AUTOEQUIVALENCES OF THE TENSOR CATEGORY OF $U_q\mathfrak{g}$-MODULES

SERGEY NESHVEYEV AND LARS TUSET

Abstract. We prove that for $q \in \mathbb{C}^*$ not a nontrivial root of unity the cohomology group defined by invariant 2-cocycles in a completion of $U_q\mathfrak{g}$ is isomorphic to $H^2(P/Q;\mathbb{T})$, where P and Q are the weight and root lattices of $\mathfrak{g}$. This implies that the group of autoequivalences of the tensor category of $U_q\mathfrak{g}$-modules is the semidirect product of $H^2(P/Q;\mathbb{T})$ and the automorphism group of the based root datum of $\mathfrak{g}$. For $q = 1$ we also obtain similar results for all compact connected separable groups.

For a tensor category $\mathcal{C}$ a natural object to study is its group of symmetries, i.e., the group $\mathrm{Aut}^{\otimes}(\mathcal{C})$ of monoidal autoequivalences of $\mathcal{C}$ identified up to monoidal natural isomorphisms. A more refined version of this group is the tensor category of autoequivalences of $\mathcal{C}$. It is, for example, used to define what is meant by an action of a group on $\mathcal{C}$, which in turn leads to such constructions as equivariantization and crossed products, see e.g. [8] for applications. At the same time there are not many examples for which the group $\mathrm{Aut}^{\otimes}(\mathcal{C})$ is explicitly computed. The aim of this note is to calculate it for the category of representations of the q-deformation G_q of a simply connected semisimple compact Lie group G. Part of the information about the group of autoequivalences in this case is contained in the work of McMullen [3], who showed that that the group of automorphisms of the fusion ring of G is isomorphic to $\mathrm{Out}(G)$, that is, to the automorphism group of the based root datum of $\mathfrak{g}$. The remaining part is determined by the possible tensor structures one can have on the identity functor, and these are described by the cohomology group defined by invariant 2-cocycles on the dual $\hat{G}_q$ of the quantum group G_q. Another motivation for computing this cohomology group is the problem of classifying Drinfeld twists that do not necessarily respect braiding; the ones that do respect braiding have been classified in [5].

In a previous paper [7] we showed that if G is a compact connected group then the cohomology group defined by invariant unitary 2-cocycles on $\hat{G}$ is isomorphic to $H^2(\widehat{Z(G)};\mathbb{T})$ and we conjectured that for semisimple Lie groups a similar result holds for the q-deformation of G. We will prove that this is indeed the case using techniques from our earlier paper [5], where we considered symmetric cocycles and were inspired by the proof of Kazhdan and Lusztig of the equivalence of the Drinfeld category and the category of $U_q\mathfrak{g}$-modules [2]. For $q = 1$ this gives an alternative proof of the main results in [7, Section 2] and allows us, at least in the separable case, to extend those results to non-unitary cocycles relying neither on ergodic actions nor on reconstruction theorems. At the same time this proof is less transparent than that in [7] and, as opposed to [7], relies heavily on the structure and representation theory of compact Lie groups.

We will follow the notation and conventions in [5]. Let G be a simply connected semisimple compact Lie group, $\mathfrak{g}$ its complexified Lie algebra, $q \in \mathbb{C}^*$ not a nontrivial root of unity. Fix a Cartan subalgebra of $\mathfrak{g}$ and a system $\{\alpha_1, \dots, \alpha_r\}$ of simple roots. The weight and root lattices are denoted by P and Q, respectively. For weight $\lambda \in P$ denote by $\lambda(i)$ the coefficients of λ in the basis consisting of fundamental weights. Take the ad-invariant symmetric form on $\mathfrak{g}$ such that $(\alpha, \alpha) = 2$ for every short root in every simple component of $\mathfrak{g}$, and put $d_i = (\alpha_i, \alpha_i)/2$ and $q_i = q^{d_i}$.* For $q \neq 1$

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*Our main result, Theorem 1, is valid for any ad-invariant symmetric form on $\mathfrak{g}$ such that its restriction to the real Lie algebra of G is negative definite, under the assumption that either $q = 1$ (in which case the choice of a form does not matter) or that q_i is not a root of unity for all i.

