NON-BERNOULLIAN QUANTUM K-SYSTEMS

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Dedicated to Professor Walter Thirring on his 70th birthday.

Abstract

We construct an uncountable family of pairwise non-conjugate non-Bernoullian K-systems of type III_1 with the same finite CNT-entropy. We also investigate clustering properties of multiple channels entropies for strong asymptotically abelian systems of type II and III. We prove that a wide enough class of systems has the K-property. In particular, such systems as the space translations of a one-dimensional quantum lattice with the Gibbs states of Araki, the space translations of the CCR-algebra and the even part of the CAR-algebra with the quasi-free states of Park and Shin, non-commutative Markov shifts in Accardi sense are entropic K-systems.

Introduction.

Entropy for transformations of a measure space introduced by Kolmogorov and Sinai is an important invariant in the ergodic theory. Connes, Narnhofer, Størmer and Thirring [11, 9, 10] defined and investigated dynamical entropy for automorphisms of an operator algebra. A more detailed bibliography and applications of dynamical entropy (or CNT-entropy) to mathematical physics can be found in the monographs [21],[3].

In the last years a lot of interesting results in computation of CNT-entropy in the models of mathematical physics was obtained. Let us consider some of them. The dynamical entropy of the space translation for Gibbs states of one-dimensional quantum lattice systems had been studied by Araki [2], was investigated in [10]. Størmer and Voiculescu [31] found a nice formula (predicted by A.Connes for the tracial state) for the entropy of Bogoliubov automorphisms of the CAR-algebra, preserving a quasi-free state the modular operator of which has pure point spectrum. Bezuglyi and Golodets [5] obtained the same formula for the entropy of Bogoliubov actions on the CAR-algebra of the groups $\mathbb{Z}^n$, $n \in \mathbb{N}$, and $\mathbb{Z} \oplus \mathbb{Z} \oplus \dots$. Important results belong to Park and Shin. They proved that the CNT-entropy of the space translation of CAR- and CCR-algebras in n-dimensional ($n < \infty$) continuous spaces with respect to an invariant quasi-free state is equal to the mean entropy and derived a simple formula for the CNT-entropy [24]. Similar results were obtained by Petz for quantum spin lattices with Markov states [21],[25]. Pimsner and Popa [26], Yin [32] and Choda [8] computed the CNT-entropy of the shifts of Temperley-Lieb algebras. Golodets and Størmer [14], Price [27] computed the entropy for a wide enough class of binary shifts. Narnhofer, Størmer and Thirring [18] proved the existence of a binary shift with zero entropy (see [14] for a bibliography about binary shifts).

The progress in computation of CNT-entropy gives the possibility of investigating new problems. The concept of K-system introduced by Rohlin and Sinai [28] is very important in classical theory. Narnhofer and Thirring [19] suggested a non-commutative, or quantum, version of K-systems as systems with "complete memory loss" (see Definition 1.2 below). It is natural to expect that these systems should have interesting properties and applications. They were studied in [19],[20],[3] (see [3] for a more detailed bibliography). In particular, Benatti and Narnhofer [4] proved that K-systems of type II_1 are asymptotically abelian. In [14] it was obtained a description of K-systems defined by bitstreams.

The simplest examples of K-systems can be constructed as follows. Let N be a von Neumann algebra and ψ be a normal faithful state of N. For each integer n let (N_n, ψ_n) be a copy of (N, ψ). Denote by (M, ϕ) the W^*-tensor product of $(N_n, \psi_n)_n$, that is $(M, \phi) = \otimes_{n\in\mathbb{Z}}(N_n, \psi_n)$, and by γ the right shift automorphism of M. Then (M, ϕ, γ) is a K-system (see Theorem 3.1 below). We shall call such systems as Bernoullian systems.

A natural problem is to prove the existence of K-systems which are non-isomorphic to Bernoulli shift. In the commutative case the problem was solved by Ornstein [22] and Ornstein and Shields [23].

In this paper we construct a quasi-free state ω of the CCR-algebra $\mathcal{U}$ and an uncountable family of Bogoliubov automorphisms $\tau_\theta,\ \theta \in [0, 2\pi)$, of $\mathcal{U}$ such that (see Theorem 5.5 below)

(i) if $x \mapsto \pi_\omega(x)$ is the GNS-representation of $\mathcal{U}$ with respect to ω $(x \in \mathcal{U})$, then $M = \pi_\omega(\mathcal{U})''$ is the injective factor of type III_1;

(ii) $\omega \circ \tau_\theta = \omega,\ \theta \in [0, 2\pi)$;

(iii) (M, ω, τ_θ) is a non-Bernoullian K-system;

(iv) the CNT-entropy $h_\omega(\tau_\theta)$ of τ_θ is finite, positive and does not depend on $\theta \in [0, 2\pi)$;

(v) the systems $(M, \omega, \tau_{\theta_1})$ and $(M, \omega, \tau_{\theta_2})$ are non-conjugate for $\theta_1 \neq \theta_2$ (see Definition 1.1).

These results are based on the properties of quasi-free states of CCR-algebras and their modular groups [7]. We also use the results of [24]. Let us note that the problem is still open for systems of type II_1.

As we mentioned, K-systems of type II_1 are asymptotically abelian according to [4]. More exactly, if (M, τ, α) is a K-system, M is an algebra of type II_1, τ is a faithful normal trace on M and $\alpha \in \operatorname{Aut} M$, $\tau \circ \alpha = \tau$, then

$$H_\tau(A, \alpha^n(A)) \to 2H_\tau(A) \quad \text{for } n \to \infty$$

for any finite dimensional subalgebra A of M. It was shown in [4] that the strong asymptotic abelianness of the system (M, τ, α) follows from this clustering property. We prove here (see Section 2 below) the reverse statement.

It is naturally to ask whether asymptotic abelianness is equivalent to the K-property for systems of type II and III. The answer is positive for the dynamical systems defined by bitstreams [14]. In the general case the question is open.

In Section 3 we present a sufficient condition for the K-property. Using this condition we prove in Sections 4 and 5 that most of the systems mentioned above are entropic K-systems. In particular, such systems as the space translations of a one-dimensional quantum spin lattice with the Gibbs state of Araki, the space translations of the CCR-algebra and the even part of the CAR-algebra with the quasi-free states of Park and Shin, non-commutative Markov shifts in Accardi sense are entropic K-systems. Thus it is true for the space translations of ideal Fermi (the even part) and Bose gases.

1 Preliminaries.

A quantum dynamical system is a triple (M, ω, α), where M is a C^*-algebra (or W^*-algebra), α is *-automorphism, and ω is an α-invariant state of M (supposed to be normal in the W^*-case).

Definition 1.1. The systems $(M_1, \omega_1, \alpha_1)$ and $(M_2, \omega_2, \alpha_2)$ are said to be conjugate (or isomorphic), if there exists a *-isomorphism $\theta\colon M_1 \to M_2$ such that $\omega_2 \circ \theta = \omega_1$ and $\theta \circ \alpha_1 = \alpha_2 \circ \theta$.

Recall the definition of CNT-entropy [10].

Let A be a finite dimensional C^*-algebra, ϕ and ψ positive linear functionals on A. The relative entropy is given by

$$S(\phi, \psi) = \mathrm{Trace}(Q_\psi(\log Q_\psi - \log Q_\phi)),$$

where $Q_\phi$and Q_ψ are the density operators corresponding to ϕ and ψ. The quantity $S(\phi) = \mathrm{Tr}\,\eta(Q_\phi)$, where $\eta(x) = -x \log x$, is called the von Neumann entropy of ϕ.

Let $\gamma_i\colon A_i \to M,\ 1 \le i \le n$, be a unital completely positive map of a finite dimensional C^*-algebra A_i to M. The quantity $H_\omega(\gamma_1, \ldots, \gamma_n)$ is defined as follows:

$$\begin{aligned} H_\omega(\gamma_1, \ldots, \gamma_n) &= \sup \sum_{i_1,\ldots,i_n} \eta\omega_{i_1\ldots i_n}(1) + \sum_{k=1}^{n} \sum_{i_k} S(\omega \circ \gamma_k, \omega_{i_k}^{(k)} \circ \gamma_k) \\ &= \sup\Big[\sum_{i_1,\ldots,i_n} \eta\omega_{i_1\ldots i_n}(1) - \sum_{k=1}^{n} \sum_{i_k} \eta\omega_{i_k}^{(k)}(1) \\ &\qquad + \sum_{k=1}^{n} \sum_{i_k} \omega_{i_k}^{(k)}(1) S(\omega \circ \gamma_k, \hat{\omega}_{i_k}^{(k)} \circ \gamma_k) \Big], \end{aligned}$$

where the supremum is taken over all finite decompositions $\omega = \sum\limits_{i_1,\ldots,i_n} \omega_{i_1\ldots i_n}$ of ω in a sum of positive linear functionals, $\omega_{i_k}^{(k)} = \sum\limits_{i_1,\ldots,i_n,\ i_k \text{ fixed}} \omega_{i_1\ldots i_n}$, $\hat{\omega}_{i_k}^{(k)} = \omega_{i_k}^{(k)}(1)^{-1}\omega_{i_k}^{(k)}$. If M is a W^*-algebra and ω is faithful, then any positive linear functional $\phi \le \omega$ can be uniquely represented in the form $\omega(\cdot\sigma_{-i/2}^{\omega}(x))$ for some $x \in M,\ x \ge 0$. Thus decompositions $\omega = \sum \omega_{i_1\ldots i_n}$ are in one-to-one correspondence to decompositions $1 = \sum x_{i_1\ldots i_n},\ x_{i_1\ldots i_n} \ge 0$.

