbb{Z}$, are mutually orthogonal.*

Proof. Consider the modular unitary $u = u_v$. If $\sigma_t(u) = ue^{ita}$ with a as in Lemma 4.4 (but now $a \in \text{Mat}_{2n}(F^\sim)$), then $u = ue^{2\pi ia}$. Hence the spectrum of a is a finite subset of \mathbb{Z} . Let p_k be the spectral projection of a corresponding to $k \in \mathbb{Z}$. Then $u_k = up_k$, and hence the partial isometries u_k have mutually orthogonal sources and ranges. We clearly have

$$u_0 = \begin{pmatrix} 1 - v^*v & v_0^* \\ v_0 & 1 - vv^* \end{pmatrix}, \quad u_k = \begin{pmatrix} 0 & v_{-k}^* \\ v_k & 0 \end{pmatrix} \quad \text{for } k \neq 0.$$

This implies that $v_k = 0$ for all but a finite number of k , and the elements v_k , $k \neq 0$, are partial isometries with mutually orthogonal sources and ranges. Consider $w = \sum_{k \neq 0} v_k$. Then w is a partial isometry and $ww^* = \sum_{k \neq 0} v_k v_k^*$, $w^*w = \sum_{k \neq 0} v_k^* v_k$. Since

$$v^*v = v_0^*v_0 + w^*w + \sum_{k \neq 0} (v_0^*v_k + v_k^*v_0)$$

is invariant, we get $v^*v = v_0^*v_0 + w^*w$. Since v^*v and w^*w are projections, it follows that $v_0^*v_0$ is a projection orthogonal to w^*w . In other words, v_0 is a partial isometry with the source projection orthogonal to $v_k^*v_k$, $k \neq 0$. Similarly one checks that the projections $v_0v_0^*$ and $v_kv_k^*$, $k \neq 0$, are orthogonal.

The converse statement is straightforward. □

Corollary 4.6. *The group $K_1(A, \sigma)$ is generated by the classes of homogeneous partial isometries.*

Proof. It suffices to observe that if v and w are modular partial isometries such that $v^*vw^*w = vv^*ww^* = 0$, then $[v + w] = [v] + [w]$. Indeed, if $R_t = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$, then

$$v_t = \left(\begin{pmatrix} 1 - ww^* & 0 \\ 0 & 1 - ww^* \end{pmatrix} + R_t w w^* \right) \begin{pmatrix} v + w & 0 \\ 0 & 0 \end{pmatrix} \left(\begin{pmatrix} 1 - w^*w & 0 \\ 0 & 1 - w^*w \end{pmatrix} + R_{-t} w^* w \right),$$

$0 \leq t \leq \pi/2$, is a modular homotopy from $\begin{pmatrix} v + w & 0 \\ 0 & 0 \end{pmatrix}$ to $\begin{pmatrix} v & 0 \\ 0 & w \end{pmatrix}$. □

Our next goal is to relate the group $K_1(A, \sigma)$ to $K_0^\mathbb{T}(M(F, A))$.

Recall that if \mathcal{K} is a finite dimensional Hilbert space considered with the trivial \mathbb{T} -module structure, we denote by $\mathcal{K}[n]$ the same space with the representation $t \mapsto e^{int}$. Assume $v \in A^\sim \otimes \mathcal{B}(\mathcal{K})$ is a partial isometry such that $v \in A_n^\sim \otimes \mathcal{B}(\mathcal{K})$, so $(\sigma_t \otimes \iota)(v) = e^{int}v$, then the partial isometry

$$w_v = \begin{pmatrix} 0 & v \\ 0 & 0 \end{pmatrix} \in A^\sim \otimes \mathcal{B}(\mathcal{K} \oplus \mathcal{K}[n])$$

is \mathbb{T} -invariant, so it defines an element of $K_0^\mathbb{T}(M)$. Sometimes we shall denote the class $[w_v] \in K_0^\mathbb{T}(M)$ by $\ll v \gg$. Note that if $n = 0$ and so v itself represents an element of $K_0^\mathbb{T}(M)$, there is no ambiguity in this notation as

$$\begin{pmatrix} 0 & v \\ 0 & 0 \end{pmatrix} \text{ is homotopic to } \begin{pmatrix} v & 0 \\ 0 & 0 \end{pmatrix},$$

and moreover, the class of v can easily be shown to be zero, see [31, Lemma 2.2(v)].

Proposition 4.7. *The map*

$$v \mapsto \sum_k \ll v_k \gg \in K_0^\mathbb{T}(M)$$

defined on modular partial isometries gives a homomorphism $T: K_1(A, \sigma) \rightarrow K_0^\mathbb{T}(M)$.

Proof. Since homotopic elements have homotopic spectral components, it is clear that the images of homotopic modular partial isometries coincide. It follows that we have a well-defined homomorphism $T: K_1(A, \sigma) \rightarrow K_0^{\mathbb{T}}(M)$; in fact, for each k the map $[u] \mapsto \ll u_k \gg$ is a homomorphism. \square

This homomorphism makes it clear why $-[v] \neq [v^*]$ in $K_1(A, \sigma)$ in general. Indeed, observe first that in the group $K_0^{\mathbb{T}}(M)$ we do have $-[w] = [w^*]$, basically because u_w is an invariant self-adjoint unitary, hence there is a homotopy from u_w to 1 consisting of invariant unitaries. In particular, for homogeneous v as above we have $-[w_v] = [w_v^*]$. The class w_v^* is represented by

$$\begin{pmatrix} 0 & v^* \\ 0 & 0 \end{pmatrix} \in A^{\sim} \otimes \mathcal{B}(\mathcal{K}[n] \oplus \mathcal{K}), \quad \text{while} \quad w_{v^*} = \begin{pmatrix} 0 & v^* \\ 0 & 0 \end{pmatrix} \in A^{\sim} \otimes \mathcal{B}(\mathcal{K} \oplus \mathcal{K}[-n]).$$

Therefore $[w_v^*] = \chi^n [w_{v^*}]$. In other words, $- \ll v \gg = \chi^n \ll v^* \gg$, so that $T(-[v]) = \chi^n T([v^*])$. Equivalently, we have

$$T([u_v]) = \ll v \gg + \ll v^* \gg = (1 - \chi^{-n}) \ll v \gg.$$

4.2. Modular index. The next Lemma justifies our definition of modular partial isometries.

Lemma 4.8. *Let $v \in \text{Mat}_n(A^{\sim})$ be a partial isometry such that $vv^*, v^*v \in \text{Mat}_n(F^{\sim})$. Then we have $v(Q \otimes 1)v^*, v^*(Q \otimes 1)v \in \text{Mat}_n(\mathcal{M})$ for every spectral projection Q of \mathcal{D} if and only if v is modular.*

Proof. Replacing v by u_v we may assume that v is unitary. Next, suppose first that v is modular. Write $\tilde{\sigma}$ for $\sigma_t \otimes \iota$ and \tilde{Q} for $Q \otimes 1$. Since $\mathcal{M} = \mathcal{N}^{\sigma}$, we need to show that $v\tilde{Q}v^*$ is $\tilde{\sigma}$ -invariant. We have

$$\tilde{\sigma}(vQv^*) = \tilde{\sigma}(v)\tilde{Q}\tilde{\sigma}(v^*) = vv^*\tilde{\sigma}(v)\tilde{Q}\tilde{\sigma}(v^*) = v\tilde{Q}v^*\tilde{\sigma}(v)\tilde{\sigma}(v^*) = v\tilde{Q}v^*.$$

A similar argument shows that $v^*\tilde{Q}v$ is invariant.

