VON NEUMANN ALGEBRAS ARISING FROM BOST-CONNES TYPE SYSTEMS

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Abstract. We show that the KMS_β-states of Bost-Connes type systems for number fields in the region $0 < \beta \le 1$, as well as of the Connes-Marcolli GL_2-system for $1 < \beta \le 2$, have type III_1. This is equivalent to ergodicity of various actions on adelic spaces. For example, the case $\beta = 2$ of the GL_2-system corresponds to ergodicity of the action of $\mathrm{GL}_2(\mathbb{Q})$ on $\mathrm{Mat}_2(\mathbb{A})$ with its Haar measure.

Introduction

The Bost-Connes system [4] is a C*-dynamical system such that for inverse temperatures $\beta > 1$ the extremal KMS_β-states carry a free transitive action of the Galois group of the maximal abelian extension of $\mathbb{Q}$, have type I and partition function $\zeta(\beta)$, while for every $\beta \in (0, 1]$ there exists a unique KMS_β-state of type III_1. The uniqueness and the type of the KMS_β-states in the critical interval $(0, 1]$ are the most difficult parts of the analysis of the system. The result is equivalent to ergodicity of certain measures on the space $\mathbb{A}$ of adeles with respect to the action of $\mathbb{Q}^*$. In particular, the case $\beta = 1$ corresponds to a Haar measure μ_1. To see why ergodicity of μ_1 is nontrivial, observe that as $\mathbb{Q}$ is discrete in $\mathbb{A}$, the orbit of any point in $\mathbb{A}^*$ is discrete in $\mathbb{A}$, while the ergodicity implies that almost every orbit in $\mathbb{A}$ is dense. There is of course no contradiction since $\mathbb{A}^*$ is a subset of $\mathbb{A}$ of measure zero, but one does realize that it is difficult to immediately see a single dense orbit in $\mathbb{A}$.

Recently the construction of Bost and Connes has been generalized first to imaginary quadratic fields [8] and then to arbitrary number fields [11]. The crucial step of imaginary quadratic fields was achieved by introducing a universal system of quadratic fields, the so called GL_2-system of Connes and Marcolli [7]. In [18] and [16] we analyzed these systems in the critical intervals $0 < \beta \le 1$ for number fields and $1 < \beta \le 2$ for the GL_2-system, and showed that there exist unique KMS_β-states. The aim of the present paper is to prove that these KMS_β-states have type III_1. For number fields the proof is similar to the one for the original Bost-Connes system [4, 20]. The interesting case is that of the GL_2-system. It amounts to proving that the action of $\mathrm{GL}_2(\mathbb{Q})$ on $\mathrm{PGL}_2(\mathbb{R}) \times \mathrm{Mat}_2(\mathbb{A}_f)$ has type III_1 with respect to certain product-measures. After passing to the quotient space $\mathrm{GL}_2(\mathbb{Z})\backslash(\mathrm{PGL}_2(\mathbb{R}) \times \mathrm{Mat}_2(\mathbb{A}_f))/\mathrm{GL}_2(\hat{\mathbb{Z}})$, we essentially use an argument showing that a nonzero element of the asymptotic ratio set is contained in the ratio set, but we write this argument in terms of L^2-spaces rather than measure spaces and use a representation of the Hecke algebra $\mathcal{H}(\mathrm{GL}_2(\mathbb{Q}), \mathrm{GL}_2(\mathbb{Z}))$ instead of group actions. An additional difficulty is that we do not have a product decomposition of the representation of the Hecke algebra. In other words, the Hecke operators defined by elements of $\mathrm{GL}_2(\mathbb{Z}[p^{-1}])$ act nontrivially on $\mathrm{GL}_2(\mathbb{Z})\backslash(\mathrm{PGL}_2(\mathbb{R}) \times \prod_{q\ne p} \mathrm{Mat}_2(\mathbb{Z}_q))/\mathrm{GL}_2(\hat{\mathbb{Z}})$. As a side remark, a similar problem would not arise for the finite part of the GL_2-system [17]. What saves the day for the full system is that this action is mixing on large subsets, which is a consequence of a variant of equidistribution of Hecke points [6].

Apart from some trivial cases when there is a subgroup of measure preserving transformations acting ergodically (for example, for $\mathrm{GL}_2(\mathbb{Q})$ acting on $\mathbb{R}^2$), computations of ratio sets are usually quite hard, see e.g. [5, 12, 13]. A large class of type III_1 actions can be obtained as follows [22]. Let G be a connected non-compact simple Lie group with finite center, $\Gamma \subset G$ a lattice and $P \subset G$

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a parabolic subgroup. Then the action of Γ on G/P has type III_1. This is proved by identifying the underlying space of the associated flow with the measure-theoretic quotient $\Gamma\backslash G/P_0$, where P_0 is the kernel of the modular function of P, and using that the action of P_0 on $\Gamma\backslash G$ is mixing by Howe-Moore's theorem [22]. With these examples in mind, it seems only natural that to compute the type of the states of the GL_2-system a form of adelic mixing is needed.

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1. Actions of type III_1

Assume a countable group G acts ergodically on a measure space (X,μ). The ratio set of the action [15] consists of all numbers $\lambda\ge0$ such that for any $\varepsilon>0$ and any subset $A\subset X$ of positive measure there exists $g\in G$ such that

$$\mu\left(\left\{x\in gA\cap A : \left|\frac{dg\mu}{d\mu}(x)-\lambda\right|<\varepsilon\right\}\right)>0,$$

where $g\mu$ is the measure defined by $g\mu(Z)=\mu(g^{-1}Z)$. The ratio set depends only on the orbit equivalence relation $\mathcal{R}=\{(x,gx)\mid x\in X,\ g\in G\}\subset X\times X$ and the measure class of μ. We will denote it by $r(\mathcal{R},\mu)$. The set $r(\mathcal{R},\mu)\setminus\{0\}$ is a closed subgroup of $\mathbb{R}^*_+$. The action is said to be of type III_1 if this subgroup coincides with the whole group $\mathbb{R}^*_+$.

Denote by λ_∞ the Lebesgue measure on $\mathbb{R}$. We have two commuting actions of $\mathbb{R}$ and G on $(\mathbb{R}_+\times X,\lambda_\infty\times\mu)$,

$$g(t,x)=\left(\frac{dg\mu}{d\mu}(gx)t,gx\right)\quad\text{for}\quad g\in G,\quad s(t,x)=(e^{-s}t,x)\quad\text{for}\quad s\in\mathbb{R}.$$

The flow of weights [9] of the von Neumann algebra $W^*(\mathcal{R})$ is the flow induced by the above action of $\mathbb{R}$ on the measure-theoretic quotient of $(\mathbb{R}_+\times X,\lambda_\infty\times\mu)$ by the action of G. The original action of G on (X,μ) has type III_1 if and only if the flow of weights is trivial, that is, the action of G on $(\mathbb{R}_+\times X,\lambda_\infty\times\mu)$ is ergodic.

Let $\{(X_n,\mu_n)\}_{n=1}^\infty$ be a sequence of at most countable probability spaces. Put $(X,\mu)=\prod_n(X_n,\mu_n)$, and define an equivalence relation $\mathcal{R}$ on X by

$$x\sim y\quad\text{if}\quad x_n=y_n\quad\text{for all}\quad n\quad\text{large enough.}$$

Since the equivalence relation $\mathcal{R}$ is countable, it is generated by an action of a countable group. Explicitly, we may assume that the sets X_n are cyclic groups. Then $\mathcal{R}$ is generated by the action of $\oplus_nX_n$ on $\prod_nX_n$ by translations. In particular, the ratio set $r(\mathcal{R},\mu)$ is well-defined.

For a finite subset $I\subset\mathbb{N}$ and $a\in\prod_{n\in I}X_n$ put

$$Z(a)=\{x\in X\mid x_n=a_n\ \text{for}\ n\in I\}.$$

The asymptotic ratio set $r_\infty(\mathcal{R},\mu)$ consists by definition [1] of all numbers $\lambda\ge0$ such that for any $\varepsilon>0$ there exist a sequence $\{I_n\}_{n=1}^\infty$ of mutually disjoint finite subsets of $\mathbb{N}$, disjoint subsets $K_n,L_n\subset\prod_{k\in I_n}X_k$ and bijections $\varphi_n\colon K_n\to L_n$ such that

$$\left|\frac{\mu(Z(\varphi_n(a)))}{\mu(Z(a))}-\lambda\right|<\varepsilon\ \text{for all}\ a\in K_n\ \text{and}\ n\ge1,\quad\text{and}\quad\sum_{n=1}^\infty\sum_{a\in K_n}\mu(Z(a))=\infty.$$

It is known that $r_\infty(\mathcal{R},\mu)\setminus\{0\}=r(\mathcal{R},\mu)\setminus\{0\}$. We will only need the rather obvious inclusion $\subset$.

