Ergodicity of the action of the positive rationals on the group of finite adeles and the Bost-Connes phase transition theorem

Sergey Neshveyev[1]

Abstract

We study relatively invariant measures with the multiplicators $\mathbb{Q}^*_+ \ni q \mapsto q^{-\beta}$ on the space $\mathcal{A}_f$ of finite adeles. We prove that for $\beta \in (0,1]$ such measures are ergodic, and then deduce from this the uniqueness of KMS$_\beta$-states for the Bost-Connes system. Combining this with a result of Blackadar and Boca-Zaharescu, we obtain also ergodicity of the action of $\mathbb{Q}^*$ on the full adeles.

Bost and Connes [BC] constructed a remarkable C*-dynamical system which has a phase transition with spontaneous symmetry breaking involving an action of the Galois group Gal$(\mathbb{Q}^{ab}/\mathbb{Q})$, and whose partition function is the Riemann ζ function. In their original definition the underlying algebra arises as the Hecke algebra associated with an inclusion of certain $ax+b$ groups. Recently Laca and Raeburn [LR, L2] have realized the Bost-Connes algebra as a full corner of the crossed product algebra $C_0(\mathcal{A}_f) \rtimes \mathbb{Q}^*_+$. This new look at the system has allowed to simplify significantly the proof of the existence of KMS-states for all temperatures, and the classification of KMS$_\beta$-states for $\beta > 1$ [L1]. On the other hand, for $\beta \le 1$ the uniqueness of KMS$_\beta$-states implies ergodicity of the action of $\mathbb{Q}^*_+$ on $\mathcal{A}_f$ for certain measures (in particular, for the Haar measure). The aim of this note is to give a direct proof of the ergodicity, and then to show that the uniqueness of KMS$_\beta$-states easily follows from it.

So let $\mathcal{P}$ be the set of prime numbers, $\mathcal{A}_f$ the restricted product of the fields $\mathbb{Q}_p$ with respect to $\mathbb{Z}_p$, $p \in \mathcal{P}$, $\mathcal{R} = \prod_p \mathbb{Z}_p$ its maximal compact subring, $W = \mathcal{R}^* = \prod_p \mathbb{Z}^*_p$. The group $\mathbb{Q}^*_+$ of positive rationals is embedded diagonally into $\mathcal{A}_f$, and so acts by multiplications on the additive group of finite adeles. Then the Bost-Connes algebra $\mathcal{C}_\mathbb{Q}$ is the full corner of $C_0(\mathcal{A}_f) \rtimes \mathbb{Q}^*_+$ determined by the characteristic function of $\mathcal{R}$ [L2]. The dynamics σ_t is defined as follows [L1]: it is trivial on $C_0(\mathcal{A}_f)$, and $\sigma_t(u_q) = q^{it}u_q$, where u_q is the multiplier of $C_0(\mathcal{A}_f) \rtimes \mathbb{Q}^*_+$ corresponding to $q \in \mathbb{Q}^*_+$. Then ([L1]) there is a one-to-one correspondence between (β, σ_t)-KMS-states on $\mathcal{C}_\mathbb{Q}$ and measures μ on $\mathcal{A}_f$ such that

$$\mu(\mathcal{R}) = 1 \ \text{ and } \ q_*\mu = q^\beta \mu \ \text{ for all } q \in \mathbb{Q}^*_+ \ \ (\text{i.e., } \mu(q^{-1}X) = q^\beta \mu(X)). \tag{1β}$$

Namely, the KMS-state corresponding to μ is the restriction of the dual weight on $C_0(\mathcal{A}_f) \rtimes \mathbb{Q}^*_+$ to $\mathcal{C}_\mathbb{Q}$.

[1]This research was partially supported by Award No UM1-2092 of the Civilian Research & Development Foundation.
1991 Mathematics Subject Classification: Primary 46L55, Secondary 28D15

Note ([L1]) that if $\beta > 1$ and μ is a measure with property (1β) then $\mu(W) = \prod_{p\in\mathcal{P}}(1 - p^{-\beta}) = \frac{1}{\zeta(\beta)} > 0$, since $W = \mathcal{R}\setminus \cup_p p\mathcal{R}$. Moreover, the sets qW, $q \in \mathbb{Q}^*_+$, are disjoint, and their union is a set of full measure (since $\sum_{n\in\mathbb{N}} \mu(nW) = \frac{1}{\zeta(\beta)} \sum_{n\in\mathbb{N}} n^{-\beta} = 1$). Thus there exists a one-to-one correspondence between probability measures on W and measures on $\mathcal{A}_f$ satisfying (1β) [L1]. On the other hand, if $\beta \le 1$ then $\mu(W) = 0$.