On the other hand, if

$$v\tilde{Q}v^* = \tilde{\sigma}(v\tilde{Q}v^*) = \tilde{\sigma}(v)\tilde{Q}\tilde{\sigma}(v^*),$$

then $v^*\tilde{\sigma}(v)$ commutes with $\tilde{Q} = Q \otimes 1$. If this is true for every spectral projection Q of the generator \mathcal{D} of σ , then $v^*\tilde{\sigma}(v)$ is $(\sigma \otimes \iota)$ -invariant. Similarly $v\tilde{\sigma}(v^*)$ is invariant. Hence v is modular. \square

Next we show that the spectral flow is indeed well-defined for modular partial isometries.

Lemma 4.9. *For a modular partial isometry $v \in A^{\sim} \otimes \mathcal{B}(\mathcal{K})$ consider the projections*

$$P_1 = \chi_{[0, +\infty)}(vv^*(\mathcal{D} \otimes 1)) \quad \text{and} \quad P_2 = \chi_{[0, +\infty)}(v(\mathcal{D} \otimes 1)v^*).$$

Then the operator $P_1P_2 \in P_1(\mathcal{M} \otimes \mathcal{B}(\mathcal{K}))P_2$ is Breuer-Fredholm and

$$\begin{aligned} & sf_{\phi_{\mathcal{D}} \otimes \text{Tr}}(vv^*(\mathcal{D} \otimes 1), v(\mathcal{D} \otimes 1)v^*) \\ &= \sum_{k < 0} \sum_{k \leq n < 0} e^{-\beta n} (\text{Tr}_{\phi} \otimes \text{Tr})(v_k v_k^*(\Phi_n \otimes 1)) - \sum_{k > 0} \sum_{0 \leq n < k} e^{-\beta n} (\text{Tr}_{\phi} \otimes \text{Tr})(v_k v_k^*(\Phi_n \otimes 1)). \end{aligned}$$

Proof. By Lemma 4.5 the element v is a finite sum of its homogeneous components v_k which are partial isometries with mutually orthogonal sources and ranges. The operators $vv^*(\mathcal{D} \otimes 1)$ and $v(\mathcal{D} \otimes 1)v^*$ commute with $v_k v_k^*$ and

$$v_k v_k^* v v^* (\mathcal{D} \otimes 1) = v_k v_k^* (\mathcal{D} \otimes 1), \quad v_k v_k^* v (\mathcal{D} \otimes 1) v^* = v_k (\mathcal{D} \otimes 1) v_k^*.$$

This shows that without loss of generality we may assume that v is homogeneous, say $v = v_k$. Furthermore, for $k = 0$ the operators coincide, so we just have to consider the case $k \neq 0$.

Let $P = \chi_{[0,+\infty)}(\mathcal{D}) = \sum_{n \geq 0} \Phi_n$. Since vv^* and v^*v commute with \mathcal{D} , we have

$$P_1 = 1 - vv^* + vv^*(P \otimes 1) \quad \text{and} \quad P_2 = 1 - vv^* + v(P \otimes 1)v^*.$$

But using homogeneity we can actually say much more and easily express these projections in terms of vv^* and Φ_n . Namely, as $v(\Phi_n \otimes 1) = (\Phi_{n+k} \otimes 1)v$, we have

$$(7) \quad v(P \otimes 1)v^* = \sum_{n \geq k} vv^*(\Phi_n \otimes 1).$$

With this information it is easy to show that $P_1 P_2$ is Breuer-Fredholm, since this is implied by $P_1 - P_2$ being compact in $\mathcal{M} \otimes \mathcal{B}(\mathcal{K})$. However from Equation (7) we have

$$P_1 - P_2 = \sum_{n=0}^{k-1} vv^*(\Phi_n \otimes 1), \quad k > 0, \quad P_1 - P_2 = - \sum_{n=k}^{-1} vv^*(\Phi_n \otimes 1), \quad k < 0.$$

To finish the proof it therefore remains to show that for every n the projection $vv^*(\Phi_n \otimes 1)$ has finite trace with respect to $\phi_{\mathcal{D}} \otimes \text{Tr}$. For the same reason as in Lemma 3.5 we have

$$e^{-\beta n} (\text{Tr}_{\phi} \otimes \text{Tr})(vv^*(\Phi_n \otimes 1)) = (\phi_{\mathcal{D}} \otimes \text{Tr})(vv^*(\Phi_n \otimes 1)) \leq (\tau \otimes \text{Tr})(vv^*).$$

Notice now that $v \in A \otimes \mathcal{B}(\mathcal{K})$, since $v = v_k$ is homogeneous with $k \neq 0$. Hence the projection $vv^* \in F \otimes \mathcal{B}(\mathcal{K})$ is in the domain of the semifinite trace $\tau \otimes \text{Tr}$ on $F \otimes \mathcal{B}(\mathcal{K})$, since the latter domain contains the Pedersen ideal and, in particular, every projection. \square

Observe that the above proof shows that if v is a modular partial isometry then $v - v_0 \in \mathcal{A} \otimes \mathcal{B}(\mathcal{K})$. Notice also that if we have a continuous path of modular partial isometries then the corresponding projections P_1 and P_2 also form norm-continuous paths. It follows that the map

$$v \mapsto sf_{\phi_{\mathcal{D}} \otimes \text{Tr}}(vv^*(\mathcal{D} \otimes 1), v(\mathcal{D} \otimes 1)v^*)$$

defines a homomorphism $K_1(A, \sigma) \rightarrow \mathbb{R}$; this of course also follows from the explicit expression for the spectral flow. We call this homomorphism the **modular index** and denote it by $\text{Index}_{\phi_{\mathcal{D}}}$. The following theorem compares $\text{Index}_{\phi_{\mathcal{D}}}$ with our K -theoretic constructions.

