Dynamical entropy for Bogoliubov actions of torsion-free abelian groups on the CAR-algebra

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Abstract

We compute dynamical entropy in Connes, Narnhofer and Thirring sense for a Bogoliubov action of a torsion-free abelian group G on the CAR-algebra. A formula analogous to that found by Størmer and Voiculescu in the case $G = \mathbf{Z}$ is obtained. The singular part of a unitary representation of G is shown to give zero contribution to the entropy. A proof of these results requires new arguments since a torsion-free group may have no finite index proper subgroups. Our approach allows to overcome these difficulties, it differs from that of Størmer-Voiculescu.

Introduction.

Entropy is an important notion in classical statistical mechanics and information theory. Initially the notion of entropy for automorphisms of a measure space was introduced by Kolmogorov and Sinai in 1958. This invariant proved to be extremely useful, it generated an entire field in the theory of classical dynamical systems and topological dynamics. The extension of the notion of entropy onto quantum systems was treated as a difficult mathematical problem. It was solved by Connes and Størmer [CS] only in 1975 for dynamical systems of type II_1. Then Connes, Narnhofer and Thirring [CNT] extended this theory to general C^*- and W^*-dynamical systems.

The computation of dynamical entropy for specific models is one of the principal trends in the theory (see [BG],[GN] for a bibliography). One of the main results in this sphere belongs to Størmer and Voiculescu [StV]. They showed that the CNT-entropy of a Bogoliubov automorphism of the CAR-algebra is computed by a simple formula (predicted by A. Connes for the tracial state), and only the absolutely continuous part of the unitary operator defining the Bogoliubov automorphism gives a contribution to the entropy. Bezuglyi and Golodets [BG] obtained the same results for Bogoliubov actions of free abelian groups. It is quite natural to extend these results to Bogoliubov actions of arbitrary countable torsion-free abelian groups.

Note that in Størmer-Voiculescu's approach it is very important that the group $\mathbb{Z}$ has a lot of finite index subgroups (see Theorem 2.1, condition (iv), in [StV]). But, for example, the group $\mathbb{Q}$ of rational numbers contains no finite index (proper) subgroups. So the methods of [StV] and [BG] cannot be immediately applied.

It is interesting to note that the problem of studying entropic properties of actions of the group $\mathbb{Q}$ is well-known in the commutative entropic theory. As far as we know there are no methods to describe the Pinsker algebra and asymptotic properties of systems with completely positive entropy for such actions. In particular, the Conze approach [Co] does not allow to solve these problems. It strengthens our interest to Bogoliubov actions of torsion-free abelian groups.

In this paper we prove that again there is a simple formula for the CNT-entropy of a Bogoliubov action of a torsion-free abelian group, and only the absolutely continuous part of a unitary representation gives a contribution to the entropy (see Theorems 3.1, 4.5 and Corollary 4.2 below). To prove these results we apply new arguments. In particular, our Lemma 2.4 allows to manage without finite index subgroups. We apply also results and methods of [StV] and [BG].

1 Definition of entropy.

Throughout the paper G denotes a discrete countable torsion-free abelian group.

By a C^*-dynamical system we mean a triple $(\mathcal{A}, \phi, \alpha_G)$, where $\mathcal{A}$ is a C^*-algebra, α_G is an action of G on $\mathcal{A}$ by *-automorphisms, and ϕ is a G-invariant state on $\mathcal{A}$.

For given channels $\gamma_i : \mathcal{B}_i \to \mathcal{A}$, $1 \le i \le n$, i.e. completely positive unital mappings of finite-dimensional C^*-algebras $\mathcal{B}_i$, $H_\phi(\gamma_1, \ldots, \gamma_n)$ denotes their mutual entropy with respect to ϕ (see [CNT]).

If γ is a channel and A is a finite subset of G, we denote by $H_\phi(\gamma^A)$ the mutual entropy of the channels $\alpha_g \circ \gamma$, $g \in A$.

Definition 1.1. A parallelepiped in G is a finite subset A of G such that there exist $n \in \mathbb{N}$, a monomorphism $I : \mathbb{Z}^n \to G$, and $m_1, \ldots, m_n \in \mathbb{N}$ such that

$$A = I\left(\{z \in \mathbb{Z}^n \mid 0 \le z_k \le m_k,\ 1 \le k \le n\}\right).$$

Definition 1.2. The entropy of the system $(\mathcal{A}, \phi, \alpha_G)$ with respect to a channel γ is the quantity

$$h_\phi(\gamma; \alpha_G) = \inf \frac{H_\phi(\gamma^A)}{|A|},$$

where A runs over the set of all parallelepipeds in G. The dynamical entropy of the system is

$$h_\phi(\alpha_G) = \sup_\gamma h_\phi(\gamma; \alpha_G).$$

Remark 1.3. If $\mathcal{A}$ is commutative and γ is the inclusion of a finite-dimensional subalgebra $\mathcal{P}$ of $\mathcal{A}$, $h_\phi(\mathcal{P}; \alpha_G)$ may be defined as the infimum of $\frac{H_\phi(\mathcal{P}^A)}{|A|}$ over all finite subsets A of G. Then one proves that this infimum is equal to the limit along a net of Følner sets, and this holds for any amenable group G [M]. The proof relies on the strong subadditivity of the function $A \mapsto H_\phi(\mathcal{P}^A)$, i.e.

$$H_\phi(\mathcal{P}^{A\cup B}) + H_\phi(\mathcal{P}^{A\cap B}) \le H_\phi(\mathcal{P}^A) + H_\phi(\mathcal{P}^B).$$

Apparently the function $A \mapsto H_\phi(\gamma^A)$ is not strongly subadditive in the non-commutative case. But it is at least subadditive [CNT], i.e.

$$H_\phi(\gamma^{A\cup B}) \le H_\phi(\gamma^A) + H_\phi(\gamma^B).$$

The following result is an immediate consequence of the subadditivity.

Proposition 1.4. *Let $G = \mathbb{Z}^n$, $n \in \mathbb{N}$. For $N \in \mathbb{N}$, let A_N denotes the cube $\{z \in \mathbb{Z}^n \mid 0 \le z_k \le N\}$. Then, for any channel γ,*

$$h_\phi(\gamma; \alpha_G) = \lim_{N\to\infty} \frac{H_\phi(\gamma^{A_N})}{|A_N|}.$$

In particular, for $G = \mathbb{Z}^n$ our definition of entropy coincides with the usual one [CNT],[BG].

■

Remark 1.5. A statement analogous to Proposition 1.4 may be formulated for any G. Let $\{g_i\}_{i=1}^N$, $N \le \infty$, be a maximal linear independent system in G, G_1 the subgroup of G generated by this system. Since G is torsion-free, we can consider G as a subgroup of $\mathcal{G} = \mathbb{Q} \otimes_{\mathbb{Z}} G$. Then $\{g_i\}_i$ is a basis of the vector space $\mathcal{G}$ over $\mathbb{Q}$. For an element $x \in \mathcal{G}$, let $x_i \in \mathbb{Q}$ denotes the i-th coordinate of x in this basis.