The properties of H_ω ([10],[19],[21]):

1. $H_\omega(\gamma_1 \circ \theta_1, \ldots, \gamma_n \circ \theta_n) \le H_\omega(\gamma_1, \ldots, \gamma_n)$ for any completely positive unital map $\theta_i\colon B_i \to A_i,\ 1 \le i \le n$.

2. If α is an automorphism of M preserving ω, then $H_\omega(\alpha\circ\gamma_1,\dots,\alpha\circ\gamma_n)=H_\omega(\gamma_1,\dots,\gamma_n)$.
3. $H_\omega(\gamma_1,\gamma_1,\dots,\gamma_n)=H_\omega(\gamma_1,\dots,\gamma_n)$.
4. $H_\omega(\gamma_1,\dots,\gamma_p,\gamma_{p+1},\dots,\gamma_n)\le H_\omega(\gamma_1,\dots,\gamma_p)+H_\omega(\gamma_{p+1},\dots,\gamma_n)$.
5. If A is a subalgebra of the centralizer M_ω of the state ω, then $H_\omega(A)=S(\omega|_A)$ and an optimal decomposition is given by $\omega=\sum_i\omega(\cdot p_i)$, where $\{p_i\}_i$ is a set of mutually orthogonal minimal projections of A, $\sum_i p_i=1$.
6. If subalgebras $A_1,\dots,A_n$ of M pairwise commute, and there exists an ω-preserving conditional expectation $M\to A_i$, $1\le i\le n$, then $H_\omega(A_1,\dots,A_n)=S(\omega|_A)$, where A is the algebra generated by $A_1,\dots,A_n$.
7. If M is a von Neumann algebra and ω is its faithful normal state, then $H_\omega(N)>0$ unless $N=\mathbb{C}1$.

The properties 2 and 4 imply that the limit

$$h_\omega(\gamma,\alpha)=\lim_{n\to\infty}\frac{1}{n}H_\omega(\gamma,\alpha\circ\gamma,\dots,\alpha^{n-1}\circ\gamma)$$

exists for any γ.

The dynamical entropy (or the CNT-entropy) $h_\omega(\alpha)$ is the supremum of $h_\omega(\gamma,\alpha)$ over all γ.

For a commutative W^*-dynamical system (M,ω,α), where ω is a normal faithful state, the following properties are equivalent ([12],[19]).

1. For any finite dimensional subalgebra A of M, $\lim_{n\to\infty}h_\omega(A,\alpha^n)=H_\omega(A)$.
2. For any finite dimensional subalgebra A of M, $A\neq\mathbb{C}1$, we have $h_\omega(A,\alpha)>0$.
3. There exists a von Neumann subalgebra A of M such that
   (i) $A\subset\alpha(A)$;
   (ii) $\cap_n\alpha^n(A)=\mathbb{C}1$;
   (iii) $\cup_n\alpha^n(A)$ is weakly dense in M.

Definition 1.2.[19] A W^*-dynamical system (M,ω,α) is an entropic K-system (resp. has completely positive entropy, is an algebraic K-system), if the property 1 (resp. 2, 3) is satisfied.

In the non-commutative case the properties 1-3 are not equivalent. It is easy to show that an entropic K-system has completely positive entropy. The existence of a system having the property 3 and zero entropy was proved in [18]. A system with completely positive entropy and without the K-property was constructed in [14]. It should be noted that both of the mentioned systems are not asymptotically abelian. So the problem of equivalence of the properties 1-3 for asymptotically abelian systems has been not solved yet.

Remark 1.3. Let N be an α-invariant W^*-subalgebra of M, $\gamma=\alpha|_N$, $\phi=\omega|_N$. Suppose there exists a ω-preserving conditional expectation $M\to N$. Then $H_\phi(A_1,\dots,A_n)=H_\omega(A_1,\dots,A_n)$ for any subalgebras $A_1,\dots,A_n$ of N. Hence, if (M,ω,α) is an entropic K-system or has completely positive entropy, then (N,ϕ,γ) has the same property.

2 Asymptotic abelianness and clustering of entropic functions.

In this section we consider asymptotically abelian systems. In particular we reverse the Benatti-Narnhofer theorem [4, 3.1.3].

Theorem 2.1. *Let (M, ω, α) be a strongly asymptotically abelian W^*-dynamical system. Suppose ω is faithful and either ω is tracial or M is approximately finite dimensional. Suppose also that, for given $k \in \mathbb{N}$, for any $x_0, \dots, x_k \in Z(M)$ (the center of M),*

$$\lim_{n\to\infty} \omega\left(x_0\alpha^n(x_1)\dots\alpha^{kn}(x_k)\right) = \omega(x_0)\dots\omega(x_k).$$

Then, for any finite dimensional subalgebra A of M, we have

$$\lim_{n\to\infty} H_\omega\left(A, \alpha^n(A), \dots, \alpha^{kn}(A)\right) = (k+1)H_\omega(A).$$

In particular, if $(Z(M), \omega, \alpha)$ is a commutative K-system, then the above convergence holds for any $k \in \mathbb{N}$.

To prove Theorem we need the following technical result.

Let M be a von Neumann algebra and ϕ a state of M. For any von Neumann subalgebra Q of M, we introduce a semi-norm

$$\|x\|^{\#}_{\phi,Q} = \sup_{\substack{y_1,y_2\in Q\\ \|y_1\|,\|y_2\|\le 1}} (\phi(y_1^*x^*xy_1) + \phi(y_2^*xx^*y_2))^{1/2}.$$

For $\delta > 0$ and subalgebras Q and P of M, we write $Q \overset{\delta}{\underset{\phi}{\subset}} P$ if, for any $x \in Q$, $\|x\| \le 1$, there exists an element $y \in P$, $\|y\| \le 1$, such that

$$\|x - y\|^{\#}_{\phi,Q} < \delta.$$

Lemma 2.2. *Let $n > 0$ and $\varepsilon > 0$ be given. Then there exists $\delta = \delta_n(\varepsilon) > 0$ such that, for any pair of von Neumann subalgebras Q and P of M with $Q \overset{\delta}{\underset{\phi}{\subset}} P$, $\dim Q = n$, and any system of matrix units $\{e^{(m)}_{kl}\}_{k,l=1,\dots,n_m,\ m=1,\dots,s}$, $\sum_{k,m} e^{(m)}_{kk} = 1$, of Q, there exists a system of matrix units $\{p^{(m)}_{kl}\}_{k,l=1,\dots,n_m,\ m=1,\dots,s}$ in P such that*

$$\|e^{(m)}_{kl} - p^{(m)}_{kl}\|^{\#}_{\phi,Q} < \varepsilon \ \ \forall k, l, m.$$

This Lemma was used in a similar form in [10]. First, it was formulated and proved for the tracial case in [11]. The same proof holds in the general case.

Proof of Theorem 2.1. Under the assumptions of Theorem, for any $\varepsilon > 0$, we can find a finite dimensional subalgebra B of M and positive elements $x_1, \dots, x_l$ in B, $\sum_i x_i = 1$, such that

$$H_\omega(A) \le \varepsilon + \sum_j \eta\omega(x_j) + \sum_j S\left(\omega|_A, \omega\left(\cdot\sigma_{-i/2}(x_j)\right)|_A\right).$$

We construct subalgebras $B(0, n), \dots, B(k, n)$ of M and *-homomorphisms $F_{in}\colon B \to B(i, n)$, $0 \le i \le k$, such that

(i) $B(i,n) = F_{in}(B) + \mathbb{C}(1 - F_{in}(1)),\ 0 \le i \le k$;
(ii) $B(0,n), \dots, B(k,n)$ pairwise commute;
(iii) for any $x \in B,\ F_{in}(x) - \alpha^{in}(x) \to 0$ in s-topology, as $n \to \infty$.
Let $B(0,n) = B$ and $F_{0n} = \mathrm{Id}_B$.

Suppose algebras $B(0,n), \dots, B(i,n)$ and *-homomorphisms $F_{0n}, \dots, F_{in}$ are constructed for any n. Let $\{e^{(m)}_{kl}\}_{k,l=1,\dots,n_m,\ m=1,\dots,s}$ be a system of matrix units of B. Define $G_{i+1,n}\colon M \to M$ by (see [17])

$$G_{i+1,n}(x) = \sum_{\substack{m_0,\dots,m_i\\ k_0,\dots,k_i}} F_{0n}(e^{(m_0)}_{k_0 1}) \dots F_{in}(e^{(m_i)}_{k_i 1}) x F_{in}(e^{(m_i)}_{1 k_i}) \dots F_{0n}(e^{(m_0)}_{1 k_0}).$$

The map $G_{i+1,n}$ has the following properties:
(i) $\|G_{i+1,n}\| \le 1$;
(ii) $G_{i+1,n}(x) \in (\cup_{0 \le j \le i} B(j,n))' \cap M$.
We assert that, for any $x \in B$,

$$\|\alpha^{(i+1)n}(x) - G_{i+1,n}(\alpha^{(i+1)n}(x))\|^{\#}_{\omega,\alpha^{(i+1)n}(B)} \to 0,\ \text{as } n \to \infty.$$

In other words, for any $x, y \in B$,

$$\|\Big(\alpha^{(i+1)n}(x) - G_{i+1,n}(\alpha^{(i+1)n}(x))\Big)\alpha^{(i+1)n}(y)\xi_\omega\| \to 0,\ \text{as } n \to \infty,$$

where ξ_ω is the cyclic vector in the GNS-representation corresponding to ω.

