ON UNIQUENESS OF ARITHMETIC MODELS

SERGEY NESHVEYEV

The goal of this appendix is to show that the endomotive $\mathcal{E}_K$ constructed in the paper is, in an appropriate sense, the unique endomotive that provides an arithmetic model for the BC-system $\mathcal{A}_K$. We will also give an alternative proof of the existence of $\mathcal{E}_K$.

Assume $\mathcal{E} = E \rtimes S$ is an algebraic endomotive such that the analytic endomotive $\mathcal{E}^{an}$ is $A_K = C(Y_K) \rtimes I_K$. By this we mean that $S = I_K$ and there exists a $Gal(\overline{K}/K)$- and I_K-equivariant homeomorphism of $Hom_{K\text{-alg}}(E, \overline{K})$ onto $Y_K = \widehat{\mathcal{O}}_K \times_{\widehat{\mathcal{O}}_K^\times} Gal(K^{ab}/K)$. Then E considered as a K-subalgebra of $C(Y_K)$ has the following properties:

(a) every function in E is locally constant;
(b) E separates points of Y_K;
(c) E contains the idempotents $\rho_{\mathfrak{a}}^n(1)$ for all $\mathfrak{a} \in I_K$ and $n \in \mathbb{N}$;
(d) for every $f \in E$ we have $f(Y_K) \subset K^{ab}$ and the map $f\colon Y_K \to K^{ab}$ is $Gal(K^{ab}/K)$-equivariant.

Recall that the endomorphism $\rho_{\mathfrak{a}}$ is defined by $\rho_{\mathfrak{a}}(f) = f(\mathfrak{a}^{-1}\cdot)$, with the convention that $\rho_{\mathfrak{a}}(f)(y) = 0$ if $y \notin \mathfrak{a}Y_K$.

Theorem 1. *The subalgebra $E_K = \varinjlim E_{\mathfrak{f}}$ of $C(Y_K)$ constructed in the paper is the unique K-subalgebra of $C(Y_K)$ with properties (a)-(d). It is, therefore, the K-algebra of all locally constant K^{ab}-valued $Gal(K^{ab}/K)$-equivariant functions on Y_K.*

Proof. We have to show that if a K-subalgebra $E \subset C(Y_K)$ satisfies properties (a)-(d), then it contains every locally constant K^{ab}-valued $Gal(K^{ab}/K)$-equivariant function f.
Fix a point $y \in Y_K$. Let $L \subset K^{ab}$ be the field of elements fixed by the stabilizer G_y of y in $Gal(K^{ab}/K)$. Then $f(y) \in L$ by equivariance.

Lemma 2. *The map $E \ni h \mapsto h(y) \in L$ is surjective.*

Proof. Let L' be the image of E under the map $h \mapsto h(y)$. Since E is a K-algebra, L' is a subfield of L. If $L' \neq L$ then there exists a nontrivial element of $Gal(L/L') \subset Gal(L/K) = Gal(K^{ab}/K)/G_y$. Lift this element to an element g of $Gal(K^{ab}/K)$. Then, on the one hand, $gy \neq y$, and, on the other hand, for every $h \in E$ we have $h(gy) = gh(y) = h(y)$. This contradicts property (b). □

Therefore there exists $h \in E$ such that $h(y) = f(y)$. Since the functions f and h are locally constant, there exists a neighbourhood W of y such that f and h coincide on W. We may assume that W is the image of an open set of the form

$$\left(\prod_{v\in F} W_v \times \widehat{\mathcal{O}}_{K,F}\right) \times W' \subset \widehat{\mathcal{O}}_K \times Gal(K^{ab}/K)$$

in Y_K, where F is a finite set of finite places of K; here we use the notation $\widehat{\mathcal{O}}_K = \prod_{v\in V_{K,f}} \mathcal{O}_{K,v}$, $\widehat{\mathcal{O}}_{K,F} = \prod_{v\in V_{K,f}\setminus F} \mathcal{O}_{K,v}$. Furthermore, we may assume that $F = F' \sqcup F''$ and for $v \in F'$ we have $W_v \subset \mathfrak{p}_v^{n_v}\mathcal{O}_{K,v}^\times$, while for $v \in F''$ we have $W_v = \mathfrak{p}_v^{n_v}\mathcal{O}_{K,v}$. Since the functions f and h are equivariant, they coincide on the set $U = Gal(K^{ab}/K)W$. The equality

$$Gal(K^{ab}/K)W = \left(\prod_{v\in F'} \mathfrak{p}_v^{n_v}\mathcal{O}_{K,v}^\times \times \prod_{v\in F''} \mathfrak{p}_v^{n_v}\mathcal{O}_{K,v} \times \widehat{\mathcal{O}}_{K,F}\right) \times_{\widehat{\mathcal{O}}_K^\times} Gal(K^{ab}/K)$$

Date: May 30, 2011; revised on January 6, 2012.

shows that the characteristic function p of U belongs to E: it is the product of $\rho^{n_v}_{\mathfrak{p}_v}(1) - \rho^{n_v+1}_{\mathfrak{p}_v}(1)$, $v \in F'$, and $\rho^{n_v}_{\mathfrak{p}_v}(1)$, $v \in F''$. Therefore $fp = hp \in E$.
Thus we have proved that for every point $y \in Y_K$ there exists a neighbourhood U of y such that the characteristic function p of U belongs to E and $fp \in E$. By compactness we conclude that $f \in E$. □

The following consequence of the above theorem shows that the arithmetic subalgebra $\mathcal{E}_K = E_K \rtimes I_K$ of the BC-system is unique within a class of algebras not necessarily arising from endomotives.

