ENTROPY OF BOGOLIUBOV AUTOMORPHISMS OF CAR AND CCR ALGEBRAS WITH RESPECT TO QUASI-FREE STATES

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Abstract

We compute the dynamical entropy of Bogoliubov automorphisms of CAR and CCR algebras with respect to arbitrary gauge-invariant quasi-free states. This completes the research started by Størmer and Voiculescu, and continued in works of Narnhofer-Thirring and Park-Shin.

1 Introduction and formulation of main result

One of the most beautiful results in the theory of dynamical entropy is the formula for the entropy of Bogoliubov automorphisms of the CAR-algebra with respect to quasi-free states obtained by Størmer and Voiculescu [SV] in 1990. They proved it under the assumption that the operator determining the quasi-free state has pure point spectrum. Since then several papers devoting to the computation of the entropy of Bogoliubov automorphisms have appeared. Narnhofer and Thirring [NT2] and Park and Shin [PS] proved the formula for some operators with continuous spectrum. The latter paper contains also a similar result for the CCR-algebra. On the other hand, Bezuglyi and Golodets [BG] proved an analogous formula for Bogoliubov actions of free abelian groups.

While the cases considered in [NT2] and [PS] required a non-trivial analysis, the proof of Størmer and Voiculescu is very elegant. It relies on an axiomatization of certain entropy functionals on the set of multiplicity functions. The main axiom there stems from the equality $h_\omega(\alpha) = \frac{1}{n}h_\omega(\alpha^n)$. Thus their method can not be directly applied to groups without finite-index subgroups. Instead, we can "cut and move" multiplicity functions without changing the entropy (see Lemma 5.1 below). This observation together with the methods developed in [BG] allowed to prove (under the same restrictions on quasi-free states) an analogue of Størmer-Voiculescu's formula for Bogoliubov actions of arbitrary torsion-free abelian groups [GN2]. In this paper we will show that, in fact, the formula holds without any restrictions on the operator determining the quasi-free state. We will prove also an analogous result for the CCR-algebra.

We will consider only the case of single automorphism, since in view of the methods of [GN2] the case of arbitrary torsion-free abelian group gives nothing but more complicated notations. So the main result of the paper is as follows.

Theorem 1.1 *Let U be a unitary operator on a Hilbert space H, α_U the corresponding Bogoliubov automorphism of the CAR or the CCR algebra over H, A a bounded ($A \le 1$ for CAR) positive operator commuting with U and determining a quasi-free state ω_A. Let $U_a = U|_{H_a}$ be* *the absolutely continuous part of U,*

$$H_a = \int_{\mathbb{T}}^{\oplus} H_z d\lambda(z), \quad U_a = \int_{\mathbb{T}}^{\oplus} z\, d\lambda(z), \quad A|_{H_a} = \int_{\mathbb{T}}^{\oplus} A_z d\lambda(z)$$

a direct integral decomposition, where λ is the Lebesgue measure on the torus $\mathbb{T}$ ($\lambda(\mathbb{T}) = 1$). Then

CAR: $h_{\omega_A}(\alpha_U) = \int_{\mathbb{T}} \mathrm{Tr}(\eta(A_z) + \eta(1 - A_z))d\lambda(z),$

CCR: $h_{\omega_A}(\alpha_U) = \int_{\mathbb{T}} \mathrm{Tr}(\eta(A_z) - \eta(1 + A_z))d\lambda(z).$

Corollary 1.2 *The necessary condition for the finiteness of the entropy is that A_z has pure point spectrum for almost all $z \in \mathbb{T}$.*

Corollary 1.3 *If the spectrum of the unitary operator is singular, then the entropy is zero.*

For CAR, the latter corollary is already known from [SV].

Finally, for systems considered in [NT2] and [PS], Theorem 1.1 may be reformulated as

Corollary 1.4 *Let I be an open subset of $\mathbb{R}$, ω a locally absolutely continuous function on I, ρ a bounded ($\rho \le 1$ for CAR) positive measurable function on I. Let U and A be the operators on $L^2(I, dx)$ of multiplication by the functions $e^{i\omega}$ and ρ, respectively. Then*

CAR: $h_{\omega_A}(\alpha_U) = \frac{1}{2\pi} \int_I [\eta(\rho(x)) + \eta(1 - \rho(x))]|\omega'(x)|dx,$

CCR: $h_{\omega_A}(\alpha_U) = \frac{1}{2\pi} \int_I [\eta(\rho(x)) - \eta(1 + \rho(x))]|\omega'(x)|dx.$

The paper is organized as follows. Section 2 contains some preliminaries on entropy and algebras of canonical commutation and anti-commutation relations. In Section 3 we prove that the entropies don't exceed the values of the integrals in Theorem 1.1. The opposite inequality is proved in Sections 4 and 5. In Section 4 we obtain a lower bound for the entropy in the case where the unitary operator has Lebesgue spectrum and the operator determining the quasi-free state is close to a scalar operator. In Section 5, first, using the observation mentioned above we extend the estimate of Section 4 to arbitrary unitaries, and then prove the required inequality.

There are also two appendices to the paper. The results of [GN1] show that modular automorphisms can have the K-property (in the sense of Narnhofer and Thirring [NT1]). This observation combined with the results of the present paper allow to construct on the hyperfinite III$_1$-factor a simple example of non-conjugate K-systems with the same finite entropy. This is done in Appendix A. Appendix B contains an auxiliary result on decomposable operators.

2 Preliminaries

Recall the definition of dynamical entropy [CNT]. Let (A, ϕ, α) be a C*-dynamical system, where A is a C*-algebra, ϕ a state on A, α a ϕ-preserving automorphism of A. By a channel in A we mean a unital completely positive mapping $\gamma\colon B \to A$ of a finite-dimensional C*-algebra B. The mutual entropy of channels $\gamma_i\colon B_i \to A$, $i = 1, \dots, n$, with respect to ϕ is given by

$$H_\phi(\gamma_1, \dots, \gamma_n) = \sup \sum_{i_1,\dots,i_n} \eta(\phi_{i_1\dots i_n}(1)) + \sum_{k=1}^{n} \sum_{i_k} S(\phi \circ \gamma_k, \phi_{i_k}^{(k)} \circ \gamma_k),$$

where $\eta(t) = -t\log t$, $S(\cdot\,,\,\cdot)$ the relative entropy, $\phi^{(k)}_{i_k} = \sum_{\substack{i_1,\ldots,i_n \\ i_k \text{fixed}}} \phi_{i_1\ldots i_n}$, and the supremum is taken over all finite decompositions $\phi = \sum \phi_{i_1\ldots i_n}$ of ϕ in the sum of positive linear functionals. If A is a W*-algebra and ϕ is a normal faithful state, then any positive linear functional $\psi \le \phi$ on A is of the form $\phi(\cdot\,\sigma^{\phi}_{-i/2}(x))$ for some $x \in A$, $0 \le x \le 1$, where σ^{ϕ}_t is the modular group corresponding to ϕ. Thus

$$H_\phi(\gamma_1,\ldots,\gamma_n) = \sup \sum_{i_1,\ldots,i_n} \eta(\phi(x_{i_1\ldots i_n})) + \sum_{k=1}^{n}\sum_{i_k} S(\phi(\gamma_k(\cdot)), \phi(\gamma_k(\cdot)\sigma_{-i/2}(x^{(k)}_{i_k}))),$$

where the supremum is taken over all finite partitions of unit.

The entropy of the automorphism α with respect to a channel γ and the state ϕ is given by

$$h_\phi(\gamma;\alpha) = \lim_{n\to\infty} \frac{1}{n} H_\phi(\gamma, \alpha\circ\gamma, \ldots, \alpha^{n-1}\circ\gamma).$$

The entropy $h_\phi(\alpha)$ of the system (A,ϕ,α) is the supremum of $h_\phi(\gamma;\alpha)$ over all channels γ in A.

We refer the reader to [CNT], [OP], [SV], [NT1] for general properties of entropy.

Lemma 2.1 *Let (A,ϕ,α) be a C*-dynamical system, $\{A_n\}_{n=1}^\infty$ a sequence of α-invariant subalgebras of A, $\{F_n\}_{n=1}^\infty$ a sequence of completely positive unital mappings $F_n\colon A \to A_n$ such that $||F_n(x) - x||_\phi \to 0$ as $n \to \infty$, for any $x \in A$. Then*

$$h_\phi(\alpha) \le \liminf_{n\to\infty} h_\phi(\alpha|_{A_n}).$$

Proof. The result follows from the continuity of mutual entropy in $||\ ||_\phi$-topology: see the proof of Lemma 3.3 in [SV].

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Though the possibility of $A_n \not\subset A_{n+1}$ is important for applications to actions of more general groups (see the proof of Theorem 4.1 in [GN2]), we will use this lemma only when $A_n \subset A_{n+1}$. Then the existence of F_n's is not necessary, as the following result shows.

Lemma 2.2 *Let A be a C*-algebra, ϕ a state on A, $\{A_n\}_{n=1}^\infty$ an increasing sequence of C*-subalgebras such that $\cup_n \pi_\phi(A_n)$ is weakly dense in $\pi_\phi(A)$. Then, for any channel $\gamma\colon B \to A$ and any $\varepsilon > 0$, there exist $n \in \mathbb{N}$ and a channel $\tilde\gamma\colon B \to A_n$ such that $||\gamma - \tilde\gamma||_\phi < \varepsilon$.*

Proof. This follows from the identification of completely positive maps $\mathrm{Mat}_d(\mathbb{C}) \to A$ with positive elements in $\mathrm{Mat}_d(A)$ [CE] and, in fact, is implicitly contained in [CNT]. We include a proof for the convenience of the reader.