consider the quantized universal enveloping algebra $U_q\mathfrak{g}$ with generators E_i, F_i and K_i, $1\le i\le r$, so that we in particular have

$$K_iE_jK_i^{-1}=q_i^{a_{ij}}E_j,\ \ K_iF_jK_i^{-1}=q_i^{-a_{ij}}F_j,\ \ E_iF_j-F_jE_i=\delta_{ij}(K_i-K_i^{-1})/(q_i-q_i^{-1}).$$

Recall that a $U_q\mathfrak{g}$-module V is called admissible if $V=\oplus_{\lambda\in P}V(\lambda)$, where $V(\lambda)$ consists of vectors $v\in V$ such that $K_iv=q_i^{\lambda(i)}v$ for all i. Denote by $\mathcal{C}_q(\mathfrak{g})$ the tensor category of admissible finite dimensional $U_q\mathfrak{g}$-modules. For $q=1$ denote by $\mathcal{C}(\mathfrak{g})=\mathcal{C}_1(\mathfrak{g})$ the usual tensor category of finite dimensional $U\mathfrak{g}$-modules. Let $\mathcal{U}(G_q)$ be the endomorphism ring of the forgetful functor $\mathcal{C}_q(\mathfrak{g})\to\mathcal{V}ec$. If for every dominant integral weight $\mu\in P_+$ we fix an irreducible $U_q\mathfrak{g}$-module V_μ with highest weight μ, then the ring $\mathcal{U}(G_q)$ can be identified with $\prod_{\mu\in P_+}\operatorname{End}(V_\mu)$. The comultiplication on $U_q\mathfrak{g}$ extends to a homomorphism $\hat\Delta_q\colon\mathcal{U}(G_q)\to\mathcal{U}(G_q\times G_q)=\prod_{\mu,\eta\in P_+}\operatorname{End}(V_\mu\otimes V_\eta)$.

An invertible element $\mathcal{E}\in\mathcal{U}(G_q\times G_q)$ is called a 2-cocycle on $\hat G_q$ if

$$(\mathcal{E}\otimes 1)(\hat\Delta_q\otimes\iota)(\mathcal{E})=(1\otimes\mathcal{E})(\iota\otimes\hat\Delta_q)(\mathcal{E}).$$

A cocycle is called invariant if it commutes with elements in the image of $\hat\Delta_q$. The set of invariant 2-cocycles forms a group under multiplication, which we denote by $Z^2_{G_q}(\hat G_q;\mathbb{C}^*)$. Cocycles of the form $(a\otimes a)\hat\Delta_q(a)^{-1}$, where a is an invertible element in the center of $\mathcal{U}(G_q)$, form a subgroup of the center of $Z^2_{G_q}(\hat G_q;\mathbb{C}^*)$. The quotient of $Z^2_{G_q}(\hat G_q;\mathbb{C}^*)$ by this subgroup is denoted by $H^2_{G_q}(\hat G_q;\mathbb{C}^*)$.

The center of $\mathcal{U}(G_q)=\prod_{\mu\in P_+}\operatorname{End}(V_\mu)$ is identified with the algebra of functions on the set P_+ of dominant integral weights. By [5, Proposition 4.5] a function on P_+ is a group-like element of $\mathcal{U}(G_q)$ if and only if it is defined by a character of P/Q. Therefore the Hopf algebra of functions on P/Q embeds into the center of $\mathcal{U}(G_q)$. Hence every 2-cocycle c on P/Q can be considered as an invariant 2-cocycle $\mathcal{E}_c$ on $\hat G_q$. Explicitly, $\mathcal{E}_c$ acts on $V_\mu\otimes V_\eta$ as multiplication by $c(\mu,\eta)$. We can now formulate our main result.