Theorem 4.10. *The modular index map $\text{Index}_{\phi_{\mathcal{D}}} : K_1(A, \sigma) \rightarrow \mathbb{R}$ is the composition of the maps*

$$K_1(A, \sigma) \xrightarrow{[v] \mapsto \sum_k \ll v_k \gg} K_0^{\mathbb{T}}(M) \xrightarrow{-\text{Index}_{\mathcal{D}}} K_0^{\mathbb{T}}(F) = K_0(F)[\chi, \chi^{-1}] \xrightarrow{\tau_*} \mathbb{R}[\chi, \chi^{-1}] \xrightarrow{\text{Ev}(e^{-\beta})} \mathbb{R},$$

where $\text{Ev}(e^{-\beta})$ is the evaluation at $\chi = e^{-\beta}$.

Proof. This is a matter of bookkeeping. Let $v = v_k \in A \otimes \mathcal{B}(\mathcal{K})$ be a homogeneous partial isometry, $k \neq 0$. Recall that $\ll v \gg$ is represented by $w_v = \begin{pmatrix} 0 & v \\ 0 & 0 \end{pmatrix} \in A \otimes \mathcal{B}(\mathcal{K} \oplus \mathcal{K}[k])$. To compute $-\text{Index}_{\mathcal{D}}([w_v])$ first observe that the projection Q_n onto the χ^n -homogeneous component of $\mathcal{H} \otimes (\mathcal{K} \oplus \mathcal{K}[k])$ is $\begin{pmatrix} \Phi_n & 0 \\ 0 & \Phi_{n-k} \end{pmatrix}$. Therefore by Theorem 2.10 we have, with $P = \sum_{n \geq 0} \Phi_n$, that

$$\begin{aligned} -\text{Index}_{\mathcal{D}}([w_v]) &= \sum_n ([w_v^* w_v Q_n(P \otimes 1)] - [w_v w_v^* Q_n(P \otimes 1)]) \chi^n \\ &= \sum_n ([v^* v (\Phi_{n-k} P \otimes 1)] - [vv^* (\Phi_n P \otimes 1)]) \chi^n. \end{aligned}$$

The projection $v^* v (\Phi_{n-k} \otimes 1) = v^* (\Phi_n \otimes 1) v$ is equivalent to the projection $vv^* (\Phi_n \otimes 1)$. It follows that the n th summand in the above expression is nonzero only when $n - k$ and n have different signs.

More precisely, for $k < 0$ we get

$$\sum_{k \leq n < 0} [v^*v(\Phi_{n-k} \otimes 1)]\chi^n = \sum_{k \leq n < 0} [vv^*(\Phi_n \otimes 1)]\chi^n,$$

and for $k > 0$ we get

$$- \sum_{0 \leq n < k} [vv^*(\Phi_n \otimes 1)]\chi^n.$$

Since $\tau_*([p]) = (\text{Tr}_\phi \otimes \text{Tr})(p)$ for projections $p \in \mathcal{K}(X \otimes \mathcal{K})$, we see that upon applying τ_* and letting $\chi = e^{-\beta}$ the above expressions coincide with those in Lemma 4.9. \square

Remarks 4.11. The map $K_0^{\mathbb{T}}(M) \rightarrow \mathbb{R}[\chi, \chi^{-1}]$ obtained as the composition of τ_* and $-\text{Index}_{\hat{\mathcal{D}}}$, can be interpreted as the equivariant spectral flow from $vv^*(\mathcal{D} \otimes 1)$ to $v(\mathcal{D} \otimes 1)v^*$ with respect to the trace $\text{Tr}_\phi \otimes \text{Tr}$, that is, it gives a Laurent polynomial with coefficients defined by the spectral flow of the restrictions of the operators to isotypic components of the circle action. Therefore the passage from the trace Tr_ϕ on \mathcal{N} to the trace $\phi_{\mathcal{D}}$ on \mathcal{M} corresponds to evaluating the equivariant spectral flow at $\chi = e^{-\beta}$ instead of $\chi = 1$.

The equivariant spectral flow can be shown to be defined without the SSA (which is needed to define $\hat{\mathcal{D}}$), but this requires some work.

5. THE ANALYTIC INDEX FROM SPECTRAL FLOW

5.1. The residue formula for one-summable semifinite flow. Our method of computing numerical invariants from KMS states exploits semifinite spectral flow and so we need to review the spectral flow formula of [6]. There are two versions of this formula in the unbounded setting, one for θ -summable spectral triples, and the other for finitely summable triples. It is the latter that we will want to use. First we quote [6, Corollary 8.11].

Proposition 5.1. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D}_0)$ be an odd unbounded θ -summable semifinite spectral triple relative to (\mathcal{M}, τ) , where τ is a faithful semifinite normal trace on \mathcal{M} . For any $\epsilon > 0$ we define a one-form α^ϵ on the affine space $\mathcal{M}_0 = \mathcal{D}_0 + \mathcal{M}_{sa}$ by*

$$\alpha^\epsilon(A) = \sqrt{\frac{\epsilon}{\pi}} \tau(Ae^{-\epsilon \mathcal{D}^2})$$

for $\mathcal{D} \in \mathcal{M}_0$ and $A \in T_{\mathcal{D}}(\mathcal{M}_0) = \mathcal{M}_{sa}$ (here $T_{\mathcal{D}}(\mathcal{M}_0)$ is the tangent space to \mathcal{M}_0 at \mathcal{D}). Then the integral of α^ϵ is independent of the piecewise C^1 path in \mathcal{M}_0 and if $\{\mathcal{D}_t = \mathcal{D}_a + A_t\}_{t \in [a,b]}$ is any piecewise C^1 path in \mathcal{M}_0 joining \mathcal{D}_a and \mathcal{D}_b then

$$sf(\mathcal{D}_a, \mathcal{D}_b) = \sqrt{\frac{\epsilon}{\pi}} \int_a^b \tau(\mathcal{D}'_t e^{-\epsilon \mathcal{D}_t^2}) dt + \frac{1}{2} \eta_\epsilon(\mathcal{D}_b) - \frac{1}{2} \eta_\epsilon(\mathcal{D}_a) + \frac{1}{2} \tau([\ker(\mathcal{D}_b)] - [\ker(\mathcal{D}_a)]).$$

Here the truncated eta invariant is given for $\epsilon > 0$ by

$$\eta_\epsilon(\mathcal{D}) = \frac{1}{\sqrt{\pi}} \int_\epsilon^\infty \tau(\mathcal{D}e^{-t\mathcal{D}^2}) t^{-1/2} dt.$$

We want to employ this formula in a finitely summable setting, so we need to Laplace transform the various terms appearing in the formula. In fact we were able in [13] to translate the formula in [6] for the spectral flow into a residue type formula. The importance of such a formula lies in the drastic simplification of computations, since we may throw away terms that are holomorphic in a neighbourhood of the point where we take a residue.

We introduce the notation

$$C_r := \frac{\sqrt{\pi}\Gamma(r-1/2)}{\Gamma(r)} = \int_{-\infty}^{\infty} (1+x^2)^{-r} dx.$$

Observe that C_r is analytic for $\Re(r) > 1/2$ and has an analytic continuation to a neighbourhood of $1/2$ where it has a simple pole (cf [11]) with residue equal to 1.