First, we note that if a bounded sequence $\{x_n\}_n$ in M converges to zero in s-topology, then, for any $y_1, \dots, y_l \in M$,

$$\|x_n \alpha^{m_1}(y_1) \dots \alpha^{m_l}(y_l)\xi_\omega\| \to 0,\ \text{as } n \to \infty,$$

uniformly on $(m_1, \dots, m_l) \in \mathbb{Z}^l$. Indeed, for any sequences $\{m^{(j)}_n\}_n,\ 1 \le j \le l$, of integers, we have $\Big(x_n\xi_\omega \to 0 \Rightarrow \alpha^{-m^{(1)}_n}(x_n)\xi_\omega \to 0 \Rightarrow \alpha^{-m^{(1)}_n}(x_n)y_1\xi_\omega \to 0 \Rightarrow x_n\alpha^{m^{(1)}_n}(y_1)\xi_\omega \to 0 \Rightarrow \dots \Rightarrow x_n\alpha^{m^{(1)}_n}(y_1) \dots \alpha^{m^{(l)}_n}(y_l)\xi_\omega \to 0\Big)$.

Second, $F_{jn}(x)\alpha^{(i+1)n}(y) - \alpha^{(i+1)n}(y)\alpha^{jn}(x) \to 0$ in s-topology for $j < i+1$, since $F_{jn}(x) - \alpha^{jn}(x) \to 0,\ [x, \alpha^{(i-j+1)n}(y)] \to 0$, and
$F_{jn}(x)\alpha^{(i+1)n}(y) - \alpha^{(i+1)n}(y)\alpha^{jn}(x) =$

$$= \alpha^{(i+1)n}\Big(\alpha^{-(i+1)n}(F_{jn}(x) - \alpha^{jn}(x))y\Big) + \alpha^{jn}([x, \alpha^{(i-j+1)n}(y)]).$$

Using these two observations we conclude that
$\lim_n \|\Big(G_{i+1,n}(\alpha^{(i+1)n}(x))\alpha^{(i+1)n}(y) - \alpha^{(i+1)n}(xy)\Big)\xi_\omega\| =$

$$\begin{aligned}
= \ & \lim_n \|\Big(\sum_{\substack{m_0,\dots,m_i\\ k_0,\dots,k_i}} F_{0n}(e^{(m_0)}_{k_0 1}) \dots F_{in}(e^{(m_i)}_{k_i 1})\alpha^{(i+1)n}(x)F_{in}(e^{(m_i)}_{1k_i}) \dots F_{0n}(e^{(m_0)}_{1k_0})\alpha^{(i+1)n}(y) - \\
& -\alpha^{(i+1)n}(xy)\Big)\xi_\omega\| \\
= \ & \lim_n \|\Big(\sum F_{0n}(e^{(m_0)}_{k_0 1}) \dots F_{in}(e^{(m_i)}_{k_i 1})\alpha^{(i+1)n}(x)F_{in}(e^{(m_i)}_{1k_i}) \dots F_{1n}(e^{(m_1)}_{1k_1})\alpha^{(i+1)n}(y)e^{(m_0)}_{1k_0} - \\
& -\alpha^{(i+1)n}(xy)\Big)\xi_\omega\| = \dots =
\end{aligned}$$

$$= \lim_n \|\Big(\sum F_{0n}(e^{(m_0)}_{k_0 1}) \dots F_{in}(e^{(m_i)}_{k_i 1})\alpha^{(i+1)n}(xy)\alpha^{in}(e^{(m_i)}_{1k_i}) \dots e^{(m_0)}_{1k_0} -$$

$$-\alpha^{(i+1)n}(xy)\Big)\xi_\omega\| = \dots =$$

$$= \lim_n \|\Big(\sum \alpha^{(i+1)n}(xy)e^{(m_0)}_{k_0 1} \dots \alpha^{in}(e^{(m_i)}_{k_i 1})\alpha^{in}(e^{(m_i)}_{1k_i}) \dots e^{(m_0)}_{1k_0} - \alpha^{(i+1)n}(xy)\Big)\xi_\omega\| = 0,$$

and our assertion is proved.

By Lemma 2.2 there exists a system of matrix units $\{p^{(m)}_{kl}(n)\}$ in $(\cup_{0\le j\le i} B(j,n))' \cap M$ such that

$$\alpha^{(i+1)n}(e^{(m)}_{kl}) - p^{(m)}_{kl}(n) \underset{n\to\infty}{\longrightarrow} 0$$

in s-topology.

We define a homomorphism $F_{i+1,n}: B \to M$ by $F_{i+1,n}(e^{(m)}_{kl}) = p^{(m)}_{kl}(n)$ and an algebra $B(i+1,n)$ by $B(i+1,n) = F_{i+1,n}(B) + \mathbb{C}(1 - F_{i+1,n}(1))$.

Then, denoting $F_{in}(x_j) + \omega(x_j)(1 - F_{in}(1))$ by $x^{(i)}_j(n)$, we obtain $H_\omega(A, \alpha^n(A), \dots, \alpha^{kn}(A)) \ge$

$$\ge \sum_{j_0,\dots,j_k} \eta\omega(x^{(0)}_{j_0}(n) \dots x^{(k)}_{j_k}(n)) + \sum_{m=0}^{k} \sum_j S\Big(\omega|_{\alpha^{mn}(A)}, \omega\Big(\cdot\sigma_{-i/2}(x^{(m)}_j(n))\Big)|_{\alpha^{mn}(A)}\Big)$$

$$= \sum_{j_0,\dots,j_k} \eta\omega(x^{(0)}_{j_0}(n) \dots x^{(k)}_{j_k}(n)) + \sum_{m=0}^{k} \sum_j S\Big(\omega|_A, \omega\Big(\cdot\sigma_{-i/2}\Big(\alpha^{-mn}(x^{(m)}_j(n))\Big)\Big)|_A\Big).$$

Hence
$\lim_n H_\omega(A, \alpha^n(A), \dots, \alpha^{kn}(A)) \ge$

$$\ge \lim_n \sum_{j_0,\dots,j_k} \eta\omega(x_{j_0}\alpha^n(x_{j_1}) \dots \alpha^{kn}(x_{j_k})) + \sum_{m=0}^{k} \sum_j S\Big(\omega|_A, \omega(\cdot\sigma_{-i/2}(x_j))|_A\Big)$$

$$\ge \lim_n \sum_{j_0,\dots,j_k} \Big(\eta\omega(x_{j_0}\alpha^n(x_{j_1}) \dots \alpha^{kn}(x_{j_k})) - \eta\omega(x_{j_0}) \dots \omega(x_{j_k})\Big)$$

$$+(k+1)(H_\omega(A) - \varepsilon).$$

It remains to show that, for any $y_0, \dots, y_k$, we have

$$\lim_n \omega(y_0\alpha^n(y_1) \dots \alpha^{kn}(y_k)) = \omega(y_0) \dots \omega(y_k).$$

Let $E: M \to Z(M)$ be an ω-preserving conditional expectation. Then, for any central sequence $\{x_n\}_n$ in M and any $y \in M$,

$$\lim_n (\omega(yx_n) - \omega(E(y)x_n)) = 0.$$

Indeed, if $z \in Z(M)$ is a w-limit point for $\{x_n\}_n$, then $\omega(yz) = \omega(E(y)z)$ is the corresponding limit point for $\{\omega(yx_n)\}_n$ and $\{\omega(E(y)x_n)\}_n$.

Since the sequence $\{\alpha^n(x_1)\alpha^{2n}(x_2)\dots\alpha^{ln}(x_l)\}$ is central for any $l \in \mathbb{N}$ and any $x_1,\dots,x_l \in M$, we obtain

$$\begin{aligned}\lim_n \omega(y_0\alpha^n(y_1)\dots\alpha^{kn}(y_k)) &= \lim_n \omega(E(y_0)\alpha^n(y_1)\dots\alpha^{kn}(y_k)) \\ &= \lim_n \omega(y_1\alpha^{-n}(E(y_0))\alpha^n(y_2)\dots\alpha^{(k-1)n}(y_k)) \\ &= \dots = \lim_n \omega\Big(E(y_k)\alpha^{-n}(E(y_{k-1}))\dots\alpha^{-kn}(E(y_0))\Big) \\ &= \omega(y_0)\dots\omega(y_k).\end{aligned}$$

The last assertion of Theorem follows from the fact that any commutative K-system is mixing of multiplicity k for any $k \in \mathbb{N}$. $\Box$

3 Sufficient condition for the K-property.

We present a sufficient condition for the K-property. This condition allows to show that many well-known quantum systems are entropic K-systems.

Theorem 3.1. *Let (M,ω,α) be a W^*-dynamical system. Suppose ω is faithful, and there exists a W^*-subalgebra M_0 in M such that*
(i) $M_0 \subset \alpha(M_0)$;
(ii) $\cap_n \alpha^n(M_0) = \mathbb{C}1$;
(iii) $\cup_{n\in\mathbb{N}} (\alpha^{-n}(M_0)' \cap \alpha^n(M_0))$ *is weakly dense in M.*
Then the system (M,ω,α) is an entropic K-system.