2. Bost-Connes type systems for number fields

Suppose K is an algebraic number field with subring of integers $\mathcal{O}$. Denote by V_K the set of places of K, and by $V_{K,f} \subset V_K$ the subset of finite places. For $v \in V_K$ denote by K_v the corresponding completion of K. If v is finite, let $\mathcal{O}_v$ be the closure of $\mathcal{O}$ in K_v. Denote also by $K_\infty = \prod_{v|\infty} K_v$ the completion of K at all infinite places. The adele ring $\mathbb{A}_K$ is the restricted product of the rings K_v, $v \in V_K$, with respect to $\mathcal{O}_v \subset K_v$, $v \in V_{K,f}$. When the product is restricted to $v \in V_{K,f}$, we get the ring $\mathbb{A}_{K,f}$ of finite adeles. The ring of finite integral adeles is $\hat{\mathcal{O}} = \prod_{v\in V_{K,f}} \mathcal{O}_v \subset \mathbb{A}_{K,f}$. We identify $\mathbb{A}^*_{K,f}$ with the subgroup of $\mathbb{A}^*_K$ consisting of elements with coordinates 1 for all infinite places.

Consider the topological space $\mathcal{G}(K^{ab}/K) \times \mathbb{A}_{K,f}$, where $\mathcal{G}(K^{ab}/K)$ is the Galois group of the maximal abelian extension of K. On this space there is an action of the group $\mathbb{A}^*_{K,f}$ of finite ideles, via the Artin map $s\colon \mathbb{A}^*_K \to \mathcal{G}(K^{ab}/K)$ on the first component and via multiplication on the second component:

$$j(\gamma, m) = (\gamma s(j)^{-1}, jm) \ \text{ for } \ j \in \mathbb{A}^*_{K,f}, \ \ \gamma \in \mathcal{G}(K^{ab}/K), \ \ m \in \mathbb{A}_{K,f}.$$

Consider the quotient space $\mathcal{G}(K^{ab}/K) \times_{\hat{\mathcal{O}}^*} \mathbb{A}_{K,f}$ by the action of $\hat{\mathcal{O}}^* \subset \mathbb{A}^*_K$. On this space we have a quotient action of the group $\mathbb{A}^*_{K,f}/\hat{\mathcal{O}}^*$, which is isomorphic to the group J_K of fractional ideals.

One can define a Bost-Connes type system for K [8, 11, 18] as the corner pAp of the crossed product

$$A := C_0(\mathcal{G}(K^{ab}/K) \times_{\hat{\mathcal{O}}^*} \mathbb{A}_{K,f}) \rtimes J_K,$$

where p is the characteristic function of the clopen set $\mathcal{G}(K^{ab}/K) \times_{\hat{\mathcal{O}}^*} \hat{\mathcal{O}} \subset \mathcal{G}(K^{ab}/K) \times_{\hat{\mathcal{O}}^*} \mathbb{A}_{K,f}$. The dynamics is defined by

$$\sigma_t(fu_g) = N(g)^{it} fu_g \ \text{ for } \ f \in C_0(\mathcal{G}(K^{ab}/K) \times_{\hat{\mathcal{O}}^*} \mathbb{A}_{K,f}) \ \text{ and } \ g \in J_K,$$

where u_g denotes the element of the multiplier algebra of the crossed product corresponding to g, and $N\colon J_K \to (0, +\infty)$ is the absolute norm.

In [18] we showed that for every $\beta \in (0, 1]$ there exists a unique KMS$_\beta$-state φ_β of the system. It is defined by the measure μ_β on $\mathcal{G}(K^{ab}/K) \times_{\hat{\mathcal{O}}^*} \mathbb{A}_{K,f}$ which is the push-forward of the product measure $\mu_\mathcal{G} \times \prod_{v\in V_{K,f}} \mu_{\beta,v}$ on $\mathcal{G}(K^{ab}/K) \times \mathbb{A}_{K,f}$, where $\mu_\mathcal{G}$ is the normalized Haar measure on $\mathcal{G}(K^{ab}/K)$, and $\mu_{\beta,v}$ is the unique measure on K_v such that $\mu_{\beta,v}(\mathcal{O}_v) = 1$ and

$$\mu_{\beta,v}(gZ) = \|g\|^\beta_v \mu_{\beta,v}(Z) \ \text{ for } \ g \in K^*_v,$$

where $\|\cdot\|_v$ is the normalized valuation in the class v, so $\|\pi\|_v = |\mathcal{O}_v/\mathfrak{p}_v|^{-1}$ for any element π generating the maximal ideal $\mathfrak{p}_v \subset \mathcal{O}_v$. The state φ_β is the composition of the state on $C(\mathcal{G}(K^{ab}/K) \times_{\hat{\mathcal{O}}^*} \hat{\mathcal{O}})$ defined by μ_β with the canonical conditional expectation $pAp \to C(\mathcal{G}(K^{ab}/K) \times_{\hat{\mathcal{O}}^*} \hat{\mathcal{O}})$.

Theorem 2.1. *The KMS$_\beta$-states φ_β, $\beta \in (0, 1]$, have type III$_1$. In other words, for every $\beta \in (0, 1]$ the action of J_K on $(\mathcal{G}(K^{ab}/K) \times_{\hat{\mathcal{O}}^*} \mathbb{A}_{K,f}, \mu_\beta)$ is of type III$_1$.*

By considering the flow of weights we can reformulate the result as follows. Denote by $K^*_+ \subset K^*$ the subgroup of totally positive elements, that is, elements $k \in K^*$ such that $\alpha(k) > 0$ for any real embedding $\alpha\colon K \hookrightarrow \mathbb{R}$. Denote by $\mu_{\beta,f}$ the measure $\prod_{v\in V_{K,f}} \mu_{\beta,v}$ on $\mathbb{A}_{K,f}$. Observe that for $\beta = 1$ we get a Haar measure on the additive group $\mathbb{A}_{K,f}$.

Corollary 2.2. *For every $\beta \in (0, 1]$, the action of K^*_+ on $(\mathbb{R}_+ \times \mathbb{A}_{K,f}, \lambda_\infty \times \mu_{\beta,f})$ defined by $k(t, x) = (N(k)t, kx)$, is ergodic.*

Proof. For any $g \in J_K$ we have $dg\mu_\beta/d\mu_\beta = N(g)^\beta$. Since the action of J_K on $(\mathcal{G}(K^{ab}/K) \times_{\hat{\mathcal{O}}^*} \mathbb{A}_{K,f}, \mu_\beta)$ is of type III$_1$, it follows that the action of J_K on $(\mathbb{R}_+ \times (\mathcal{G}(K^{ab}/K) \times_{\hat{\mathcal{O}}^*} \mathbb{A}_{K,f}), \lambda_\infty \times \mu_\beta)$ defined by $g(t, x) = (N(g)t, gx)$ is ergodic. In other words, the action of $\mathbb{A}^*_{K,f}$ on

$$(\mathbb{R}_+ \times \mathcal{G}(K^{ab}/K) \times \mathbb{A}_{K,f}, \lambda_\infty \times \mu_\mathcal{G} \times \mu_{\beta,f})$$

defined by $g(t,x,y)=(N(g)t,s(g)^{-1}x,gy)$, is ergodic.