Theorem 3. *The K-subalgebra $\mathcal{E}_K$ of A_K constructed in the paper is the unique arithmetic subalgebra that is generated by some locally constant functions on Y_K and by the elements $U_\mathfrak{a}$ and $U^*_\mathfrak{a}$, $\mathfrak{a} \in I_K$.*

Proof. Assume $\mathcal{E}$ is such an arithmetic subalgebra. Consider the K-algebra $E = \mathcal{E} \cap C(Y_K)$. It satisfies properties (a)-(c), while (d) a priori holds only on the subset $Y^\times_K \subset Y_K$. However, the algebra E is invariant under the endomorphisms $\sigma_\mathfrak{a}$, $\mathfrak{a} \in I_K$, defined by $\sigma_\mathfrak{a}(f) = f(\mathfrak{a}\,\cdot) = U^*_\mathfrak{a} f U_\mathfrak{a}$. Hence property (d) holds on the subsets $\mathfrak{a}Y^\times_K$ of Y_K. Since $\cup_{\mathfrak{a}\in I_K}\mathfrak{a}Y^\times_K$ is dense in Y_K and the functions in E are locally constant, it follows that (d) holds on the whole set Y_K. Therefore $E = E_K$ by the previous theorem, and so $\mathcal{E} = \mathcal{E}_K$. □

Let E be the K-algebra of locally constant K^{ab}-valued $Gal(K^{ab}/K)$-equivariant functions on Y_K. Let us now show directly that $E \rtimes I_K$ is an arithmetic subalgebra of A_K.
In order to prove the density of the $\mathbb{C}$-algebra generated by $E \rtimes I_K$ in A_K, it suffices to show that the $\mathbb{C}$-algebra generated by E is equal to the algebra of complex valued locally constant functions on Y_K. Since Y_K is a projective limit of finite $Gal(K^{ab}/K)$-sets, this follows from the following simple statement: if L is a finite Galois extension of K and Y is a finite $Gal(L/K)$-set, then the L-linear span of the K-algebra of $Gal(L/K)$-equivariant functions $Y \to L$ coincides with the L-algebra of all L-valued functions on Y.
In particular, E separates points of Y_K. The property that K^{ab} is generated by the values $f(y)$, $f \in E$, for any $y \in Y^\times_K$, follows now from Lemma 2, as $Gal(K^{ab}/K)$ acts freely on $Y^\times_K$.
Thus $E \rtimes I_K \subset A_K$ is indeed an arithmetic subalgebra. Furthermore, it is easy to see that E is an inductive limit of étale K-algebras and $Hom_{K\text{-alg}}(E, \overline{K}) = Y_K$. Therefore $\mathcal{E} = E \rtimes I_K$ is, in fact, an endomotive and $\mathcal{E}^{an} = A_K$.

We finish by making a few remarks about general arithmetic subalgebras of the BC-system $\mathcal{A}_K$. Assume $\mathcal{E} \subset A_K$ is an arithmetic subalgebra. Also assume that it contains the elements $U_\mathfrak{a}$ and $U^*_\mathfrak{a}$ for all $\mathfrak{a} \in I_K$. Consider the image of $\mathcal{E}$ under the canonical conditional expectation $A_K \to C(Y_K)$, and let E be the K-algebra generated by this image. Then E satisfies the following properties:

- (a′) every function in E is continuous;
- (b′) the $\mathbb{C}$-algebra generated by E is dense in $C(Y_K)$; in particular, E separates points of Y_K;
- (c′) E is invariant under the endomorphisms $\rho_\mathfrak{a}$ and $\sigma_\mathfrak{a}$ for all $\mathfrak{a} \in I_K$;
- (d′) for every $f \in E$ we have $f(Y^\times_K) \subset K^{ab}$ and the map $f\colon Y^\times_K \to K^{ab}$ is $Gal(K^{ab}/K)$-equivariant.

Conversely, if E is a unital K-algebra of functions on Y_K with properties (a′)-(d′), then $\mathcal{E} = E \rtimes I_K$ is an arithmetic subalgebra of A_K and the intersection $\mathcal{E} \cap C(Y_K)$, as well as the image of $\mathcal{E}$ under the conditional expectation onto $C(Y_K)$, coincides with E. Note again that the property that K^{ab} is generated by the values $f(y)$, $f \in E$, for any $y \in Y^\times_K$, follows from the proof of Lemma 2. The largest algebra satisfying properties (a′)-(d′) is the K-algebra of continuous functions such that their restrictions to $\mathfrak{a}Y^\times_K$ are K^{ab}-valued and $Gal(K^{ab}/K)$-equivariant for all $\mathfrak{a} \in I_K$. This algebra is strictly larger than the algebra E_K. Indeed, it, for example, contains the functions of the form $\sum^\infty_{n=0} q_n \rho^n_{\mathfrak{p}_v}(1)$, where $\sum_n q_n$ is any convergent series of rational numbers. Such a function takes value $\sum^\infty_{n=0} q_n$, which can be any real number, at every point $y \in \cap_{n\ge0}\mathfrak{p}^n_v Y_K$.

Department of Mathematics, University of Oslo, P.O. Box 1053 Blindern, N-0316 Oslo, Norway.
E-mail address: sergeyn@math.uio.no