Without loss of generality we may suppose that $B = \mathrm{Mat}_d(\mathbb{C})$. The channels $B \to A$ are in one-to-one correspondence with positive elements $Q \in \mathrm{Mat}_d(A)$ such that $\sum_k Q_{kk} = 1$. By Kaplansky's density theorem, there exists a net $\{\tilde Q_i\}_i \subset \cup_n \mathrm{Mat}_d(\pi_\phi(A_n))$ such that

$$0 \le \tilde Q_i \le 1, \quad \tilde Q_i \underset{i}{\rightarrow} \pi_\phi(Q) \ \text{ strongly.}$$

We can lift $\tilde Q_i$ to an element $Q_i \in \cup_n \mathrm{Mat}_d(A_n)$, $0 \le Q \le 1$. For $\delta > 0$, set

$$Q(i;\delta)_{kl} = \left(\sum_j Q(i)_{jj} + d\delta\right)^{-1/2} (Q(i)_{kl} + \delta_{kl}\delta) \left(\sum_j Q(i)_{jj} + d\delta\right)^{-1/2}.$$

Let $\gamma_{i,\delta}: B \to \cup_n A_n$ be the corresponding channel, $\gamma_{i,\delta}(e_{kl}) = Q(i;\delta)_{kl}$. Then

$$\lim_{\delta\to 0}\lim_i ||\gamma - \gamma_{i,\delta}||_\phi = 0.$$

■

Now recall some facts concerning CAR and CCR algebras [BR2].

Let H be a Hilbert space. The CAR-algebra $\mathcal{A}(H)$ over H is a C*-algebra generated by elements $a(f)$ and $a^*(f)$, $f \in H$, such that the mapping $f \mapsto a^*(f)$ is linear, $a(f)^* = a^*(f)$ and

$$a^*(f)a(g) + a(g)a^*(f) = (f,g)1, \quad a(f)a(g) + a(g)a(f) = 0.$$

Each unitary operator U on H defines a Bogoliubov automorphism α_U of $\mathcal{A}(H)$, $\alpha_U(a(f)) = a(Uf)$. The fixed point algebra $\mathcal{A}(H)_e = \mathcal{A}(H)^{\alpha_U}$ for $U = -1$ is called the even part of $\mathcal{A}(H)$.

Each operator A on H, $0 \le A \le 1$, defines a quasi-free state ω_A on $\mathcal{A}(H)$,

$$\omega_A(a^*(f_1)\dots a^*(f_n)a(g_m)\dots a(g_1)) = \delta_{nm}\det((Af_i, g_j))_{i,j}.$$

If $\operatorname{Ker} A = \operatorname{Ker}(1-A) = 0$, then ω_A is a KMS-state with respect to the group given by

$$\sigma_t^{\omega_A}(a(f)) = a(B^{it}f), \quad \text{where} \quad B = \frac{A}{1-A}. \tag{2.1}$$

If U and A commute, then ω_A is α_U-invariant.

If $H = K \oplus L$, then $\mathcal{A}(K)$ and $\mathcal{A}(L)_e$ commute, and we have

$$\mathcal{A}(H)^{\alpha_{1\oplus -1}} = \mathcal{A}(K) \vee \mathcal{A}(L)_e \cong \mathcal{A}(K) \otimes \mathcal{A}(L)_e.$$

If K is an invariant subspace for A, then

$$\omega_A|_{\mathcal{A}(K)\otimes\mathcal{A}(L)_e} = \omega_A|_{\mathcal{A}(K)} \otimes \omega_A|_{\mathcal{A}(L)_e}.$$

In particular, there exists an ω_A-preserving conditional expectation

$$\left(\mathrm{Id}_{\mathcal{A}(K)} \otimes \omega_A(\cdot)|_{\mathcal{A}(L)_e}\right) \circ \frac{1 + \alpha_{1\oplus -1}}{2}$$

onto $\mathcal{A}(K)$. If $\dim K = n < \infty$, then $\mathcal{A}(K)$ is a full matrix algebra of dimension 2^{2n}. In particular, for any $f \in H$, $||f|| = 1$, the algebra $\mathcal{A}(\mathbb{C}f)$ is isomorphic to $\mathrm{Mat}_2(\mathbb{C})$, and we define matrix units for it as

$$e_{11}(f) = a(f)a^*(f),\ e_{22}(f) = a^*(f)a(f),\ e_{12}(f) = a(f),\ e_{21}(f) = a^*(f). \tag{2.2}$$

The restriction of a quasi-free state ω_A to $\mathcal{A}(\mathbb{C}f)$ is given by the matrix

$$\begin{pmatrix} 1-\lambda & 0 \\ 0 & \lambda \end{pmatrix}, \quad \text{where} \quad \lambda = (Af, f). \tag{2.3}$$

The CCR-algebra $\mathcal{U}(H)$ over H is a C*-algebra generated by unitaries $W(f)$, $f \in H$, such that

$$W(f)W(g) = e^{i\frac{\operatorname{Im}(f,g)}{2}} W(f+g).$$

A representation π of $\mathcal{U}(H)$ is called regular, if the mapping $\mathbb{R} \ni t \mapsto \pi(W(tf))$ is strongly continuous for each $f \in H$. For any such a representation, the generator $\Phi_\pi(f)$ of the group

$\{\pi(W(tf))\}_t$ is defined, $\pi(W(tf)) = e^{it\Phi_\pi(f)}$. Then annihilation and creation operators are defined as

$$a_\pi(f) = \frac{\Phi_\pi(f) + i\Phi_\pi(if)}{\sqrt{2}}, \quad a_\pi^*(f) = \frac{\Phi_\pi(f) - i\Phi_\pi(if)}{\sqrt{2}}.$$

These are closed unbounded operators affiliated with $\pi(\mathcal{U}(H))''$, $a_\pi(f)^* = a_\pi^*(f)$, $a_\pi^*(f)$ depends on f linearly, and for any $f, g \in H$ we have the commutation relations

$$a_\pi(g)a_\pi^*(f) - a_\pi^*(f)a_\pi(g) = (f,g)1, \quad a_\pi(g)a_\pi(f) - a_\pi(f)a_\pi(g) = 0$$

on a dense subspace. In the sequel we will suppress π in the notations of annihilation and creation operators.

Each unitary operator U on H defines a Bogoliubov automorphism α_U of $\mathcal{U}(H)$, $\alpha_U(W(f)) = W(Uf)$.

Each positive operator A on H defines a quasi-free state ω_A on $\mathcal{U}(H)$,

$$\omega_A(W(f)) = e^{-\frac{1}{4}||f||^2 - \frac{1}{2}(Af,f)}.$$

The cyclic vector ξ_{ω_A} in the GNS-representation belongs to the domain of any operator of the form $a^\#(f_1)\dots a^\#(f_n)$, where $a^\#$ means either a^* or a, and

$$(a^*(f)a(g)\xi_{\omega_A}, \xi_{\omega_A}) = (Af, g).$$

If $\operatorname{Ker} A = 0$, then ω_A is separating (i. e., ξ_{ω_A} is separating for $\pi_{\omega_A}(\mathcal{U}(H))''$), and

$$\sigma_t^{\omega_A}(W(f)) = W(B^{it}f), \quad \text{where} \quad B = \frac{A}{1+A},$$

so that

$$\Delta_{\omega_A}^{it} a^\#(f_1)\dots a^\#(f_n)\xi_{\omega_A} = a^\#(B^{it}f_1)\dots a^\#(B^{it}f_n)\xi_{\omega_A}. \tag{2.4}$$

If $H = K \oplus L$, then $\mathcal{U}(H) \cong \mathcal{U}(K) \otimes \mathcal{U}(L)$. If K is an invariant subspace for A, then

$$\omega_A = \omega_A|_{\mathcal{U}(K)} \otimes \omega_A|_{\mathcal{U}(L)},$$

so that there exists an ω_A-preserving conditional expectation $\mathrm{Id}_{\mathcal{U}(K)} \otimes \omega_A|_{\mathcal{U}(L)}$ onto $\mathcal{U}(K)$.

If K is finite-dimensional, then every regular representation π of $\mathcal{U}(K)$ is quasi-equivalent to the Fock representation, in particular, $\pi(\mathcal{U}(K))''$ is a factor of type I_∞ (if $K \neq 0$). Thus for any regular state ω on $\mathcal{U}(K)$ (so that the mapping $t \mapsto \omega(W(tf))$ is continuous) the von Neumann entropy of the continuation $\bar\omega$ of the state ω to $\pi_\omega(\mathcal{U}(K))''$ is defined. We will denote it by $S(\omega)$ (in fact, the notion of entropy of state can be defined for all C*-algebras, and then $S(\omega) = S(\bar\omega)$ [OP]). If $K = \mathbb{C}f$, $||f|| = 1$, we define a system of matrix units $\{e_{ij}(f)\}_{i,j=0}^\infty$ for $\pi(\mathcal{U}(K))''$ as follows:

$e_{kk}(f)$ is the spectral projection of $a^*(f)a(f)$ corresponding to $\{k\}$,

$$e_{k+n,k}(f) = \left(\frac{k!}{(k+n)!}\right)^{1/2} a^*(f)^n e_{kk}(f) = \left(\frac{k!}{(k+n)!}\right)^{1/2} \overline{e_{k+n,k+n}(f)a^*(f)^n}. \tag{2.5}$$

In particular, if ω_A is a quasi-free state on $\mathcal{U}(H)$, for any $f \in H$, $||f|| = 1$, we obtain a system of matrix units $\{e_{ij}(f)\}_{i,j}$ in $\pi_{\omega_A}(\mathcal{U}(H))''$, and

$$\omega_A(e_{ij}(f)) = \delta_{ij}\frac{\lambda^i}{(1+\lambda)^{i+1}}, \quad \text{where} \quad \lambda = (Af, f). \tag{2.6}$$

(This is equivalent to the fact that if A is of trace class, then the quasi-free state ω_A is given in the Fock representation by the density operator $\frac{\Gamma(B)}{\mathrm{Tr}\,\Gamma(B)}$, where Γ is the operator of second quantization.) For future use, note that

$$\sum_{k=0}^{\infty}\eta(\lambda^k(1+\lambda)^{-k-1}) = \eta(\lambda) - \eta(1+\lambda)\,. \tag{2.7}$$

In the sequel we will write $\mathcal{C}(H)$ instead of $\mathcal{A}(H)$ and $\mathcal{U}(H)$ in the arguments that are identical for CAR and CCR.