Choose a set $\{s_{kn}\}_{1\le k\le N, n\in\mathbb{N}}$ of non-negative numbers such that
(i) $\lim_{n\to\infty} s_{kn} = \infty$ for any k;
(ii) for any n, only finitely many of s_{kn}'s are non-zero. Set

$$A_n = \left\{ x \in \left(\frac{1}{n!} G_1\right) \cap G \mid 0 \le x_k < s_{kn} \ (x_k = 0 \text{ if } s_{kn} = 0) \right\}$$

(note that A_n is not a parallelepiped in the sense of Definition 1.1). Then

$$h_\phi(\gamma; \alpha_G) = \inf_n \frac{H_\phi(\gamma^{A_n})}{|A_n|} = \lim_n \frac{H_\phi(\gamma^{A_n})}{|A_n|}.$$

This result will not be used in the sequel.

The next lemma follows from the definitions.

Lemma 1.6. *Let $\{G_n\}_{n=1}^\infty$ be an increasing sequence of subgroups of G such that $\cup_n G_n = G$. Then $h_\phi(\gamma; \alpha_{G_n}) \searrow h_\phi(\gamma; \alpha_G)$.* ■

Proposition 1.7. *Let $(\mathcal{A}, \phi, \alpha_G)$ be a C^*-dynamical system, $\{G_n\}_{n=1}^\infty$ a sequence of subgroups of G, $\mathcal{A}_n$ a G_n-invariant C^*-subalgebra of $\mathcal{A}$, $F_n\colon \mathcal{A} \to \mathcal{A}_n$ a completely positive unital mapping, $F_n \to id$ pointwise-norm (we don't require $\mathcal{A}_n \subset \mathcal{A}_{n+1}$). Then*
(i) $h_\phi(\alpha_G) \le \liminf_{n\to\infty} h_\phi(\alpha_{G_n}|_{\mathcal{A}_n})$;
(ii) if $\mathcal{A}_n$'s are G-invariant and F_n's are ϕ-preserving conditional expectations then

$$h_\phi(\alpha_G) = \lim_{n\to\infty} h_\phi(\alpha_G|_{\mathcal{A}_n}).$$

Proof. See the proof of Lemma 3.3 in [StV].

(i) For a fixed channel $\gamma\colon \mathcal{B} \to \mathcal{A}$, let $\varepsilon_n = ||F_n \circ \gamma - \gamma||$. By [CNT], Proposition IV.3,

$$|H_\phi(\gamma^A) - H_\phi((F_n \circ \gamma)^A)| \le |A| \delta(\varepsilon_n, d)$$

for any finite $A \subset G$, where $\delta(\varepsilon_n, d)$ depends only on ε_n and $d = \dim \mathcal{B}$, and $\delta(\varepsilon, d) \to 0$ as $\varepsilon \to 0$. Thus, for any parallelepiped A in G_n, we have

$$\frac{H_{\phi|\mathcal{A}_n}((F_n \circ \gamma)^A)}{|A|} \ge \frac{H_\phi((F_n \circ \gamma)^A)}{|A|} \ge \frac{H_\phi(\gamma^A)}{|A|} - \delta(\varepsilon_n, d) \ge h_\phi(\gamma; \alpha_G) - \delta(\varepsilon_n, d),$$

so that

$$h_\phi(\alpha_{G_n}|_{\mathcal{A}_n}) \geq h_\phi(F_n \circ \gamma; \alpha_{G_n}|_{\mathcal{A}_n}) \geq h_\phi(\gamma; \alpha_G) - \delta(\varepsilon_n, d),$$

whence $\liminf h_\phi(\alpha_{G_n}|_{\mathcal{A}_n}) \geq h_\phi(\gamma; \alpha_G)$.

(ii) follows from (i) ($G_n = G\ \forall n \in \mathbb{N}$) and the fact that if there exists a ϕ-preserving conditional expectation onto a G-invariant subalgebra $\mathcal{D}$ of $\mathcal{A}$ then $H_{\phi|\mathcal{D}}(\gamma_1, \ldots, \gamma_n) = H_\phi(\gamma_1, \ldots, \gamma_n)$ for any channels $\gamma_1, \ldots, \gamma_n$ in $\mathcal{D}$, hence $h_\phi(\alpha_G|_\mathcal{D}) \leq h_\phi(\alpha_G)$. ■

If H is a subgroup of G, then $h_\phi(\gamma; \alpha_H) \geq h_\phi(\gamma; \alpha_G)$, whence $h_\phi(\alpha_H) \geq h_\phi(\alpha_G)$. The following proposition makes this relation more precise.

Proposition 1.8. *Let $(\mathcal{A}, \phi, \alpha_G)$ be a C^*-dynamical system, $\mathcal{A}$ nuclear, H a subgroup of G. Then*

(i) if $[G : H] < \infty$, then $h_\phi(\alpha_H) = [G : H]\, h_\phi(\alpha_G)$;

(ii) if $[G : H] = \infty$ and $h_\phi(\alpha_G) > 0$, then $h_\phi(\alpha_H) = \infty$.

Proof.

(i) Let $p = [G : H] < \infty$. First prove that $h_\phi(\gamma; \alpha_H) \leq p\, h_\phi(\gamma; \alpha_G)$.

Choose an increasing sequence $\{G_n\}_{n=1}^\infty$ of finitely generated subgroups of G such that $\cup_n G_n = G$. By Lemma 1.6, for a given $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that $h_\phi(\gamma; \alpha_{G_n}) < h_\phi(\gamma; \alpha_G) + \varepsilon$. Since G_n is a finite rank, free abelian group and $H_n = H \cap G_n$ is a subgroup of G_n of index$\leq p$, there exist a basis $g_1, \ldots, g_m$ in G_n and $k_1, \ldots, k_m \in \mathbb{N}$ such that $k_1 g_1, \ldots, k_m g_m$ is a basis in H_n [L]. For $N \in \mathbb{N}$, let A_N be the cube

$$\{l_1 g_1 + \ldots + l_m g_m \mid 0 \leq l_i \leq N - 1\}$$

in G_n. The set $A_N \cap H_n$ is a parallelepiped in H, and if $k_1, \ldots, k_m$ divide N, then $|A_N| = [G_n : H_n]|A_N \cap H_n|$, hence

$$h_\phi(\gamma; \alpha_H) \leq \frac{H_\phi(\gamma^{A_N \cap H_n})}{|A_N \cap H_n|} \leq \frac{H_\phi(\gamma^{A_N})}{|A_N \cap H_n|} = [G_n : H_n]\frac{H_\phi(\gamma^{A_N})}{|A_N|} \leq p\frac{H_\phi(\gamma^{A_N})}{|A_N|},$$

and using Proposition 1.4 we obtain

$$h_\phi(\gamma; \alpha_H) \leq p\, h_\phi(\gamma; \alpha_{G_n}) \leq p\, h_\phi(\gamma; \alpha_G) + p\,\varepsilon.$$

Thus we have proved that $h_\phi(\alpha_H) \leq p\, h_\phi(\alpha_G)$, and the assumption of nuclearity has not been used yet.