Theorem 1. *The homomorphism $c\mapsto\mathcal{E}_c$ induces an isomorphism*

$$H^2(P/Q;\mathbb{T})\cong H^2_{G_q}(\hat G_q;\mathbb{C}^*).$$

In particular, if $\mathfrak{g}$ is simple and $\mathfrak{g}\not\cong\mathfrak{so}_{4n}(\mathbb{C})$ then $H^2_{G_q}(\hat G_q;\mathbb{C}^*)$ is trivial, and if $\mathfrak{g}\cong\mathfrak{so}_{4n}(\mathbb{C})$ then $H^2_{G_q}(\hat G_q;\mathbb{C}^*)\cong\mathbb{Z}/2\mathbb{Z}$.

The last statement follows from the fact that for simple Lie algebras the group P/Q is cyclic unless $\mathfrak{g}\cong\mathfrak{so}_{4n}(\mathbb{C})$, in which case $P/Q\cong\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$, see e.g. Table IV on page 516 in [1].

Note that for $q>0$ the same result holds for unitary cocycles. This easily follows by polar decomposition, see [5, Lemma 1.1].

In the proof of the theorem we will assume that $q\ne 1$, the case $q=1$ is similar and for unitary cocycles is also proved by a different method in [7].

Our first goal will be to construct a homomorphism $H^2_{G_q}(\hat G_q;\mathbb{C}^*)\to H^2(P/Q;\mathbb{T})$. For every $\mu\in P_+$ fix a highest weight vector $\xi_\mu\in V_\mu$. Recall [5, Section 2] that for $\mu,\eta\in P_+$ there exists a unique morphism

$$T_{\mu,\eta}\colon V_{\mu+\eta}\to V_\mu\otimes V_\eta\ \text{ such that }\ \xi_{\mu+\eta}\mapsto\xi_\mu\otimes\xi_\eta.$$

The image of $T_{\mu,\eta}$ is the isotypic component of $V_\mu\otimes V_\eta$ with highest weight $\mu+\eta$. Hence if $\mathcal{E}$ is an invariant 2-cocycle then it acts on this image as multiplication by a nonzero scalar $c_\mathcal{E}(\mu,\eta)$. As in the proof of [5, Lemma 2.2], the identity $(T_{\mu,\eta}\otimes\iota)T_{\mu+\eta,\nu}=(\iota\otimes T_{\eta,\nu})T_{\mu,\eta+\nu}$ immediately implies that $c_\mathcal{E}$ is a 2-cocycle on P_+. Furthermore, the cohomology class $[c_\mathcal{E}]$ of $c_\mathcal{E}$ in $H^2(P_+;\mathbb{C}^*)$ depends only on the class of $\mathcal{E}$ in $H^2_{G_q}(\hat G_q;\mathbb{C}^*)$, since if $a\in\mathcal{U}(G_q)$ is a central element acting on V_μ as multiplication by a scalar $a(\mu)$ then the action of $(a\otimes a)\hat\Delta_q(a)^{-1}$ on the image of $T_{\mu,\eta}$ is multiplication by $a(\mu)a(\eta)a(\mu+\eta)^{-1}$. Thus the map $\mathcal{E}\mapsto c_\mathcal{E}$ defines a homomorphism $H^2_{G_q}(\hat G_q;\mathbb{C}^*)\to H^2(P_+;\mathbb{C}^*)$.

Given a cocycle on P/Q, we can consider it as a cocycle on P and then get a cocycle on P_+ by restriction. Thus we have a homomorphism $H^2(P/Q;\mathbb{T}) \to H^2(P_+;\mathbb{C}^*)$. It is injective since the quotient map $P_+ \to P/Q$ is surjective and a cocycle on P/Q is a coboundary if it is symmetric.