The following result is known, but we will give a proof for the reader's convenience.

Lemma 2.3 *Let H be finite-dimensional, ω_A a quasi-free state on $\mathcal{C}(H)$. Then*
(i) CAR: $S(\omega_A) = \mathrm{Tr}(\eta(A) + \eta(1-A))$, CCR: $S(\omega_A) = \mathrm{Tr}(\eta(A) - \eta(1+A))$;
(ii) if $H = H_1 \oplus H_2$, then $S(\omega_A) \le S(\omega_A|_{\mathcal{C}(H_1)}) + S(\omega_A|_{\mathcal{C}(H_2)})$.

Proof. Let P_i be the projection onto H_i, $A_i = P_iA|_{H_i}$. Set $M_i = \pi_{\omega_{A_i}}(\mathcal{U}(H_i))''$, $M = \pi_{\omega_A}(\mathcal{U}(H))''$. Since all regular representations of $\mathcal{U}(H_i)$ are quasi-equivalent, we may consider M_i as a subalgebra of M. Since M_1 is a type I factor, we have $M = M_1 \otimes (M_1' \cap M)$, whence $M = M_1 \otimes M_2$. Thus the assertion (ii) for CCR is the usual subadditivity of von Neumann entropy.

Turning to CAR, let us first note that if M is a full matrix algebra, ω a state on M and α an automorphism of M, then $S(\omega) \le S(\omega|_{M^\alpha})$, and the equality holds iff ω is α-invariant. Indeed, let Q (resp. $\tilde{Q}$) be the density operator for ω (resp. $\omega|_{M^\alpha}$). Since the canonical trace on M^α is given by the restriction of the canonical trace Tr on M, we have $\mathrm{Tr}\,\tilde{Q} = 1$, hence

$$S(\omega|_{M^\alpha}) - S(\omega) = \mathrm{Tr}\,Q(\log Q - \log\tilde{Q}) \ge 0,$$

and the equality holds iff $Q = \tilde{Q}$, i. e., $Q \in M^\alpha$.

Applying this to CAR, we obtain

$$S(\omega_A) \le S(\omega_A|_{\mathcal{A}(H_1)\otimes\mathcal{A}(H_2)_e}) \le S(\omega_A|_{\mathcal{A}(H_1)}) + S(\omega_A|_{\mathcal{A}(H_2)_e}) = S(\omega_A|_{\mathcal{A}(H_1)}) + S(\omega_A|_{\mathcal{A}(H_2)}).$$

We see also that if H_i is an invariant subspace for A, then

$$S(\omega_A) = S(\omega_A|_{\mathcal{C}(H_1)}) + S(\omega_A|_{\mathcal{C}(H_2)}).$$

So, in proving (i) it is enough to consider one-dimensional spaces, for which the result follows immediately from (2.3), (2.6) and (2.7).

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Lemma 2.4 *Let U be a unitary operator on H, $\{P_n\}_{n=1}^\infty$ a sequence of projections in $B(H)$, $P_nU = UP_n$, $P_n \to 1$ strongly, $H_n = P_nH$. Then, for the Bogoliubov automorphism α_U and any α_U-invariant quasi-free state ω_A on $\mathcal{C}(H)$, we have*

$$h_{\omega_A}(\alpha_U) \le \liminf_{n\to\infty} h_{\omega_A}(\alpha_U|_{\mathcal{C}(H_n)}).$$

Proof. Let C be an operator commuting with P_n for all $n \in \mathbb{N}$. Let E_n be the ω_C-preserving conditional expectation of $\mathcal{C}(H)$ onto $\mathcal{C}(H_n)$ defined above. Then $||E_n(x) - x||_{\omega_A} \to 0$ for any

$x \in \mathcal{C}(H)$. Indeed, for CAR we have even the convergence in norm, that follows from $||E_n|| = 1$ and $||a(f)|| = ||f||$: if $x = a^{\#}(f_1) \dots a^{\#}(f_m)$ then

$$x = a^{\#}(P_n f_1) \dots a^{\#}(P_n f_m) + \sum_{k=1}^{m} a^{\#}(P_n f_1) \dots a^{\#}(P_n f_{k-1}) a^{\#}(f_k - P_n f_k) a^{\#}(f_{k+1}) \dots a^{\#}(f_m),$$

whence $||E_n(x) - x|| \le 2 \sum_{k=1}^{m} ||P_n f_1|| \dots ||P_n f_{k-1}|| \cdot ||f_k - P_n f_k|| \cdot ||f_{k+1}|| \dots ||f_m||$.

For CCR, the assertion follows from the equalities

$$E_n(W(f)) = e^{-\frac{1}{4}||(1-P_n)f||^2 - \frac{1}{2}(C(1-P_n)f,(1-P_n)f)} W(P_n f),$$

$$||W(f) - W(g)||_{\omega_A}^2 = 2 - 2\mathrm{Re}\left(e^{i\frac{\mathrm{Im}(f,g)}{2}} \omega_A(W(f-g))\right).$$

Thus we can apply Lemma 2.1.

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3 Upper bound for the entropy

In this section we will prove that the entropies do not exceed the values of the integrals in Theorem 1.1.

There exists a Hilbert space K and a unitary operator V on K such that $U_a \oplus V$ has countably multiple Lebesgue spectrum. Set

$$\tilde{H} = H \oplus K, \quad \tilde{U} = U \oplus V, \quad \tilde{A} = A \oplus 0.$$

Then, due to the existence of an $\omega_{\tilde{A}}$-preserving conditional expectation $\mathcal{C}(\tilde{H}) \to \mathcal{C}(H)$, we have $h_{\omega_A}(\alpha_U) \le h_{\omega_{\tilde{A}}}(\alpha_{\tilde{U}})$. On the other hand, the passage to $(\tilde{H}, \tilde{U}, \tilde{A})$ does not change the value of the integral in Theorem 1.1. So, without loss of generality we may suppose that U_a has countably multiple Lebesgue spectrum. If the value of the integral is finite, then A_z has pure point spectrum for almost all $z \in \mathbb{T}$. Then we can represent H_a as the sum of a countable set of copies of $L^2(\mathbb{T}, d\lambda)$ in such a way that U and A act on the n-th copy as multiplications by functions z and $\lambda_n(z)$, respectively (see Appendix B). By Lemma 2.4, we may restrict ourselves to the sum of a finite number of copies of $L^2(\mathbb{T})$. Thus we suppose

$$H_a = \mathop{\oplus}_{m=1}^{m_0} L^2(\mathbb{T}), \quad U_a = \mathop{\oplus}_{m=1}^{m_0} z, \quad A|_{H_a} = \mathop{\oplus}_{m=1}^{m_0} \lambda_m(z),$$

and we have to prove that

<u>CAR</u>: $h_{\omega_A}(\alpha_U) \le \sum_{m=1}^{m_0} \int_{\mathbb{T}} (\eta(\lambda_m(z)) + \eta(1 - \lambda_m(z))) d\lambda(z),$

<u>CCR</u>: $h_{\omega_A}(\alpha_U) \le \sum_{m=1}^{m_0} \int_{\mathbb{T}} (\eta(\lambda_m(z)) - \eta(1 + \lambda_m(z))) d\lambda(z).$

Let H_0 be the m_0-dimensional subspace of H spanned by constant functions in each copy of $L^2(\mathbb{T})$. Then $H_a = \oplus_{n \in \mathbb{Z}} U^n H_0$. For $n \in \mathbb{N}$, set $H_n = \oplus_{k=0}^{n} U^k H_0$. We state that

$$h_{\omega_A}(\alpha_U) \le \lim_{n \to \infty} \frac{1}{n} S(\omega_A|_{\mathcal{C}(H_{n-1})}). \tag{3.1}$$

For CAR, this is implicitly contained in the proof of Lemma 5.3 in [SV]. So we will consider CCR only.

For a finite set X, we denote by $\mathrm{Mat}(X)$ the C*-algebra of linear operators on $l^2(X)$. Let $\{e_{xy}\}_{x,y\in X}$ be the canonical system of matrix units for $\mathrm{Mat}(X)$. Following Voiculescu (see Lemmas 5.1 and 6.1 in [V]), for $X\subset H$, we introduce unital completely positive mappings

$$i_X\colon \mathrm{Mat}(X)\to\mathcal{U}(H),\quad j_X\colon\mathcal{U}(H)\to\mathrm{Mat}(X),$$

$$i_X(e_{xy})=\frac{1}{|X|}W(x)W(y)^*,\quad j_X(a)=P_X\pi_\tau(a)P_X,$$

where τ denotes the unique trace on $\mathcal{U}(H)$ ($\tau(W(f))=0$ for $f\neq 0$), P_X is the projection onto the subspace $\mathrm{Lin}\{\pi_\tau(W(x))\xi_\tau\,|\,x\in X\}\subset H_\tau$ identified with $l^2(X)$, and $|X|$ is the cardinality of X. Then

$$(i_X\circ j_X)(W(f))=\frac{|X\cap(X-f)|}{|X|}W(f)\ \ \forall f\in H.$$

Hence, for any subspace K of H, there exists a net $\{X_i\}_i$ of finite subsets of K such that $||(i_{X_i}\circ j_{X_i})(a)-a||\underset{i}{\to}0\ \forall a\in\mathcal{U}(K)$.