To prove the inequality $h_\phi(\alpha_G) \leq \frac{1}{p} h_\phi(\alpha_H)$ choose representatives $\bar{g}_1, \ldots, \bar{g}_p$ for cosets G/H. Due to the nuclearity, for a fixed $\varepsilon > 0$ and a channel $\gamma\colon \mathcal{B} \to \mathcal{A}$, $d = \dim \mathcal{B}$, there exist a channel $\theta\colon \mathcal{D} \to \mathcal{A}$ and channels $\theta_1, \ldots, \theta_p\colon \mathcal{B} \to \mathcal{D}$ such that

$$||\theta \circ \theta_i - \alpha_{\bar{g}_i} \circ \gamma|| < \varepsilon,\ 1 \leq i \leq p.$$

Let $\{G_n\}_{n=1}^\infty$ be an increasing sequence of finitely generated subgroups of G such that $\cup_n G_n = G$ and $\bar{g}_1, \ldots, \bar{g}_p \in G_n$ for any n. For a fixed n, there exist a basis $g_1, \ldots, g_m$ in G_n and numbers $k_1, \ldots, k_m \in \mathbb{N}$ such that $k_1 g_1, \ldots, k_m g_m$ is a basis in $H_n = H \cap G_n$. The absolute values of coordinates of $\bar{g}_1, \ldots, \bar{g}_p$ in this basis don't exceed a number N_0. For $N \in \mathbb{N}$, let

$$\begin{aligned} A_N &= \{l_1 g_1 + \ldots + l_m g_m \mid 0 \leq l_i \leq k_i N - 1\}, \\ \tilde{A}_N &= \{l_1 g_1 + \ldots + l_m g_m \mid N_0 \leq l_i \leq k_i N - N_0 - 1\}, \\ B_N &= A_N \cap H = \{l_1 k_1 g_1 + \ldots + l_m k_m g_m \mid 0 \leq l_i \leq N - 1\}. \end{aligned}$$

The sets $\bar{g}_i + B_N$, $1 \le i \le p$, are mutually disjoint and $\tilde{A}_N \subset \cup_i(\bar{g}_i + B_N)$. Thus

$$h_\phi(\gamma;\alpha_G) \le \frac{H_\phi(\gamma^{A_N})}{|A_N|} \le \frac{H_\phi(\gamma^{\tilde{A}_N})}{|A_N|} + \left(1 - \frac{|\tilde{A}_N|}{|A_N|}\right) H_\phi(\gamma),$$

$$\begin{aligned} H_\phi(\gamma^{\tilde{A}_N}) \le H_\phi(\gamma^{\cup_i(\bar{g}_i+B_N)}) &\le H_\phi(\{\alpha_g \circ \theta \circ \theta_i \mid g \in B_N,\ 1 \le i \le p\}) + p|B_N|\delta(\varepsilon, d) \\ &\le H_\phi(\theta^{B_N}) + p|B_N|\delta(\varepsilon, d), \end{aligned}$$

where we have used [CNT], Proposition IV.3 and Proposition III.6(a),(c).

Since $\frac{|A_N|}{|B_N|} = [G_n : H_n] = p$, we obtain

$$h_\phi(\gamma;\alpha_G) \le \frac{1}{p}\frac{H_\phi(\theta^{B_N})}{|B_N|} + \delta(\varepsilon, d) + \left(1 - \frac{|\tilde{A}_N|}{|A_N|}\right) H_\phi(\gamma).$$

Letting $N \to \infty$ and using Proposition 1.4 we conclude that

$$h_\phi(\gamma;\alpha_G) \le \frac{1}{p} h_\phi(\theta;\alpha_{H_n}) + \delta(\varepsilon, d),$$

and by Lemma 1.6,

$$h_\phi(\gamma;\alpha_G) \le \frac{1}{p} h_\phi(\theta;\alpha_H) + \delta(\varepsilon, d).$$

So, due to the arbitrariness of ε, $h_\phi(\gamma;\alpha_G) \le \frac{1}{p} h_\phi(\alpha_H)$.

(ii) Suppose $[G : H] = \infty$ and $h_\phi(\alpha_H) < \infty$, and prove that $h_\phi(\alpha_G) = 0$. Consider two cases.

a) The group G/H is periodic.

There exists an increasing sequence $\{H_n\}_{n=1}^\infty$ of subgroups of G such that $H \subset H_n$, $[H_n : H] < \infty$, $[H_n : H] \to \infty$. Then

$$h_\phi(\alpha_G) \le h_\phi(\alpha_{H_n}) = \frac{1}{[H_n : H]} h_\phi(\alpha_H) \to 0.$$

b) The rank of G/H is non-zero.

Let $g \in G$ be an element, whose image in G/H has infinite order. Let $H_n = H + n\mathbb{Z}g$. Then

$$h_\phi(\alpha_G) \le h_\phi(\alpha_{H_1}) = \frac{1}{n} h_\phi(\alpha_{H_n}) \le \frac{1}{n} h_\phi(\alpha_H) \to 0.$$

■

2 Bogoliubov actions on the CAR-algebra.

Let H be a complex Hilbert space. Recall (see [StV],[BR2]) that the CAR-algebra $\mathcal{A}(H)$ over H is a C^*-algebra generated by elements $a(f)$, $f \in H$, such that $f \mapsto a(f)$ is a linear map and

$$a(f)a(g)^* + a(g)^*a(f) = (f, g)1, \quad a(f)a(g) + a(g)a(f) = 0.$$

If K is a closed subspace of H, we consider $\mathcal{A}(K)$ as a subalgebra of $\mathcal{A}(H)$.

The even part of the CAR-algebra is the C^*-subalgebra $\mathcal{A}(H)_e$ generated by even products of $a(f)$'s and $a(g)^*$'s.

If H_1 and H_2 are mutually orthogonal subspaces of H, then $\mathcal{A}(H_1)$ and $A(H_2)_e$ commute and the C^*-algebra they generate is identified with $\mathcal{A}(H_1)\otimes\mathcal{A}(H_2)_e$.

If $0\le A\le 1$ is an operator on H, then the quasi-free state ω_A on $\mathcal{A}(H)$ is given by

$$\omega_A(a(f_n)^*\dots a(f_1)^*a(g_1)\dots a(g_m))=\delta_{nm}\det((Ag_i,f_j)).$$

We will write ω_λ instead of $\omega_{\lambda I}$. The state $\omega_{\frac{1}{2}}$ is the unique tracial state on $\mathcal{A}(H)$.