First, we need the following known result. We prove it for the reader's convenience.

Lemma 3.2. *Let (X,μ) be a Lebesgue space, ξ and η its measurable partitions, $\xi = (X_1,\dots,X_d)$. Suppose*

$$|\int_{X_i} g\,d\mu - \mu(X_i)\int_X g\,d\mu| \le \varepsilon||g||_\infty \quad \forall g \in L^\infty(X/\eta),\ i = 1,\dots,d.$$

Then $H(\xi|\eta) \ge H(\xi) - \delta(\varepsilon,d)$, where $\delta(\varepsilon,d) = (\varepsilon\, d)^{1/2}(\frac{3}{2} + 2\log d + 3\log(1+(\frac{d}{\varepsilon})^{1/2})) \underset{\varepsilon\to 0}{\to} 0$.

Proof. Let $Y = X/\eta$, ν the measure on Y induced by μ, $\mu = \int_Y \mu_y d\nu(y)$ the disintegration of μ with respect to ν. If we denote by ω (resp. ω_y) the state on $L^\infty(X/\xi)$ determined by μ (resp. μ_y), then

$$H(\xi|\eta) = \int_Y S(\omega_y)\,d\nu(y), \qquad H(\xi) = S(\omega),$$

and the assumption of Lemma means that

$$|\int_Y \omega_y(p_i)g(y)d\nu(y) - \omega(p_i)\int_Y g(y)d\nu(y)| \le \varepsilon||g||_\infty,$$

where p_i is the characteristic function of the set X_i, hence

$$\int_Y |\omega_y(p_i) - \omega(p_i)|d\nu(y) \le \varepsilon,$$

so that

$$\int_Y ||\omega_y - \omega|| d\nu(y) \le \varepsilon\, d.$$

Let $Z = \{y \in Y \mid ||\omega_y - \omega|| \ge (\varepsilon\, d)^{1/2}\}$. Then $\nu(Z) \le (\varepsilon\, d)^{1/2}$, $|S(\omega_y) - S(\omega)| \le 2\log d$ for any $y \in Z$, and $|S(\omega_y) - S(\omega)| \le 3(\varepsilon\, d)^{1/2}(1/2 + \log(1 + d^{1/2}/\varepsilon^{1/2}))$ for any $y \in Y\backslash Z$ by [10, Lemma IV.1]. Thus we obtain the desired. □

Proof of Theorem 3.1. Let N be a finite dimensional subalgebra of M. For any $\varepsilon > 0$ there exist $m \in \mathbb{N}$ and elements $x_1, \dots, x_d \in \alpha^{-m}(M_0)' \cap \alpha^m(M_0)$, $\sum x_i = 1$, such that

$$H_\omega(N) < \varepsilon + \sum \eta\omega(x_j) + \sum S(\omega|_N, \omega(\cdot\sigma_{-i/2}(x_j))|_N).$$

Choose $\varepsilon_1 > 0$ such that $\delta(\varepsilon_1, d) < \varepsilon$. By [6, 2.6.1] there exists $n_0 \ge 2m$ such that

$$|\omega(x_j y) - \omega(x_j)\omega(y)| \le \varepsilon_1 ||y|| \quad \forall y \in \alpha^{m-n_0}(M_0),\ j = 1, \dots, d.$$

Let us fix $n \ge n_0$. For each $j \in \mathbb{Z}$, let A_j be a copy of a finite dimensional abelian C^*-algebra A_0 with minimal projections $p_1, \dots, p_d$, and for a finite subset $J = \{j_1, \dots, j_m\}$ of $\mathbb{Z}$, $A_J = A_{j_1} \otimes \dots \otimes A_{j_m}$. We define a unital positive map $F_J\colon A_J \to M$ by

$$F_J(p_{i_1} \otimes \dots \otimes p_{i_m}) = \alpha^{nj_1}(x_{i_1}) \dots \alpha^{nj_m}(x_{i_m}).$$

Let A be the infinite C^*-tensor product $\otimes_{j\in\mathbb{Z}} A_j$. Since A is the inductive limit of $\{A_J\}_J$, the coherent system $\{F_J\}_J$ defines a positive unital map $F\colon A \to M$.

Let $\mu = \omega \circ F$, γ the right shift automorphism of A, π_μ the GNS-representation corresponding to μ, $\bar{A} = \pi_\mu(A)''$, $\bar{\mu}$ and $\bar{\gamma}$ the state and the automorphism of $\bar{A}$ corresponding to μ and γ respectively.

Since ω is faithful, F induces a normal unital positive map $\bar{F}\colon \bar{A} \to M$. Indeed, any bounded linear map of A (in particular F, π_μ, μ) can be uniquely extended to a normal map of the W^*-enveloping algebra A^{**} of the algebra A which we denote by the same letter. Then $\bar{A} = \pi_\mu(A^{**})$, and we only have to show that $\mathrm{Ker}\,\pi_\mu \subset \mathrm{Ker}F$. This follows from the Schwarz inequality $F(x)^*F(x) \le F(x^*x)$: $\mathrm{Ker}\,\pi_\mu = \{x \in A^{**} \mid \mu(x^*x) = 0\} = \{x \in A^{**} \mid F(x^*x) = 0\} \subset \{x \in A^{**} \mid F(x) = 0\} = \mathrm{Ker}F$.

For any subset J of $\mathbb{Z}$, we denote by $\bar{A}_J$ the von Neumann subalgebra of $\bar{A}$ generated by $\pi_\mu(A_j)$, $j \in J$. Then

1) $\bar{F}(\bar{A}_{(-\infty,k]}) \subset \alpha^{m+nk}(M_0)$;
2) if $a \in \bar{A}_{J_1}$, $b \in \bar{A}_{J_2}$ and $J_1 \cap J_2 = \emptyset$, then $\bar{F}(ab) = \bar{F}(a)\bar{F}(b)$;
3) $\bar{\mu} = \omega \circ \bar{F}$ and $\bar{F} \circ \bar{\gamma} = \alpha^n \circ \bar{F}$.

For any $a \in \bar{A}_{(-\infty,-1]}$ and $i \in \{1, \dots, d\}$ we have

$$|\bar{\mu}(p_i a) - \bar{\mu}(p_i)\bar{\mu}(a)| = |\omega(x_i\bar{F}(a)) - \omega(x_i)\omega(\bar{F}(a))| \le \varepsilon_1 ||\bar{F}(a)|| \le \varepsilon_1 ||a||.$$

By Lemma 3.2 $H_{\bar{\mu}}(\bar{A}_0 | \bar{A}_{(-\infty,-1]}) \ge H_{\bar{\mu}}(\bar{A}_0) - \delta(\varepsilon_1, d)$. On the other hand, $H_{\bar{\mu}}(\bar{A}_0) = \sum_j \eta\omega(x_j)$ and

$$\begin{aligned} H_{\bar{\mu}}(\bar{A}_0 | \bar{A}_{(-\infty,-1]}) &= h_{\bar{\mu}}(\bar{A}_0, \gamma) = \lim_{k\to\infty} \frac{1}{k} H_{\bar{\mu}}(\bar{A}_{[0,k-1]}) \\ &= \lim_{k\to\infty} \frac{1}{k} \sum_{i_1,\dots,i_k} \eta\omega(x_{i_1}\alpha^n(x_{i_2}) \dots \alpha^{n(k-1)}(x_{i_k})). \end{aligned}$$

By the definition of H_ω we have
$H_\omega(N, \alpha^n(N), \dots, \alpha^{n(k-1)}(N)) \geq$

$$\geq \sum_{i_1,\dots,i_k} \eta\omega(x_{i_1}\alpha^n(x_{i_2})\dots\alpha^{n(k-1)}(x_{i_k}))+$$

$$+\sum_{l=1}^{k}\sum_{i_l} S\Big(\omega|_{\alpha^{n(l-1)}(N)}, \omega\Big(\cdot\sigma_{-i/2}(\alpha^{n(l-1)}(x_{i_l}))\Big)|_{\alpha^{n(l-1)}(N)}\Big),$$

so that
$h_\omega(N, \alpha^n) \geq$

$$\begin{aligned}
&\geq \lim_{k\to\infty}\frac{1}{k}\sum_{i_1,\dots,i_k}\eta\omega(x_{i_1}\alpha^n(x_{i_2})\dots\alpha^{n(k-1)}(x_{i_k})) + \sum_j S(\omega|_N, \omega(\cdot\sigma_{-i/2}(x_j))|_N)\\
&\geq \sum_j \eta\omega(x_j) - \delta(\varepsilon_1, d) + \sum_j S(\omega|_N, \omega(\cdot\sigma_{-i/2}(x_j))|_N)\\
&\geq H_\omega(N) - \varepsilon - \delta(\varepsilon_1, d) \geq H_\omega(N) - 2\varepsilon. \qquad \square
\end{aligned}$$

4 Entropic properties of quantum systems with Markov states.

An example of a system, for which the conditions of Theorem 3.1 are satisfied, is the space translation for the Gibbs state of a one-dimensional quantum lattice system corresponding to a finite range interaction. Such a state is always factorial and has exponential decay of correlations [2].

In this section we study entropic properties of a quantum spin system with a Markov state and its subsystem given by the restriction to the centralizer of the Markov state. We prove that these systems are entropic K-systems too.