Let $H_s=H\ominus H_a$ be the subspace corresponding to the singular part of the spectrum of U. By Lemma 2.2, in computing the entropy we may consider only the channels in $\cup_m\mathcal{U}(H_s\oplus H_m)$. If γ is a channel in $\mathcal{U}(H_s\oplus H_m)=\mathcal{U}(H_s)\otimes\mathcal{U}(H_m)$, then it can be approximated in norm by a channel of the form $(i_X\otimes i_Z)\circ(j_X\otimes j_Z)\circ\gamma$, where $X\subset H_s$ and $Z\subset H_m$. Hence, it suffices to consider only the channels $i_X\otimes i_Z$.

So, let $\gamma=i_X\otimes i_Z\colon\mathrm{Mat}(X)\otimes\mathrm{Mat}(Z)\to\mathcal{U}(H_s)\otimes\mathcal{U}(H_m)=\mathcal{U}(H_s\oplus H_m)$. Set $L=\mathrm{Lin}\,X$. Fix $\varepsilon>0$. By Lemma 5.1 in [SV], there exist $n_0\in\mathbb{N}$ and a sequence of projections $\{Q_n\}_{n=n_0}^\infty$ in $B(H_s)$ such that $\dim Q_n\le\varepsilon n$ and $||(U^k-Q_nU^k)|_L||\le\varepsilon$ for $k=0,\dots,n-1$. Define a channel $i_X^{(n,k)}\colon\mathrm{Mat}(X)\to\mathcal{U}(H_s)$,

$$i_X^{(n,k)}(e_{xy})=\frac{1}{|X|}W(Q_nU^kx)W(Q_nU^ky)^*=\frac{1}{|X|}e^{-\frac{i}{2}\mathrm{Im}(Q_nU^kx,Q_nU^ky)}W(Q_nU^k(x-y)).$$

On the other hand, we have

$$(\alpha_U^k\circ i_X)(e_{xy})=\frac{1}{|X|}W(U^kx)W(U^ky)^*=\frac{1}{|X|}e^{-\frac{i}{2}\mathrm{Im}(U^kx,U^ky)}W(U^k(x-y)).$$

We may conclude that there exists an upper bound for $||\alpha_U^k\circ i_X-i_X^{(n,k)}||_{\omega_A}$ depending only on ε, $||A||$, $|X|$ and $||X||=\max\{||x||\,|\,x\in X\}$. Set

$$\gamma_{n,k}=i_X^{(n,k)}\otimes(\alpha_U^k\circ i_Z).$$

Then $||\alpha_U^k\circ\gamma-\gamma_{n,k}||_{\omega_A}$ is bounded by a value depending only on ε, $||A||$, $||X||$, $|X|$ and $|Z|$. By Proposition IV.3 in [CNT],

$$|H_{\omega_A}(\gamma,\alpha_U\circ\gamma,\dots,\alpha_U^{n-1}\circ\gamma)-H_{\omega_A}(\gamma_{n,0},\gamma_{n,1},\dots,\gamma_{n,n-1})|<n\delta,\tag{3.2}$$

where $\delta=\delta(\varepsilon,||A||,||X||,|X|,|Z|)\underset{\varepsilon\to0}{\to}0$. Since $\gamma_{n,k}$'s are channels in $\mathcal{U}(Q_nH_s\oplus H_{m+n-1})$, we have

$$H_{\omega_A}(\gamma_{n,0},\gamma_{n,1},\dots,\gamma_{n,n-1})\le S(\omega_A|_{\mathcal{U}(Q_nH_s\oplus H_{m+n-1})}).\tag{3.3}$$

By Lemma 2.3,

$$S(\omega_A|_{\mathcal{U}(Q_nH_s\oplus H_{m+n-1})})\le S(\omega_A|_{\mathcal{U}(Q_nH_s)})+S(\omega_A|_{\mathcal{U}(H_{m+n-1})})\tag{3.4}$$

and

$$S(\omega_A|_{\mathcal{U}(Q_nH_s)}) \leq (\eta\,(||A||) - \eta\,(1+||A||))\dim Q_nH_s \leq \varepsilon n(\eta\,(||A||) - \eta\,(1+||A||)). \tag{3.5}$$

From (3.2)-(3.5) we conclude that

$$h_{\omega_A}(\gamma;\alpha_U) \leq \delta + \varepsilon(\eta\,(||A||) - \eta\,(1+||A||)) + \lim_{n\to\infty}\frac{1}{n}S(\omega_A|_{\mathcal{U}(H_{n-1})}).$$

Because of the arbitrariness of ε, the proof of (3.1) is complete.

Applying Lemma 2.3, we obtain

CAR: $h_{\omega_A}(\alpha_U) \leq S(\omega_A|_{\mathcal{A}(H_0)}) = \sum_{m=1}^{m_0}(\eta\,(\lambda_m) + \eta\,(1-\lambda_m))$,

CCR: $h_{\omega_A}(\alpha_U) \leq S(\omega_A|_{\mathcal{U}(H_0)}) = \sum_{m=1}^{m_0}(\eta\,(\lambda_m) - \eta\,(1+\lambda_m))$,

where $\lambda_m = \int_{\mathbb{T}}\lambda_m(z)d\lambda(z)$. Applying these inequalities to the operator U^n and using the equality $h_{\omega_A}(\alpha_U) = \frac{1}{n}h_{\omega_A}(\alpha_{U^n})$, we may conclude that

$$\underline{\text{CAR}}: h_{\omega_A}(\alpha_U) \leq \frac{1}{n}\sum_{m=1}^{m_0}\sum_{k=1}^{n}(\eta\,(\lambda_{mnk}) + \eta\,(1-\lambda_{mnk})), \tag{3.6}$$

$$\underline{\text{CCR}}: h_{\omega_A}(\alpha_U) \leq \frac{1}{n}\sum_{m=1}^{m_0}\sum_{k=1}^{n}(\eta\,(\lambda_{mnk}) - \eta\,(1+\lambda_{mnk})), \tag{3.7}$$

where $\lambda_{mnk} = n\int_{\frac{k-1}{n}}^{\frac{k}{n}}\lambda_m\left(e^{2\pi it}\right)dt$.

To complete the proof of our assertion, we apply the following lemma to (3.6) and (3.7).

Lemma 3.1 *Let g be a bounded measurable function, f a continuous function. Then*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}f\left(n\int_{\frac{k-1}{n}}^{\frac{k}{n}}g(t)dt\right) = \int_0^1 f(g(t))dt.$$

Proof. Define a linear operator F_n on $L^1(0,1)$,

$$(F_nh)(\tau) = n\int_{\frac{k-1}{n}}^{\frac{k}{n}}h(t)dt \quad \text{on} \quad \left[\frac{k-1}{n},\frac{k}{n}\right].$$

Then $F_n \to \mathrm{id}$ pointwise-norm. Indeed, since $||F_n|| = 1$, it suffices to prove the assertion for continuous functions, for which it is obvious. Thus $F_ng \to g$ in mean, hence in measure. By virtue of the uniform continuity of f, we conclude that $f\circ F_ng \to f\circ g$ in measure, whence $\int_0^1 f\circ F_ng\,dt \to \int_0^1 f\circ g\,dt$.

■

4 Lower bound for the entropy: basic estimate

The aim of this section is to prove the following estimate.

Proposition 4.1 *For given $\varepsilon > 0$ and $C > 0$ ($C < 1$ for CAR), there exists $\delta > 0$ such that if* $\operatorname{Spec} A \subset (\lambda_0 - \delta, \lambda_0 + \delta)$ *for some $\lambda_0 \in (0, C)$ and the spectrum of U^n has Lebesgue component for some $n \in \mathbb{N}$, then*

<u>CAR</u>: $h_{\omega_A}(\alpha_U|_{\mathcal{A}(H)_e}) \geq \dfrac{1}{n}(\eta(\lambda_0) + \eta(1-\lambda_0) - \varepsilon)$;

<u>CCR</u>: $h_{\omega_A}(\alpha_U) \geq \dfrac{1}{n}(\eta(\lambda_0) - \eta(1+\lambda_0) - \varepsilon)$.

First, we will prove that if $f \in H$ is close to be an eigenvector for A, then the state ω_A on $\mathcal{C}(H) = \mathcal{C}(\mathbb{C}f) \otimes (\mathcal{C}(\mathbb{C}f)' \cap \mathcal{C}(H))$ is close to $\omega_A|_{\mathcal{C}(\mathbb{C}f)} \otimes \omega_A|_{\mathcal{C}(\mathbb{C}f)' \cap \mathcal{C}(H)}$.

Lemma 4.2 *Let $\{e_{ij}\}_{i,j}$ be a system of matrix units in a W^*-algebra M, $e = \sum_k e_{kk}$, ω a normal faithful state on M. Then, for any $x \in M$ commuting with the matrix units, we have*

$$|\omega(e_{kk}x) - \lambda_k\omega(x)| \leq 2(\lambda_k^{1/2}||1-e||_\omega + \sum_j ||\lambda_j^{1/2}\sigma_{-i/2}(e_{kj}) - \lambda_k^{1/2}e_{kj}||_\omega)||x||_\omega^\#,$$

where $||x||_\omega^\# = (\omega(x^*x) + \omega(xx^*))^{1/2}$ and $\lambda_k = \omega(e_{kk})$.