PROPOSITION. *For $\beta \in (0, 1]$ and any measure μ satifying (1β), the action of $\mathbb{Q}^*_+$ on $(\mathcal{A}_f, \mu)$ is ergodic.*

Proof. Consider the space $L^2(\mathcal{R}, d\mu)$ and the subspace H of it consisting of the functions that are constant on $\mathbb{N}$-orbits. In other words, $H = \{f \in L^2(\mathcal{R}, d\mu) \,|\, V_n f = f,\ n \in \mathbb{N}\}$, where $(V_n f)(x) = f(nx)$. Since any $\mathbb{Q}^*_+$-invariant subset of $\mathcal{A}_f$ is completely determined by its intersection with $\mathcal{R}$, it suffices to prove that H consists of constant functions. For this we will compute the action of the projection $P\colon L^2(\mathcal{R}, d\mu) \to H$ on functions spanning a dense subspace of $L^2(\mathcal{R}, d\mu)$.

Let B be a finite subset of $\mathcal{P}$. Consider the projection $\pi_B\colon \mathcal{R} \to \prod_{p\in B} \mathbb{Z}_p$, and set $\mu_B = (\pi_B)_*\mu$. Then $L^2(\prod_{p\in B} \mathbb{Z}_p, d\mu_B)$ can be considered as a subspace of $L^2(\mathcal{R}, d\mu)$, and the union of these subspaces over all finite B is dense in $L^2(\mathcal{R}, d\mu)$. The characters of $\prod_{p\in B} \mathbb{Z}^*_p$ span a dense subspace of $L^2(\prod_{p\in B} \mathbb{Z}^*_p, d\mu_B)$. Let $\mathbb{N}_B$ be the unital multiplicative subsemigroup of $\mathbb{N}$ generated by $p \in B$. Note that the sets $n\prod_{p\in B} \mathbb{Z}^*_p$, $n \in \mathbb{N}_B$, are disjoint, their union is a subset of $\prod_{p\in B} \mathbb{Z}_p$ of full measure (condition (1β) implies that the set $\{x \in \mathcal{R} \,|\, x_p = 0\}$ has zero measure), and the operator $n^{-\beta/2} V^*_n$ maps isometrically $L^2(\prod_{p\in B} \mathbb{Z}^*_p, d\mu_B)$ onto $L^2(n\prod_{p\in B} \mathbb{Z}^*_p, d\mu_B)$ for any $n \in \mathbb{N}_B$. Hence the functions $V^*_n\chi$, $n \in \mathbb{N}_B$, $\chi \in (\prod_{p\in B} \mathbb{Z}^*_p)\hat{}$, span a dense subspace of $L^2(\prod_{p\in B} \mathbb{Z}_p, d\mu_B)$. So we have to compute $PV^*_n\chi$. But if $g \in H$ then $(V^*_n\chi, g) = (\chi, g)$, whence $PV^*_n\chi = P\chi$. Thus we have only to compute $P\chi$.

For a finite subset A of $\mathcal{P}$, let H_A be the subspace consisting of the functions that are constant on $\mathbb{N}_A$-orbits, P_A the projection onto H_A. Then $P_A \searrow P$ as $A \nearrow \mathcal{P}$. Set

$$W_A = \prod_{p\in A} \mathbb{Z}^*_p \times \prod_{q\in\mathcal{P}\setminus A} \mathbb{Z}_q \subset \mathcal{R}.$$

Note, as above, that $\cup_{n\in\mathbb{N}_A} nW_A$ is a subset of $\mathcal{R}$ of full measure. We assert that

$$P_A f|_{\mathbb{N}_A x} \equiv \frac{1}{\zeta_A(\beta)} \sum_{n\in\mathbb{N}_A} n^{-\beta} f(nx) \quad \text{for} \quad x \in W_A, \tag{2}$$

where $\zeta_A(\beta) = \sum_{n\in\mathbb{N}_A} n^{-\beta} = \prod_{p\in A}(1 - p^{-\beta})^{-1}$. Indeed, denoting the right hand part of (2) by f_A, for $g \in H_A$ we obtain

$$\begin{aligned}
(f_A, g) &= \sum_{n\in\mathbb{N}_A} \int_{nW_A} f_A(x)\overline{g(x)}d\mu(x) = \sum_{n\in\mathbb{N}_A} n^{-\beta} \int_{W_A} f_A(x)\overline{g(x)}d\mu(x) \\
&= \zeta_A(\beta) \int_{W_A} f_A(x)\overline{g(x)}d\mu(x) = \sum_{n\in\mathbb{N}_A} n^{-\beta} \int_{W_A} f(nx)\overline{g(x)}d\mu(x) \\
&= \sum_{n\in\mathbb{N}_A} \int_{nW_A} f(x)\overline{g(x)}d\mu(x) = (f, g).
\end{aligned}$$