Lemma 2. *For every invariant 2-cocycle $\mathcal{E}$ on $\hat{G}_q$ the class of $c_\mathcal{E}$ in $H^2(P_+;\mathbb{C}^*)$ is contained in the image of $H^2(P/Q;\mathbb{T})$.*

Proof. Consider the skew-symmetric bi-quasicharacter $b\colon P_+ \times P_+ \to \mathbb{C}^*$ defined by

$$b(\mu,\eta) = c_\mathcal{E}(\mu,\eta)c_\mathcal{E}(\eta,\mu)^{-1}.$$

It extends uniquely to a skew-symmetric bi-quasicharacter on P. To prove the lemma it suffices to show that the root lattice Q is contained in the kernel of this extension. Indeed, since $H^2(P/Q;\mathbb{T})$ is isomorphic to the group of skew-symmetric bi-characters on P/Q, it then follows that there exists a cocycle c on P/Q such that the cocycle $c_\mathcal{E}c^{-1}$ on P_+ is symmetric. Then by [6, Lemma 4.2] the cocycle $c_\mathcal{E}c^{-1}$ is a coboundary, so $c_\mathcal{E}$ and the restriction of c to P_+ are cohomologous.

To prove that Q is contained in the kernel of b, recall [5, Section 2] that for every simple root α_i and weights $\mu,\eta \in P_+$ with $\mu(i),\eta(i) \ge 1$ we can define a morphism

$$\tau_{i;\mu,\eta}\colon V_{\mu+\eta-\alpha_i} \to V_\mu \otimes V_\eta \ \text{ such that }\ \xi_{\mu+\eta-\alpha_i} \mapsto [\mu(i)]_{q_i}\xi_\mu \otimes F_i\xi_\eta - q_i^{\mu(i)}[\eta(i)]_{q_i}F_i\xi_\mu \otimes \xi_\eta.$$

The image of $\tau_{i;\mu,\eta}$ is the isotypic component of $V_\mu \otimes V_\eta$ with highest weight $\mu+\eta-\alpha_i$. Since the element $\mathcal{E}$ is invariant, it acts on this image as multiplication by a nonzero scalar $c_i(\mu,\eta)$. As in the proof of [5, Lemma 2.3], consider now another weight ν with $\nu(i) \ge 1$. The isotypic component of $V_\mu \otimes V_\eta \otimes V_\nu$ with highest weight $\mu+\eta+\nu-\alpha_i$ has multiplicity two, and is spanned by the images of $(\iota \otimes T_{\eta,\nu})\tau_{i;\mu,\eta+\nu}$ and $(\iota \otimes \tau_{i;\eta,\nu})T_{\mu,\eta+\nu-\alpha_i}$, as well as by the images of $(T_{\mu,\eta}\otimes\iota)\tau_{i;\mu+\eta,\nu}$ and $(\tau_{i;\mu,\eta}\otimes\iota)T_{\mu+\eta-\alpha_i,\nu}$. We have

$$[\eta(i)]_{q_i}(T_{\mu,\eta}\otimes\iota)\tau_{i;\mu+\eta,\nu} - [\nu(i)]_{q_i}(\tau_{i;\mu,\eta}\otimes\iota)T_{\mu+\eta-\alpha_i,\nu} = [\mu(i)+\eta(i)]_{q_i}(\iota\otimes\tau_{i;\eta,\nu})T_{\mu,\eta+\nu-\alpha_i}. \tag{1}$$

Apply the element

$$\Omega := (\mathcal{E}\otimes 1)(\hat{\Delta}_q \otimes \iota)(\mathcal{E}) = (1\otimes\mathcal{E})(\iota\otimes\hat{\Delta}_q)(\mathcal{E})$$

to this identity. The morphisms $(T_{\mu,\eta}\otimes\iota)\tau_{i;\mu+\eta,\nu}$, $(\tau_{i;\mu,\eta}\otimes\iota)T_{\mu+\eta-\alpha_i,\nu}$ and $(\iota\otimes\tau_{i;\eta,\nu})T_{\mu,\eta+\nu-\alpha_i}$ are eigenvectors of the operator of multiplication by Ω on the left with eigenvalues $c_\mathcal{E}(\mu,\eta)c_i(\mu+\eta,\nu)$, $c_i(\mu,\eta)c_\mathcal{E}(\mu+\eta-\alpha_i,\nu)$ and $c_i(\eta,\nu)c_\mathcal{E}(\mu,\eta+\nu-\alpha_i)$, respectively. Since the morphisms $(T_{\mu,\eta}\otimes\iota)\tau_{i;\mu+\eta,\nu}$ and $(\tau_{i;\mu,\eta}\otimes\iota)T_{\mu+\eta-\alpha_i,\nu}$ are linearly independent, by applying Ω to (1) we conclude that these three eigenvalues coincide. In particular,

$$c_i(\mu,\eta)c_\mathcal{E}(\mu+\eta-\alpha_i,\nu) = c_i(\eta,\nu)c_\mathcal{E}(\mu,\eta+\nu-\alpha_i).$$