Proof. Let $\xi = \xi_\omega$ and $J = J_\omega$ be the cyclic vector and the modular involution corresponding to ω. We have

$$\lambda_j\omega(e_{kk}x) = \lambda_j^{1/2}((\lambda_j^{1/2}Je_{jk} - \lambda_k^{1/2}e_{kj})\xi, Je_{jk}x\xi) + \lambda_k^{1/2}(e_{kj}Jx^*\xi, (\lambda_j^{1/2}Je_{jk} - \lambda_k^{1/2}e_{kj})\xi) +$$
$$+\lambda_k(x\xi, Je_{jj}\xi),$$

whence

$$\begin{aligned}|\lambda_j\omega(e_{kk}x) - \lambda_k(x\xi, Je_{jj}\xi)| &\leq (\lambda_j^{1/2}||x||_\omega + \lambda_k^{1/2}||x^*||_\omega)||\lambda_j^{1/2}\sigma_{-i/2}(e_{kj}) - \lambda_k^{1/2}e_{kj}||_\omega \\ &\leq 2||x||_\omega^\#||\lambda_j^{1/2}\sigma_{-i/2}(e_{kj}) - \lambda_k^{1/2}e_{kj}||_\omega. \qquad (4.1)\end{aligned}$$

Further,

$$|\omega(e_{kk}x) - \sum_j \lambda_j\omega(e_{kk}x)| = \omega(1-e)|\omega(e_{kk}x)| \leq ||1-e||_\omega\lambda_k^{1/2}||x||_\omega, \qquad (4.2)$$

and

$$|\sum_j \lambda_k(x\xi, Je_{jj}\xi) - \lambda_k\omega(x)| = \lambda_k|(x\xi, J(1-e)\xi)| \leq \lambda_k^{1/2}||1-e||_\omega||x||_\omega. \qquad (4.3)$$

Summing up (4.1)-(4.3), we obtain the desired estimate.

■

Recall that in Section 2 we introduced a system of matrix units $\{e_{ij}(f)\}_{i,j}$ in $\pi_{\omega_A}(\mathcal{C}(H))''$ ($f \in H$, $||f|| = 1$). In the sequel we will identify $\mathcal{C}(H)$ with its image in $B(H_{\omega_A})$.

Lemma 4.3
<u>CAR</u>: *For given $\varepsilon > 0$, there exists $\delta > 0$ such that if* $\operatorname{Spec} A \subset (0,1)$ *and*

$$\left|\left|\left(\frac{A}{1-A}\right)^{1/2} f - \left(\frac{\lambda}{1-\lambda}\right)^{1/2} f\right|\right| < \delta \;\; \textit{for some } f,\ ||f|| = 1,\ \textit{where } \lambda = (Af, f),$$

then $||\lambda_j^{1/2}\sigma_{-i/2}(e_{kj}(f)) - \lambda_k^{1/2}e_{kj}(f)||_{\omega_A} \le \varepsilon(\lambda_j\lambda_k)^{1/4}$, $k,j = 1,2$, *where* $\lambda_1 = 1-\lambda$, $\lambda_2 = \lambda$.
CCR: *For given* $\varepsilon > 0$, $C > 0$ *and* $k, j \in \mathbb{Z}_+$, *there exists* $\delta > 0$ *such that if* $\operatorname{Spec} A \subset (0, C)$ *and*

$$\left\|\left(\frac{A}{1+A}\right)^{1/2} f - \left(\frac{\lambda}{1+\lambda}\right)^{1/2} f\right\| < \delta \quad \text{for some } f,\ ||f|| = 1,\ \text{where } \lambda = (Af, f),$$

then $||\lambda_j^{1/2}\sigma_{-i/2}(e_{kj}(f)) - \lambda_k^{1/2}e_{kj}(f)||_{\omega_A} \le \varepsilon(\lambda_j\lambda_k)^{1/4}$, *where* $\lambda_m = \dfrac{\lambda^m}{(1+\lambda)^{m+1}}$.

Proof. For brevity we will write e_{kj} for $e_{kj}(f)$. We have

$$||\lambda_j^{1/2}\sigma_{-i/2}(e_{kj}) - \lambda_k^{1/2}e_{kj}||_{\omega_A}^2 = 2(\lambda_j\lambda_k)^{1/2}((\lambda_j\lambda_k)^{1/2} - \omega_A(e_{jk}\sigma_{-i/2}(e_{kj}))).$$

So we must prove that $\omega_A(e_{jk}\sigma_{-i/2}(e_{kj}))$ is close to $(\lambda_j\lambda_k)^{1/2}$ when δ is sufficiently small.

CAR: Set $B = \dfrac{A}{1-A}$ and $\beta = \dfrac{\lambda}{1-\lambda}$. We have

$$\omega_A(e_{12}\sigma_{-i/2}(e_{21})) = \omega_A(e_{21}\sigma_{-i/2}(e_{12})),$$

$$\lambda_1 - \omega_A(e_{11}\sigma_{-i/2}(e_{11})) = \omega_A(e_{11}\sigma_{-i/2}(e_{22})) = \lambda_2 - \omega_A(e_{22}\sigma_{-i/2}(e_{22})).$$

By virtue of (2.1) and (2.2), $\sigma_{-i/2}(e_{21}) = \sigma_{-i/2}(a^*(f)) = a^*(B^{1/2}f)$, so

$$\|\sigma_{-i/2}(e_{21}) - \beta^{1/2}e_{21}\| = \|B^{1/2}f - \beta^{1/2}f\| < \delta,$$

whence

$$|\omega_A(e_{12}\sigma_{-i/2}(e_{21})) - \lambda^{1/2}(1-\lambda)^{1/2}| = |\omega_A(e_{12}(\sigma_{-i/2}(e_{21}) - \beta^{1/2}e_{21}))| < \delta$$

and

$$|\omega_A(e_{11}\sigma_{-i/2}(e_{22}))| = |\omega_A(e_{11}(\sigma_{-i/2}(e_{21}) - \beta^{1/2}e_{21})\sigma_{-i/2}(e_{12}))| < \delta.$$

CCR: Set $B = \dfrac{A}{1+A}$ and $\beta = \dfrac{\lambda}{1+\lambda}$. First consider the case $k = j$. We have to prove that

$$\lambda_k - \omega_A(e_{kk}\sigma_{-i/2}(e_{kk})) = \omega_A(e_{kk}\sigma_{-i/2}(1 - e_{kk})) = \sum_{m \ne k} \omega_A(e_{kk}\sigma_{-i/2}(e_{mm}))$$

is small if δ is small enough. Since $\left\|\sum_{m=m_0}^{\infty} e_{mm}\right\|_{\omega_A} = \beta^{\frac{m_0}{2}} \le \left(\dfrac{C}{1+C}\right)^{\frac{m_0}{2}} \underset{m_0\to\infty}{\to} 0$, it suffices to prove that $\omega_A(e_{kk}\sigma_{-i/2}(e_{mm}))$ can be made arbitrary small for any fixed $m \ne k$. Since $\omega_A(e_{kk}\sigma_{-i/2}(e_{mm})) = \omega_A(e_{mm}\sigma_{-i/2}(e_{kk}))$, we may suppose that $m > k$, i. e., $m = k+n$ for some $n \in \mathbb{N}$. We have (see (2.5))

$$e_{k+n,k+n} = c_{kn}a^*(f)^{k+1}e_{n-1,k+n}, \quad \text{where} \quad c_{kn} = \left(\frac{(n-1)!}{(k+n)!}\right)^{1/2}.$$

Using (2.4), we obtain

$$\Delta^{1/2}e_{k+n,k+n}\xi = c_{kn}Je_{k+n,n-1}Ja^*(B^{1/2}f)^{k+1}\xi.$$

Since $||(a^*(f_1)^{k+1} - a^*(f_2)^{k+1})\xi||$ is bounded by a value which depends only on k, $||A||$, $||f_i||$ and $||f_1 - f_2||$ (this is most easily seen from the explicit description of the GNS-representation in terms of the Fock representation, see Example 5.2.18 in [BR2]), we conclude that $\sigma_{-i/2}(e_{k+n,k+n})\xi$ is close to

$$\beta^{\frac{k+1}{2}} c_{kn} J e_{k+n,n-1} J a^*(f)^{k+1}\xi$$

when δ is sufficiently small. But then $e_{kk}\sigma_{-i/2}(e_{k+n,k+n})\xi$ is close to

$$\beta^{\frac{k+1}{2}} c_{kn} J e_{k+n,n-1} J e_{kk} a^*(f)^{k+1}\xi = 0.$$

It remains to consider the case $j \neq k$. As above, we may suppose that $j > k$, $j = k + n$ for some $n \in \mathbb{N}$. We have

$$e_{k+n,k} = d_{kn} a^*(f)^n e_{kk}, \quad \text{where} \quad d_{kn} = \left(\frac{k!}{(k+n)!}\right)^{1/2}.$$

As above, we conclude that $\sigma_{-i/2}(e_{k+n,k})\xi$ is close to

$$\beta^{\frac{n}{2}} d_{kn} J e_{kk} J a^*(f)^n \xi$$

for sufficiently small δ, so $\omega_A(e_{k,k+n}\sigma_{-i/2}(e_{k+n,k}))$ is close to

$$\beta^{\frac{n}{2}} d_{kn} (e_{k,k+n} a^*(f)^n \xi, J e_{kk}\xi) = \beta^{\frac{n}{2}} (e_{kk}\xi, J e_{kk}\xi) = \beta^{\frac{n}{2}} \omega_A(e_{kk}\sigma_{-i/2}(e_{kk})).$$

As we have proved, $\omega_A(e_{kk}\sigma_{-i/2}(e_{kk}))$ can be made close to $\lambda_k = \dfrac{\lambda^k}{(1+\lambda)^{k+1}}$, but then $\beta^{\frac{n}{2}}\omega_A(e_{kk}\sigma_{-i/2}(e_{kk}))$ is close to $\left(\dfrac{\lambda}{1+\lambda}\right)^{\frac{n}{2}} \cdot \dfrac{\lambda^k}{(1+\lambda)^{k+1}} = (\lambda_k\lambda_{k+n})^{1/2}$.