Lemma 5.2. *Let \mathcal{D} be a self-adjoint operator on the Hilbert space \mathcal{H} , affiliated to the semifinite von Neumann algebra \mathcal{M} . Suppose that for a fixed faithful, normal, semifinite trace τ on \mathcal{M} we have*

$$(1 + \mathcal{D}^2)^{-1/2} \in \mathcal{L}^{(p,\infty)}(\mathcal{M}, \tau), \quad p \geq 1.$$

Then the Laplace transform of $\eta_\epsilon(\mathcal{D})$, the eta invariant of \mathcal{D} , is given by $\frac{1}{C_r}\eta_{\mathcal{D}}(r)$ where

$$\eta_{\mathcal{D}}(r) = \int_1^\infty \tau(\mathcal{D}(1+s\mathcal{D}^2)^{-r})s^{-1/2}ds, \quad \Re(r) > 1/2 + p/2.$$

Proof. We need to Laplace transform the ‘ θ summable formula’ for the truncated η invariant:

$$\eta_\epsilon(\mathcal{D}) = \frac{1}{\sqrt{\pi}} \int_\epsilon^\infty \tau(\mathcal{D}e^{-t\mathcal{D}^2})t^{-1/2}dt.$$

This integral converges for all $\epsilon > 0$. First we rewrite the formula as

$$\eta_\epsilon(\mathcal{D}) = \frac{\sqrt{\epsilon}}{\sqrt{\pi}} \int_1^\infty \tau(\mathcal{D}e^{-\epsilon s\mathcal{D}^2})s^{-1/2}ds.$$

Using

$$1 = \frac{1}{\Gamma(r-1/2)} \int_0^\infty \epsilon^{r-3/2}e^{-\epsilon}d\epsilon$$

for $\Re(r) > p/2 + 1/2$, the Laplace transform of $\eta_\epsilon(\mathcal{D})$ is

$$\begin{aligned} \frac{1}{C_r}\eta_{\mathcal{D}}(r) &= \frac{1}{\sqrt{\pi}\Gamma(r-1/2)} \int_0^\infty \epsilon^{r-1}e^{-\epsilon} \int_1^\infty \tau(\mathcal{D}e^{-\epsilon s\mathcal{D}^2})s^{-1/2}dsd\epsilon \\ &= \frac{1}{\sqrt{\pi}\Gamma(r-1/2)} \int_1^\infty s^{-1/2}\tau(\mathcal{D} \int_0^\infty \epsilon^{r-1}e^{-\epsilon(1+s\mathcal{D}^2)}d\epsilon)ds \\ (8) \quad &= \frac{\Gamma(r)}{\sqrt{\pi}\Gamma(r-1/2)} \int_1^\infty s^{-1/2}\tau(\mathcal{D}(1+s\mathcal{D}^2)^{-r})ds. \end{aligned}$$

□

For our final formula we restrict to $p = 1$, which is the case of interest in this paper.

Proposition 5.3. *Let \mathcal{D}_a be a self-adjoint densely defined unbounded operator on the Hilbert space \mathcal{H} , affiliated to the semifinite von Neumann algebra \mathcal{M} . Suppose that for a fixed faithful, normal, semifinite trace τ on \mathcal{M} we have $(1 + \mathcal{D}_a^2)^{-1/2} \in \mathcal{L}^{(1,\infty)}(\mathcal{M}, \tau)$. Let \mathcal{D}_b differ from \mathcal{D}_a by a bounded self adjoint operator in \mathcal{M} . Then for any piecewise C^1 path $\{\mathcal{D}_t = \mathcal{D}_a + A_t\}$, $t \in [a, b]$ in $\mathcal{M}_0 = \mathcal{D}_a + \mathcal{M}_{sa}$ joining \mathcal{D}_a and \mathcal{D}_b , the spectral flow $sf_\tau(\mathcal{D}_a, \mathcal{D}_b)$ is given by the formula*

$$\begin{aligned} \text{Res}_{r=1/2} C_r sf_\tau(\mathcal{D}_a, \mathcal{D}_b) &= \text{Res}_{r=1/2} \left(\int_a^b \tau(\dot{\mathcal{D}}_t(1 + \mathcal{D}_t^2)^{-r})dt + \frac{1}{2}(\eta_{\mathcal{D}_b}(r) - \eta_{\mathcal{D}_a}(r)) \right) \\ (9) \quad &+ \frac{1}{2}(\tau(P_{\ker \mathcal{D}_b}) - \tau(P_{\ker \mathcal{D}_a})), \end{aligned}$$

where $\eta_{\mathcal{D}}(r) := \int_1^\infty \tau(\mathcal{D}(1+s\mathcal{D}^2)^{-r})s^{-1/2}ds$, $\Re(r) > 1$. The meaning of (9) is that the function of r on the right hand side has a meromorphic continuation to a neighbourhood of $r = 1/2$ with a simple pole at $r = 1/2$ where we take the residue.

Proof. We apply the Laplace transform to the general spectral flow formula. The computation of the Laplace transform of the eta invariants is above, and the Laplace transform of the other integral is in [6]. The existence of the residue follows from the equality, for $\Re(r)$ large,

$$C_r sf_\tau(\mathcal{D}_a, \mathcal{D}_b) = \int_a^b \tau(\dot{\mathcal{D}}_t(1 + \mathcal{D}_t^2)^{-r}) dt + \frac{1}{2}(\eta_{\mathcal{D}_b}(r) - \eta_{\mathcal{D}_a}(r)) + C_r \frac{1}{2}(\tau(P_{\ker \mathcal{D}_b}) - \psi(P_{\ker \mathcal{D}_a}))$$

which shows that the sum of the integral and the eta terms has a meromorphic continuation as claimed. \square

This is the formula for spectral flow we will employ in the sequel.

5.2. Residue type formulas for analytic spectral flow. Let $v \in A^\sim$ be a modular partial isometry. Recall that (as we observed after Lemma 4.9) we automatically have $v_k \in \mathcal{A}$ for $k \neq 0$. Furthermore, the same Lemma shows that v_0 does not contribute to the spectral flow. In other words, we have the following.

Lemma 5.4. *Given the modular spectral triple for (A, σ, ϕ) , let $v \in A^\sim$ be a modular partial isometry so that $p = vv^* - v_0v_0^* \in \mathcal{F}$, where $v_0 \in A_0$ is the σ -invariant part of v . Then p commutes with \mathcal{D} and $v\mathcal{D}v^*$, and*

$$sf_{\phi_{\mathcal{D}}}(vv^*\mathcal{D}, v\mathcal{D}v^*) = sf_{\phi_{\mathcal{D},p}}(p\mathcal{D}, pv\mathcal{D}v^*),$$

where $\phi_{\mathcal{D},p} = \phi_{\mathcal{D}}|_{p\mathcal{M}p}$ is the trace on $p\mathcal{M}p$.