Returning to the computation of $P\chi$, we see that

$$P_A\chi|_{\mathbb{N}_A x} \equiv \frac{\chi(x)}{\zeta_A(\beta)} \sum_{n\in\mathbb{N}_A} n^{-\beta}\chi(n) = \chi(x) \prod_{p\in A} \frac{1 - p^{-\beta}}{1 - \chi(p)p^{-\beta}} \quad \text{for} \quad x \in W_A. \tag{3}$$

Hence

$$\|P\chi\|_2 = \lim_{A \nearrow \mathcal{P}} \prod_{p\in A} \left| \frac{1-p^{-\beta}}{1-\chi(p)p^{-\beta}} \right|. \tag{4}$$

If χ is trivial, then using (3) we see that $P_A\chi \equiv \prod_{p\in B}(1-p^{-\beta})$ for all $A \supset B$, hence $P\chi$ is a constant. Suppose χ is non-trivial. The limit in (4) is an increasing function in β on $(0,+\infty)$ (because each factor is increasing), which is equal to $|L(\beta,\chi)|\zeta(\beta)^{-1}$ for $\beta > 1$, where $L(\beta,\chi) = \sum_{n=1}^{\infty} \chi(n)n^{-\beta}$ is the Dirichlet L-function corresponding to the number character χ [S]. By elementary properties of Dirichlet series, $|L(\beta,\chi)|$ tends to a finite value as $\beta \to 1+0$, while $\zeta(\beta) \to \infty$. Thus $P\chi = 0$.

■

Since the set of measures satisfying (1β) is convex and consists of ergodic measures, there exists at most one measure satisfying (1β). Such a measure does exist. In fact, for each $\beta \in (0,+\infty)$ there is a unique W-invariant measure μ_β satisfying (1β) [BC, L1]. Explicitly, $\mu_\beta = \prod_p \mu_{\beta,p}$, where $\mu_{\beta,p}$ is the measure on $\mathbb{Q}_p$ such that $\mu_{1,p}$ is the Haar measure ($\mu_{1,p}(\mathbb{Z}_p) = 1$), and

$$\frac{d\mu_{\beta,p}}{d\mu_{1,p}}(a) = \frac{1-p^{-\beta}}{1-p^{-1}}|a|_p^{\beta-1} \quad \text{for} \quad a \in \mathbb{Q}_p.$$

Let ϕ_β be the (β,σ_t)-KMS state on $\mathcal{C}_\mathbb{Q}$ corresponding to μ_β. Then $\pi_{\phi_\beta}(\mathcal{C}_\mathbb{Q})''$ is a factor, which is a reduction of the factor $L^\infty(\mathcal{A}_f,\mu_\beta) \rtimes \mathbb{Q}_+^*$. It is easy to describe its flow of weights [CT]. Consider a standard measure space X_β with the measure algebra consisting of $\mathbb{Q}_+^*$-invariant $(\lambda \times \mu_\beta)$-measurable subsets of $\mathbb{R}_+ \times \mathcal{A}_f$, where λ is the Lebesgue measure, and the flow F_t^β on it coming from the action $t(x,a) = (e^{-t/\beta}x, a)$ of $\mathbb{R}$ on $\mathbb{R}_+ \times \mathcal{A}_f$. Then F_t^β is ergodic by Proposition, and it is the flow of weights of the factors $L^\infty(\mathcal{A}_f,\mu_\beta) \rtimes \mathbb{Q}_+^*$ and $\pi_{\phi_\beta}(\mathcal{C}_\mathbb{Q})''$.