■

Lemma 4.4 *For given $N \in \mathbb{N}$ and $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon, N) > 0$ such that if A is an abelian W*-algebra, ω a normal faithful state on A, $B \subset A$ a W*-subalgebra, and $\{x_i\}_{i=1}^N$ a family of projections in A such that $\sum_i x_i = 1$ and*

$$|\omega(x_i y) - \omega(x_i)\omega(y)| \le \delta||y|| \quad \forall y \in B,\ i = 1, \dots, N,$$

then

$$\sum_{i,j} \eta(\omega(x_i y_j)) \ge \sum_i \eta(\omega(x_i)) + \sum_j \eta(\omega(y_j)) - \varepsilon$$

for any finite family of projections $\{y_j\}_j$ in B with $\sum_j y_j = 1$.

Proof. Cf. [GN1, Lemma 3.2].

■

The proof of Theorem 3.1 in [GN1] shows that Lemma 4.4 is also valid for non-abelian A (with $x_i \in B' \cap A$) and without the requirement that x_i's and y_j's are projections, but we will not use this fact.

Proof of Proposition 4.1. Consider the case of CCR-algebra.

There exists $\delta_1 > 0$ such that

$$|\eta(\lambda) - \eta(1+\lambda) - \eta(\lambda_0) + \eta(1+\lambda_0)| < \frac{\varepsilon}{6} \quad \forall \lambda_0 \in (0, C)\ \forall \lambda \ge 0 : |\lambda - \lambda_0| < \delta_1.$$

We can find $N \in \mathbb{N}$ such that

$$\sum_{k=N}^{\infty} \eta\left(\frac{\lambda^k}{(1+\lambda)^{k+1}}\right) < \frac{\varepsilon}{6} \quad \forall \lambda \in (0, C+\delta_1).$$

Then using (2.7), we obtain

$$\sum_{k=0}^{N-1} \eta\left(\frac{\lambda^k}{(1+\lambda)^{k+1}}\right) > \eta(\lambda_0) - \eta(1+\lambda_0) - \frac{\varepsilon}{3} \quad \forall \lambda_0 \in (0, C)\ \forall \lambda \ge 0 : |\lambda - \lambda_0| < \delta_1. \tag{4.4}$$

By assumptions of Proposition, there exists $f \in H$ such that $\{U^{kn} f\}_{k \in \mathbb{Z}}$ is an orthonormal system in H. Set $p_k = e_{kk}(f)$, $k = 0, \ldots, N-1$, and $p_N = 1 - \sum_{k=0}^{N-1} p_k$. Let $\mathcal{P}$ be the algebra generated by p_k, $k = 0, \ldots, N$. Then

$$\begin{aligned} h_{\omega_A}(\alpha_U) &\ge \lim_{k\to\infty} \frac{1}{kn} H_{\omega_A}(\mathcal{P}, \alpha_U(\mathcal{P}), \ldots, \alpha_U^{kn-1}(\mathcal{P})) \ge \lim_{k\to\infty} \frac{1}{kn} H_{\omega_A}(\mathcal{P}, \alpha_U^{n}(\mathcal{P}), \ldots, \alpha_U^{k(n-1)}(\mathcal{P})) \\ &\ge \lim_{k\to\infty} \frac{1}{kn} \sum_{i_0,\ldots,i_{k-1}=0}^{N} \eta(\omega_A(p_{i_0} \alpha_U^n(p_{i_1}) \ldots \alpha_U^{(k-1)n}(p_{i_{k-1}}))) + \\ &\qquad + \frac{1}{n} \sum_{j=0}^{N} S(\omega_A|_{\mathcal{P}}, \omega_A(\cdot\, \sigma_{-i/2}(p_j))|_{\mathcal{P}}). \end{aligned} \tag{4.5}$$

We want to prove that if $\operatorname{Spec} A \subset (\lambda_0 - \delta, \lambda_0 + \delta)$ with sufficiently small δ, then the first term in (4.5) is close to $\frac{1}{n}(\eta(\lambda_0) - \eta(1+\lambda_0))$ to within $\frac{\varepsilon}{n}$, while the second term is close to zero.

Start with the second term. We have

$$\begin{aligned} \sum_{j=0}^{N} S(\omega_A|_{\mathcal{P}}, \omega_A(\cdot\, \sigma_{-i/2}(p_j))|_{\mathcal{P}}) &= \sum_{j=0}^{N}\sum_{k=0}^{N} \omega_A(p_k \sigma_{-i/2}(p_j))(\log \omega_A(p_k \sigma_{-i/2}(p_j)) - \log \omega_A(p_k)) \\ &= \sum_{k=0}^{N} \left(\eta(\omega_A(p_k)) - \sum_{j=0}^{N} \eta(\omega_A(p_k \sigma_{-i/2}(p_j))) \right). \end{aligned}$$

By Lemma 4.3, $\omega_A(p_k \sigma_{-i/2}(p_j))$ can be made arbitrary close to $\delta_{kj}\omega_A(p_k)$ (more precisely, we can state that this is true for $j \le N-1$, but since $\omega_A(p_k \sigma_{-i/2}(p_N)) = \omega_A(p_N \sigma_{-i/2}(p_k))$ and

$$\omega_A(p_N) - \omega_A(p_N \sigma_{-i/2}(p_N)) = \sum_{k=0}^{N-1} \omega_A(p_N \sigma_{-i/2}(p_k)),$$

this holds for all $k, j \le N$). Hence, there exists $\delta_2 \in (0, \delta_1)$ such that if $\operatorname{Spec} A \subset (\lambda_0 - \delta_2, \lambda_0 + \delta_2)$, $\lambda_0 \in (0, C)$, then

$$\sum_{j=0}^{N} S(\omega_A|_{\mathcal{P}}, \omega_A(\cdot\, \sigma_{-i/2}(p_j))|_{\mathcal{P}}) > -\frac{\varepsilon}{3}. \tag{4.6}$$

Turning to the first term in (4.5), set

$$\varepsilon_1 = \delta\left(\frac{\varepsilon}{3}, N+1\right), \tag{4.7}$$

where $\delta(\cdot\,,\,\cdot)$ is from Lemma 4.4. Find $N_1 \in \mathbb{N}$ such that

$$\left(\frac{C+\delta_2}{1+C+\delta_2}\right)^{\frac{N_1}{2}} < \frac{\varepsilon_1}{8N}.$$

Then

$$\left\|1 - \sum_{k=0}^{N_1-1} e_{kk}(f)\right\|_{\omega_A} < \frac{\varepsilon_1}{8N} \ \text{ if } \ A \le C + \delta_2,$$

hence, by Lemmas 4.2 and 4.3 applied to $\{e_{kj}(f)\}_{k,j=0}^{N_1-1}$, there exists $\delta_3 \in (0, \delta_2)$ such that if $\operatorname{Spec} A \subset (\lambda_0 - \delta_3, \lambda_0 + \delta_3)$, $\lambda_0 \in (0, C)$, then

$$|\omega_A(p_k x) - \omega_A(p_k)\omega_A(x)| \le \frac{\varepsilon_1}{2N}||x||^{\#}_{\omega_A} \le \varepsilon_1||x|| \ \ \forall x \in \mathcal{U}(f^{\perp})'', \ k = 0, \dots, N-1. \tag{4.8}$$

We have also

$$|\omega_A(p_N x) - \omega_A(p_N)\omega_A(x)| \le \sum_{k=0}^{N-1} |\omega_A(p_k x) - \omega_A(p_k)\omega_A(x)| \le \varepsilon_1||x|| \ \ \forall x \in \mathcal{U}(f^{\perp})''. \tag{4.9}$$

From (4.7)-(4.9) and Lemma 4.4 we infer that if $\operatorname{Spec} A \subset (\lambda_0 - \delta_3, \lambda_0 + \delta_3)$, $\lambda_0 \in (0, C)$, then, for any $k \in \mathbb{N}$,

$$\sum_{i_0,\dots,i_{k-1}=0}^{N} \eta(\omega_A(p_{i_0}\alpha_U^n(p_{i_1})\dots\alpha_U^{(k-1)n}(p_{i_{k-1}}))) \ge$$

$$\ge \sum_{i_0=0}^{N} \eta(\omega_A(p_{i_0})) + \sum_{i_1,\dots,i_{k-1}=0}^{N} \eta(\omega_A(p_{i_1}\alpha_U^n(p_{i_2})\dots\alpha_U^{(k-2)n}(p_{i_{k-1}}))) - \frac{\varepsilon}{3}$$

$$\ge \ \dots \ge k\sum_{j=0}^{N} \eta(\omega_A(p_j)) - (k-1)\frac{\varepsilon}{3} > k\sum_{j=0}^{N-1} \eta\left(\frac{\lambda^j}{(1+\lambda)^{j+1}}\right) - (k-1)\frac{\varepsilon}{3}, \tag{4.10}$$

where $\lambda = (Af, f) \in (\lambda_0 - \delta_3, \lambda_0 + \delta_3)$. It follows from (4.4), (4.6) and (4.10) that we may take $\delta = \delta_3$.

The proof for CAR is similar, and we omit the details.

■

5 Lower bound for the entropy: end of the proof

In this section we will complete the proof of the lower bound for the entropy.

By virtue of the existence of an ω_A-preserving conditional expectation $\mathcal{C}(H) \to \mathcal{C}(H_a)$, we have $h_{\omega_A}(\alpha_U) \ge h_{\omega_A}(\alpha_{U_a})$. So we may suppose that U has absolutely continuous spectrum.

First, we will extend Proposition 4.1 to arbitrary unitaries. The main step here is the following observation.

Lemma 5.1 *Let U_n be a unitary operator on H_n, $n \in \mathbb{N}$, and $\{z_n\}_{n=1}^{\infty} \subset \mathbb{T}$. Consider two unitary operators U' and U'' on $H = \oplus_{n=1}^{\infty} H_n$,*

$$U' = \mathop{\oplus}_{n=1}^{\infty} U_n, \quad U'' = \mathop{\oplus}_{n=1}^{\infty} z_n U_n.$$

Then $h_{\omega_A}(\alpha_{U'}) = h_{\omega_A}(\alpha_{U''})$ for any $\alpha_{U'}$- and $\alpha_{U''}$-invariant quasi-free state ω_A on $\mathcal{C}(H)$. For CAR, the same holds for the restrictions of the automorphisms to the even part $\mathcal{A}(H)_e$ of the algebra.