We apply Proposition 5.3 to the path $\mathcal{D}_t = p\mathcal{D} + tpv[\mathcal{D}, v^*] = p\mathcal{D} + tv[\mathcal{D}, v^*]$ of operators affiliated with $p\mathcal{M}p$.

Lemma 5.5. *We have $\phi_{\mathcal{D},p}(P_{\ker \mathcal{D}_0}) - \phi_{\mathcal{D},p}(P_{\ker \mathcal{D}_1}) = 0$.*

Proof. Since we cut down by the projection p , we may assume that $v_0 = 0$ and so $v \in \mathcal{A}$ and $vv^* = p$. Then $P_{\ker \mathcal{D}_0} = vv^*\Phi_0$ and $P_{\ker \mathcal{D}_1} = v\Phi_0v^*$. As $\phi_{\mathcal{D}}(f\Phi_0) = \phi(f)$ for $f \in \mathcal{F}$ by Equation (6), we have

$$\phi_{\mathcal{D}}(vv^*\Phi_0 - v\Phi_0v^*) = \phi_{\mathcal{D}}((\sigma_{-i\beta}(v^*)v - vv^*)\Phi_0) = \phi(\sigma_{-i\beta}(v^*)v - vv^*) = 0.$$

\square

Thus the kernel correction terms vanish for modular partial isometries. Next we obtain a residue formula for the spectral flow:

Theorem 5.6. *Given the modular spectral triple for (A, σ, ϕ) let $v \in A^\sim$ be a modular partial isometry. Then $sf_{\phi_{\mathcal{D}}}(vv^*\mathcal{D}, v\mathcal{D}v^*)$ is given by*

$$\text{Res}_{r=1/2} \left(r \mapsto \phi_{\mathcal{D}}(v[\mathcal{D}, v^*](1 + \mathcal{D}^2)^{-r}) + \frac{1}{2} \int_1^\infty \phi_{\mathcal{D}}((\sigma_{-i\beta}(v^*)v - vv^*)\mathcal{D}(1 + s\mathcal{D}^2)^{-r})s^{-1/2} ds \right).$$

Proof. We apply Proposition 5.3 to the path $\mathcal{D}_t = p\mathcal{D} + tv[\mathcal{D}, v^*]$. Thus by Lemma 5.4 and Lemma 5.5 we get

$$sf_{\phi_{\mathcal{D}}}(vv^*\mathcal{D}, v\mathcal{D}v^*) = \text{Res}_{r=1/2} \left(\int_0^1 \phi_{\mathcal{D}}(v[\mathcal{D}, v^*](1 + \mathcal{D}_t^2)^{-r}) dt + \frac{1}{2}(\eta_{\mathcal{D}_1}(r) - \eta_{\mathcal{D}_0}(r)) \right).$$

First we observe that by [5, Proposition 10, Appendix B], the difference

$$(1 + (\mathcal{D} + tv[\mathcal{D}, v^*])^2)^{-r} - (1 + \mathcal{D}^2)^{-r}$$

is (uniformly) trace class in the corner $p\mathcal{M}p$ for $r \geq 1/2$. Hence in the spectral flow formula above we may exploit analyticity in r for $\Re(r) > 1/2$ as in [11] (we are working in the semifinite algebra $p\mathcal{M}p$ with trace $\phi_{\mathcal{D}}|_{p\mathcal{M}p}$) to write

$$\int_0^1 \phi_{\mathcal{D}}(v[\mathcal{D}, v^*](1 + (\mathcal{D} + tv[\mathcal{D}, v^*])^2)^{-r}) dt = \phi_{\mathcal{D}}(v[\mathcal{D}, v^*](1 + \mathcal{D}^2)^{-r}) + \text{remainder}.$$

The remainder is finite at $r = 1/2$, and in fact by [11], holomorphic at $r = 1/2$.

Next consider the eta terms. We have, for $\Re(r) > 1$,

$$\begin{aligned} \eta_{\mathcal{D}_1}(r) &= \int_1^\infty \phi_{\mathcal{D}}(pv\mathcal{D}v^*(1 + s(v\mathcal{D}v^*)^2)^{-r}) s^{-1/2} ds \\ &= \int_1^\infty \phi_{\mathcal{D}}(pv\mathcal{D}(1 + s\mathcal{D}^2)^{-r}v^*) s^{-1/2} ds \\ &= \int_1^\infty \phi_{\mathcal{D}}(\sigma_{-i\beta}(v^*)pv\mathcal{D}(1 + s\mathcal{D}^2)^{-r}) s^{-1/2} ds \end{aligned}$$

and

$$\eta_{\mathcal{D}_0}(r) = \int_1^\infty \phi_{\mathcal{D}}(p\mathcal{D}(1 + s\mathcal{D}^2)^{-r}) s^{-1/2} ds.$$

Using that $\sigma_{-i\beta}(v^*)pv = \sigma_{-i\beta}(v^*)v - v_0^*v_0$ and $p = vv^* - v_0v_0^*$, we see that to finish the proof we have to check that

$$\phi_{\mathcal{D}}((v_0^*v_0 - v_0v_0^*)\mathcal{D}(1 + s\mathcal{D}^2)^{-r}) = 0.$$

This is true since $\phi_{\mathcal{D}}(\cdot \mathcal{D}(1 + s\mathcal{D}^2)^{-r})$ is a trace on \mathcal{M} (note that if we considered partial isometries in a matrix algebra over \mathcal{A}^\sim we would have to require in addition that $v_0^*v_0 - v_0v_0^*$ is an element over \mathcal{F}). \square

Finally, when the circle action has full spectral subspaces, the eta corrections also vanish.

Corollary 5.7. *Assume the circle action σ has full spectral subspaces. Then for every modular partial isometry $v \in \mathcal{A}^\sim$ we have*

$$sf_{\phi_{\mathcal{D}}}(vv^*\mathcal{D}, v\mathcal{D}v^*) = \text{Res}_{r=1/2} \phi_{\mathcal{D}}(v[\mathcal{D}, v^*](1 + \mathcal{D}^2)^{-r}) dt.$$