If $H=H_1\oplus H_2$, $A_i\in B(H_i)$, $0\le A_i\le 1$, $A=A_1\oplus A_2$, then

$$\omega_A|_{\mathcal{A}(H_1)\otimes\mathcal{A}(H_2)_e}=\omega_{A_1}|_{\mathcal{A}(H_1)}\otimes\omega_{A_2}|_{\mathcal{A}(H_2)_e}.$$

Suppose there exists an orthonormal basis $\{f_n\}_{n=1}^N$, $N\le\infty$, such that $Af_n=\lambda_n f_n$. Then

$$(\mathcal{A}(H),\omega_A)\cong\mathop{\otimes}_{n=1}^{N}(\mathrm{Mat}_2(\mathbb{C})_n,\rho_{\lambda_n}) \tag{2.1}$$

via the homomorphism sending $a(f_n)\prod_{i=1}^{n-1}(1-2a(f_i)^*a(f_i))$ to $e_{12}^{(n)}$ (so $a(f_n)a(f_n)^*\mapsto e_{11}^{(n)}$), where ρ_λ is the state on $\mathrm{Mat}_2(\mathbb{C})$ given by

$$\rho_\lambda\left(\begin{pmatrix} a & b\\ c & d\end{pmatrix}\right)=(1-\lambda)a+\lambda d.$$

Each unitary operator U on H defines a Bogoliubov automorphism α of $\mathcal{A}(H)$ by $\alpha(a(f))=a(Uf)$. So, for any unitary representation of G on H, we obtain an action of G on $\mathcal{A}(H)$ called Bogoliubov. If U and A commute, then α preserves ω_A.

If K is an invariant subspace for A and σ is the Bogoliubov automorphism corresponding to the operator $1\oplus -1$ on $H=K\oplus K^\perp$, then $\frac{\mathrm{id}+\sigma}{2}$ is an ω_A-preserving conditional expectation of $\mathcal{A}(H)$ onto $\mathcal{A}(K)\otimes\mathcal{A}(K^\perp)_e$, and composing it with $\mathrm{id}\otimes\omega_A(\cdot)$ we obtain an ω_A-preserving conditional expectation $\mathcal{A}(H)\to\mathcal{A}(K)$.

Lemma 2.1. *Let $\{P_n\}_{n=1}^\infty$ be a sequence of projections in $B(H)$, $P_n\to 1$ strongly. Let E_n be a conditional expectation of $\mathcal{A}(H)$ onto $\mathcal{A}(P_nH)$. Then $E_n\to\mathrm{id}$ pointwise-norm.*

Proof. The result easily follows from the facts that E_n is a projection of norm one and $||a(f)||=||f||$ for any $f\in H$. ■

Using the existence of conditional expectations and (2.1) we obtain also the following.

Lemma 2.2. *Let $H_1,\dots,H_n$ be mutually orthogonal finite-dimensional subspaces of H invariant for A. Then*

$$H_{\omega_A}(\mathcal{A}(H_1),\dots,\mathcal{A}(H_n))=S(\omega_A|_{\mathcal{A}(H_1\oplus\dots\oplus H_n)})=\sum_{i=1}^n\mathrm{Tr}_{H_i}(\eta(A)+\eta(1-A)),$$

where $\eta(x)=-x\log x$.

Proof. Cf. [CNT], Corollary VIII.8. ■

Lemma 2.3. *Let $U\colon G\to B(H)$ be a unitary representation, α_G the corresponding Bogoliubov action on $\mathcal{A}(H)$, $\{G_n\}_{n=1}^\infty$ a sequence of subgroups of G, $\{P_n\}_{n=1}^\infty$ a sequence of projections in $B(H)$ such that $H_n=P_nH$ is G_n-invariant and $P_n\to 1$ strongly. Then* (*i*) *for any* G*-invariant state* ϕ *on* $\mathcal{A}(H)$*, we have*

$$h_\phi(\alpha_G) \le \liminf_{n\to\infty} h_\phi(\alpha_{G_n}|_{\mathcal{A}(H_n)}) \ ;$$

(*ii*) *if* H_n*'s are* G*-invariant then, for* $A \in B(H)$*,* $0 \le A \le 1$*,* $U_gA = A\,U_g$*,* $A\,H_n \subset H_n$*, we have*

$$h_{\omega_A}(\alpha_G) = \lim_{n\to\infty} h_{\omega_A}(\alpha_G|_{\mathcal{A}(H_n)}).$$

Proof. This is a consequence of Lemma 2.1 and Proposition 1.7. ∎

The next simple observation plays the central role in the subsequent computations.

Lemma 2.4. *Let* $U^{(n)}\colon G \to B(H_n)$ *be a unitary representation (*$n \in \mathbb{N}$*),* $\{\chi_n\}_{n=1}^\infty \subset \hat{G}$ *a sequence of characters of* G*. Consider two unitary representations of* G *on* $H = \oplus_{n=1}^\infty H_n$*,*

$$U'_g = \mathop{\oplus}_{n=1}^{\infty} U_g^{(n)}, \quad U''_g = \mathop{\oplus}_{n=1}^{\infty} \chi_n(g) U_g^{(n)},$$

and let α'_G *and* α''_G *be the corresponding Bogoliubov actions. Then, for any* α'*- and* α''*-invariant state* ϕ *on* $\mathcal{A}(H)$*, we have*

$$h_\phi(\alpha'_G) = h_\phi(\alpha''_G).$$

Proof. For $n \in \mathbb{N}$, let $\{f_{kn}\}_{k=1}^\infty$ be an orthonormal basis in H_n. Let $\mathcal{A}_m$ be the C^*-subalgebra of $\mathcal{A}(H)$ generated by $a(f_{kn})$, $1 \le k, n \le m$. Then

$$h_\phi(\alpha'_G) = \lim_{m\to\infty} h_\phi(\mathcal{A}_m; \alpha'_G), \quad h_\phi(\alpha''_G) = \lim_{m\to\infty} h_\phi(\mathcal{A}_m; \alpha''_G)$$

by the proof of [CNT], Theorem V.2. On the other hand, since $\alpha''_g(a(f_{kn})) = \chi_n(g)\alpha'_g(a(f_{kn}))$, we have $\alpha''_g(\mathcal{A}_m) = \alpha'_g(\mathcal{A}_m)$ for any $g \in G$, hence $h_\phi(\mathcal{A}_m; \alpha''_G) = h_\phi(\mathcal{A}_m; \alpha'_G)$. ∎

3 Entropy formula: the case of absolutely continuous spectrum.

Recall some notions of the theory of representations that will be used below (see [K]).

Let $U\colon G \to B(H)$ be a unitary representation. Considering elements of G as characters of the dual group $\hat{G}$ we can extend it to a *-representation $f \mapsto U_f$ of the algebra of bounded Borel functions on $\hat{G}$. Then the spectral projection for a Borel subset $X \subset \hat{G}$ is the projection $U_{\mathbb{1}_X}$.

For a vector $\xi \in H$, the spectral measure μ_ξ is a positive Borel measure on $\hat{G}$ such that

$$(U_g\xi, \xi) = \int_{\hat{G}} \chi(g) d\mu_\xi(\chi), \quad g \in G.$$

The representation U is decomposed into a direct sum $U = U^a \oplus U^s$, $H = H^a \oplus H^s$, of two representations such that, for any $\xi \in H^a$ (resp. $\xi \in H^s$), the spectral measure μ_ξ is absolutely continuous (resp. singular) with respect to the Haar measure λ on $\hat{G}$. We say that U^a has absolutely continuous spectrum and U^s has singular spectrum.

The representation U^a is decomposed in a direct integral

$$H = \int_{\hat{G}}^{\oplus} H_\chi d\lambda(\chi), \quad U_g^a = \int_{\hat{G}}^{\oplus} \chi(g) d\lambda(\chi).$$

The function $m(\chi) = \dim H_\chi$ is called the multiplicity function of the representation U^a.

