

On the K-property of quantized Arnold cat maps

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Corrected version

Abstract

We prove that some quantized Arnold cat maps are entropic K-systems. This result was formulated by H. Narnhofer [1], but the fact that the optimal decomposition for the multi-channel entropy constructed there is not strictly local was not appropriately taken care of. We propose a strictly local decomposition based on a construction of Voiculescu.

I Introduction

The concept of K-system is very important in ergodic theory. Narnhofer and Thirring [2] introduced a non-commutative analogue of this notion. In [3] V.Ya. Golodets and the author proved the following sufficient condition for the K-property: a W^* -system (M, ϕ, α) is an entropic K-system if there exists a W^* -subalgebra M_0 of M such that $M_0 \subset \alpha(M_0)$, $\bigcap_{n \in \mathbb{Z}} \alpha^n(M_0) = \mathbb{C}1$, $\bigcup_{n \in \mathbb{N}} (\alpha^{-n}(M_0)' \cap \alpha^n(M_0))$ is weakly dense in M . This condition and the observation that a subsystem of a K-system invariant under the modular group is a K-system too allow to construct a large class of quantum K-systems (see, in particular, [3, 4, 5]). We know only one class of quantum systems for which the K-property is obtained by different arguments. This is quantized Arnold cat maps. This result was formulated in Narnhofer's paper [1]. The decompositions constructed in the course of the proof there are not strictly local, that leads to a factor that again could only be controlled by using asymptotic abelian arguments. So the essential interest lies in the construction of a completely positive map that is strictly local and can be well controlled and generalized in a larger context.

II The K-property of quantized cat maps

Let G be a discrete abelian group, $\omega: G \times G \rightarrow \mathbb{T}$ a bicharacter. Consider the twisted group C^* -algebra $C^*(G, \omega)$ generated by unitaries u_g , $g \in G$, such that

$$u_g u_h = \omega(g, h) u_{g+h}.$$

The canonical trace τ on $C^*(G, \omega)$ is given by $\tau(u_g) = 0$ for $g \neq 0$. It is known that the uniqueness of the trace is equivalent to the simplicity of $C^*(G, \omega)$, and is also equivalent to the non-degeneracy of the pairing $(g, h) \mapsto \omega(g, h) \bar{\omega}(h, g)$. In particular, if G is countable and the pairing is non-degenerate, then $\pi_\tau(C^*(G, \omega))''$ is the hyperfinite II_1 -factor. Each ω -preserving automorphism T of G defines an automorphism α_T of $C^*(G, \omega)$, $\alpha_T(u_g) = u_{Tg}$.

The non-commutative torus A_θ ($\theta \in [0, 1)$) is the algebra $C^*(\mathbb{Z}^2, \omega_\theta)$, where

$$\omega_\theta(g, h) = e^{i\pi\theta\sigma(g, h)}, \quad \sigma(g, h) = g_1h_2 - g_2h_1.$$

The following theorem was formulated in [1].

Theorem 1. *Let $T \in \mathrm{SL}_2(\mathbb{Z})$, $\mathrm{Spec} T = \{\lambda, \lambda^{-1}\}$. Suppose $|\lambda| > 1$ (so that λ is real) and $\theta \in [0, 1) \cap (2\mathbb{Z}\lambda^2 + 2\mathbb{Z})$. Then $(\pi_\tau(A_\theta)'' , \tau, \alpha_T)$ is an entropic K -system.*

We will prove the following more general result.

Theorem 2. *Let T be an aperiodic ω -preserving automorphism of G . Suppose that*

$$\sum_{n \in \mathbb{Z}} |1 - \omega(g, T^n h)| < \infty \quad \forall g, h \in G.$$

Then $(\pi_\tau(C^(G, \omega))'' , \tau, \alpha_T)$ is an entropic K -system.*

It was proved in [1, Theorem 3.8] that under the assumptions of Theorem 1, for any $g, h \in \mathbb{Z}^2$, we have

$$|1 - \omega(g, T^n h)| \leq C|\lambda|^{-|n|},$$

so Theorem 1 really follows from Theorem 2. The key observation for that estimate was the equality

$$T^n h = \frac{1}{\lambda^2 - 1} \sum_{i=0}^2 (\lambda^{n+i} + \lambda^{-n-i}) \bar{h}_i + \lambda^{-n} \bar{h} \quad (n \in \mathbb{N}),$$