Proof. For CAR, this was proved in [GN2, Lemma 2.4]. Though for CCR the result is valid by similar reasons, this case requires some additional arguments, since the local algebras are not finite-dimensional.

Consider the unitary operator $V = \oplus_{n=1}^{\infty} z_n$. We state that there exists a set $\{\mathcal{C}_i\}_i$ of finite-dimensional C*-subalgebras of $\mathcal{C}(H)''(\subset B(H_{\omega_A}))$ such that $\alpha_V(\mathcal{C}_i) = \mathcal{C}_i$ and

$$h_{\omega_A}(\alpha) = \sup_i h_{\omega_A}(\mathcal{C}_i; \alpha)$$

for any ω_A-preserving automorphism α. Suppose the statement is proved. Then, since $\alpha_{U''} = \alpha_V \alpha_{U'} = \alpha_{U'} \alpha_V$, we have $\alpha^k_{U''}(\mathcal{C}_i) = \alpha^k_{U'}(\mathcal{C}_i)$ $\forall k \in \mathbb{Z}$, and hence $h_{\omega_A}(\mathcal{C}_i; \alpha_{U''}) = h_{\omega_A}(\mathcal{C}_i; \alpha_{U'})$ $\forall i$, whence $h_{\omega_A}(\alpha_{U''}) = h_{\omega_A}(\alpha_{U'})$.

So it remains to prove the existence of $\mathcal{C}_i$'s. For each $n \in \mathbb{N}$, choose an increasing sequence $\{H_{nk}\}_{k=1}^{\infty}$ of finite-dimensional subspaces of H_n such that $\cup_k H_{nk}$ is dense in H_n. Set

$$K_n = H_{1n} \oplus \ldots \oplus H_{nn}.$$

Then K_n is finite-dimensional, $K_n \subset K_{n+1}$, $\cup K_n$ is dense in H. Since $VK_n = K_n$, for CAR we may take $\mathcal{C}_n = \mathcal{A}(K_n)$ (respectively, for the even part we may take $\mathcal{A}(K_n)_e$) [GN2, Lemma 2.4]. For CCR, we can not take $\mathcal{U}(K_n)$'s, since they are infinite-dimensional. However there exist finite-dimensional subalgebras of $\mathcal{U}(K_n)''$ that are still invariant under α_V. Namely, for any finite-dimensional subspace K of H and any $n \in \mathbb{N}$, we define a finite-dimensional C*-subalgebra $\mathcal{U}_n(K)$ of $\mathcal{U}(H)''$ as follows. Let N_K be the number operator corresponding to K, i. e.,

$$N_K = a^*(f_1)a(f_1) + \ldots + a^*(f_m)a(f_m),$$

where $f_1, \ldots, f_m$ is an orthonormal basis in K. This is a selfadjoint operator affiliated with $\mathcal{U}(K)''$, its spectrum is $\mathbb{Z}_+$ (see [BR2]). Let $P_n(K)$ be the spectral projection of N_K corresponding to $[0, n-1]$. Set

$$\mathcal{U}_n(K) = P_n(K)\mathcal{U}(K)''P_n(K) + \mathbb{C}(1 - P_n(K)).$$

The algebra $\mathcal{U}_n(K)$ is finite-dimensional, since in the Fock representation of $\mathcal{U}(K)$ the projection $P_n(K)$ is the projection onto the first n components $\oplus_{k=0}^{n-1} S^k K$ of the symmetric Fock space over K, and any regular representation of $\mathcal{U}(K)$ is quasi-equivalent to the Fock representation. If $VK = K$, then $\alpha_V(N_K) = N_K$ and $\alpha_V(\mathcal{U}(K)) = \mathcal{U}(K)$, hence $\alpha_V(\mathcal{U}_n(K)) = \mathcal{U}_n(K)$. Since $\cup_n \mathcal{U}(K_n)''$ is weakly dense in $\mathcal{U}(H)''$, and $\cup_m \mathcal{U}_m(K_n)$ is weakly dense in $\mathcal{U}(K_n)''$, by Lemma 2.2 we conclude that any channel in $\mathcal{U}(H)''$ can be approximated in strong operator topology by a channel γ in $\mathcal{U}_m(K_n)$ for some $m, n \in \mathbb{N}$. But then $h_{\omega_A}(\gamma; \alpha) \le h_{\omega_A}(\mathcal{U}_m(K_n); \alpha)$. Thus we may take $\mathcal{C}_{mn} = \mathcal{U}_m(K_n)$.

∎

Lemma 5.2 *Let X_1, X_2 be measurable subsets of $\mathbb{T}$, $\lambda(X_1)$, $\lambda(X_2) > 0$. Then there exist a measurable subset Y of X_1, $\lambda(Y) > 0$, and $z \in \mathbb{T}$ such that $zY \subset X_2$.*

Proof. See, for example, Lemma 3.5 in [GN2].

∎

Now we can extend Proposition 4.1 to arbitrary unitaries (with absolutely continuous spectrum). Consider a direct integral decomposition

$$H = \int_{\mathbb{T}}^{\oplus} H_z d\lambda(z), \quad U = \int_{\mathbb{T}}^{\oplus} z\, d\lambda(z),$$

and set $X = \{z \in \mathbb{T} \mid H_z \neq 0\}$.

Lemma 5.3 *For given $\varepsilon > 0$ and $C > 0$ ($C < 1$ for CAR) there exists $\delta > 0$ such that if* $\mathrm{Spec}\, A \subset (\lambda_0 - \delta, \lambda_0 + \delta)$ *for some $\lambda_0 \in (0, C)$, then*
CAR: $h_{\omega_A}(\alpha_U|_{\mathcal{A}(H)_e}) \geq \lambda(X)(\eta(\lambda_0) + \eta(1-\lambda_0) - \varepsilon)$;
CCR: $h_{\omega_A}(\alpha_U) \geq \lambda(X)(\eta(\lambda_0) - \eta(1+\lambda_0) - \varepsilon)$.

Proof. Consider the case of CAR-algebra. Choose $\delta > 0$ as in the formulation of Proposition 4.1. Let $\{n_k\}_{k=1}^{\infty} \subset \mathbb{N}$ be a sequence such that $\lambda(X) = \sum_k \frac{1}{n_k}$. The Zorn lemma and Lemma 5.2 ensure the existence of an at most countable set $\{X_{1m}\}_m$ of disjoint measurable subsets of X and a set $\{z_{1m}\}_m \subset \mathbb{T}$ such that

$$\exp\left(2\pi i \left[0, \frac{1}{n_k}\right]\right) = \bigsqcup_m z_{km} X_{km} \mod 0 \tag{5.1}$$

holds for $k = 1$. Proceeding by induction, we obtain a countable measurable partition $\{X_{km}\}_{k,m}$ of X and a countable subset $\{z_{km}\}_{k,m}$ of $\mathbb{T}$ such that (5.1) holds for all $k \in \mathbb{N}$. Let H_{km} be the spectral subspace for U corresponding to the set X_{km}. Set $H_k = \oplus_m H_{km}$, and define a unitary operator U_k on H_k,

$$U_k = \bigoplus_m z_{km} U|_{H_{km}}.$$

By Lemma 5.1 and Proposition 4.1, we have

$$h_{\omega_A}(\alpha_U|_{\mathcal{A}(H_k)_e}) = h_{\omega_A}(\alpha_{U_k}|_{\mathcal{A}(H_k)_e}) \geq \frac{1}{n_k}(\eta(\lambda_0) + \eta(1-\lambda_0) - \varepsilon).$$

For any $k_0 \in \mathbb{N}$, there exists an ω_A-preserving conditional expectation $\mathcal{A}(H) \to \otimes_{k=1}^{k_0} \mathcal{A}(H_k)_e$ (see Remark 4.2 in [SV]). By virtue of the superadditivity of the entropy [SV, Lemma 3.4], we conclude that

$$h_{\omega_A}(\alpha_U|_{\mathcal{A}(H)_e}) \geq \sum_{k=1}^{k_0} h_{\omega_A}(\alpha_U|_{\mathcal{A}(H_k)_e}) \geq \left(\sum_{k=1}^{k_0} \frac{1}{n_k}\right)(\eta(\lambda_0) + \eta(1-\lambda_0) - \varepsilon).$$

Letting $k_0 \to \infty$, we obtain the estimate we need.

The proof for CCR is similar, and we omit it.

■

We are already able to complete the proof in the case where A_z has continuous spectrum on a set of positive measure, i. e., we now show Corollary 1.2.

Proof of Corollary1.2. We will consider only the case of CAR-algebra. Fix $\delta_0 \in (0, \frac{1}{2})$ and take $\varepsilon \in (0, \eta(\delta_0))$. Let δ be as in the formulation of Lemma 5.3 with $C = 1 - \delta_0$. For any Borel subset X of $\mathbb{R}$, let $\mathbf{1}_X(A)$ be the spectral projection of A corresponding to X. Then

$$\mathbf{1}_X(A) = \int_{\mathbb{T}}^{\oplus} \mathbf{1}_X(A_z) d\lambda(z).$$

Define a measurable function ϕ_X on $\mathbb{T}$,

$$\phi_X(z) = \begin{cases} 1, & \mathbf{1}_X(A_z) \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

By Lemma 5.3, we conclude that if X is a Borel subset of $(\lambda_0-\delta, \lambda_0+\delta)$ for some $\lambda_0 \in (\delta_0, 1-\delta_0)$, then

$$h_{\omega_A}(\alpha_U|_{\mathcal{A}(\mathbf{1}_X(A)H)_e}) \geq (\eta(\lambda_0)+\eta(1-\lambda_0)-\varepsilon)\int_{\mathbb{T}}\phi_X(z)d\lambda(z) \geq \eta(1-\delta_0)\cdot\int_{\mathbb{T}}\phi_X(z)d\lambda(z), \quad (5.2)$$

where we have used the inequality $\eta(\lambda_0)+\eta(1-\lambda_0) \geq \eta(\delta_0)+\eta(1-\delta_0)$.