The Artin map $s\colon \mathbb{A}_K^*\to\mathcal{G}(K^{ab}/K)$ is surjective with kernel $\overline{K^*(K_\infty^*)^o}$, where $(K_\infty^*)^o$ is the connected component of the identity in K_∞^*. Since $\mathbb{A}_K^*=K_\infty^*\times\mathbb{A}_{K,f}^*$, it follows that the action of $K^*\times\mathbb{A}_{K,f}^*$ on

$$(\mathbb{R}_+\times(K_\infty^*/(K_\infty^*)^o)\times\mathbb{A}_{K,f}^*\times\mathbb{A}_{K,f},\lambda_\infty\times\nu\times\mu\times\mu_{\beta,f}),$$

defined by $(k,g)(t,x,y,z)=(N(g)t,k^{-1}x,k^{-1}g^{-1}y,gz)$, is ergodic, where ν and μ are Haar measures on $K_\infty^*/(K_\infty^*)^o$ and $\mathbb{A}_{K,f}^*$, respectively. In other words, the action of K^* on

$$(\mathbb{R}_+\times(K_\infty^*/(K_\infty^*)^o)\times\mathbb{A}_{K,f},\lambda_\infty\times\nu\times\mu_{\beta,f}),$$

defined by $k(t,x,z)=(N(k)^{-1}t,k^{-1}x,k^{-1}z)$, is ergodic. Since the group $K_\infty^*/(K_\infty^*)^o$ is finite and the homomorphism $K^*\to K_\infty^*/(K_\infty^*)^o$ is surjective with kernel K_+^*, this is equivalent to ergodicity of the action of K_+^* on $(\mathbb{R}_+\times\mathbb{A}_{K,f},\lambda_\infty\times\mu_{\beta,f})$. ∎

Observe that we can equivalently say that the action of K_+^* on $(\mathbb{A}_{K,f},\mu_{\beta,f})$ is ergodic of type III_1.

The action of J_K on $(\mathcal{G}(K^{ab}/K)\times_{\hat{\mathcal{O}}^*}\mathbb{A}_{K,f},\mu_\beta)$ is ergodic by the proof of [18, Theorem 2.1]. The computation of the ratio set will be based on the following lemma.

Lemma 2.3. *For any $\beta\in(0,1]$, $\lambda>1$ and $\varepsilon>0$ there exists a set $\{\mathfrak{p}_n,\mathfrak{q}_n\}_{n\ge1}$ of distinct prime ideals in $\mathcal{O}$ such that*

$$\left|\frac{N(\mathfrak{q}_n)^\beta}{N(\mathfrak{p}_n)^\beta}-\lambda\right|<\varepsilon\ \ \textit{for all}\ \ n\ge1,\quad\textit{and}\quad\sum_{n=1}^\infty N(\mathfrak{q}_n)^{-\beta}=\infty.$$

Proof. The proof is the same as in [2, 3] for $K=\mathbb{Q}$, the only difference is that instead of the prime number theorem one has to use the prime ideal theorem.

It suffices to consider the case $\beta=1$. For $x>0$ denote by $\pi(x)$ the number of prime ideals $\mathfrak{p}$ in $\mathcal{O}$ with $N(\mathfrak{p})\le x$. Then

$$\pi(x)\sim\frac{x}{\log x}\quad\text{as}\quad x\to\infty,$$

see e.g. [19, Theorem 1.3], where one can also find an estimate of the error term.

Choose $\delta>0$ such that $1+\delta<\lambda$ and $\lambda\delta<\varepsilon$. Since

$$\frac{(1+\delta)x}{\log((1+\delta)x)}-\frac{x}{\log x}\sim\frac{\delta x}{\log x},$$

we get

$$\pi((1+\delta)x)-\pi(x)\sim\frac{\delta x}{\log x}. \tag{2.1}$$

In other words, if we put $B(x)=\{\mathfrak{p}\mid x<N(\mathfrak{p})\le(1+\delta)x\}$, then $|B(x)|\sim\dfrac{\delta x}{\log x}$. In particular, there exists $x_0>0$ such that $|B(\lambda x)|>|B(x)|$ for all $x\ge x_0$. Let $\mathfrak{p}_1,\mathfrak{p}_2,\dots$ be an enumeration of the set $\cup_{m\ge0}B(\lambda^{2m}x_0)$ such that $N(\mathfrak{p}_1)\le N(\mathfrak{p}_2)\le\dots$. For each $m\ge0$ choose a subset $C_m\subset B(\lambda^{2m+1}x_0)$ such that $|C_m|=|B(\lambda^{2m}x_0)|$. Let $\mathfrak{q}_1,\mathfrak{q}_2,\dots$ be an enumeration of the set $\cup_{m\ge0}C_m$ such that $N(\mathfrak{q}_1)\le N(\mathfrak{q}_2)\le\dots$. Since $1+\delta<\lambda$, the sets $B(\lambda^kx_0)$ are disjoint. Therefore, for every $n\ge1$, if $\mathfrak{p}_n\in B(\lambda^{2m}x_0)$ then $\mathfrak{q}_n\in C_m\subset B(\lambda^{2m+1}x_0)$, so that

$$\lambda-\varepsilon<\frac{\lambda}{1+\delta}<\frac{N(\mathfrak{q}_n)}{N(\mathfrak{p}_n)}<\lambda(1+\delta)<\lambda+\varepsilon.$$

By (2.1) we also have

$$\sum_{n=1}^\infty N(\mathfrak{q}_n)^{-1}\ge\sum_{m=0}^\infty\frac{|B(\lambda^{2m}x_0)|}{(1+\delta)\lambda^{2m+1}x_0}=\infty.$$

∎

Proof of Theorem 2.1. Since $\mathcal{G}(K^{ab}/K)$ is compact and totally disconnected, it suffices to show that the action of J_K on

$$((\mathcal{G}(K^{ab}/K)\times_{\hat{\mathcal{O}}^*}\mathbb{A}_{K,f})/\mathcal{G}(K^{ab}/K),\mu_\beta)=(\mathbb{A}_{K,f}/\hat{\mathcal{O}}^*,\mu_{\beta,f})$$

is of type III_1, see the proof of the main theorem and Remark (ii) in [20], as well as [16, Proposition 4.6] for a more general statement.

The measure space $(\mathcal{O}_v^\times/\mathcal{O}_v^*,\mu_{\beta,v})$, where $\mathcal{O}_v^\times=\mathcal{O}_v\setminus\{0\}$, can be identified with $(\mathbb{Z}_{\ge0},\nu_{\beta,v})$, where $\nu_{\beta,v}$ is the measure defined by

$$\nu_{\beta,v}(n)=N(\mathfrak{p}_v)^{-n\beta}(1-N(\mathfrak{p}_v)^{-\beta})\quad\text{for}\quad n\ge0.$$

Therefore, modulo sets of measure zero, the measure space $(\hat{\mathcal{O}}/\hat{\mathcal{O}}^*,\mu_{\beta,f})$ can be identified with $\prod_{v\in V_{K,f}}(\mathbb{Z}_{\ge0},\nu_{\beta,v})$, and the equivalence relation $\mathcal{R}$ induced on $\hat{\mathcal{O}}/\hat{\mathcal{O}}^*$ by the action of J_K on $\mathbb{A}_{K,f}/\hat{\mathcal{O}}^*$ is exactly the equivalence relation considered in our discussion of the asymptotic ratio set.

To compute $r_\infty(\mathcal{R},\mu_{\beta,f})$, fix $\lambda>1$ and $\varepsilon>0$. Let $\{\mathfrak{p}_n,\mathfrak{q}_n\}_{n\ge1}$ be the set of prime ideals given by Lemma 2.3. Let v_n and w_n be the places corresponding to $\mathfrak{p}_n$ and $\mathfrak{q}_n$, respectively. Then we define the sets $I_n\subset V_{K,f}$ and $K_n,L_n\subset\prod_{v\in I_n}\mathbb{Z}_{\ge0}$ required by the definition of the asymptotic ratio set by

$$I_n=\{v_n,w_n\},\quad K_n=\{(0,1)\},\quad L_n=\{(1,0)\},$$

and denote by $\varphi_n\colon K_n\to L_n$ the unique bijection. For $a=(0,1)\in K_n$ and $b=(1,0)\in L_n$ we have

$$\frac{\mu_{\beta,f}(Z(b))}{\mu_{\beta,f}(Z(a))}=\frac{\nu_{\beta,v_n}(1)\nu_{\beta,w_n}(0)}{\nu_{\beta,v_n}(0)\nu_{\beta,w_n}(1)}=\frac{N(\mathfrak{p}_n)^{-\beta}}{N(\mathfrak{q}_n)^{-\beta}},$$

which is close to λ up to ε. We also have

$$\sum_{n=1}^\infty\sum_{a\in K_n}\mu_{\beta,f}(Z(a))=\sum_{n=1}^\infty N(\mathfrak{q}_n)^{-\beta}(1-N(\mathfrak{p}_n)^{-\beta})(1-N(\mathfrak{q}_n)^{-\beta})=\infty.$$

It follows that $\lambda\in r_\infty(\mathcal{R},\mu_{\beta,f})$. Since this is true for all $\lambda>1$, we conclude that the action is of type III_1. ☐

Remark 2.4. In view of Corollary 2.2 it is natural to ask whether the action of K^* on $(K_\infty\times\mathbb{A}_{K,f},\bar\lambda_\infty\times\mu_{\beta,f})$ is ergodic, where $\bar\lambda_\infty$ is a Haar measure on $K_\infty\cong\mathbb{R}^{[K:\,\mathbb{Q}]}$; see also Remark 3.7(ii) below for a more general question. Assume for simplicity that K is an imaginary quadratic field of class number one. Then one can try to prove that the action of K^* on $\mathbb{T}\times\mathbb{A}_{K,f}$ given by $k(z,x)=(k|k|^{-1}z,kx)$, is ergodic, e.g. by adapting the strategy in [20, 18], and then compute the ratio set of this action. However, for the latter one would need information about the distribution of the angles of prime ideals, that is, of the values of the homomorphism $J_K\to\mathbb{T}/\mathcal{O}^*$, $(k)\to k|k|^{-1}$. We are not aware of any result of this sort.