So, let $B = Mat_s(\mathbb{C})$ be a full matrix algebra. For every $i \in \mathbb{Z}$ a copy A_i of B is associated and A is the infinite C^*-tensor product $\otimes_i A_i$. The right shift automorphism of the algebra A will be denoted by γ.

For each subset J of $\mathbb{Z}$, let A_J be the C^*-subalgebra of A generated by $A_i, i \in J$. Recall that a state ϕ of A is called locally faithful provided its restriction to A_J is faithful for any finite J. We restrict ourselves to locally faithful states.

According to the definition of Accardi [1] a translation invariant state ϕ of A is called Markov state if the following condition is satisfied.

> For every $n \in \mathbb{N}$ there exists a completely positive unital mapping $F_n: A_{[0,n+2]} \to A_{[0,n+1]}$ which preserves the state ϕ and leaves $A_{[0,n]}$ pointwise invariant.

Petz proved that the latter condition is equivalent to the next one:

$$S(\phi|_{A_{[0,n+2]}}) + S(\phi|_{A_{n+1}}) = S(\phi|_{A_{[0,n+1]}}) + S(\phi|_{A_{[n+1,n+2]}})$$

This equality implies that the mean entropy $s(\phi)$ of a Markov state ϕ is equal to $S(\phi|_{A_{[0,1]}}) - S(\phi|_{A_0})$.

Theorem 4.1. *Let ϕ be a Markov state. Then*

1) ϕ is separating, i. e. the cyclic vector ξ_ϕ is separating for $M = \pi_\phi(A)''$;

2) M is a factor;

3) the centralizer M_ϕ of the state ϕ is the hyperfinite II_1-factor.

Proof. 1) Define $\phi_0 = \phi|_{A_{[0,\infty)}}$. The state ϕ_0 is separating and, for any $n \geq 1$, there exists a $\sigma_t^{\phi_0}$-invariant *-subalgebra N_n of $A_{[0,n+1]}$ such that $A_{[0,n]} \subset N_n$ (see [15]).

Let E_n be a ϕ_0-preserving conditional expectation of $A_{[0,\infty]}$ onto $N_n \cap N_1' \subset A_{[0,n+1]} \cap A'_{[0,1]} = A_{[2,n+1]}$. The map $\gamma^{-k} \circ E_{m+2k} \circ \gamma^k\colon A_{[-k,\infty)} \to A_{[-k+2,\infty)}$ leaves $N_m \cap N_1'$ pointwise invariant for any $k \geq 0$, since $\gamma^k(N_m \cap N_1') \subset A_{[k+2,k+m+1]} \subset N_{[2k+m]} \cap N_1'$ for $k \geq 1$. Hence the formula

$$E_{n,m} = E_m \circ \gamma^{-1} \circ E_{m+2} \circ \gamma^{-1} \circ \ldots \circ E_{m+2(n-1)} \circ \gamma^{-1} \circ E_{m+2n} \circ \gamma^n$$

defines a ϕ-preserving conditional expectation $A_{[-n,\infty)} \to N_m \cap N_1'$. Then $\{E_{n,m}\}_n$ defines a ϕ-preserving conditional expectation of A onto $N_m \cap N_1'$.

So, the algebra $A_\infty = \cup_n A_{[-n,n]}$ is the union of such finite dimensional subalgebras that there exists a ϕ-preserving conditional expectation onto each of them. Hence ϕ is separating [15].

2) Since $A_{[0,n]}$ is a type I subfactor of N_n, the algebra N_n is generated by $A_{[0,n]}$ and its relative commutant $N_n \cap A_{n+1}$ in N_n. So, taking $\tilde N_n = \gamma^{-n-1}(N_n \cap A_{n+1}) \subset A_0$, we have $N_n = A_{[0,n]} \otimes \gamma^{n+1}(\tilde N_n)$.

A ϕ_0-preserving conditional expectation $A_{[0,\infty)} \to N_n$ maps $A_{[m,\infty)}$ to $A'_{[0,m-1]} \cap N_n = A_{[m,n]} \otimes \gamma^{n+1}(\tilde N_n)$ for $m \leq n$, and $A_{[n+1,\infty)}$ to $\gamma^{n+1}(\tilde N_n)$. Hence the algebras $\tilde N_n$ and $A_{[0,m]} \otimes \gamma^{m+1}(\tilde N_n)$, $m \leq n$, are the images of ϕ_0-preserving conditional expectations.

Let $N = \cup_{n=1}^\infty \cap_{m=n}^\infty \tilde N_m$. Then N is a subalgebra of A_0, and there exist ϕ_0-preserving conditionnal expectations onto N and $A_{[0,n]} \otimes \gamma^{n+1}(N)$, $n \geq 0$.

Let $E\colon A_{[0,\infty)} \to N$ be a ϕ_0-preserving conditional expectation. Since, for any n,

$$\gamma^{n+1} \circ E \circ \gamma^{-n-1}\colon A_{[n+1,\infty)} \to \gamma^{n+1}(N)$$

is a ϕ_0-preserving conditional expectation, the unique ϕ_0-preserving conditional expectation

$$A_{[0,\infty)} = A_{[0,n]} \otimes A_{[n+1,\infty)} \to A_{[0,n]} \otimes \gamma^{n+1}(N)$$

coincides with $\mathrm{Id}_{A_{[0,n]}} \otimes (\gamma^{n+1} \circ E \circ \gamma^{-n-1})$. Hence, for $a_0, \ldots, a_n \in A_0$, we have

$$\begin{aligned} E(a_0\gamma(a_1)\ldots\gamma^n(a_n)) &= \Big(E \circ (\mathrm{Id}_{A_0} \otimes \gamma \circ E \circ \gamma^{-1})\Big)(a_0 \ldots \gamma^n(a_n)) \\ &= E\Big(a_0\gamma\Big(E(a_1 \ldots \gamma^{n-1}(a_n))\Big)\Big) \\ &= \ldots = (E_{a_0} \circ \ldots \circ E_{a_n})(1), \end{aligned} \tag{4.1}$$

where $E_a\colon N \to N$, $a \in A_0$, maps $b \in N$ to $E(a\gamma(b))$. (In other words, ϕ is a C^*-finitely correlated state, see [13]).

E_1 maps N to the center $Z(N)$ of the algebra N. If $p_1, \ldots, p_n$ is the list of minimal projections of $Z(N)$, then the matrix $\big(\phi(p_i)^{-1}\phi(p_i\gamma(p_j))\big)_{ij}$ of the mapping $E_1|_{Z(N)}$ with respect to this basis is a stochastic matrix with strictly positive elements. The probability distribution $(\phi(p_1),\dots,\phi(p_n))$ is invariant for the corresponding Markov process, and the Markov dynamical system so obtained is simply (Z,ϕ,γ), where $Z=(\cup_{n\in\mathbb{Z}}\gamma^n(Z(N)))''$. We want to use mixing properties of this system. We need two lemmas to do it.

Lemma 4.2. *Let z be a minimal projection of $Z(N)$, $a\in A_{(-\infty,-1]}$, $b\in A_{[1,\infty)}$. Then*

$$\phi(z)\phi(azb)=\phi(az)\phi(zb).$$

Proof. Suppose $a=\gamma^{-n}(a_n)\dots\gamma^{-1}(a_1)$ and $b=\gamma(b_1)\dots\gamma^n(b_n)$, where $a_1,\dots,a_n,b_1,\dots,b_n\in A_0$. Then

$$\phi(azb)=\phi((E_{a_n}\circ\dots\circ E_{a_1}\circ E_z\circ E_{b_1}\circ\dots\circ E_{b_n})(1)).$$

Since z is minimal in $Z(N)$, the element $E_z(n)=zE_1(n)$ is a scalar multiple of z for any $n\in N$, so

$$(E_z\circ E_{b_1}\circ\dots\circ E_{b_n})(1)=\frac{\phi(zb)}{\phi(z)}z.$$

Then

$$\phi(azb)=\frac{\phi(zb)}{\phi(z)}\phi((E_{a_n}\circ\dots\circ E_{a_1})(z))=\frac{\phi(zb)}{\phi(z)}\phi(az).\quad\square$$

Lemma 4.3. *The subalgebra Z of M lies in the centralizer M_ϕ of ϕ. In particular, there exists a ϕ-preserving conditional expectation $G\colon M\to Z$. We have:*

(i) if $a\in A_{[n,m]}$, then $G(a)\in A_{[n-1,m+1]}$;

(ii) if $a\in A_{(-\infty,n]}$, $b\in A_{[n+2,\infty)}$, then $G(ab)=G(a)G(b)$.

Proof. If $a=\gamma^{-n}(a_{-n})\dots\gamma^n(a_n)$, $z\in Z(N)$, where $a_{-n},\dots,a_n\in A_0$, then by (4.1)

$$\phi(az)=\phi((E_{a_{-n}}\circ\dots E_{a_{-1}}\circ E_{a_0z}\circ E_{a_1}\circ\dots E_{a_n})(1)).$$

Since $E_{a_0z}=E_{za_0}$, this implies that $Z(N)$ lies in the centralizer of ϕ.