Our main result in this section is as follows.

Theorem 3.1. *Let $U\colon G \to B(H)$ be a unitary representation with absolutely continuous spectrum and the multiplicity function m. Then, for the corresponding Bogoliubov action α_G and $\beta \in [0, 1]$,*

$$h_{\omega_\beta}(\alpha_G) = (\eta(\beta) + \eta(1-\beta)) \int_{\hat{G}} m(\chi)\, d\lambda(\chi),$$

where $\eta(x) = -x \log x$.

The proof of Theorem is divided onto several lemmas.

First note that if $\beta = 0$ or $\beta = 1$ then the state ω_β is pure, so that the entropy of any channel is zero and there is nothing to prove. Thus we can suppose $\beta \in (0, 1)$.

Lemma 3.2. *Let $q, l \in \mathbb{N}$, $X = \cup_{k=0}^{l-1} [\frac{k}{l}, \frac{k}{l} + \frac{1}{ql}]$. Then*

$$\{q^{1/2} e^{2\pi i(lqk+r)t} \mid k \in \mathbb{Z},\ 0 \le r \le l-1\}$$

is an orthonormal basis in $L^2(X, dt)$.

Proof. Note that

$$\int_X e^{2\pi i n t} dt = \left(\sum_{k=0}^{l-1} e^{2\pi i \frac{nk}{l}} \right) \int_0^{\frac{1}{ql}} e^{2\pi i n t} dt.$$

This expression is zero if either l does not divide n or n is divided by ql. This implies the orthonormality.

The mapping $\exp(2\pi i l k t) \mapsto \exp(2\pi i k t)$ defines a unitary operator from

$$\overline{\mathrm{Lin}}\{e^{2\pi i l k t} \mid k \in \mathbb{Z}\} \subset L^2(X)$$

onto $L^2(0, \frac{1}{q})$. Hence, for any $p \in \mathbb{Z}$, $\exp(2\pi i l p t)$ belongs to the closed subspace of $L^2(X)$ spanned by $\exp(2\pi i l q k t)$, $k \in \mathbb{Z}$ (since $\{\exp(2\pi i q k t) \mid k \in \mathbb{Z}\}$ is an orthogonal basis in $L^2(0, \frac{1}{q})$). Then $\exp(2\pi i(lp + r)t)$, $0 \le r \le l-1$, lies in the closed subspace spanned by $\exp(2\pi i(lqk + r)t)$, $k \in \mathbb{Z}$. ∎

Lemma 3.3. *Let $G = \mathbb{Z}^n$. Suppose the multiplicity function m equals to $p\mathbb{1}_X$, where $p \in \mathbb{N}$ and*

$$X = \exp\left(2\pi i \left(\bigcup_{k=0}^{l-1} \left[\frac{k}{l}, \frac{k}{l} + \frac{1}{ql}\right]\right)\right) \times \underbrace{\mathbb{T} \times \ldots \times \mathbb{T}}_{n-1}.$$

So the space H of the representation can be identified with the sum of p copies of $L^2(X, d\lambda)$,

$$H = \bigoplus_{i=1}^{p} L^2(X, d\lambda)_i.$$

Let K be the subspace of H spanned by $\mathbb{1}_X \in L^2(X)_i$, $1 \le i \le p$. Then

$$h_{\omega_\beta}(\alpha_G) = h_{\omega_\beta}(\mathcal{A}(K); \alpha_G) = (\eta(\beta) + \eta(1-\beta))\frac{p}{q}.$$

Proof. The equality $h_{\omega_\beta}(\alpha_G) = (\eta(\beta) + \eta(1-\beta))\frac{p}{q}$ is known: see [StV], Lemma 4.5, or [BG], Lemma 3.4. Thus we have only to prove that $h_{\omega_\beta}(\mathcal{A}(K); \alpha_G) \ge (\eta(\beta) + \eta(1-\beta))\frac{p}{q}$. For $N \in \mathbb{N}$, let

$$\begin{aligned} A_N &= \{z \in \mathbb{Z}^n \mid 0 \le z_k \le N\}, \\ \tilde{A}_N &= A_N \cap \{z \in \mathbb{Z}^n \mid z_1 = lqk + r,\ k \in \mathbb{Z},\ 0 \le r \le l-1\}. \end{aligned}$$

By Lemma 3.2, the subspaces $U_g K$, $g \in \tilde{A}_N$, are mutually orthogonal. So by Lemma 2.2,

$$H_{\omega_\beta}(\mathcal{A}(K)^{\tilde{A}_N}) = |\tilde{A}_N|(\eta(\beta) + \eta(1-\beta))p,$$

and using Proposition 1.4 we obtain

$$h_{\omega_\beta}(\mathcal{A}(K); \alpha_G) \ge \lim_{N\to\infty} \frac{|\tilde{A}_N|}{|A_N|}(\eta(\beta) + \eta(1-\beta))p = (\eta(\beta) + \eta(1-\beta))\frac{p}{q}.$$

■

It is worth to note that the inequality $h_{\omega_\beta}(\alpha_G) \le (\eta(\beta) + \eta(1-\beta))\frac{p}{q}$ may also be deduced from the completeness assertion of Lemma 3.2.

Lemma 3.4. *Let $g \in G\backslash\{0\}$, $X = g^{-1}(\exp(2\pi i[0, \frac{1}{q}])) \subset \hat{G}$, where we consider g as a character of $\hat{G}$, $m = p\mathbb{1}_X$. Then*

$$h_{\omega_\beta}(\alpha_G) = (\eta(\beta) + \eta(1-\beta))\frac{p}{q}.$$

Proof. The space H of the representation is identified with the sum of p copies of $L^2(X, d\lambda)$, $H = \oplus_{i=1}^p L^2(X, d\lambda)_i$. Let K be the subspace of H spanned by $\mathbb{1}_X \in L^2(X)_i$, $1 \le i \le p$.

There exists an increasing sequence $\{G_n\}_{n=1}^\infty$ of finitely generated subgroups of G such that $\cup_n G_n = G$ and $g \in G_n$ for any n. Let H_n be the minimal G_n-invariant subspace of H containing K.