3. The Connes-Marcolli GL_2-system

Let G be a discrete group and Γ be a subgroup of G. Recall that (G,Γ) is called a Hecke pair if every double coset of Γ contains finitely many right cosets of Γ, so that

$$R_\Gamma(g):=|\Gamma\backslash\Gamma g\Gamma|<\infty\quad\text{for any}\quad g\in G.$$

Then the space $\mathcal{H}(G,\Gamma)$ of finitely supported functions on $\Gamma\backslash G/\Gamma$ is a $*$-algebra with product

$$(f_1*f_2)(g)=\sum_{h\in\Gamma\backslash G}f_1(gh^{-1})f_2(h)$$

and involution $f^*(g)=\overline{f(g^{-1})}$. Denote by $[g]\in\mathcal{H}(G,\Gamma)$ the characteristic function of the double coset $\Gamma g\Gamma$.

If G acts on a space X then every element $g\in G$ defines a Hecke operator T_g acting on functions on $\Gamma\backslash X$, which we also consider as Γ-invariant functions on X:

$$(T_gf)(x)=\frac{1}{R_\Gamma(g)}\sum_{h\in\Gamma\backslash\Gamma g\Gamma}f(hx).$$

Then $[g^{-1}]\mapsto R_\Gamma(g)T_g$ is a representation of the Hecke algebra $\mathcal{H}(G,\Gamma)$ on the space of Γ-invariant functions.

If X is locally compact and the action of Γ on X is proper, one can define a C*-algebra $C^*_r(\Gamma\backslash G\times_\Gamma X)$ which can be thought of as a crossed product of $C_0(\Gamma\backslash X)$ by $\mathcal{H}(G,\Gamma)$, see [7, 16]. It is a completion of the algebra $C_c(\Gamma\backslash G\times_\Gamma X)$ of continuous compactly supported functions on $\Gamma\backslash G\times_\Gamma X$ with convolution product

$$(f_1*f_2)(g,x)=\sum_{h\in\Gamma\backslash G}f_1(gh^{-1},hx)f_2(h,x)$$

and involution $f^*(g,x)=\overline{f(g^{-1},gx)}$. If the action of Γ is free then $\Gamma\backslash G\times_\Gamma X$ is a groupoid and $C^*_r(\Gamma\backslash G\times_\Gamma X)$ is the usual groupoid C*-algebra.

Consider now the Hecke pair $(\mathrm{GL}_2^+(\mathbb{Q}),\mathrm{SL}_2(\mathbb{Z}))$, where $\mathrm{GL}_2^+(\mathbb{Q})$ is the group of rational matrices with positive determinant. The group $\mathrm{GL}_2^+(\mathbb{Q})$ acts by multiplication on $\mathrm{Mat}_2(\mathbb{Q}_p)$ for every prime p. It also acts by Möbius transformations on the upper half-plane $\mathbb{H}$. The GL_2-system of Connes and Marcolli [7] is the corner of the C*-algebra

$$C^*_r(\mathrm{SL}_2(\mathbb{Z})\backslash\mathrm{GL}_2^+(\mathbb{Q})\times_{\mathrm{SL}_2(\mathbb{Z})}(\mathbb{H}\times\mathrm{Mat}_2(\mathbb{A}_f)))$$

defined by the projection corresponding to the subspace $\mathbb{H}\times\mathrm{Mat}_2(\hat{\mathbb{Z}})\subset\mathbb{H}\times\mathrm{Mat}_2(\mathbb{A}_f)$, where $\mathbb{A}_f=\mathbb{A}_{\mathbb{Q},f}$. We denote this algebra by $C^*_r(\mathrm{SL}_2(\mathbb{Z})\backslash\mathrm{GL}_2^+(\mathbb{Q})\boxtimes_{\mathrm{SL}_2(\mathbb{Z})}(\mathbb{H}\times\mathrm{Mat}_2(\hat{\mathbb{Z}})))$. The dynamics on it is defined by

$$\sigma_t(f)(g,x)=\det(g)^{it}f(g,x).$$

In [16] we showed that for every $\beta\in(1,2]$ there exists a unique KMS$_\beta$-state φ_β on the GL_2-system. It is defined by the product-measure $\mu_{\mathbb{H}}\times\prod_p\mu_{\beta,p}$ on $\mathbb{H}\times\mathrm{Mat}_2(\mathbb{A}_f)$, where $\mu_{\mathbb{H}}$ is the unique $\mathrm{GL}_2^+(\mathbb{R})$-invariant measure on $\mathbb{H}$ such that $\mu_{\mathbb{H}}(\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H})=2$, and $\mu_{\beta,p}$ is the unique measure on $\mathrm{Mat}_2(\mathbb{Q}_p)$ such that $\mu_{\beta,p}(\mathrm{Mat}_2(\mathbb{Z}_p))=1$ and

$$\mu_{\beta,p}(gZ)=|\det(g)|_p^\beta\mu_{\beta,p}(Z)\quad\text{for}\quad g\in\mathrm{GL}_2(\mathbb{Q}_p).$$

Denote the measure $\prod_p\mu_{\beta,p}$ by $\mu_{\beta,f}$. Observe that $\mu_{2,f}$ is the Haar measure of the additive group $\mathrm{Mat}_2(\mathbb{A}_f)$ normalized so that $\mu_{2,f}(\mathrm{Mat}_2(\hat{\mathbb{Z}}))=1$.

The definition of the GL_2-system required a new type of crossed product construction because of non-freeness of the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}\times\mathrm{Mat}_2(\mathbb{A}_f)$. However, the set of points with nontrivial stabilizers is $\mathbb{H}\times\{0\}$, which has measure zero with respect to $\mu_{\mathbb{H}}\times\mu_{\beta,f}$. As a result the von Neumann algebra generated by the GL_2-system in the GNS-representation of φ_β is much easier to describe. It is the reduction of the von Neumann algebra crossed product $L^\infty(\mathbb{H}\times\mathrm{Mat}_2(\mathbb{A}_f),\mu_{\mathbb{H}}\times\mu_{\beta,f})\rtimes\mathrm{GL}_2^+(\mathbb{Q})$ by the projection defined by a fundamental domain of the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}\times(\mathrm{Mat}_2(\hat{\mathbb{Z}})\setminus\{0\})$. Therefore to compute the type of the algebra we have to compute the type of the action of $\mathrm{GL}_2^+(\mathbb{Q})$ on $\mathbb{H}\times\mathrm{Mat}_2(\mathbb{A}_f)$.

It is natural to consider a slightly more general problem. Namely, replace $\mathbb{H}=\mathrm{PGL}_2^+(\mathbb{R})/\mathrm{PSO}_2(\mathbb{R})$ by $\mathrm{PGL}_2(\mathbb{R})$ and $\mathrm{GL}_2^+(\mathbb{Q})$ by $\mathrm{GL}_2(\mathbb{Q})$. Denote by μ_∞ the Haar measure of $\mathrm{PGL}_2(\mathbb{R})$ normalized so that $\mu_\infty(\mathrm{GL}_2(\mathbb{Z})\backslash\mathrm{PGL}_2(\mathbb{R}))=2$. Put $\mu_\beta=\mu_\infty\times\mu_{\beta,f}$. The action of $\mathrm{GL}_2(\mathbb{Q})$ on $(\mathrm{PGL}_2(\mathbb{R})\times\mathrm{Mat}_2(\mathbb{A}_f),\mu_\beta)$ is ergodic by [16, Corollary 4.7].