THEOREM.[BC] *For $\beta \in (0,1]$, μ_β is a unique measure satisfying (1β). The action of $\mathbb{Q}_+^*$ on $(\mathcal{A}_f,\mu_\beta)$ is ergodic, moreover, the action of $\mathbb{Q}^*$ on $(\mathcal{A},\nu_\beta)$, where $\mathcal{A} = \mathbb{R} \times \mathcal{A}_f$ is the space of full adeles and $\nu_\beta = \lambda \times \mu_\beta$, is ergodic. Equivalently, ϕ_β is a unique (β,σ_t)-KMS state on $\mathcal{C}_\mathbb{Q}$, and $\pi_{\phi_\beta}(\mathcal{C}_\mathbb{Q})''$ is the hyperfinite factor of type III_1.*

Proof. In view of the above description of the flow of weights, the factor $\pi_{\phi_\beta}(\mathcal{C}_\mathbb{Q})''$ is of type III_1 if and only if the action of $\mathbb{Q}_+^*$ on $(\mathbb{R}_+ \times \mathcal{A}_f, \lambda \times \mu_\beta)$ is ergodic, or equivalently, the action of $\mathbb{Q}^*$ on $(\mathcal{A},\nu_\beta)$ is ergodic.

To prove the ergodicity, first note that the action of W on X_β is ergodic. Indeed, the induced flow on X_β/W is the flow of weights of the factor

$$L^\infty(\mathcal{A}_f/W,\mu_\beta) \rtimes \mathbb{Q}_+^* = (L^\infty(\mathcal{A}_f,\mu_\beta) \rtimes \mathbb{Q}_+^*)^W.$$

It is easy to see ([BC]) that this factor is ITPFI with eigenvalue list $\{p^{-n\beta}(1-p^{-\beta}) \,|\, n \geq 0\}_{p\in\mathcal{P}}$. Hence it is of type III_1 by [B] (see also [BZ]). Thus its flow of weights is trivial, i.e. the action of W on X_β is ergodic. Since W is compact, the action is transitive. So we may identify X_β with W/W_β for some closed subgroup W_β of W. Then the flow F_t^β is given by a continuous one-parametric subgroup of W/W_β. Since W/W_β is totally disconnected, this one-parametric subgroup is trivial, and since the flow is ergodic, $W_\beta = W$. Thus X_β is singlepoint, and the action of $\mathbb{Q}_+^*$ on $(\mathbb{R}_+ \times \mathcal{A}_f, \lambda \times \mu_\beta)$ is ergodic.

■

REMARKS.
(i) In order to prove that ϕ_β is a unique KMS$_\beta$-state, it is enough to know that μ_β is ergodic.

Indeed, this means that ϕ_β is an extremal KMS$_\beta$-state. Since ϕ_β is a unique W-invariant KMS$_\beta$-state, we can argue as in the proof of [BC, Theorem 25]: If ψ is an extremal KMS$_\beta$-state, then $\int_W w_*\psi\, dw = \phi_\beta$. Since KMS$_\beta$-states form a simplex, we conclude that $\psi = \phi_\beta$.

Thus, the uniqueness and the assertion about the type for KMS$_\beta$-states follow easily from ergodicity of the action of $\mathbb{Q}^*$ on $(\mathcal{A}, \nu_\beta)$.

(ii) A slight modification of the argument in the proof of Theorem gives the following general result, apparently well-known to specialists: if M is a factor, G a compact totally disconnected group acting on M such that M^G is a factor of type III$_1$, then M is also of type III$_1$.

Acknowledgement. The author is grateful to Marcelo Laca, Sergey Sinelshchikov and Yoshimichi Ueda for helpful discussions.

References

[B] Blackadar B., *The regular representation of restricted direct product groups*, J. Func. Anal. **25** (1977), 267–274.

[BZ] Boca F.P., Zaharescu A., *Factors of type III and the distribution of prime numbers*, Proc. London Math. Soc. **80** (2000), 145–178.

[BC] Bost J.-B., Connes A., *Hecke algebras, type III factors and phase transitions with spontaneous symmetry breaking in number theory*, Selecta Math. (New Series) **1** (1995), 411–457.

[CT] Connes A., Takesaki M., *The flow of weights on factors of type III*, Tôhoku Math. J. **29** (1977), 473–575.

[L1] Laca M., *Semigroups of *-endomorphisms, Dirichlet series, and phase transitions*, J. Func. Anal. **152** (1998), 330–378.

[L2] Laca M., *From endomorphisms to automorphisms and back: dilations and full corners*, J. London Math. Soc. **61** (2000), 893–904.

[LR] Laca M., Raeburn I., *A semigroup crossed product arising in number theory*, J. London Math. Soc. **59** (1999), 330–344.

[S] Serre J.-P. Cours d'arithmétique. P.U.F. Paris, 1970.

Institute for Low Temperature Physics & Engineering
47 Lenin ave
Kharkov 310164, Ukraine
neshveyev@ilt.kharkov.ua, neshveyev@hotmail.com