For a fixed $k \in \mathbb{N}$, consider g as a character of $\hat{G}_k$, and set $Y_k = g^{-1}(\exp(2\pi i[0, \frac{1}{q}])) \subset \hat{G}_k$. Then the representation of G_k on H_k has the multiplicity function $p\mathbb{1}_{Y_k}$. There exist a basis $g_1, \ldots, g_n$ of G_k and $l \in \mathbb{N}$ such that $g = lg_1$. Having fixed a basis we can identify $\hat{G}_k$ with $\mathbb{T}^n$. Then g maps $(t_1, \ldots, t_n) \in \mathbb{T}^n = \hat{G}_k$ to t_1^l, so that

$$Y_k = \exp\left(2\pi i \left(\bigcup_{j=0}^{l-1} \left[\frac{j}{l}, \frac{j}{l} + \frac{1}{ql}\right]\right)\right) \times \underbrace{\mathbb{T} \times \ldots \times \mathbb{T}}_{n-1},$$

By virtue of Lemma 3.3,

$$h_{\omega_\beta}(\alpha_{G_k}|_{\mathcal{A}(H_k)}) = h_{\omega_\beta}(\mathcal{A}(K); \alpha_{G_k}|_{\mathcal{A}(H_k)}) = (\eta(\beta) + \eta(1-\beta))\frac{p}{q},$$

and using Lemma 1.6 and Lemma 2.3(i) we obtain

$$h_{\omega_\beta}(\alpha_G) = h_{\omega_\beta}(\mathcal{A}(K); \alpha_G) = (\eta(\beta) + \eta(1-\beta))\frac{p}{q}.$$

■

Lemma 3.5. *Let H be a locally compact group, λ its left Haar measure, X_1 and X_2 measurable subsets of H, $0 < \lambda(X_1), \lambda(X_2) < \infty$. Then there exist a measurable subset Y of X_1, $\lambda(Y) > 0$, and $h \in H$ such that $hY \subset X_2$.*

Proof. The mapping $h \mapsto \mathbb{1}_{X_2 h} \in L^1(H, d\lambda)$ is continuous. Hence there exists a neighbourhood U of the unit such that

$$||\mathbb{1}_{X_2} - \mathbb{1}_{X_2 h^{-1}}||_1 \le \frac{1}{2}\lambda(X_2) \text{ for any } h \in U.$$

We can find $h_0 \in H$ with $\lambda(h_0 U \cap X_1) > 0$. Let $Y_1 = U \cap h_0^{-1} X_1$. Then $\lambda(Y_1) > 0$ and

$$\int_{X_2} |1 - \mathbb{1}_{X_2}(xy)| d\lambda(x) \le \frac{1}{2}\lambda(X_2)$$

for any $y \in Y_1$, hence

$$\int_{Y_1} d\lambda(y) \int_{X_2} d\lambda(x) |1 - \mathbb{1}_{X_2}(xy)| \le \frac{1}{2}\lambda(X_2)\lambda(Y_1),$$

and changing the order of integration,

$$\frac{1}{\lambda(X_2)} \int_{X_2} d\lambda(x) \int_{Y_1} d\lambda(y) |1 - \mathbb{1}_{X_2}(xy)| \le \frac{1}{2}\lambda(Y_1).$$

Hence there exists $x_0 \in X_2$ such that

$$\int_{Y_1} |1 - \mathbb{1}_{X_2}(x_0 y)| d\lambda(y) \le \frac{1}{2}\lambda(Y_1).$$

In other words, if we set $\tilde{Y} = \{y \in Y_1 \mid x_0 y \notin X_2\}$, then $\lambda(\tilde{Y}) \le \frac{1}{2}\lambda(Y_1)$. Thus, for $Y = h_0(Y_1 \backslash \tilde{Y})$, we have $\lambda(Y) \ge \frac{1}{2}\lambda(Y_1) > 0$, $Y \subset X_1$ and $x_0 h_0^{-1} Y \subset X_2$. ■

For a multiplicity function m and the corresponding Bogoliubov action α_G, set

$$\mu_\beta(m) = \frac{h_{\omega_\beta}(\alpha_G)}{\eta(\beta) + \eta(1-\beta)}.$$

Lemma 3.6. *For integrable multiplicity functions, $\mu_\beta(m)$ depends only on $\int_{\hat{G}} m \, d\lambda$.*

Proof. Suppose $\int m' d\lambda = \int m'' d\lambda < \infty$. Since m' and m'' are at most countable sums of indicator functions, Lemma 3.5 and a simple maximality argument ensure the existence of measurable subsets $Y_n \subset \hat{G}$, $n \in \mathbb{N}$, and a sequence $\{\chi_n\}_{n=1}^\infty \subset \hat{G}$ such that

$$m' = \sum_{n=1}^\infty \mathbb{1}_{Y_n} \text{ and } m'' = \sum_{n=1}^\infty \mathbb{1}_{\chi_n Y_n} \text{ a.e.}$$

Then the result follows from Lemma 2.4. ■

Lemma 3.7. *If $m_n \nearrow m$ a.e. then $\mu_\beta(m_n) \nearrow \mu_\beta(m)$.*

Proof. If H (resp. H_n) is the space of the representation U (resp. $U^{(n)}$) with the multiplicity function m (resp. m_n), then we may assume $H_1 \subset H_2 \subset \ldots \subset H$ and $U^{(n)} = U|_{H_n}$. It remains to apply Lemma 2.3(ii). ∎

Proof of Theorem. We have to prove that $\mu_\beta(m) = \int m\, d\lambda$.

First suppose $\int m\, d\lambda < \infty$. Choose $g \in G\backslash\{0\}$ and set $X(r) = g^{-1}(\exp(2\pi i[0,r]))$, $r \in [0,1]$. Then, for $p, q, n \in \mathbb{N}$,

$$\begin{aligned}\mu_\beta(p\mathbb{1}_{X(\frac{n}{q})}) &= \mu_\beta(pn\mathbb{1}_{X(\frac{1}{q})}) \text{ (Lemma 3.6)}\\ &= \frac{pn}{q} \qquad\qquad \text{(Lemma 3.4)}.\end{aligned}$$

By Lemma 3.7, $\mu_\beta(p\mathbb{1}_{X(r)}) = pr\ \forall p \in \mathbb{N}\ \forall r \in [0,1]$. Finding $p \in \mathbb{N}$ and $r \in [0,1]$ with $\int m\, d\lambda = pr$ we obtain $\mu_\beta(m) = \int m\, d\lambda$ by Lemma 3.6.

Suppose $\int m\, d\lambda = \infty$. Letting $m_n = m \wedge (n\mathbb{1})$ and applying Lemma 3.7 we obtain

$$\mu_\beta(m) = \lim_{n\to\infty} \mu_\beta(m_n) = \lim_{n\to\infty} \int m_n\, d\lambda = \infty.$$

∎

Remark 3.8. An inspection of the proof shows that the same entropy formula is valid for the restriction of a Bogoliubov action to the even part of the CAR-algebra.

4 Entropy formula.

Theorem 4.1. *Let $U^{(i)}: G \to B(H_i)$ be a unitary representation, $i = 1,2$, $A_1 \in B(H_1)$, $0 \le A_1 \le 1$, $A_1U_g^{(1)} = U_g^{(1)}A_1$, $m^{(2)}$ the multiplicity function of the absolutely continuous part of $U^{(2)}$. Suppose the representation $U^{(1)}$ has absolutely continuous spectrum and A_1 has pure point spectrum. Let $H_1 = \int_{\hat G}^{\oplus} H_\chi d\lambda(\chi)$ and $A_1 = \int_{\hat G}^{\oplus} A(\chi)d\lambda(\chi)$ be direct integral decompositions (λ is the Haar measure on $\hat G$). Then, for the Bogoliubov action α_G corresponding to the representation $U^{(1)} \oplus U^{(2)}$ and for any G-invariant state ϕ on $\mathcal{A}(H_1 \oplus H_2)$ such that $\phi|_{\mathcal{A}(H_1)} = \omega_{A_1}$, we have*

$$h_\phi(\alpha_G) \le \int_{\hat G} \operatorname{Tr}\left(\eta(A(\chi) + \eta(1 - A(\chi))\right) d\lambda(\chi) + (\log 2)\int_{\hat G} m^{(2)}(\chi)d\lambda(\chi).$$

Corollary 4.2. *If the spectrum of a unitary representation of G is singular then the entropy of the corresponding Bogoliubov action is zero with respect to any invariant state.* ∎

The proof of Theorem is a slight modification of the method used in [BG] to handle the case of singular spectrum. First, we need two lemmas.