Theorem 3.1. *For every $\beta\in(1,2]$, the action of $\mathrm{GL}_2(\mathbb{Q})$ on $(\mathrm{PGL}_2(\mathbb{R})\times\mathrm{Mat}_2(\mathbb{A}_f),\mu_\beta)$ has type III$_1$. In particular, the KMS$_\beta$-states φ_β, $\beta\in(1,2]$, of the Connes-Marcolli GL_2-system have type III$_1$.*

As we already remarked in [16], the flows of weights of the above actions are easy to describe, and then the result takes the following essentially equivalent form. Denote by λ_∞ the usual Lebesgue measure on $\mathrm{Mat}_2(\mathbb{R}) \cong \mathbb{R}^4$, and put $\lambda_\beta = \lambda_\infty \times \mu_{\beta,f}$. For $\beta = 2$ we get a Haar measure on the additive group $\mathrm{Mat}_2(\mathbb{A}) = \mathrm{Mat}_2(\mathbb{R}) \times \mathrm{Mat}_2(\mathbb{A}_f)$, where $\mathbb{A} = \mathbb{A}_\mathbb{Q}$.

Corollary 3.2. *For every $\beta \in (1, 2]$, the action of $\mathrm{GL}_2(\mathbb{Q})$ on $(\mathrm{Mat}_2(\mathbb{A}), \lambda_\beta)$ is ergodic.*

Proof. If the action of $\mathrm{GL}_2(\mathbb{Q})$ on $(\mathrm{PGL}_2(\mathbb{R}) \times \mathrm{Mat}_2(\mathbb{A}_f), \mu_\beta)$ is of type III_1 then clearly also the action of $\mathrm{GL}_2^+(\mathbb{Q})$ on $(\mathrm{PGL}_2^+(\mathbb{R}) \times \mathrm{Mat}_2(\mathbb{A}_f), \mu_\beta)$ is of type III_1. As we discussed in [16, Remark 4.9], using the isomorphism $\mathrm{GL}_2^+(\mathbb{R})/\{\pm1\} \cong \mathbb{R}^*_+ \times \mathrm{PGL}_2^+(\mathbb{R})$ we can identify the underlying space of the flow of weights of this action with the quotient of $((\mathrm{GL}_2^+(\mathbb{R})/\{\pm1\}) \times \mathrm{Mat}_2(\mathbb{A}_f), \lambda_\beta)$ by the diagonal action of $\mathrm{GL}_2^+(\mathbb{Q})$. Therefore this diagonal action is ergodic. Since $\mathrm{GL}_2^+(\mathbb{R})$ is connected and $\{\pm1\}$ is finite, by [16, Proposition 4.6] we conclude that the action of $\mathrm{GL}_2^+(\mathbb{Q})$ on $(\mathrm{GL}_2^+(\mathbb{R}) \times \mathrm{Mat}_2(\mathbb{A}_f), \lambda_\beta)$ is ergodic. But then the action of $\mathrm{GL}_2(\mathbb{Q})$ on $(\mathrm{GL}_2(\mathbb{R}) \times \mathrm{Mat}_2(\mathbb{A}_f), \lambda_\beta)$ is also ergodic. □

To simplify notation from now on we write G for $\mathrm{GL}_2(\mathbb{Q})$, Γ for $\mathrm{GL}_2(\mathbb{Z})$ and X for $\mathrm{PGL}_2(\mathbb{R}) \times \mathrm{Mat}_2(\mathbb{A}_f)$.

Recall, see e.g. [14], that the group G is generated by Γ and the matrices $\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$, $p \in \mathcal{P}$, where $\mathcal{P}$ is the set of prime numbers. We have

$$\{m \in \mathrm{Mat}_2(\mathbb{Z}) : |\det(m)| = p\} = \Gamma \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \Gamma \quad \text{and} \quad R_\Gamma\left(\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}\right) = p + 1. \tag{3.1}$$

Recall also, see [16, Section 3], that

$$\mu_{\beta,p}(\mathrm{GL}_2(\mathbb{Z}_p)) = (1 - p^{-\beta})(1 - p^{-\beta+1}), \tag{3.2}$$

Using the scaling property of $\mu_{\beta,p}$ and (3.1) we then conclude that

$$\mu_{\beta,p}(\{m \in \mathrm{Mat}_2(\mathbb{Z}_p) : |\det(m)|_p = p^{-1}\}) = p^{-\beta}(p + 1)(1 - p^{-\beta})(1 - p^{-\beta+1}). \tag{3.3}$$

One of the key ingredients of the proof of the theorem will be the following version of equidistribution of Hecke points. I am indebted to Hee Oh for explaining me how to put it in the general setup of equidistibution of Hecke points for reductive groups.

Denote by G_p the group generated by Γ and $\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$. Equivalently, G_p is the group $\mathrm{GL}_2(\mathbb{Z}[p^{-1}])$. For a nonempty subset F of primes denote by G_F the group generated by G_p for all $p \in F$. For finite F denote by X_F the space $\mathrm{PGL}_2(\mathbb{R}) \times \mathrm{Mat}_2(\mathbb{Q}_F)$, where $\mathbb{Q}_F = \prod_{p\in F} \mathbb{Q}_p$, and by $\bar\mu_{\beta,F}$ the measure $\mu_\infty \times \prod_{p\in F} \mu_{\beta,p}$. Consider also the measure $\bar\nu_{\beta,F}$ on $\Gamma\backslash X_F$ defined by $\bar\mu_{\beta,F}$. We shall write $\mathbb{Z}_F$ for $\prod_{p\in F} \mathbb{Z}_p$.

Lemma 3.3. *Let F be a finite set of primes, $r \in \mathrm{GL}_2(\mathbb{Q}_F)$. Assume f is a compactly supported continuous right $\mathrm{GL}_2(\mathbb{Z}_F)$-invariant function on $Z = \Gamma\backslash(\mathrm{PGL}_2(\mathbb{R}) \times \mathrm{GL}_2(\mathbb{Z}_F) r \mathrm{GL}_2(\mathbb{Z}_F)) \subset \Gamma\backslash X_F$. Then for any $\varepsilon > 0$ and any compact subset C of Z there exists $M > 0$ such that if $g \in G_{F^c}$ ($F^c = \mathcal{P} \setminus F$) and $R_\Gamma(g) \ge M$ then*

$$\left| T_g f(x) - \bar\nu_{\beta,F}(Z)^{-1} \int_Z f \, d\bar\nu_{\beta,F} \right| < \varepsilon \quad \textit{for all} \quad x \in C.$$

Proof. The $\mathrm{GL}_2(\mathbb{Z}_F)$-space $\mathrm{GL}_2(\mathbb{Z}_F) r \mathrm{GL}_2(\mathbb{Z}_F)/\mathrm{GL}_2(\mathbb{Z}_F)$ can be identified with $\mathrm{GL}_2(\mathbb{Z}_F)/H$, where $H = r\mathrm{GL}_2(\mathbb{Z}_F)r^{-1} \cap \mathrm{GL}_2(\mathbb{Z}_F)$. Therefore f can be considered as a function on $\Gamma\backslash(\mathrm{PGL}_2(\mathbb{R}) \times \mathrm{GL}_2(\mathbb{Z}_F))/H$. Consider the compact open subgroup $U = H \times \prod_{p\in F^c} \mathrm{GL}_2(\mathbb{Z}_p)$ of $\mathrm{GL}_2(\hat{\mathbb{Z}})$. By considering $\mathrm{GL}_2(\mathbb{Z}_F)$ as the subgroup of $\mathrm{GL}_2(\hat{\mathbb{Z}})$ consisting of elements with coordinates 1 for $p \in F^c$, we get a homeomorphism

$$\Gamma\backslash(\mathrm{PGL}_2(\mathbb{R}) \times \mathrm{GL}_2(\mathbb{Z}_F))/H \to G\backslash(\mathrm{PGL}_2(\mathbb{R}) \times \mathrm{GL}_2(\mathbb{A}_f))/U,$$