Proof. Consider the modular partial isometry $w = v - v_0$. Then $w \in \mathcal{A}$, so we can apply the previous Theorem. Since the spectral flow corresponding to v and w coincide and $v[\mathcal{D}, v^*] = w[\mathcal{D}, w^*]$, all we have to do is to show that the eta term defined by w vanishes. By the assumption of full spectral subspaces we have

$$\phi_{\mathcal{D}}((\sigma_{-i\beta}(w^*)w - ww^*)\Phi_k) = \phi(\sigma_{-i\beta}(w^*)w - ww^*) = 0,$$

for all $k \in \mathbb{Z}$, and as

$$\phi_{\mathcal{D}}((\sigma_{-i\beta}(w^*)w - ww^*)\mathcal{D}(1 + s\mathcal{D}^2)^{-r}) = \sum_{k \in \mathbb{Z}} \phi_{\mathcal{D}}((\sigma_{-i\beta}(w^*)w - ww^*)\Phi_k) k(1 + sk^2)^{-r},$$

the eta term is indeed zero. \square

5.3. Twisted cyclic cocycles. This subsection is motivated by the observation of [9] that when there are no eta or kernel correction terms we can define a functional on $\mathcal{A} \otimes \mathcal{A}$ by

$$(a_0, a_1) \mapsto \omega\text{-}\lim_{s \rightarrow \infty} \frac{1}{s} \phi_{\mathcal{D}}(a_0[\mathcal{D}, a_1](1 + \mathcal{D}^2)^{-1/s-1/2})$$

which is, at least formally, a twisted (by $\sigma_{-i\beta}$) cyclic cocycle. However we saw in [13] that in the case of $SU_q(2)$ the eta corrections created a subtle difficulty in that individually they do not have the same

holomorphy properties as the term in the previous equation and that only by combining them do we obtain something we can understand in cohomological terms. Thus we set, for $a_0, a_1 \in \mathcal{A}$,

$$\eta_{\mathcal{D}}^r(a_0, a_1) = \frac{1}{2} \int_1^\infty \phi_{\mathcal{D}}((\sigma_{-i\beta}(a_1)a_0 - a_0a_1)\mathcal{D}(1+s\mathcal{D}^2)^{-r})s^{-1/2}ds.$$

This is well-defined for $\Re(r) > 1$, and as we shall see later, extends analytically to $\Re(r) > 1/2$. When we pair with a modular partial isometry we necessarily have $(r - 1/2)\eta_{\mathcal{D}}^r(v, v^*)$ bounded, since the sum of the eta term and $\phi_{\mathcal{D}}(v[\mathcal{D}, v^*](1 + \mathcal{D}^2)^{-r})$ has a simple pole by Proposition 5.3.

Throughout this Section, b^σ, B^σ denote the twisted Hochschild and Connes coboundary operators in twisted cyclic theory, [24]. The twisting will always come from the regular automorphism $\sigma := \sigma_{-i\beta} = \sigma_i^{\phi_{\mathcal{D}}}$ of \mathcal{A} (recall that an algebra automorphism σ is regular if $\sigma(a)^* = \sigma^{-1}(a^*)$, [24]).

In order to be able to describe the index pairing of Theorem 5.6 as the pairing of a twisted b^σ, B^σ cocycle with the modular K_1 group, we need to address the analytic difficulties we have just described. This is done in the next Lemma.

Lemma 5.8. *For $a_0, a_1 \in \mathcal{A}$, let*

$$\psi^r(a_0, a_1) = \phi_{\mathcal{D}}(a_0[\mathcal{D}, a_1](1 + \mathcal{D}^2)^{-r}) + \eta_{\mathcal{D}}^r(a_0, a_1).$$

Then for $a_0, a_1, a_2 \in \mathcal{A}$ the functions $r \mapsto \phi_{\mathcal{D}}(a_0[\mathcal{D}, a_1](1 + \mathcal{D}^2)^{-r})$ and $r \mapsto \eta_{\mathcal{D}}^r(a_0, a_1)$ are analytic for $\Re(r) > 1/2$, while $r \mapsto (b^\sigma \psi^r)(a_0, a_1, a_2)$ is analytic for $\Re(r) > 0$.

Proof. Recall that the algebra \mathcal{A} consists of finite sums of homogeneous elements in the domain of ϕ . Therefore we may assume that a_0, a_1, a_2 are homogeneous. Consider the conditional expectation $\Psi: \mathcal{N} \rightarrow \mathcal{N}^\sigma$, $\Psi(x) = \sum_n \Phi_n x \Phi_n$. Then $\phi_{\mathcal{D}} = \phi_{\mathcal{D}} \circ \Psi$. It follows that if $a_0 \in A_k$ and $a_1 \in A_m$ then $\phi_{\mathcal{D}}(a_0[\mathcal{D}, a_1](1 + \mathcal{D}^2)^{-r}) = 0$ unless $k = -m$, and in the latter case we have

$$\phi_{\mathcal{D}}(a_0[\mathcal{D}, a_1](1 + \mathcal{D}^2)^{-r}) = \sum_{n \in \mathbb{Z}} \frac{s_n}{(1 + n^2)^r},$$

where $s_n = m \phi_{\mathcal{D}}(a_0 a_1 \Phi_n)$. By Lemma 3.5 the sequence $\{s_n\}_n$ is bounded. Hence the function $\phi_{\mathcal{D}}(a_0[\mathcal{D}, a_1](1 + \mathcal{D}^2)^{-r})$ is analytic for $\Re(r) > 1/2$.

Consider now $\eta^r(a_0, a_1)$. If $a_0 \in A_k$ and $a_1 \in A_m$ then $\eta^r(a_0, a_1) = 0$ unless $k = -m$. In the latter case put $s_n = \phi_{\mathcal{D}}(a_0 a_1 \Phi_n)$. Notice that

$$\phi_{\mathcal{D}}(\sigma(a_1)a_0\Phi_n) = \phi_{\mathcal{D}}(a_0\Phi_n a_1) = \phi_{\mathcal{D}}(a_0 a_1 \Phi_{n-m}) = s_{n-m}.$$

The sequence $\{s_n\}_n$ is bounded. Assume $m \geq 0$. Then for $\Re(r) > 1$ we have

$$\int_1^\infty \phi_{\mathcal{D}}((\sigma(a_1)a_0 - a_0a_1)\mathcal{D}(1+s\mathcal{D}^2)^{-r})s^{-1/2}ds = \sum_{n \in \mathbb{Z}} \int_1^\infty \frac{(s_{n-m} - s_n)n}{(1 + sn^2)^r} s^{-1/2}ds$$

which we may write as

$$\begin{aligned} & 2 \sum_{n>0} (s_{n-m} - s_n) \int_n^\infty \frac{dt}{(1+t^2)^r} - 2 \sum_{n<0} (s_{n-m} - s_n) \int_{-n}^\infty \frac{dt}{(1+t^2)^r} \\ &= 2 \sum_{n=-m+1}^0 s_n \int_{n+m}^\infty \frac{dt}{(1+t^2)^r} - 2 \sum_{n>0} s_n \int_n^{n+m} \frac{dt}{(1+t^2)^r} \\ &+ 2 \sum_{n=-m}^{-1} s_n \int_{-n}^\infty \frac{dt}{(1+t^2)^r} - 2 \sum_{n<-m} s_n \int_{-n-m}^{-n} \frac{dt}{(1+t^2)^r}. \end{aligned}$$

The above series of functions analytic on $\Re(r) > 1/2$ converge uniformly on $\Re(r) > 1/2 + \epsilon$ for every $\epsilon > 0$. A similar argument works for $m \leq 0$. Hence the function $r \mapsto \eta^r(a_0, a_1)$ extends analytically to $\Re(r) > 1/2$.