where $\bar{h}_i \in \mathbb{Z}^2$ and $\bar{h} \in \mathbb{R}^2$ depend only on h and T , which is obtained by computations in a basis diagonalizing T . Since $\sigma(g, \bar{h}_i)$, $\sigma(g, h)$, $\lambda^{n+i} + \lambda^{-n-i} = \mathrm{Tr} T^{n+i}$ are all integers and $\theta \equiv 2s(\lambda^2 - 1) \pmod{2\mathbb{Z}}$ for some $s \in \mathbb{Z}$, we have

$$\theta\sigma(g, T^n h) \equiv \lambda^{-n} 2s(\lambda^2 - 1)\sigma(g, \bar{h}) \pmod{2\mathbb{Z}},$$

whence $|1 - \omega(g, T^n h)| \leq |\lambda|^{-n} 2\pi|s|(\lambda^2 - 1)|\sigma(g, \bar{h})|$.

Starting the proof of Theorem 2, consider a unital completely positive mapping $\gamma: A \rightarrow \pi_\tau(C^*(G, \omega))''$ of a finite-dimensional C^* -algebra A . By definition [2], we have to prove that

$$\lim_{n \rightarrow \infty} \lim_{k \rightarrow \infty} \frac{1}{k} H_\tau(\gamma, \alpha_T^n \circ \gamma, \dots, \alpha_T^{n(k-1)} \circ \gamma) = H_\tau(\gamma).$$

For a finite set X , we denote by $\mathrm{Mat}(X)$ the C^* -algebra of linear operators on $l^2(X)$. Let $\{e_{xy}\}_{x, y \in X}$ be the canonical system of matrix units in $\mathrm{Mat}(X)$. For $X \subset G$, we define a unital completely positive mapping $i_X: \mathrm{Mat}(X) \rightarrow C^*(G, \omega)$ by

$$i_X(e_{xy}) = \frac{1}{|X|} u_x u_y^* = \frac{\bar{\omega}(x - y, y)}{|X|} u_{x-y}.$$

As follows from [6] (see Lemmas 5.1 and 6.1 there), there exist a net $\{X_i\}_i$ of finite subsets in G and, for each i , a unital completely positive mapping $j_{X_i}: C^*(G, \omega) \rightarrow \mathrm{Mat}(X_i)$ such that $\|(i_{X_i} \circ j_{X_i})(a) - a\|_i \rightarrow 0 \quad \forall a \in C^*(G, \omega)$. From this we may conclude that any partition of unit in $\pi_\tau(C^*(G, \omega))''$ can be approximated in strong operator topology by a partition of the form $\{i_X(a_k)\}_k$, where $\{a_k\}_k$ is a partition of unit in $\mathrm{Mat}(X)$. Hence, for any $\varepsilon > 0$, there exist

a finite subset $X \subset G$ and a finite partition of unit $1 = \sum_{i \in I} a_i$ in $\text{Mat}(X)$ such that, for $b_i = i_X(a_i)$, we have

$$H_\tau(\gamma) < \varepsilon + \sum_i \eta \tau(b_i) + \sum_i S(\tau(\gamma(\cdot)), \tau(\gamma(\cdot)b_i)),$$

where $\eta x = -x \log x$. Set $X_{nk} = \sum_{l=1}^k T^{n(l-1)}(X)$.

The following lemma was proved in [1] for $G = \mathbb{Z}^2$.

Lemma. *Let G be a discrete abelian group, T an aperiodic endomorphism of G , $\text{Ker } T = 0$, Y a finite subset of G , $0 \in Y$. Then there exists $n_0 \in \mathbb{N}$ such that if*

$$\sum_{l=1}^k T^{n(l-1)} y_l = 0 \tag{1}$$

for some $y_1, \dots, y_k \in Y$, $n \geq n_0$, $k \in \mathbb{N}$, then $y_1 = \dots = y_k = 0$.

Proof. First consider the case where G is finitely generated. Then the periodic part of G is finite. Since T acts on it aperiodically, it is trivial, so $G \cong \mathbb{Z}^n$ for some $n \in \mathbb{N}$. Then T is defined by a non-degenerate matrix with integral entries, which we denote by the same letter T . It is known that the aperiodicity is equivalent to the fact that T has no roots of unity as eigenvalues. Let $\lambda_1, \dots, \lambda_m$ be all eigenvalues of T not lying on the unit circle, $V_i \subset \mathbb{C}^n$ the root space corresponding to λ_i and P_i the spectral projection onto V_i . Let $P = P_1 + \dots + P_m$, $V_0 = \text{Ker } P$. First note that $\mathbb{Z}^n \cap V_0 = 0$. Indeed, if \tilde{V}_0 is the linear span of $\mathbb{Z}^n \cap V_0 \cong \mathbb{Z}^l$, then $T|_{\tilde{V}_0}$ is represented by an l by l matrix with integral coefficients. Since $\tilde{V}_0 \subset V_0$, all eigenvalues of $T|_{\tilde{V}_0}$ are on the unit circle. But then they are roots of unity by the Dirichlet theorem. Hence $\tilde{V}_0 = 0$. Then (1) is equivalent to the system of equalities