Lemma 4.3. *Under the assumptions of Theorem 4.1 suppose that we are given a one more representation $U^{(3)}: G \to B(H_3)$ and $A_3 \in B(H_3)$, $0 \le A_3 \le 1$, $A_3U_g^{(3)} = U_g^{(3)}A_3$. Let $\tilde\alpha_G$ be the Bogoliubov action corresponding to $U^{(1)} \oplus U^{(2)} \oplus U^{(3)}$. Then there exists a*

G-invariant state ψ on $\mathcal{A}(H_1 \oplus H_2 \oplus H_3)$ such that $\psi|_{\mathcal{A}(H_1\oplus H_3)} = \omega_{A_1\oplus A_3}$ and $h_\psi(\tilde\alpha_G) \geq h_\phi(\alpha_G)$.

Proof. Let σ be the Bogoliubov automorphism corresponding to the operator $1 \oplus 1 \oplus -1$. Then $E = \frac{\mathrm{id}+\sigma}{2}$ is a G-invariant conditional expectation of $\mathcal{A}(H_1 \oplus H_2 \oplus H_3)$ onto $\mathcal{A}(H_1 \oplus H_2) \otimes \mathcal{A}(H_3)_e$ (see Section 2). Set $\psi = (\phi \otimes \omega_{A_3}) \circ E$. Then ψ is G-invariant, $\psi|_{\mathcal{A}(H_1\oplus H_3)} = \omega_{A_1\oplus A_3}$, and since there exists a ψ-preserving conditional expectation onto $\mathcal{A}(H_1 \oplus H_2)$ (namely $(\mathrm{id}_{\mathcal{A}(H_1\oplus H_2)} \otimes \omega_{A_3}(\cdot)) \circ E$), we have $h_\psi(\tilde\alpha_G) \geq h_\phi(\alpha_G)$. ∎

Lemma 4.4. *Let U be a unitary representation of G on H, m the multiplicity function of the absolutely continuous part of U, $\{G_n\}_{n=1}^\infty$ an increasing sequence of subgroups of G with $\cup_n G_n = G$. Then there exist a sequence $k_n \nearrow \infty$ and, for each $n \in \mathbb{N}$, a G_{k_n}-invariant subspace H_n of H such that*

(i) if P_n is the projection onto H_n, then $P_n \to \mathrm{id}$ strongly ;

(ii) if m_n is the multiplicity function of the absolutely continuous part of the representation $U_{G_{k_n}}|_{H_n}$, then

$$\limsup_{n\to\infty} \int_{\hat G_{k_n}} m_n \, d\lambda_{k_n} \leq \int_{\hat G} m \, d\lambda.$$

Proof. It suffices to prove Lemma for a cyclic representation. So let $\xi \in H$ be a cyclic vector, μ its spectral measure, $\mu = \mu_a + \mu_s$ the decomposition into the sum of the absolutely continuous and the singular parts,

$$X = \left\{ \chi \in \hat G \mid \frac{d\mu_a}{d\lambda}(\chi) > 0 \right\}.$$

Then the representation U is equivalent to the canonical representation on

$$L^2(\hat G, d\mu) = L^2(\hat G, d\mu_s) \oplus L^2(\hat G, d\mu_a) \cong L^2(\hat G, d\mu_s) \oplus L^2(X, d\lambda).$$

In particular, $m = \mathbb{I}_X$.

For each $n \in \mathbb{N}$, there exists a compact subset X_n of $\hat G$ such that

$$\lambda(X \bigtriangleup X_n) < \frac{1}{n} \quad \text{and} \quad \mu(\hat G \backslash X_n) < \frac{1}{n}.$$

Then we can find an open $Y_n \subset \hat G$ such that

$$X_n \subset Y_n \quad \text{and} \quad \lambda(Y_n \backslash X_n) < \frac{1}{n}.$$

Denote by I_n the inclusion $G_n \hookrightarrow G$. Due to the compactness of X_n and the equality $\hat G = \varprojlim \hat G_n$, there exist $k_n \geq n$ and a compact $\tilde Z_n \subset \hat G_{k_n}$ such that, for $Z_n = \hat I_{k_n}^{-1}(\tilde Z_n)$, we have $X_n \subset Z_n \subset Y_n$. Then

$$\lambda(Z_n \bigtriangleup X) \leq \lambda(Z_n \bigtriangleup X_n) + \lambda(X_n \bigtriangleup X) < \frac{2}{n} \quad \text{and} \quad \mu(\hat G \backslash Z_n) < \frac{1}{n}.$$

Let $E(Z_n)$ be the spectral projection corresponding to Z_n. Set $\xi_n = E(Z_n)\xi$, and let H_n be the minimal G_{k_n}-invariant subspace containing ξ_n. Then the spectral measure of ξ_n (with respect to G_{k_n}) is supported by $\tilde Z_n$, so if m_n is the multiplicity function of the absolutely continuous part of the representation of G_{k_n} on H_n, we have

$$\int_{\hat G_{k_n}} m_n \, d\lambda_{k_n} \leq \lambda_{k_n}(\tilde Z_n) = \lambda(Z_n) \leq \frac{2}{n} + \lambda(X) = \frac{2}{n} + \int_{\hat G} m \, d\lambda.$$

Thus the condition (ii) is satisfied. (i) follows from the estimate

$$||\xi - \xi_n||^2 = \mu(\hat{G}\backslash Z_n) < \frac{1}{n},$$

since ξ is cyclic and any $g \in G$ is eventually contained in G_{k_n}. ∎

Proof of Theorem. Let $\{\lambda_n\}_{n=1}^N$, $N \le \infty$, be the point spectrum of A_1, and e_n the spectral projection corresponding to λ_n. Note that if m_n is the multiplicity function of the representation $U^{(1)}|_{e_n H_1}$ then

$$\int_{\hat{G}} \mathrm{Tr}\,(\eta(A(\chi) + \eta(1 - A(\chi)))\,d\lambda(\chi) = \sum_{n=1}^N (\eta(\lambda_n) + \eta(1-\lambda_n)) \int_{\hat{G}} m_n(\chi) d\lambda(\chi).$$

Choosing an increasing sequence of finitely generated subgroups of G and applying Lemma 4.4 to each subspace H_2, $e_n H_1$, $1 \le n \le N$, we infer from Lemma 2.3(i) that it suffices to consider the case where $G \cong \mathbb{Z}^n$ and $N < \infty$. By Lemma 2.3(i), we can also suppose that the multiplicity functions $m^{(2)}$, m_n, $1 \le n \le N$, are finite sums of indicator functions of compact sets.