Turning to $b^\sigma \psi^r$, first notice that $b^\sigma \eta_{\mathcal{D}}^r = 0$, since $r \mapsto b^\sigma \eta_{\mathcal{D}}^r$ is analytic for $\Re(r) > 1/2$ and $\eta_{\mathcal{D}}^r = b^\sigma \theta_{\mathcal{D}}^r$ for $\Re(r) > 1$, where

$$\theta_{\mathcal{D}}^r(a_0) = -\frac{1}{2} \int_1^\infty \phi_{\mathcal{D}}(a_0 \mathcal{D}(1 + s\mathcal{D}^2)^{-r}) s^{-1/2} ds.$$

It follows that $(b^\sigma \psi^r)(a_0, a_1, a_2)$ is given by

$$\begin{aligned} & \phi_{\mathcal{D}}(a_0 a_1 [\mathcal{D}, a_2] (1 + \mathcal{D}^2)^{-r} - \phi_{\mathcal{D}}(a_0 [\mathcal{D}, a_1 a_2] (1 + \mathcal{D}^2)^{-r}) + \phi_{\mathcal{D}}(\sigma(a_2) a_0 [\mathcal{D}, a_1] (1 + \mathcal{D}^2)^{-r}) \\ & = -\phi_{\mathcal{D}}(a_0 [\mathcal{D}, a_1] a_2 (1 + \mathcal{D}^2)^{-r}) + \phi_{\mathcal{D}}(\sigma(a_2) a_0 [\mathcal{D}, a_1] (1 + \mathcal{D}^2)^{-r}). \end{aligned}$$

If $a_0 \in A_k$, $a_1 \in A_l$ and $a_2 \in A_m$, then the above expression is zero unless $k + l + m = 0$. In the latter case put $s_n = l \phi_{\mathcal{D}}(a_0 a_1 a_2 \Phi_n)$. Then a computation similar to that for η^r yields, for $\Re(r) > 1/2$,

$$\begin{aligned} (b^\sigma \psi^r)(a_0, a_1, a_2) &= \sum_{n \in \mathbb{Z}} s_n ((1 + (n + m)^2)^{-r} - (1 + n^2)^{-r}) \\ &= \sum_{n \in \mathbb{Z}} s_n (1 + n^2)^{-r} \left(\left(1 + \frac{2mn + m^2}{1 + n^2} \right)^{-r} - 1 \right) \end{aligned}$$

Using that if Ω is a compact subset of $\Re(r) > 0$ then $|(1 + x)^{-r} - 1| \leq C|x|$ for some $C > 0$, sufficiently small x and all $r \in \Omega$, we see that the above series converges uniformly on Ω . Hence $(b^\sigma \psi^r)(a_0, a_1, a_2)$ extends analytically to $\Re(r) > 0$. \square

The following result links our analytic constructions to twisted cyclic theory.

Theorem 5.9. *Given the modular spectral triple for (A, σ, ϕ) define a bilinear functional on \mathcal{A} with values in the functions holomorphic for $\Re(r) > 1$ by*

$$a_0, a_1 \mapsto \left(r \mapsto \left(\phi_{\mathcal{D}}(a_0 [\mathcal{D}, a_1] (1 + \mathcal{D}^2)^{-r}) + \frac{1}{2} \int_1^\infty \phi_{\mathcal{D}}((\sigma(a_1) a_0 - a_0 a_1) \mathcal{D}(1 + s\mathcal{D}^2)^{-r}) s^{-1/2} ds \right) \right)$$

This functional continues analytically to $\Re(r) > 1/2$ and is a twisted b, B -cocycle modulo functions holomorphic for $\Re(r) > 0$. The twisting is given by the regular automorphism $\sigma := \sigma_{-i\beta} = \sigma_i^{\phi_{\mathcal{D}}}$.

Proof. As before, for $\Re(r) > 1$ we define the functional ψ^r by the formula

$$\psi^r(a_0, a_1) = \phi_{\mathcal{D}}(a_0 [\mathcal{D}, a_1] (1 + \mathcal{D}^2)^{-r}) + \frac{1}{2} \int_1^\infty \phi_{\mathcal{D}}((\sigma(a_1) a_0 - a_0 a_1) \mathcal{D}(1 + s\mathcal{D}^2)^{-r}) s^{-1/2} ds,$$

and then extend ψ^r analytically to $\Re(r) > 1/2$, which is possible by Lemma 5.8. Then $(B^\sigma \psi^r)(a_0) = \psi^r(1, a_0)$ and for $\Re(r) > 1$ is given by

$$(B^\sigma \psi^r)(a_0) = \phi_{\mathcal{D}}([\mathcal{D}, a_0] (1 + \mathcal{D}^2)^{-r}) + \frac{1}{2} \int_1^\infty \phi_{\mathcal{D}}((\sigma(a_0) - a_0) \mathcal{D}(1 + s\mathcal{D}^2)^{-r}) s^{-1/2} ds.$$

The first term vanishes since $\Psi([\mathcal{D}, a_0]) = 0$ for any $a_0 \in \mathcal{A}$, while the second term vanishes by σ_t -invariance of $\phi_{\mathcal{D}}$. That $b^\sigma \psi^r$ is analytic for $\Re(r) > 0$ was proved in the last Lemma. \square

Corollary 5.10. *If the circle action has full spectral subspaces then for all $a_0, a_1 \in \mathcal{A}$ the residue*

$$\phi_1(a_0, a_1) := \text{Res}_{r=1/2} \phi_{\mathcal{D}}(a_0 [\mathcal{D}, a_1] (1 + \mathcal{D}^2)^{-r})$$

exists and equals $\phi(a_0[\mathcal{D}, a_1])$. It defines a twisted cyclic cocycle on \mathcal{A} , and for any modular partial isometry $v \in \mathcal{A}$

$$sf_{\phi_{\mathcal{D}}}(vv^*\mathcal{D}, v\mathcal{D}v^*) = \phi_1(v, v^*) = \phi(v[\mathcal{D}, v^*]).$$

Proof. Under the full spectral subspaces assumption we have $\phi_{\mathcal{D}}(f\Phi_n) = \phi(f)$ for $f \in \mathcal{F}$, whence

$$\phi_{\mathcal{D}}(a_0[\mathcal{D}, a_1](1 + \mathcal{D}^2)^{-r}) = \phi_{\mathcal{D}}(\Psi(a_0[\mathcal{D}, a_1])(1 + \mathcal{D}^2)^{-r}) = \phi(a_0[\mathcal{D}, a_1]) \sum_{n \in \mathbb{Z}} \frac{1}{(1 + n^2)^r}.$$

This shows that the residue exists and equals $\phi(a_0[\mathcal{D}, a_1])$. That it defines a twisted cyclic cocycle follows from the proof of Theorem 5.9. That $\phi_1(v, v^*)$ computes the spectral flow follows from Corollary 5.7. \square

Remarks 5.11. It is of course easy to see directly that $\phi(a_0[\mathcal{D}, a_1])$ is a twisted cyclic cocycle, while the fact that it computes the spectral flow agrees with Lemma 4.9.