$$\sum_{l=1}^k T^{n(l-1)} P_i y_l = 0, \tag{2}$$

$i = 1, \dots, m$. Fix i . Suppose, for definiteness, that $|\lambda_i| < 1$, and choose δ , $0 < \delta < 1 - |\lambda_i|$. Since $T|_{V_i}$ is a sum of Jordan cells, there exists a constant C such that

$$\|T^n|_{V_i}\| \leq C(|\lambda_i| + \delta)^n \quad \forall n \in \mathbb{N}.$$

There exists also a constant $M > 0$ such that, for $y \in Y$, we have either $P_i y = 0$ or $M^{-1} \leq \|P_i y\| \leq M$. Finally, choose $n_i \in \mathbb{N}$ such that

$$\sum_{n=n_i}^{\infty} MC(|\lambda_i| + \delta)^n < M^{-1}.$$

Then if equality (2) holds with $n \geq n_i$, then $P_i y_1 = 0$. Since $\text{Ker } T = 0$, we can rewrite (2) as $\sum_{l=1}^{k-1} T^{n(l-1)} P_i y_{l+1} = 0$. Thus we sequentially obtain $P_i y_1 = \dots = P_i y_k = 0$. So we may take $n_0 = \max_i n_i$.

We prove the general case by induction on $|Y|$ using the same method as in [7] to reduce the proof to the case considered above.

Let H_0 be the group generated by Y, TY, T^2Y, \dots . Set $H_n = T^n H_0$, $H_\infty = \bigcap_n H_n$, $Y' = Y \cap H_\infty$. Suppose $Y' \neq Y$. There exists $n_1 \in \mathbb{N}$ such that $Y' = Y \cap H_{n_1}$. If equality

(1) holds with $n \geq n_1$, then $y_1 \in H_{n_1} \cap Y = Y' \subset H_\infty$. Then $\sum_{l=2}^k T^{n(l-1)} y_l \in H_\infty$. Since $\text{Ker } T = 0$ and $TH_\infty = H_\infty$, we conclude that $\sum_{l=1}^{k-1} T^{n(l-1)} y_{l+1} \in H_\infty$. Thus we sequentially obtain that $y_1, \dots, y_k \in Y'$. Since $|Y'| < |Y|$, we may apply the inductive assumption.

If $Y' = Y$, then $Y \subset H_1$, hence there exists $n \in \mathbb{N}$ such that if \bar{H} is the group generated by $Y, TY, \dots, T^n Y$, then $Y \subset T\bar{H}$. Then \bar{H} is a finitely generated group, T^{-1} an aperiodic endomorphism of \bar{H} . For this case Lemma is already proved. ■

Applying Lemma to the set $Y = X - X$ we see that the mapping

$$X^k \rightarrow X_{nk}, \quad (x_1, \dots, x_k) \mapsto \sum_{l=1}^k T^{n(l-1)} x_l,$$

is a bijection for all $k \in \mathbb{N}$ and for all $n \in \mathbb{N}$ sufficiently large. This bijection induces an isomorphism of $\text{Mat}(X^k)$ onto $\text{Mat}(X_{nk})$. Composing it with $i_{X_{nk}}: \text{Mat}(X_{nk}) \rightarrow C^*(G, \omega)$ and identifying $\text{Mat}(X^k)$ with $\text{Mat}(X)^{\otimes k}$ we obtain a unital completely positive mapping

$$\sigma_{nk}: \text{Mat}(X)^{\otimes k} \rightarrow C^*(G, \omega).$$

Set $b(n, k)_{i_1 \dots i_k} = \sigma_{nk}(a_{i_1} \otimes \dots \otimes a_{i_k})$. By definition [8], we obtain

$$\begin{aligned} & \frac{1}{k} H_\tau(\gamma, \alpha_T^n \circ \gamma, \dots, \alpha_T^{n(k-1)} \circ \gamma) \geq \\ & \geq \frac{1}{k} \sum_{i_1, \dots, i_k} \eta \tau(b(n, k)_{i_1 \dots i_k}) + \frac{1}{k} \sum_{l=1}^k \sum_{i_l} S\left(\tau(\gamma(\cdot)), \tau(\gamma(\cdot) \alpha_T^{-n(l-1)}(b(n, k)_{i_l}^{(l)}))\right), \end{aligned}$$

where $b(n, k)_{i_l}^{(l)} = \sum_{i_1, \dots, \hat{i}_l, \dots, i_k} b(n, k)_{i_1 \dots i_k}$.