since $G\cap \mathrm{GL}_2(\hat{\mathbb{Z}})=\Gamma$ and $\mathrm{GL}_2(\mathbb{A}_f)=G\mathrm{GL}_2(\hat{\mathbb{Z}})$. Furthermore, we have $\mathrm{GL}_2(\mathbb{A}_f)=GU$. To see this recall that $\mathrm{SL}_2(\mathbb{Q})$ is dense in $\mathrm{SL}_2(\mathbb{A}_f)$ by the strong approximation theorem. It follows that GU contains $\mathrm{SL}_2(\mathbb{A}_f)$, and in particular $\mathrm{SL}_2(\hat{\mathbb{Z}})$. Since $\mathrm{GL}_2(\mathbb{A}_f)=G\mathrm{GL}_2(\hat{\mathbb{Z}})$, it is therefore enough to check that $\mathrm{SL}_2(\hat{\mathbb{Z}})U=\mathrm{GL}_2(\hat{\mathbb{Z}})$, that is, $\mathrm{SL}_2(\mathbb{Z}_F)H=\mathrm{GL}_2(\mathbb{Z}_F)$. For this we have to show that the determinant map $\det\colon H\to\mathbb{Z}_F^*$ is surjective. Since every double coset of $\mathrm{GL}_2(\mathbb{Z}_F)$ contains a diagonal matrix, without loss of generality we may assume that r is diagonal. But then H contains all the diagonal matrices of $\mathrm{GL}_2(\mathbb{Z}_F)$, and surjectivity of the determinant is immediate.

We can then proceed as in [6], see Remark (3) following [6, Theorem 1.7], as well as Sections 2 and 3 in [10]. □

For a Γ-invariant measurable subset A of $\mathrm{PGL}_2(\mathbb{R})\times\mathrm{Mat}_2(\mathbb{A}_f)$, denote by $m(A)$ the operator of multiplication by the characteristic function of A on $L^2(\Gamma\backslash X,d\nu_\beta)$, where ν_β is the measure on $\Gamma\backslash X$ defined by μ_β.

Lemma 3.4. *For a prime p, consider the sets*

$$A_p=\{x\in\mathrm{PGL}_2(\mathbb{R})\times\mathrm{Mat}_2(\hat{\mathbb{Z}}):x_p\in\mathrm{GL}_2(\mathbb{Z}_p)\},$$
$$B_p=\{x\in\mathrm{PGL}_2(\mathbb{R})\times\mathrm{Mat}_2(\hat{\mathbb{Z}}):|\det(x_p)|_p=p^{-1}\},$$

and the element $g=\begin{pmatrix}1&0\\0&p\end{pmatrix}$. Then for the operator $m(A_p)T_gm(B_p)=m(A_p)T_g$ on $L^2(\Gamma\backslash X,d\nu_\beta)$ we have

$$\|m(A_p)T_gm(B_p)\|=p^{\beta/2}(p+1)^{-1/2}=\nu_\beta(\Gamma\backslash A_p)^{1/2}\nu_\beta(\Gamma\backslash B_p)^{-1/2}.$$

Proof. Since $B_p=\Gamma gA_p$, $R_\Gamma(g)=p+1$ and $|T_g(f)|^2\le T_g(|f|^2)$ pointwise by convexity, this follows from [16, Lemma 2.7], but we will sketch a proof for the reader's convenience.

Fix representatives $h_1,\dots,h_{p+1}$ of $\Gamma\backslash\Gamma g\Gamma$. Choose a fundamental domain C for the action of Γ on A_p. Using that the action of Γ on A_p is free and that $G_p\cap\mathrm{GL}_2(\mathbb{Z}_p)=\Gamma$ one can easily check that the sets Γh_iC are mutually disjoint and the factor-map $p\colon X\to\Gamma\backslash X$ is injective on the sets h_iC. Consider the operators S_i defined by

$$(S_if)(p(x))=\begin{cases}f(p(h_ix)), & \text{if } x\in C,\\ 0, & \text{if } x\notin A_p.\end{cases}$$

Then $p^{-\beta/2}S_i$ is a partial isometry with initial space $L^2(p(h_iC),d\nu_\beta)$ and range $L^2(p(C),d\nu_\beta)$, and $m(A_p)T_gm(B_p)=(S_1+\dots+S_{p+1})/(p+1)$. Since the spaces $L^2(p(h_iC),d\nu_\beta)$ are mutually orthogonal, we have

$$\|p^{-\beta/2}S_1+\dots+p^{-\beta/2}S_{p+1}\|=(p+1)^{1/2},$$

which gives the first equality in the statement. The second equality follows from (3.2) and (3.3). □

For $x\in X$ denote by $\bar{x}_F$ the image of x under the factor-map $X\to X_F$. For a function f on X_F consider the function f_F on X defined by

$$f_F(x)=\begin{cases}f(\bar{x}_F), & \text{if } x_p\in\mathrm{Mat}_2(\mathbb{Z}_p) \text{ for all } p\notin F,\\ 0, & \text{otherwise.}\end{cases}$$

Lemma 3.5. *For any $\beta\in(1,2]$ and $\lambda>1$ there exists $c>0$ such that for any $\varepsilon>0$, any finite set F of primes and any positive compactly supported continuous right $\mathrm{GL}_2(\mathbb{Z}_F)$-invariant function f on $\Gamma\backslash(\mathrm{PGL}_2(\mathbb{R})\times\mathrm{Mat}_2(\mathbb{Z}_F))\subset\Gamma\backslash X_F$ with $\int_{\Gamma\backslash X_F}f\,d\bar{\nu}_{\beta,F}=1$, there exist a subset $\{p_n,q_n\}_{n\ge1}$ of F^c and Γ-invariant measurable subsets X_{1n}, X_{2n}, Y_{1n} and Y_{2n}, $n\ge1$, of X such that*

(i) $\left|\dfrac{q_n^\beta}{p_n^\beta}-\lambda\right|<\varepsilon$ *for all $n\ge1$;*

(ii) *the sets Y_{1n}, $n\ge1$, as well as the sets Y_{2n}, $n\ge1$, are mutually disjoint;*

(iii) $\sum_{n=1}^{\infty}\left(\frac{m(X_{1n})T_{g_n}m(Y_{1n})}{\|m(X_{1n})T_{g_n}m(Y_{1n})\|}f_F, \frac{m(X_{2n})T_{h_n}m(Y_{2n})}{\|m(X_{2n})T_{h_n}m(Y_{2n})\|}f_F\right)_{L^2(\Gamma\backslash X, d\nu_\beta)} > c$, *where* $g_n = \begin{pmatrix}1 & 0\\ 0 & p_n\end{pmatrix}$ *and* $h_n = \begin{pmatrix}1 & 0\\ 0 & q_n\end{pmatrix}$.

Proof. Fix $\delta \in (0,1)$. Choose representatives r_k, $k \geq 1$, of the double cosets

$$\mathrm{GL}_2(\mathbb{Z}_F)\backslash(\mathrm{GL}_2(\mathbb{Q}_F) \cap \mathrm{Mat}_2(\mathbb{Z}_F))/\mathrm{GL}_2(\mathbb{Z}_F),$$

and put $Z_k = \Gamma\backslash(\mathrm{PGL}_2(\mathbb{R}) \times \mathrm{GL}_2(\mathbb{Z}_F) r_k \mathrm{GL}_2(\mathbb{Z}_F))$. Let N be such that

$$\sum_{k=1}^{N}\int_{Z_k} f d\bar\nu_{\beta,F} > \int_{\Gamma\backslash X_F} f d\bar\nu_{\beta,F} - \delta = 1 - \delta.$$

Next choose compact subsets C_k of Z_k such that

$$\bar\nu_{\beta,F}(C_k) > (1-\delta)\bar\nu_{\beta,F}(Z_k) \quad \text{for} \quad k = 1,\dots,N.$$

By Lemma 3.3 there exists M such that for any element $g \in G_{F^c}$ with $R_\Gamma(g) \geq M$ we have

$$T_g f(x) \geq \frac{1-\delta}{\bar\nu_{\beta,F}(Z_k)}\int_{Z_k} f d\bar\nu_{\beta,F} \quad \text{for} \quad x \in C_k, \quad k = 1,\dots,N.$$