6. EXAMPLES

Our first two examples are covered in detail in [9, 13] so we will only present a summary here.

Example 1. For the algebra \mathcal{O}_n (with generators S_1, \dots, S_n) we write S_α for the product $S_{\mu_1} \dots S_{\mu_k}$ and $k = |\alpha|$. We take the usual gauge action σ , and the unique KMS state ϕ for this circle action.

In the Cuntz algebra case we have full spectral subspaces. Due to the absence of eta terms, the analytic formula is the easiest to apply, so we can compute the pairing with $S_\alpha S_\beta^*$ using the residue cocycle, Corollary 5.10, and get

$$sf(S_\alpha S_\alpha^* \mathcal{D}, S_\alpha S_\beta^* \mathcal{D} S_\beta S_\alpha^*) = (|\beta| - |\alpha|) \frac{1}{n^{|\alpha|}}.$$

Example 2. For $SU_q(2)$ we used the graph algebra description of Hong and Szymanski, [18], and we use the notation and computations from [13]. There we introduced a new set of generators T_k, \tilde{T}_k, U_n for this algebra. The generators T_k and \tilde{T}_k are non-trivial homogenous partial isometries for the modular group of the Haar state, h , which is a $\text{KMS}_{-\log q^2}$ state.

For $SU_q(2)$ there are eta correction terms. Given the explicit computations in [13] and the description of the fixed point algebra as the unitization of an infinite direct sum of copies of $C(S^1)$ (that is the C^* -algebra of the one point compactification of an infinite union of circles of radius q^{2k} , $k \geq 0$) it is not hard to see that our SSA is satisfied for $SU_q(2)$.

The presence of the eta corrections makes the analytic computation of spectral flow from the twisted cocycle harder (it can still be done explicitly as in [13]). Instead we employ the factorisation through the KK -pairing. Taking the value of the trace $\text{Tr}_\phi(T_k^* T_k \Phi_j) = q^{2(|j|+1)}$ from [13] we have

$$sf_{\phi_{\mathcal{D}}}(T_k^* T_k \mathcal{D}, T_k^* \mathcal{D} T_k) = \text{Ev}(e^{\log q^2}) \circ \tau_* \left(\sum_{j=-k}^{-1} [T_k^* T_k \Phi_j] \chi^j \right) = \sum_{j=-k}^{-1} \text{Tr}_\phi(T_k^* T_k \Phi_j) q^{2j} = kq^2.$$

The point of this example is that there are naturally occurring examples satisfying the SSA but without full spectral subspaces.

Example 3: the Araki-Woods factors. We will follow the treatment of the Araki-Woods factors in Pedersen [29], Subsection 8.12 and the subsequent discussion. We let A be the Fermion algebra,

that is the C^* -inductive limit of the matrix algebras $\text{Mat}_{2^n}(\mathbb{C})$ which is the n -fold tensor product of the matrix algebra of 2×2 matrices $\text{Mat}_2(\mathbb{C})$.

For $0 < \lambda < 1/2$ let

$$h_n = \otimes_{k=1}^n \begin{pmatrix} 2(1-\lambda) & 0 \\ 0 & 2\lambda \end{pmatrix}$$

Let ϕ be the tracial state on A (given by the tensor product of the normalised traces on $\text{Mat}_2(\mathbb{C})$) and define

$$\phi_\lambda(x) = \phi(h_n x), \quad x \in \text{Mat}_{2^m}(\mathbb{C}), \quad m \leq n.$$

Then ϕ_λ is a state on $\text{Mat}_{2^m}(\mathbb{C})$ and is independent of n . By continuity it extends to a state on A . Consider the automorphism group defined by $\text{Ad } h_n^{-it}$. It is not hard to see that ϕ_λ satisfies the KMS condition with respect to $\text{Ad } h_n^{-it}$ at 1 for this group or equivalently at $\beta = \ln \frac{1-\lambda}{\lambda}$ for the gauge action $\sigma_t = \text{Ad } h_n^{-it/\beta}$. Everything extends by continuity to A . Then the GNS representation corresponding to ϕ_λ generates a type III $_{\lambda'}$ factor, where $\lambda' = \lambda/(1-\lambda)$ (for a proof see [29] 8.15.13).

The simplest way to see that we have full spectral subspaces for the circle action σ is to replace this version of the Fermion algebra by the isomorphic copy given by annihilation and creation operators, see e.g. [17].

To describe the isomorphism, we let σ_j , $j = 1, 2, 3$ be the Pauli matrices in their usual representation:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then the isomorphism is given by defining $a_j = \sigma_3 \otimes \dots \otimes \sigma_3 \otimes (\sigma_1 + i\sigma_2)/2$, where the last term is in the j -th tensorial factor. Then the a_j , $j \in \mathbb{N}$, and their adjoints a_j^* satisfy the usual relations of the C^* -algebra of the canonical anticommutation relations (i.e. the Fermion algebra):

$$a_j a_k^* + a_k^* a_j = \delta_{jk}, \quad a_j a_k + a_k a_j = 0.$$

The gauge invariant algebra is generated by monomials in the a_j, a_k^* which have equal numbers of creation and annihilation operators. Clearly A_1 is generated by monomials with one more creation operator than annihilation operator. From the anticommutation relations above it is now clear that $A_1^* A_1$ and $A_1 A_1^*$ are dense in the gauge invariant subalgebra. So we have full spectral subspaces. Thus the main results of the paper apply to this example.

Modular partial isometries are easy to find, since each a_j is an homogenous partial isometry in A_1 . For a single a_j we can employ the twisted cyclic cocycle to get the index

$$sf_{\phi_{\mathcal{D}}}(a_j a_j^* \mathcal{D}, a_j \mathcal{D} a_j^*) = -\phi(a_j a_j^*) = -\lambda = -(1 + e^\beta)^{-1}.$$

Similarly if we have the partial isometry v formed by taking the product of n distinct a_j 's we obtain

$$sf_{\phi_{\mathcal{D}}}(v v^* \mathcal{D}, v \mathcal{D} v^*) = -n(1 + e^\beta)^{-n}.$$

In [9] we made the observation that for modular unitaries u_v , $sf_{\phi_{\mathcal{D}}}(\mathcal{D}, u_v \mathcal{D} u_v^*)$ is just Araki's relative entropy [1] of the two KMS weights $\phi_{\mathcal{D}}$ and $\phi_{\mathcal{D}} \circ \text{Ad } u_v$. In this example of the Fermion algebra we see that the relative entropy depends on two physical parameters, the inverse temperature β and the modulus of the charge n carried by the product of Fermion annihilation or creation operators appearing in v .

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