If we denote by τ_Y the unique tracial state on $\text{Mat}(Y)$, then $\tau_Y = \tau \circ i_Y$, so that $\tau \circ \sigma_{nk} = \tau_X^{\otimes k}$, whence

$$\tau(b(n, k)_{i_1 \dots i_k}) = \prod_{l=1}^k \tau_X(a_{i_l}) = \prod_{l=1}^k \tau(b_{i_l}).$$

So the first term in the inequality above is equal to $\sum_i \eta \tau(b_i)$, and in order to prove Theorem it remains to show that

$$\|\alpha_T^{-n(l-1)}(b(n, k)_{i_l}^{(l)}) - b_{i_l}\| \xrightarrow{n \rightarrow \infty} 0$$

uniformly on $k, l \in \mathbb{N}$ ($l \leq k$) and $i_l \in I$. Let θ_l be the embedding of $\text{Mat}(X)$ into $\text{Mat}(X)^{\otimes k}$ defined by

$$\theta_l(a) = \underbrace{1 \otimes \dots \otimes 1}_{l-1} \otimes a \otimes \underbrace{1 \otimes \dots \otimes 1}_{k-l}.$$

Then $b(n, k)_{i_l}^{(l)} = (\sigma_{nk} \circ \theta_l)(a_{i_l})$. Thus we just have to estimate

$$\|\alpha_T^{-n(l-1)} \circ \sigma_{nk} \circ \theta_l - i_X\|.$$

Using the facts that ω is bilinear and T -invariant we obtain

$$\begin{aligned}
\sigma_{nk}(e_{x_1x_1} \otimes \dots \otimes e_{x_{l-1}x_{l-1}} \otimes e_{xy} \otimes e_{x_{l+1}x_{l+1}} \otimes \dots \otimes e_{x_kx_k}) &= \\
= \frac{1}{|X|^k} \bar{\omega} \left(T^{n(l-1)}(x-y), T^{n(l-1)}y + \sum_{i=1, i \neq l}^k T^{n(i-1)}x_i \right) u_{T^{n(l-1)}(x-y)} & \\
= \left(\prod_{i=1, i \neq l}^k \frac{\bar{\omega}(x-y, T^{n(i-1)}x_i)}{|X|} \right) \frac{\bar{\omega}(x-y, y)}{|X|} u_{T^{n(l-1)}(x-y)}, &
\end{aligned}$$

so that

$$\begin{aligned}
\|(\alpha_T^{-n(l-1)} \circ \sigma_{nk} \circ \theta_l - i_X)(e_{xy})\| &= \\
= \frac{1}{|X|} \left| \sum_{x_1, \dots, \hat{x}_l, \dots, x_k} \left(\prod_{i=1, i \neq l}^k \frac{\bar{\omega}(x-y, T^{n(i-1)}x_i)}{|X|} \right) - 1 \right| & \\
= \frac{1}{|X|} \left| \prod_{i=1, i \neq l}^k \left(\frac{1}{|X|} \sum_{z \in X} \bar{\omega}(x-y, T^{n(i-1)}z) \right) - 1 \right|. &
\end{aligned}$$

We must show that the latter expression tends to zero as $n \rightarrow \infty$ uniformly on $k, l \in \mathbb{N}$ ($l \leq k$). This follows from

$$\sum_{n \in \mathbb{Z}} \left| 1 - \frac{1}{|X|} \sum_{z \in X} \omega(x-y, T^n z) \right| < \infty.$$

So the proof of Theorem 2 is complete.

III Classical case

If $\omega \equiv 1$, then $C^*(G, \omega) = C(\hat{G})$, the algebra of continuous functions on the dual group \hat{G} . It is known that an automorphism T of G is aperiodic iff the dual automorphism of \hat{G} is ergodic. Thus we obtain a classical Rohlin's result [7] stating that ergodic automorphisms of compact abelian groups have completely positive entropy. Note that in this case we have

$$b(n, k)_{i_1 \dots i_k} = b_{i_1} \alpha_T^n(b_{i_2}) \dots \alpha_T^{n(k-1)}(b_{i_k}),$$

so what is really necessary for the proof is Lemma above and the possibility of approximating in mean measurable partitions of unit by partitions consisting of trigonometric polynomials, which can be proved by elementary methods without appealing to Voiculescu's completely positive mappings.

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