Let $t_0 = \delta_0 < t_1 < \ldots < t_m = 1-\delta_0$, $t_k - t_{k-1} < \delta$. Then by the same reasons as in the proof of Lemma 5.3, we obtain from (5.2) the inequality

$$h_{\omega_A}(\alpha_U) \geq \eta(1-\delta_0)\cdot\int_{\mathbb{T}}\sum_{k=1}^{m}\phi_{(t_{k-1},t_k]}(z)d\lambda(z).$$

Letting $\max(t_k - t_{k-1}) \to 0$, we conclude that if $h_{\omega_A}(\alpha_U) < \infty$, then $(\delta_0, 1-\delta_0)\cap \operatorname{Spec} A_z$ is finite for almost all $z \in \mathbb{T}$. Since δ_0 is arbitrary, A_z has pure point for almost all z provided the entropy is finite.

■

It remains to consider the case where A_z has pure point spectrum for almost all z. Then (see Appendix B)

$$H = \bigoplus_{n=1}^{N} L^2(X_n, d\lambda),$$

where X_n is a measurable subset of $\mathbb{T}$, $N \leq \aleph_0$, U and A act on $L^2(X_n)$ as multiplications by functions z and $\lambda_n(z)$, respectively. We must prove that

<u>CAR</u>: $h_{\omega_A}(\alpha_U) \geq \sum_{n=1}^{N}\int_{X_n}(\eta(\lambda_n(z))+\eta(1-\lambda_n(z)))d\lambda(z)$,

<u>CCR</u>: $h_{\omega_A}(\alpha_U) \geq \sum_{n=1}^{N}\int_{X_n}(\eta(\lambda_n(z))-\eta(1+\lambda_n(z)))d\lambda(z)$.

Again, consider only the case of CAR-algebra. Using the superadditivity as above, we see that it suffices to estimate $h_{\omega_A}(\alpha_U|_{\mathcal{A}(H)_e})$ supposing $N=1$. As in the proof of Corollary 1.2, fixing $\delta_0 > 0$, $\varepsilon > 0$ and choosing $t_0 = \delta_0 < t_1 < \ldots < t_m = 1-\delta_0$, we obtain

$$h_{\omega_A}(\alpha_U|_{\mathcal{A}(H)_e}) \geq \sum_{k=1}^{m}\int_{\{t_{k-1}<\lambda_1(z)\leq t_k\}}(\eta(t_k)+\eta(1-t_k)-\varepsilon)d\lambda(z)$$

if $\max(t_k - t_{k-1})$ is small enough. Letting $\max(t_k - t_{k-1}) \to 0$, we obtain

$$h_{\omega_A}(\alpha_U|_{\mathcal{A}(H)_e}) \geq \int_{\{\delta_0<\lambda_1(z)\leq 1-\delta_0\}}(\eta(\lambda_1(z))+\eta(1-\lambda_1(z)))d\lambda(z)-\varepsilon.$$

In view of the arbitrariness of δ_0 and ε, the proof is complete.

Appendix A

The results of the paper allow to construct a simple example of non-conjugate K-systems with the same finite entropy (see also Section 5 in [GN1]).

Theorem A.1 *Let U be a unitary operator on H with absolutely continuous spectrum, $A \in B(H)$, $A \geq 0$, $\mathrm{Ker} A = 0$, $AU = UA$. Suppose*

$$\left(\frac{A}{1+A}\right)^{it_0} = U \quad \textit{for some} \quad t_0 \in \mathbb{R}\backslash\{0\}.$$

Let ω and τ_θ, $\theta \in \mathbb{R}$, be the quasi-free state and the Bogoliubov automorphism of the CCR-algebra $\mathcal{U}(H)$ corresponding to A and $e^{i\theta}U$, respectively. Set $M = \pi_\omega(\mathcal{U}(H))''$. Then

(i) M is the hyperfinite III_1-factor;

(ii) (M, ω, τ_θ), $\theta \in [0, 2\pi)$, are pairwise non-conjugate entropic K-systems with the same entropy.

Proof. There exist a larger space $K \supset H$ and a unitary operator V on K with homogeneous Lebesgue spectrum such that $U = V|_H$. Let C be a non-singular bounded positive operator on K commuting with V such that $A = C|_H$. Set $\phi = \omega_C$, $\beta_\theta = \alpha_{e^{i\theta}V}$ and $N = \pi_\phi(\mathcal{U}(K))''$. Since ϕ is separating, we may consider M as a subalgebra of N. The algebras M and N are hyperfinite III_1-factors, moreover, the centralizer M_ω is trivial (see, for example, [GN1], p. 227). There exists a subspace K_0 of K such that $K_0 \subset VK_0$, $\cap_n V^n K_0 = 0$, $\cup_n V^n K_0$ is dense in K. Let N_0 be the W*-subalgebra of N generated by $\mathcal{U}(K_0)$. Then $N_0 \subset \beta_\theta(N_0)$, $\cup_{n\in\mathbb{N}}(\beta_\theta^{-n}(N_0)' \cap \beta_\theta^n(N_0)) \supset \cup_{n\in\mathbb{N}}\mathcal{U}(V^n K_0 \ominus V^{-n}K_0)$ is weakly dense in N, $\cap_n \beta_\theta^n(N_0) = \mathbb{C}1$ since N is a factor. Hence, (N, ϕ, β_θ) is an entropic K-system by [GN1, Theorem 3.1]. Since (M, ω, τ_θ) is a subsystem, and there exists a ϕ-preserving conditional expectation $N \to M$, it is an entropic K-system too.

The fact that $h_\omega(\tau_\theta)$ does not depend on θ follows either from the formula for the entropy or directly from Lemma 5.1.

It remains to prove the non-conjugacy. Let $\theta \mapsto \gamma_\theta$ be the gauge action. Since $\tau_\theta = \gamma_\theta\tau_0$, it suffices to prove that (M, ω, τ_0) and (M, ω, τ_θ) are non-conjugate for $\theta \in (0, 2\pi)$. Since $\tau_0 = \sigma^\omega_{t_0}$, any ω-preserving automorphism of M commutes with τ_0 and can not conjugate τ_0 with an automorphism different from τ_0.

■

Note that any K-automorphism is ergodic, and for any ergodic automorphism there exists at most one invariant normal state. Hence, any automorphism of M conjugating τ_{θ_1} with τ_{θ_2} preserves ω. Thus the automorphisms τ_θ, $\theta \in [0, 2\pi)$, are pairwise non-conjugate (but their restrictions to $\mathcal{U}(H)$ are conjugate).

To obtain finite entropy we may take, for example, unitaries with finitely multiple spectrum. We see also that if the unitary has homogeneous Lebesgue spectrum, then the systems constructed above have the algebraic K-property.

Appendix B

The following result was used in Sections 3 and 5.

Theorem B.1 *Let (Z, ν) be a Lebesgue space, $Z \ni z \mapsto H_z$ a measurable field of Hilbert spaces, $d(z) = \dim H_z$, $A = \int_Z^\oplus A_z d\nu(z)$ a decomposable selfadjoint operator on $H = \int_Z^\oplus H_z d\nu(z)$.*

Suppose that A_z has pure point spectrum ν-a. e. Then there exist measurable vector fields $e_1(z), e_2(z), \dots$, such that $\{e_n(z)\}_{n=1}^{d(z)}$ is an orthonormal basis of H_z consisting of eigenvectors of A_z for almost all z, and $e_n(z) = 0$ for $n > d(z)$ if $d(z) < \aleph_0$.

Proof. First, prove that there exists a measurable vector field e such that $e(z)$ is an eigenvector of norm one for A_z for almost all z. By Luzin's theorem, in proving this we may suppose that Z is a compact metric space, $\{H_z\}_z$ the constant field defined by a separable Hilbert space H_0, and $z \mapsto A_z \in B(H_0)$ a weakly continuous mapping. Consider the subset X of $Z \times H_0 \times \mathbb{R}$ defined by

$$X = \{(z, e, \lambda) \mid ||e|| = 1,\, A_z e = \lambda e\}.$$

Since X is closed, there exists a measurable section for the projection $X \to Z$, and our statement is proved.

Let $\{e_i\}_{i \in I}$ be a maximal family of vectors in H such that $e_i(z)$ and $e_j(z)$ are mutually orthogonal a. e. for $i \neq j$, and $e_i(z)$ is an eigenvector of norm one for A_z for almost all z. Since H is separable, I is at most countable. Hence, if P_z is the projection onto the space spanned by $e_i(z), i \in I$, then $z \mapsto P_z$ is a measurable field of projections, whence $z \mapsto (1 - P_z)H_z$ is a measurable field of subspaces. By the maximality, $\{e_i(z)\}_{i \in I}$ is an orthonormal basis of H_z consisting of eigenvectors of A_z on a subset of Z of positive measure. Thus the conclusion of Theorem holds on a subset of positive measure. Applying the maximality argument once again, we obtain an at most countable measurable partition of Z such that vector fields with the required properties exist over each element of the partition. Gluing them, we get the conclusion.

■

Note that if it was a priori known that there exist measurable functions $\lambda_1(z), \lambda_2(z), \dots$, such that the point spectrum of A_z coincides with $\{\lambda_n(z)\}_n$ (counting with multiplicities), then the conclusion of Theorem would follow directly from Lemma 2 on p.166 in [D].

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