It follows that if we take two elements $g, h \in G_{F^c}$ with $R_\Gamma(g), R_\Gamma(h) \geq M$, then

$$\begin{aligned}
\int_{\Gamma\backslash X_F} T_g f\, T_h f d\bar\nu_{\beta,F} &\geq \sum_{k=1}^{N}\int_{C_k} T_g f\, T_h f d\bar\nu_{\beta,F}\\
&\geq \sum_{k=1}^{N}\left(\frac{1-\delta}{\bar\nu_{\beta,F}(Z_k)}\int_{Z_k} f d\bar\nu_{\beta,F}\right)^2 \bar\nu_{\beta,F}(C_k)\\
&\geq (1-\delta)^3\sum_{k=1}^{N}\left(\frac{1}{\bar\nu_{\beta,F}(Z_k)}\int_{Z_k} f d\bar\nu_{\beta,F}\right)^2 \bar\nu_{\beta,F}(Z_k)\\
&\geq (1-\delta)^3\left(\sum_{k=1}^{N}\int_{Z_k} f d\bar\nu_{\beta,F}\right)^2,
\end{aligned}$$

since the function $t \mapsto t^2$ is convex and $\sum_{k=1}^{N}\bar\nu_{\beta,F}(Z_k) \leq \bar\nu_{\beta,F}(\Gamma\backslash(\mathrm{PGL}_2(\mathbb{R}) \times \mathrm{Mat}_2(\mathbb{Z}_F))) = 1$. Therefore

$$\int_{\Gamma\backslash X_F} T_g f\, T_h f d\bar\nu_{\beta,F} \geq (1-\delta)^5. \tag{3.4}$$

Using Lemma 2.3 choose a subset $\{p_n, q_n\}_{n\geq 1}$ of F^c such that $q_n > p_n \geq M$ and $|q_n^\beta/p_n^\beta - \lambda| < \varepsilon$ for all n, and

$$\sum_n p_n^{1-\beta} = \infty. \tag{3.5}$$

Consider the sets A_p and B_p from Lemma 3.4 and put

$$X_{1n} = A_{p_n}\backslash(B_{p_1}\cup\dots\cup B_{p_{n-1}}), \quad Y_{1n} = B_{p_n}\backslash(B_{p_1}\cup\dots\cup B_{p_{n-1}}),$$
$$X_{2n} = A_{q_n}\backslash(B_{q_1}\cup\dots\cup B_{q_{n-1}}), \quad Y_{2n} = B_{q_n}\backslash(B_{q_1}\cup\dots\cup B_{q_{n-1}}).$$

Let $g_n = \begin{pmatrix}1 & 0\\ 0 & p_n\end{pmatrix}$ and $h_n = \begin{pmatrix}1 & 0\\ 0 & q_n\end{pmatrix}$.

We claim that if $g \in \Gamma g_n \Gamma$ and $x \in X_{1n}$ then $gx \in Y_{1n}$. Indeed, we clearly have $gA_{p_n} \subset B_{p_n}$, so that $gx \in B_{p_n}$. Furthermore, if $gx \in B_{p_k}$ for some $k < n$ then $p_k^{-1} = |\det(gx_{p_k})|_{p_k} = |\det(x_{p_k})|_{p_k}$, since $g_n \in \mathrm{GL}_2(\mathbb{Z}_{p_k})$. Therefore $x \in B_{p_k}$, which contradicts the assumption that $x \in X_{1n}$. Hence $gx \in Y_{1n}$.

It follows that $m(X_{1n})T_{g_n}m(Y_{1n})f_F = (T_{g_n}f)_F\mathbb{1}_{\Gamma\backslash X_{1n}}$. For similar reasons, $m(X_{2n})T_{h_n}m(Y_{2n})f_F = (T_{h_n}f)_F\mathbb{1}_{\Gamma\backslash X_{2n}}$. Therefore

$$
\begin{aligned}
&(m(X_{1n})T_{g_n}m(Y_{1n})f_F, m(X_{2n})T_{h_n}m(Y_{2n})f_F)_{L^2(\Gamma\backslash X,d\nu_\beta)}\\
&\qquad = (T_{g_n}f, T_{h_n}f)_{L^2(\Gamma\backslash X_F,d\bar\nu_{\beta,F})}\nu_\beta(\Gamma\backslash(X_{1n}\cap X_{2n})) \ge (1-\delta)^5\nu_\beta(\Gamma\backslash(X_{1n}\cap X_{2n}))
\end{aligned}
$$

by (3.4).

By Lemma 3.4 we have

$$\|m(X_{1n})T_{g_n}m(Y_{1n})\| \le \nu_\beta(\Gamma\backslash B_{p_n})^{-1/2} \quad\text{and}\quad \|m(X_{2n})T_{h_n}m(Y_{2n})\| \le \nu_\beta(\Gamma\backslash B_{q_n})^{-1/2}.$$

If follows that

$$
\begin{aligned}
&\sum_{n=1}^\infty \left(\frac{m(X_{1n})T_{g_n}m(Y_{1n})}{\|m(X_{1n})T_{g_n}m(Y_{1n})\|}f_F, \frac{m(X_{2n})T_{h_n}m(Y_{2n})}{\|m(X_{2n})T_{h_n}m(Y_{2n})\|}f_F\right)_{L^2(\Gamma\backslash X,d\nu_\beta)}\\
&\qquad \ge (1-\delta)^5\sum_{n=1}^\infty(\nu_\beta(\Gamma\backslash B_{p_n})\nu_\beta(\Gamma\backslash B_{q_n}))^{1/2}\nu_\beta(\Gamma\backslash(X_{1n}\cap X_{2n})). \qquad (3.6)
\end{aligned}
$$

We have

$$\nu_\beta(\Gamma\backslash(X_{1n}\cap X_{2n})) = \nu_\beta(\Gamma\backslash A_{p_n})\nu_\beta(\Gamma\backslash A_{q_n})\prod_{k=1}^{n-1}(1-\nu_\beta(\Gamma\backslash(B_{p_k}\cup B_{q_k}))). \qquad (3.7)$$

We may assume that M is so large that

$$\nu_\beta(\Gamma\backslash A_{p_n})\nu_\beta(\Gamma\backslash A_{q_n}) = (1-p_n^{-\beta})(1-p_n^{-\beta+1})(1-q_n^{-\beta})(1-q_n^{-\beta+1}) > 1-\delta, \qquad (3.8)$$

see (3.2). Since

$$\nu_\beta(B_p) = p^{-\beta}(p+1)(1-p^{-\beta})(1-p^{-\beta+1}) \sim p^{1-\beta}$$

by (3.3), we may also assume that M is so large and the ratios q_n^β/p_n^β are so close to λ that

$$
\begin{aligned}
(\nu_\beta(\Gamma\backslash B_{p_n})\nu_\beta(\Gamma\backslash B_{q_n}))^{1/2} &> (1-\delta)c_0\big(\nu_\beta(\Gamma\backslash B_{p_n}) + \nu_\beta(\Gamma\backslash B_{q_n}) - \nu_\beta(\Gamma\backslash(B_{p_n})\nu_\beta(\Gamma\backslash B_{q_n})\big)\\
&= (1-\delta)c_0\nu_\beta(\Gamma\backslash(B_{p_n}\cup B_{q_n})), \qquad (3.9)
\end{aligned}
$$

where

$$c_0 = \frac{\lambda^{(1-\beta)/2\beta}}{1+\lambda^{(1-\beta)/\beta}}.$$

Combining (3.7)-(3.9) we conclude that the right hand side of (3.6) is not smaller than

$$(1-\delta)^7c_0\sum_{n=1}^\infty\nu_\beta(\Gamma\backslash(B_{p_n}\cup B_{q_n}))\prod_{k=1}^{n-1}(1-\nu_\beta(\Gamma\backslash(B_{p_k}\cup B_{q_k}))) = (1-\delta)^7c_0,$$

because

$$\sum_{n=1}^\infty\nu_\beta(\Gamma\backslash(B_{p_n}\cup B_{q_n})) \ge \sum_{n=1}^\infty\nu_\beta(\Gamma\backslash B_{p_n}) = \infty$$

by (3.5). Since δ can be arbitrarily small, we see that we can take any $c < c_0$. □

Proof of Theorem 3.1. Similarly to the proof of Theorem 2.1, since $\mathrm{GL}_2(\hat{\mathbb{Z}})$ is compact and totally disconnected, it suffices to show that the action of $\mathrm{GL}_2(\mathbb{Q})$ on $(\mathrm{PGL}_2(\mathbb{R})\times\mathrm{Mat}_2(\mathbb{A}_f)/\mathrm{GL}_2(\hat{\mathbb{Z}}), \mu_\beta)$ is of type III_1. In other words, in computing the ratio set it suffices to consider right $\mathrm{GL}_2(\hat{\mathbb{Z}})$-invariant sets.