Let $a\in A_{[-n,-1]}$, $b\in A_{[1,n]}$. It is sufficient to prove that if $\tilde a$ (resp. $\tilde b$) is the image of a (resp. b) under a ϕ-preserving conditional expectation $A_{[-n-1,0]}\to\vee_{k=0}^{n+1}\gamma^{-k}(Z)$ (resp. $A_{[0,n+1]}\to\vee_{k+0}^{n+1}\gamma^k(Z)$), then $G(ab)=\tilde a\tilde b$.

For this it is enough to show that, for any system $z_{-k},\dots,z_0,\dots,z_k$, $k\ge n+1$, of minimal projections of Z, we have

$$\phi(ab\gamma^{-k}(z_{-k})\gamma^{-k+1}(z_{-k+1})\dots\gamma^k(z_k))=\phi(\tilde a\tilde b\gamma^{-k}(z_{-k})\dots\gamma^k(z_k)).$$

Apply Lemma 4.2:

$$\begin{aligned}\phi(ab\gamma^{-k}(z_{-k})\dots\gamma^k(z_k)) &= \frac{1}{\phi(z_0)}\phi(a\gamma^{-k}(z_{-k})\dots z_0)\phi(bz_0\dots\gamma^k(z_k))\\ &= \frac{1}{\phi(z_0)}\frac{1}{\phi(z_{-n-1})}\phi(\gamma^{-k}(z_{-k})\dots\gamma^{-n-1}(z_{-n-1}))\times\\ &\qquad \phi(a\gamma^{-n-1}(z_{-n-1})\dots z_0)\times\\ &\qquad \frac{1}{\phi(z_{n+1})}\phi(bz_0\dots\gamma^{n+1}(z_{n+1}))\times\\ &\qquad \phi(\gamma^{n+1}(z_{n+1})\dots\gamma^k(z_k)).\end{aligned}$$

Since $\phi(a\gamma^{-n-1}(z_{-n-1})\dots z_0) = \phi(\tilde{a}\gamma^{-n-1}(z_{-n-1})\dots z_0)$ and $\tilde{a}\gamma^{-n-1}(z_{-n-1})\dots z_0$ is a scalar multiple of $\gamma^{-n-1}(z_{-n-1})\dots z_0$, we obtain the same result for $\phi(\tilde{a}\tilde{b}\gamma^{-k}(z_{-k})\dots\gamma^k(z_k))$. □

For a subset J of Z, let Z_J be the W^*-subalgebra of Z generated by $\gamma^j(Z(N))$, $j \in J$. Since (Z, ϕ, γ) is a classical mixing Markov dynamical system, $\cap_{n\in\mathbb{N}} Z_{(-\infty,-n]\cup[n,\infty)} = \mathbb{C}1$. By [6, 2.6.1] it is equivalent to the convergence

$$\sup_{b\in Z_{(-\infty,-n]\cup[n,\infty)}} \frac{|\phi(ab)-\phi(a)\phi(b)|}{\|b\|} \underset{n\to\infty}{\to} 0, \quad \forall a \in Z.$$

By virtue of Lemma 4.3 it follows that

$$\sup_{b\in A_{(-\infty,-n]\cup[n,\infty)}} \frac{|\phi(ab)-\phi(a)\phi(b)|}{\|b\|} \underset{n\to\infty}{\to} 0, \quad \forall a \in \cup_{m\in\mathbb{N}} A_{[-m,m]}.$$

Hence ϕ is factorial by [6, 2.6.10].

3) We prove that the center $Z(M_\phi)$ of the centralizer M_ϕ is contained in the center of the algebra M.

As we showed above, the *-algebra A_∞ is the union of the finite dimensional σ_t^ϕ-invariant subalgebras. Hence the linear span of elements $b \in A_\infty$ such that $\Delta_\phi^{1/2} b\xi_\phi = \lambda^{1/2} b\xi_\phi$ for some $\lambda > 0$ is s-dense in M. So, it is sufficient to prove that any such an element commutes with $Z(M_\phi)$.

First, we prove that $[\gamma^n(b), a] \to 0$ in s-topology for any $a \in M$.

Let $\varepsilon > 0$. There exists an $a_\varepsilon \in A_\infty$ such that $\|(a - a_\varepsilon)\xi_\phi\| < \varepsilon$. Then

$$(a - a_\varepsilon)\gamma^n(b)\xi_\phi = \lambda^{-1/2}(a - a_\varepsilon) j\gamma^n(b^*)\xi_\phi = \lambda^{-1/2}(j\gamma^n(b^*)j)(a - a_\varepsilon)\xi_\phi,$$

so that

$$\|[a, \gamma^n(b)]\xi_\phi\| \le (1 + \lambda^{-1/2})\|b\|\varepsilon + \|[a_\varepsilon, \gamma^n(b)]\xi_\phi\|.$$

Since $[a_\varepsilon, \gamma^n(b)] = 0$ for n sufficiently large, our assertion is proved.

For any n, $\gamma^n(b^*)b \in M_\phi$. Hence, for $z \in Z(M_\phi)$, $z\gamma^n(b^*)b = \gamma^n(b^*)bz$. Then

$$\gamma^n(b)z\gamma^n(b^*)b = \gamma^n(bb^*)bz. \tag{4.2}$$

Since M is a factor and $\{\gamma^n(bb^*)\}_n$ is central, letting $n \to \infty$, at the right hand side of (4.2) we obtain $\phi(bb^*)bz$. The left hand side of (4.2) is equal to $\gamma^n(b)[z, \gamma^n(b^*)]b + \gamma^n(bb^*)zb$, so it weakly converges to $\phi(bb^*)zb$. Hence z lies in the center of M, which is trivial.

Hyperfiniteness of the factor M_ϕ is evident: if $\{M_n\}_n$ is an increasing sequence of σ_t^ϕ-invariant finite dimensional subalgebras of M such that $\cup_n M_n$ is weakly dense in M, then $\cup_n(M_n \cap M_\phi)$ is weakly dense in M_ϕ. □

Remark 4.4. It follows from the Perron-Frobenius theorem and the above considerations that a Markov state has exponential decay of correlations. More precisely, if λ is the maximum of $|\mu|$ over all eigenvalues μ of E_1 different from 1, then there exists a constant $C > 0$ such that

$$|\phi(ab) - \phi(a)\phi(b)| \le C\lambda^n \phi(|G(a)|)\phi(|G(b)|), \quad \forall a \in A_{[k,l]}\ \forall b \in A_{(-\infty,k-n]\cup[l+n,\infty)}.$$

Theorem 4.5. *Let (M, ϕ, γ) be as in Theorem 4.1, then the systems (M, ϕ, γ) and $(M_\phi, \phi|_{M_\phi}, \gamma|_{M_\phi})$ are entropic K-systems and*

$$h_\phi(\gamma|_{M_\phi}) = h_\phi(\gamma) = s(\phi) = S(\phi|_{A_{[0,1]}}) - S(\phi|_{A_0}).$$

Proof. Since ϕ is factorial, the K-property follows from Theorem 3.1.

The equality $h_\phi(\gamma) = s(\phi)$ was obtained by Petz [25]. Equality of $h_\phi(\gamma|_{M_\phi})$ and $h_\phi(\gamma)$ also follows from his proof, but for the sake of completeness we give a proof.

The inequalities $h_\phi(\gamma|_{M_\phi}) \leq h_\phi(\gamma) \leq s(\phi)$ always hold.

We will use the notations of the proof of Theorem 4.1. Let M_n be a σ_t^ϕ-invariant subalgebra of M such that $A_{[0,n]} \subset M_n \subset A_{[-1,n+1]}$ (see the proof of Theorem 4.1,1)), and let $\tilde{M}_n = M_n \cap M_\phi$. Then $h_\phi(\gamma|_{M_\phi}) = \lim_{n\to\infty} h_\phi(\tilde{M}_n, \gamma)$ by a Kolmogorov-Sinai type theorem [10].

For any k, we have

$$H_\phi(\tilde{M}_n, \gamma(\tilde{M}_n), \ldots, \gamma^{k(n+3)}(\tilde{M}_n)) \geq H_\phi(\tilde{M}_n, \gamma^{n+3}(\tilde{M}_n), \ldots, \gamma^{k(n+3)}(\tilde{M}_n)) = S(\phi|_{M_{n,k}}),$$

where $M_{n,k}$ is the algebra generated by $M_n, \gamma^{n+3}(M_n), \ldots, \gamma^{k(n+3)}(M_n)$ [10].

We need the following lemma to estimate $S(\phi|_{M_{n,k}})$:

Lemma 4.6. *Let $A \subset N \subset B$ be finite dimensional C^*-algebras, $A = Mat_p(\mathbb{C}), B = Mat_q(\mathbb{C}), \psi$ be a state of B. Then*

$$S(\psi|_A) + \log q/p \geq S(\psi|_N) \geq S(\psi) - \log q/p.$$

Proof. Let $\tau = \mathrm{Tr}_B(1)^{-1}\mathrm{Tr}_B$ be the unique tracial state of B. Let $Q_\tau \in N$ be the density matrix of $\tau|_N$, i. e. $\tau|_N = \mathrm{Tr}_N(\cdot Q_\tau)$. Then

$$S(\psi|_N) = -\psi(\log Q_\tau) - S(\tau|_N, \psi|_N).$$

Every minimal projection e of N majorizes a minimal projection of B and is equivalent to a projection which is majorized by a minimal projection of A. Hence $1/q \leq \tau(e) \leq 1/p$, so that $1/q \leq Q_\tau \leq 1/p$.