Let $G = \mathbb{Z}$. Suppose the assertion is proved under the additional assumption that $m^{(2)}$ and m_n, $1 \le n \le N$, are finite sums of indicator functions of (closed) arcs of rational length. Consider the general case. For a fixed $\varepsilon > 0$, since for any compact set $X \subset \mathbb{T}$ there exists a set Y such that $X \subset Y$, Y is a finite union of disjoint arcs of rational length and $\lambda(Y\backslash X)$ is arbitrary small, we can find multiplicity functions $\tilde{m}^{(2)}$, $\tilde{m}_n$, $1 \le n \le N$, such that the functions $m^{(2)} + \tilde{m}^{(2)}$, $m_n + \tilde{m}_n$, $1 \le n \le N$, are finite sums of indicator functions of arcs of rational length and

$$\sum_{n=1}^N (\eta(\lambda_n) + \eta(1-\lambda_n)) \int \tilde{m}_n d\lambda + (\log 2) \int \tilde{m}^{(2)} d\lambda < \varepsilon.$$

Let $\tilde{H}^{(2)}$, $\tilde{H}_n$, $1 \le n \le N$, be the spaces of the corresponding representations. Set

$$H_3 = \tilde{H}_1 \oplus \ldots \oplus \tilde{H}_N \oplus \tilde{H}^{(2)} \;\text{ and }\; A_3 = \lambda_1 1_{\tilde{H}_1} \oplus \ldots \oplus \lambda_N 1_{\tilde{H}_N} \oplus \frac{1}{2} 1_{\tilde{H}^{(2)}}.$$

Let ψ and $\tilde{\alpha}_G$ be as in the formulation of Lemma 4.3. By assumption, Theorem is true for $\tilde{\alpha}_G$, so that

$$h_\phi(\alpha_G) \le h_\psi(\tilde{\alpha}_G) \le \sum_{n=1}^N (\eta(\lambda_n) + \eta(1-\lambda_n)) \int (m_n + \tilde{m}_n) d\lambda + (\log 2) \int (m^{(2)} + \tilde{m}^{(2)}) d\lambda$$

$$< \sum_{n=1}^N (\eta(\lambda_n) + \eta(1-\lambda_n)) \int m_n d\lambda + (\log 2) \int m^{(2)} d\lambda + \varepsilon.$$

Since this holds for all $\varepsilon > 0$, we conclude that

$$h_\phi(\alpha_G) \le \sum_{n=1}^N (\eta(\lambda_n) + \eta(1-\lambda_n)) \int m_n d\lambda + (\log 2) \int m^{(2)} d\lambda.$$

It remains to consider the case where the multiplicity functions $m^{(2)}$, m_n, $1 \le n \le N$, are finite sums of indicator functions of arcs of rational length. If q is a common denominator of these rational numbers, we can pass to the subgroup $q\mathbb{Z}$ of $\mathbb{Z}$ (since $h_\phi(\alpha_{\mathbb{Z}}) = \frac{1}{q} h_\phi(\alpha_{q\mathbb{Z}})$) thus supposing $m^{(2)} = p^{(2)}\mathbb{I}$, $m_n = p_n \mathbb{I}$, $1 \le n \le N$, for certain $p^{(2)}, p_1, \dots, p_N \in \mathbb{N}$. Then the representations on $e_n H$ and H_2^a (the absolutely continuous part of $U^{(2)}$) are finite sums of bilateral shifts, so that there exist subspaces $K_n \subset e_n H_1$, $K \subset H_2^a$ such that

$\dim K_n = p_n$, the spaces $U_j^{(1)} K_n$, $j \in \mathbb{Z}$, are mutually orthogonal, $\overline{\bigcup_{j\in\mathbb{Z}} U_j^{(1)} K_n} = e_n H_1$; and analogously for $K \subset H_2^a$.

Set $Z_n = \overset{n}{\underset{j=1}{\oplus}} U_j^{(1)}(K_1 \oplus \dots \oplus K_N)$ and $X_n = Z_n \oplus \left(\overset{n}{\underset{j=1}{\oplus}} U_j^{(2)} K \right)$. Then the proof of Lemma 5.3 in [StV] shows that

$$h_\phi(\alpha_{\mathbb{Z}}) \le \liminf_{n\to\infty} \frac{1}{n} S(\phi|_{\mathcal{A}(X_n)})$$

for any invariant state ϕ. Since $\mathcal{A}(Z_n)$ and $\mathcal{A}(X_n)$ are full matrix algebras of dimensions $2^{2n(p_1+\dots+p_N)}$ and $2^{2n(p^{(2)}+p_1+\dots+p_N)}$ respectively, the subadditivity of the von Neumann entropy implies

$$S(\phi|_{\mathcal{A}(X_n)}) \le S(\phi|_{\mathcal{A}(Z_n)}) + np^{(2)} \log 2.$$

By (2.1), we have $S(\phi|_{\mathcal{A}(Z_n)}) = n \sum_{k=1}^{N} (\eta(\lambda_k) + \eta(1-\lambda_k)) p_k$, so

$$\begin{aligned} h_\phi(\alpha_{\mathbb{Z}}) &\le \sum_{k=1}^{N} (\eta(\lambda_k) + \eta(1-\lambda_k)) p_k + p^{(2)} \log 2 \\ &= \sum_{k=1}^{N} (\eta(\lambda_k) + \eta(1-\lambda_k)) \int_{\mathbb{T}} m_k \, d\lambda + (\log 2) \int_{\mathbb{T}} m^{(2)} d\lambda, \end{aligned}$$

and the proof for $G = \mathbb{Z}$ is complete.

The case $G = \mathbb{Z}^n$, $n > 1$, is analogous (see Lemma 3.7 and Theorem 3.8 in [BG]). We leave the details to the reader. ■

Theorem 4.5. *Let $U: G \to B(H)$ be a unitary representation, $U^a|_{H^a}$ its absolutely continuous part, $A \in B(H)$, $0 \le A \le 1$, $AU_g = U_g A$. Let $H^a = \int_{\hat{G}}^{\oplus} H_\chi d\lambda(\chi)$ and $A|_{H^a} = \int_{\hat{G}}^{\oplus} A(\chi) d\lambda(\chi)$ be direct integral decompositions. If $A|_{H^a}$ has pure point spectrum then, for the Bogoliubov action α_G corresponding to U, we have*

$$h_{\omega_A}(\alpha_G) = \int_{\hat{G}} \mathrm{Tr}\,(\eta(A(\chi) + \eta(1 - A(\chi)))\, d\lambda(\chi).$$

Proof. The inequality $\le$ is proved in Theorem 4.1. Let $\{\lambda_n\}_{n=1}^N$ be the point spectrum of $A|_{H^a}$, e_n the spectral projection corresponding to λ_n. Taking into account Remark 3.8 the same arguments as in [StV], Theorem 6.3, give us

$$h_{\omega_A}(\alpha_G) \ge \sum_{n=1}^{N} h_{\omega_{\lambda_n}}(\alpha_G|_{\mathcal{A}(e_n H)_e}) = \int_{\hat{G}} \mathrm{Tr}\,(\eta(A(\chi) + \eta(1 - A(\chi)))\, d\lambda(\chi).$$

■

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