Let $\lambda > 1$, $\varepsilon > 0$ and Y be a measurable right $\mathrm{GL}_2(\hat{\mathbb{Z}})$-invariant subset of X such that $\mu_\beta(Y) > 0$. Let $c > 0$ be given by Lemma 3.5. There exists $g_0 \in \mathrm{GL}_2(\mathbb{Q})$ such that the intersection $Y_0 := g_0Y\cap(\mathrm{PGL}_2(\mathbb{R})\times\mathrm{Mat}_2(\hat{\mathbb{Z}}))$ has positive measure. Let φ be the function $\nu_\beta(\Gamma\backslash\Gamma Y_0)^{-1}\mathbb{1}_{\Gamma\backslash\Gamma Y_0}$ on $\Gamma\backslash X$.

We can find a finite set F of primes and a positive compactly supported continuous right $\mathrm{GL}_2(\mathbb{Z}_F)$-invariant function f on $\Gamma\backslash(\mathrm{PGL}_2(\mathbb{R})\times \mathrm{Mat}_2(\mathbb{Z}_F))$ such that

$$\int_{\Gamma\backslash X_F} f\,d\bar\nu_{\beta,F}=1 \quad\text{and}\quad \|\varphi-f_F\|_2(\|\varphi\|_2+\|f_F\|_2)<c.$$

Let $p_n,q_n\in F^c$, $X_{1n},X_{2n},Y_{1n},Y_{2n}\subset X$ and $g_n,h_n\in G$ be given by Lemma 3.5.

Denote by T'_n and T''_n the contractions $\dfrac{m(X_{1n})T_{g_n}m(Y_{1n})}{\|m(X_{1n})T_{g_n}m(Y_{1n})\|}$ and $\dfrac{m(X_{2n})T_{h_n}m(Y_{2n})}{\|m(X_{2n})T_{h_n}m(Y_{2n})\|}$, respectively, and by e'_n and e''_n the projections $m(Y_{1n})$ and $m(Y_{2n})$. We have

$$\begin{aligned}(T'_n\varphi,T''_n\varphi)&\ge (T'_nf_F,T''_nf_F)-\|T'_n\varphi-T'_nf_F\|_2\|T''_n\varphi\|_2-\|T''_n\varphi-T''_nf_F\|_2\|T'_nf_F\|_2\\&\ge (T'_nf_F,T''_nf_F)-\|e'_n(\varphi-f_F)\|_2\|e''_n\varphi\|_2-\|e''_n(\varphi-f_F)\|_2\|e'_nf_F\|_2.\end{aligned}$$

Since the projections e'_n, as well as the projections e''_n, are mutually orthogonal, we have

$$\sum_n\|e'_n(\varphi-f_F)\|_2\|e''_n\varphi\|_2\le\left(\sum_n\|e'_n(\varphi-f_F)\|_2^2\right)^{1/2}\left(\sum_n\|e''_n\varphi\|_2^2\right)^{1/2}\le\|\varphi-f_F\|_2\|\varphi\|_2,$$

and similarly

$$\sum_n\|e''_n(\varphi-f_F)\|_2\|e'_nf_F\|_2\le\|\varphi-f_F\|_2\|f_F\|_2.$$

It follows that

$$\sum_n(T'_n\varphi,T''_n\varphi)\ge\sum_n(T'_nf_F,T''_nf_F)-\|\varphi-f_F\|_2(\|\varphi\|_2+\|f_F\|_2)>c-c=0.$$

Hence there exists n such that $(T'_n\varphi,T''_n\varphi)>0$. Then

$$\int_{\Gamma\backslash X}T_{g_n}\varphi\,T_{h_n}\varphi\,d\nu_\beta>0,$$

which means that the set $\Gamma g_n^{-1}\Gamma Y_0\cap\Gamma h_n^{-1}\Gamma Y_0$ has positive measure. It follows that there exist $g\in\Gamma g_n\Gamma$ and $h\in\Gamma h_n\Gamma$ such that $g^{-1}Y_0\cap h^{-1}Y_0$ has positive measure, and hence $g_0^{-1}hg^{-1}g_0Y\cap Y$ has positive measure. Since

$$\frac{d(g_0^{-1}hg^{-1}g_0\mu_\beta)}{d\mu_\beta}=|\det(g_0^{-1}hg^{-1}g_0)|^\beta=\det(h_ng_n^{-1})^\beta=\frac{q_n^\beta}{p_n^\beta},$$

and $|q_n^\beta/p_n^\beta-\lambda|<\varepsilon$, we conclude that λ belongs to the ratio set of the action of $\mathrm{GL}_2(\mathbb{Q})$ on $(\mathrm{PGL}_2(\mathbb{R})\times\mathrm{Mat}_2(\mathbb{A}_f)/\mathrm{GL}_2(\hat{\mathbb{Z}}),\mu_\beta)$. Since this is true for all $\lambda>1$, the action is of type III_1. □

We finish our discussion of the GL_2-system with the following simple observation.

Proposition 3.6. *For every $\beta\in(1,2]$, the action of $\mathrm{GL}_2(\mathbb{Q})$ on $(\mathrm{PGL}_2(\mathbb{R})\times\mathrm{Mat}_2(\mathbb{A}_f),\mu_\beta)$ is amenable. Therefore the von Neumann algebra generated by the GL_2-system in the GNS-representation of φ_β, $\beta\in(1,2]$, is the injective factor of type III_1.*

Proof. If F is a finite set of primes then G_F is a discrete subgroup of $\mathrm{PGL}_2(\mathbb{R})\times\mathrm{GL}_2(\mathbb{Q}_F)$, since $G_F\cap\mathrm{GL}_2(\mathbb{Z}_F)=\Gamma$ and the homomorphism $\Gamma\to\mathrm{PGL}_2(\mathbb{R})$ has discrete image and finite kernel. Since $\mathrm{PGL}_2(\mathbb{R})\times\mathrm{GL}_2(\mathbb{Q}_F)$ is a subset of X_F of full measure, it follows that $L^\infty(X_F,\bar\mu_{\beta,F})\rtimes G_F$ is a type I von Neumann algebra. Since the algebra $L^\infty(X,\mu_\beta)\rtimes G$ is the closure of an increasing union of algebras of the form $L^\infty(X_F,\bar\mu_{\beta,F})\rtimes G_F$, it is injective, and therefore the action of G on (X,μ_β) is amenable [21]. □

Remark 3.7.

(i) Since the C*-algebra A of the GL_2-system is a corner of the crossed product of $C_0(\mathrm{SL}_2(\mathbb{Z})\backslash(\mathbb{H}\times\mathrm{Mat}_2(\mathbb{A}_f)))$ by the Hecke algebra $\mathcal{H}(\mathrm{GL}_2^+(\mathbb{Q}),\mathrm{SL}_2(\mathbb{Z}))$, which is abelian, one might expect that not only the von Neumman algebras $\pi_{\varphi_\beta}(A)''$ are injective, but that A is nuclear. To see that this is not the case, consider the state φ_0 on A defined by the measure $\frac{1}{2}\mu_{\mathbb{H}}$, considered as a measure supported on $\mathbb{H}\times\{0\}\subset\mathbb{H}\times\mathrm{Mat}_2(\mathbb{A}_f)$. The algebra $\pi_{\varphi_0}(A)''$ is a reduction of the crossed product $L^\infty(\mathbb{H},\mu_{\mathbb{H}})\rtimes(\mathrm{GL}_2^+(\mathbb{Q})/\{\pm1\})$. Therefore if A were nuclear, the action of $\mathrm{GL}_2^+(\mathbb{Q})/\{\pm1\}$ on $(\mathbb{H},\mu_{\mathbb{H}})$ would be amenable, which would contradict [23, Corollary 1.2].

(ii) It is apparently straightforward to extend the above results to GL_n (with the interval $(1,2]$ replaced by $(n-1,n]$). One can however formulate a more general problem. Let K be a number field, M a finite dimensional central simple K-algebra, G the group of invertible elements in M. Is the action of $G(K)$ on $M(\mathbb{A}_K)$ with its Haar measure ergodic?

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