Using monotonicity of the relative entropy, we obtain

$$S(\psi|_N) \geq \log p - S(\tau, \psi) = S(\psi) - \log q/p.$$

Analogously

$$S(\psi|_N) \leq \log q - S(\tau|_A, \psi|_A) = S(\psi|_A) + \log q/p. \ \square$$

Applying the lemma to $A = A_{[0,n]} \otimes A_{[n+3,2n+3]} \otimes \ldots \otimes A_{[k(n+3),k(n+3)+n]}$, $N = M_{n,k}$ and $B = A_{[-1,k(n+3)+n+1]}$, we obtain

$$S(\phi|_{M_{n,k}}) \geq S(\phi|_{A_{[-1,k(n+3)+n+1]}}) - \log \frac{s^{(k+1)(n+3)}}{s^{(k+1)(n+1)}}$$

(recall that $A_0 = Mat_s(\mathbb{C})$). Hence

$$
\begin{aligned}
h_\phi(\tilde{M}_n,\gamma) &\geq \lim_{k\to\infty}\frac{1}{k(n+3)}\Big(S(\phi|_{A_{[-1,k(n+3)+n+1]}})-2(k+1)\log s\Big)\\
&= s(\phi)-\frac{2}{n+3}\log s.
\end{aligned}
$$

This ends the proof of Theorem. □

Remark 4.7. It is not difficult to construct a system (M,ϕ,γ) where ϕ is a Markov state and Z is a Cartan subalgebra of M. In [11] non-commutative Bernoulli shifts are defined. Analogously it is naturally to call the system (M_ϕ,ϕ,γ) a non-commutative Markov shift.

It is well-known that a classical mixing Markov system is conjugate to a Bernoulli shift with the same entropy. Is this true in the non-commutative case?

5 Non-isomorphic entropic K-systems.

In this section we obtain an uncountable family of non-conjugate K-systems on the injective III_1-factor all having the same finite entropy.

Besides the space translation of a one-dimensional quantum lattice system the examples of systems, for which the conditions of Theorem 3.1 are satisfied, are also the space translations of the CCR-algebra over the pre-Hilbert space $L^2_0(\mathbb{R})$ and the space translations of the even part of the CAR-algebra, when all the systems are in factor states.

Let us consider the case of space translation of CCR-algebra in more detail.

So, let $\mathcal{U}$ be the CCR-algebra over $L^2(\mathbb{R})$, τ the Bogoliubov automorphism of $\mathcal{U}$ corresponding to the space translation of 1, i. e.

$$\tau(W(f))=W(Vf),\ (Vf)(x)=f(x-1).$$

(For all the facts and the definitions concerning CCR-algebra we refer the reader to [7].) Let A be a positive bounded operator on $L^2(\mathbb{R})$ that commutes with V, and ω be the quasi-free state corresponding to A. If $\mathrm{Ker}A=0$, then the state ω is separating and

$$\sigma^\omega_t(W(f))=W(B^{it}f),\ \text{where } B=\frac{A}{1+A}. \tag{5.1}$$

The GNS-triple $(H_\omega,\pi_\omega,\xi_\omega)$ corresponding to ω can be expressed in terms of the Fock representation as follows:

$$
\begin{aligned}
H_\omega &= F_+\otimes F_+,\\
\xi_\omega &= \Omega\otimes\Omega,\\
\pi_\omega(a^*(f)) &= a^*((1+A)^{1/2}f)\otimes 1+1\otimes a(JA^{1/2}f),
\end{aligned} \tag{5.2}
$$

where F_+ is the symmetric Fock space over $L^2(\mathbb{R})$, Ω is the vacuum vector, and J is an anti-linear isometric involution on $L^2(\mathbb{R})$.

Then the automorphism τ is implemented by the unitary

$$\Gamma(V) \otimes \Gamma(JVJ), \tag{5.3}$$

where Γ is the operator of second quantization.

Using (5.1) and (5.2) one also concludes that

$$\Delta_\omega^{it} = \Gamma(B^{it}) \otimes \Gamma(JB^{it}J) = \Gamma(B^{it}) \otimes \Gamma((JBJ)^{-it}), \tag{5.4}$$

equivalently

$$\Delta_\omega = \Gamma(B) \otimes \Gamma((JBJ)^{-1}) = \Gamma(B) \otimes \Gamma(JB^{-1}J).$$

So we see that the discrete part of the spectrum of Δ_ω is the group generated by the eigenvalues of B. Moreover, if the spectrum of B is continuous then ξ_ω is the unique eigenvector of Δ_ω. In the latter case the centralizer of the state ω is trivial, and hence $M = \pi_\omega(\mathcal{U})''$ is a type III_1 factor (see [30, Theorem 29.9]). This factor is injective, since $\mathcal{U}$ is nuclear.

For a subset Λ of $\mathbb{R}$, let $\mathcal{U}_\Lambda$ be the C^*-subalgebra of $\mathcal{U}$ generated by $W(f)$, $\mathrm{supp} f \subset \Lambda$. Then $\mathcal{U}_\Lambda$ is the CCR-algebra over $L^2(\Lambda)$, $\mathcal{U}_{\Lambda_1}$ and $\mathcal{U}_{\Lambda_2}$ commute for $\Lambda_1 \cap \Lambda_2 = \emptyset$, and $\cup_{\Lambda \text{ compact}} \pi_\omega(\mathcal{U}_\Lambda)$ is weakly dense in M (though $\cup_\Lambda \mathcal{U}_\Lambda$ is not norm-dense in $\mathcal{U}$). So that the assumptions of Theorem 3.1 are satisfied with $M_0 = \mathcal{U}''_{(-\infty,0]}$.

Let us summarize what we have proved:

Proposition 5.1. *Under the above notations, let A has pure continuous spectrum. Then $M = \pi_\omega(\mathcal{U})''$ is the injective III_1-factor, the cyclic vector ξ_ω is the unique eigenvector of the modular operator Δ_ω, and the system (M, ω, τ) is an entropic K-system.* □

It is worth to note that the same result holds for the even part of the CAR-algebra.

Park and Shin considered the situation, when A is the operator of convolution with a function K. Under certain conditions on K they proved that

$$h_\omega(\tau) = \frac{1}{2\pi} \int \left(\eta \hat{K}(x) - \eta(1 + \hat{K}(x)) \right) dx, \tag{5.5}$$

where $\hat{K}(x) = \int K(y) e^{iyx} dy$ is the Fourier transform of K.

The operator A can be considered via the Fourier transform as the operator of multiplication by the function $\hat{K}$. Let us suppose that

$$K(x) = o(e^{-\alpha|x|}), \text{ as } |x| \to \infty, \text{ for certain } \alpha > 0. \tag{5.6}$$

Then $\hat{K}$ is analytic in the strip $|\mathrm{Im}\, z| < \alpha$, and hence A has pure continuous spectrum and Proposition 5.1 can be applied. The next theorem shows that such systems are usually non-conjugate.

Theorem 5.2. *Let K_i be a function satisfying* (5.6) *such that $\hat{K}_i \geq 0$, and ω_i be the state corresponding to K_i, $i = 1, 2$. Suppose the systems (M, ω_1, τ) and (M, ω_2, τ) are isomorphic. It follows that*

$$\hat{K}_2(x) = \hat{K}_1(x + 2\pi n)$$

for certain $n \in \mathbb{Z}$, equivalently $K_2(x) = e^{i2\pi nx} K_1(x)$.

Proof. It is more convenient for us to pass to the Fourier transform, i. e. we consider the automorphism τ as the Bogoliubov automorphism corresponding to the operator of multiplication by the function e^{ix} and the states ω_1 and ω_2 as the quasi-free states corresponding to the operators of multiplication by the functions $\hat{K}_1$ and $\hat{K}_2$ respectively.

The space of the GNS-representation corresponding to $\omega_i, i = 1, 2$, is identified with $F_+ \otimes F_+$ as described above, and we choose J to be the usual pointwise conjugation on $L^2(\mathbb{R})$.

An isomorphism of our systems is implemented by a unitary U on $F_+ \otimes F_+$. This operator maps ξ_{ω_1} to ξ_{ω_2}, conjugates the modular operators and the operators implementing the automorphisms. In view of the identities (5.3), (5.4) this means that

$$U(\Omega \otimes \Omega) = \Omega \otimes \Omega,$$

$$U\Gamma(e^{iX}) \otimes \Gamma(e^{-iX})U^* = \Gamma(e^{iX}) \otimes \Gamma(e^{-iX}), \tag{5.7}$$

$$U\Gamma(B_1^{it}) \otimes \Gamma(B_1^{-it})U^* = \Gamma(B_2^{it}) \otimes \Gamma(B_2^{-it}), \tag{5.8}$$

where X is the operator of multiplication by x and B_j is the operator of multiplication by the function $D_j = \hat{K}_j(1 + \hat{K}_j)^{-1}, j = 1, 2$.

For non-negative integers l, m let $P_{l,m}$ be the projection onto the subspace $(F_+)_l \otimes (F_+)_m$ of $F_+ \otimes F_+$. There exist l and m such that $l+m \geq 1$ and $T = P_{l,m}UP_{1,0} \neq 0$. Then, identifying $(F_+)_p \otimes (F_+)_q$ with the subspace of $L^2(\mathbb{R}^{p+q})$ consisting of functions $f(x_1, \ldots, x_p, y_1, \ldots, y_q)$ symmetric on each group of variables, we can rewrite the identities (5.7) and (5.8) as follows:

$$Te^{iX} = e^{i(X_1+\ldots+X_l-Y_1-\ldots-Y_m)}T, \tag{5.7$'$}$$

$$TD_1(X)^{it} = \left(\frac{D_2(X_1)\ldots D_2(X_l)}{D_2(Y_1)\ldots D_2(Y_m)}\right)^{it} T. \tag{5.8$'$}$$

The identity (5.7$'$) implies that

$$Tf(X) = f(X_1 + \ldots + X_l - Y_1 - \ldots - Y_m)T \tag{5.9}$$

for any bounded measurable 2π-periodic function f on $\mathbb{R}$. Indeed, the algebra of functions for which (5.9) holds is closed under pointwise limits of bounded sequences and contains e^{ix}.

Now we take an integer k such that $T|_{L^2(2\pi k, 2\pi(k+1))} \neq 0$. Denoting by D the 2π-periodic function that coincides with D_1 on $(2\pi k, 2\pi(k+1))$ and taking a function $\xi \in L^2(2\pi k, 2\pi(k+1))$ for which $T\xi \neq 0$, we obtain:

$$\begin{aligned}\left(\frac{D_2(X_1)\ldots D_2(X_l)}{D_2(Y_1)\ldots D_2(Y_m)}\right)^{it} T\xi &= TD_1(X)^{it}\xi \\ &= TD(X)^{it}\xi \\ &= D(X_1 + \ldots + X_l - Y_1 - \ldots - Y_m)^{it}T\xi\end{aligned}$$

Then

$$\left(\frac{D_2(x_1)\ldots D_2(x_l)}{D_2(y_1)\ldots D_2(y_m)}\right)^{it} = D(x_1 + \ldots + x_l - y_1 - \ldots - y_m)^{it} \tag{5.10}$$

for any $t \in \mathbb{R}$ for almost all $(x_1, \ldots, y_m)$ belonging to the support of $T\xi$. Hence there exists a set $\Lambda \subset \mathbb{R}^{l+m}$ of positive measure such that

$$\frac{D_2(x_1)\ldots D_2(x_l)}{D_2(y_1)\ldots D_2(y_m)} = D(x_1 + \ldots + x_l - y_1 - \ldots - y_m)$$

on Λ (we can take Λ to be the set of $(x_1, \ldots, y_m)$ for which (5.10) holds for any rational t). Replacing, if necessary, Λ by a subset of positive measure we find an integer n such that $(x_1 + \ldots + x_l - y_1 - \ldots - y_m) \in (2\pi(k-n), 2\pi(k-n+1))$ for any $(x_1, \ldots, y_m) \in \Lambda$. Then

$$\frac{D_2(x_1) \ldots D_2(x_l)}{D_2(y_1) \ldots D_2(y_m)} = D_1(x_1 + \ldots + x_l - y_1 - \ldots - y_m + 2\pi n) \tag{5.11}$$

on Λ. Using the Fubini theorem and the uniqueness theorem for meromorphic functions we conclude that (5.11) holds on $\mathbb{R}^{l+m}$. (Indeed, there exists a subset $\tilde{\Lambda}$ of $\mathbb{R}^{l+m-1}$ of positive measure such that, for any $(x_1, \ldots, x_l, y_1, \ldots, y_{m-1}) \in \tilde{\Lambda}$, the identity (5.11) holds on a set of positive measure on y_m. Then (5.11) holds on $\tilde{\Lambda} \times \mathbb{R}$, and so on.)

The following cases are possible:

1) $l + m \geq 2$.

Comparing the level sets of the functions in (5.11) corresponding to the values 0 and ∞, we see that D_1 and D_2 have neither roots nor poles on the real axis. Taking the logarithm and comparing the power-series expansions for $\log D_i,\ i = 1, 2$, one concludes that the functions $\log D_1$ and $\log D_2$ are linear. So,

$$D_2(x) = ce^{ax},\ D_1(x) = c^{l-m} e^{a(x-2\pi n)}$$

for certain real c and a. This contradicts the fact that $D_i(x) = \hat{K}_i(x)(1 + \hat{K}_i(x))^{-1} \to 0$, as $|x| \to \infty$.

2) $l = 0, m = 1$.

Then we have $D_2(y)^{-1} = D_1(2\pi n - y)$. This is impossible by the same reason as above.

3) $l = 1, m = 0$.

Then $D_2(x) = D_1(x + 2\pi n)$. Hence $\hat{K}_2(x) = D_2(x)(1 - D_2(x))^{-1} = \hat{K}_1(x + 2\pi n)$. $\square$

The simplest example of an entropic K-system is the shift automorphism of an infinite tensor product algebra with a faithful product-state. We shall call such systems Bernoullian.

Theorem 5.3. *Let N be a von Neumann algebra and ψ be a normal faithful state of N. For each integer n, let (N_n, ψ_n) be a copy of (N, ψ) and (M, ϕ) be the W^*-tensor product $\otimes_n (N_n, \psi_n)$. The right shift automorphism of M is denoted by γ. Suppose that $h_\phi(\gamma) < \infty$. Then N is at most a countable sum of factors of type I.*

Proof. Let $\tilde{S}(\psi) = \sup \sum_i \lambda_i S(\psi|_A, \psi_i|_A)$, where the supremum is taken over all finite convex decompositions $\psi = \sum_i \lambda_i \psi_i$ into states, over all finite dimensional abelian subalgebras A of N. The proof of Theorem 6.10 in [21] shows that if $\tilde{S}(\psi) < \infty$, then N is at most a countable sum of factors of type I.

On the other hand, for any finite dimensional subalgebra A of N_0, we have

$$h_\phi(A, \gamma) \geq \sup_{\psi_0 = \sum_i \lambda_i \psi_i} \sum_i \lambda_i S(\psi_0|_A, \psi_i|_A),$$

so that $h_\phi(\gamma) \geq \tilde{S}(\psi)$. $\square$

Corollary 5.4. *Let (M, ω, α) be an entropic K-system. Suppose $h_\omega(\alpha) < \infty$, ω is faithful, and the modular operator Δ_ω is not diagonalizable. Then the system (M, ω, α) is non-Bernoullian.* $\square$

Now we return to the Park-Shin systems considered above.

Let U and $V_\theta, \theta \in \mathbb{R}$, be the unitary operators on $L^2(\mathbb{R})$ defined by

$$(Uf)(x) = f(x-1), \quad (V_\theta f)(x) = e^{i\theta x} f(x),$$

and τ_θ be the Bogoliubov automorphism of $\mathcal{U}$ corresponding to the operator $V_\theta U V_{-\theta}$. (Note that $\tau_{\theta+2\pi} = \tau_\theta$.)

Theorem 5.5. *Let L be a non-zero smooth compactly supported function such that $\hat{L} \geq 0$, $K = L * L$, ω the quasi-free state of $\mathcal{U}$ corresponding to K, $M = \pi_\omega(\mathcal{U})''$. Then M is the injective III_1-factor and*

1) *for any θ, the system (M, ω, τ_θ) is a non-Bernoullian entropic K-system with the entropy*

$$h_\omega(\tau_\theta) = \frac{1}{2\pi} \int \left(\eta \hat{K}(x) - \eta(1 + \hat{K}(x)) \right) dx;$$

2) *for $0 \leq \theta < 2\pi$, the systems (M, ω, τ_θ) are pairwise non-conjugate.*

Proof. M is the injective III_1-factor and (M, ω, τ_0) is a K-system by Proposition 5.1.

(M, ω, τ_0) is non-Bernoullian by virtue of Corollary 5.4.

Let A be the operator of convolution with the function K. Since $V_{-\theta} A V_\theta$ is the operator of convolution with the function $e^{-i\theta x} K(x)$, the Bogoliubov automorphism corresponding to $V_{-\theta}$ conjugates the systems $(\mathcal{U}, \omega, \tau_\theta)$ and $(\mathcal{U}, \omega_{-\theta}, \tau_0)$, where $\omega_{-\theta}$ is the quasi-free state corresponding to the operator of convolution with the function $e^{-i\theta x} K(x)$. Hence

$$\begin{aligned}
h_\omega(\tau_\theta) &= h_{\omega_{-\theta}}(\tau_0) \\
&= \frac{1}{2\pi} \int \left(\eta \hat{K}(x-\theta) - \eta(1 + \hat{K}(x-\theta)) \right) dx \\
&= \frac{1}{2\pi} \int \left(\eta \hat{K}(x) - \eta(1 + \hat{K}(x)) \right) dx \\
&= h_\omega(\tau_0).
\end{aligned}$$

Thus our Theorem follows from what we have proved and Theorem 5.2. □

Remark 5.6. Under the assumptions of Theorem 5.5 K has a compact support, and if $\mathrm{supp} f_i \cap (\mathrm{supp} f_j + \mathrm{supp} K) = \emptyset$ and $\mathrm{supp} f_i \cap \mathrm{supp} f_j = \emptyset$ for $i \neq j$, then

$$\omega(W(f_1) \ldots W(f_n)) = \omega(W(f_1)) \ldots \omega(W(f_n)).$$

Recalling the proof of Theorem 3.1 one sees that such a clustering property simplifies the proof crucially. So that the K-property for the systems in Theorem 5.5 (as well as for Bernoullian systems) is rather evident.

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