Applying this to $\eta=\nu=\mu$ we get

$$b(2\mu-\alpha_i,\mu)=1.$$

Since b is skew-symmetric, this gives $b(\alpha_i,\mu)=1$. The latter identity holds for all $\mu\in P_+$ with $\mu(i)\ge 1$. Since every element in P can be written as a difference of two such elements μ, it follows that α_i is contained in the kernel of b. □

Therefore the map $\mathcal{E}\mapsto c_\mathcal{E}$ induces a homomorphism $H^2_{G_q}(\hat{G}_q;\mathbb{C}^*) \to H^2(P/Q;\mathbb{T})$. Clearly, it is a left inverse of the homomorphism $H^2(P/Q;\mathbb{T}) \to H^2_{G_q}(\hat{G}_q;\mathbb{C}^*)$, $[c]\mapsto[\mathcal{E}_c]$, constructed earlier. Thus it remains to prove that the homomorphism $H^2_{G_q}(\hat{G}_q;\mathbb{C}^*) \to H^2(P/Q;\mathbb{T})$ is injective.

Assume that $\mathcal{E}$ is an invariant 2-cocycle such that the cocycle $c_\mathcal{E}$ on P_+ is a coboundary. Our goal is to show that $\mathcal{E}$ is the coboundary of a central element in $\mathcal{U}(G_q)$. We will follow the strategy in [5], where this was shown under the additional assumption that $\mathcal{E}$ is symmetric, that is, $\mathcal{R}_\hbar\mathcal{E} = \mathcal{E}_{21}\mathcal{R}_\hbar$ for an R-matrix $\mathcal{R}_\hbar \in \mathcal{U}(G_q\times G_q)$, which depends on the choice of a number $\hbar\in\mathbb{C}$ such that $q=e^{\pi i\hbar}$.

The first step in [5], see the discussion following Lemma 2.2 in [5], was to show that $\mathcal{E}$ is cohomologous to a cocycle such that

$$\mathcal{E}T_{\mu,\eta} = T_{\mu,\eta} \ \text{ and }\ \mathcal{E}\tau_{i;\nu,\omega} = \tau_{i;\nu,\omega} \tag{2}$$

for all $\mu,\eta\in P_+$, $1\le i\le r$ and $\nu,\omega\in P_+$ such that $\nu(i),\omega(i)\ge 1$. This part goes through in the non-symmetic case without any changes, as the symmetry requirement was needed only to conclude that $c_\mathcal{E}$ is a coboundary.

Therefore to prove the injectivity of $H^2_{G_q}(\hat G_q;\mathbb{C}^*)\to H^2(P/Q;\mathbb{T})$ it suffices to establish the following result, which extends [5, Corollary 4.4].

Proposition 3. *If $\mathcal{E}$ is an invariant 2-cocycle on $\hat G_q$ with property* (2) *then $\mathcal{E}=1$.*

The proof of this statement in [5] for symmetric cocycles is based on considering the action of $\mathcal{E}$ on a comonoid representing the forgetful functor on $\mathcal{C}_q(\mathfrak{g})$. Recall briefly how this comonoid, essentially constructed by Kazhdan and Lusztig, is defined. For every weight $\mu\in P_+$ fix an irreducible $U_q\mathfrak{g}$-module $\bar V_\mu$ with lowest weight $-\mu$ and a lowest weight vector $\bar\xi_\mu$. For $\lambda\in P$ and $\mu,\eta\in P_+$ such that $\lambda+\mu\in P_+$, there exists a unique morphism

$$\mathrm{tr}^\eta_{\mu,\lambda+\mu}\colon \bar V_{\mu+\eta}\otimes V_{\lambda+\mu+\eta}\to \bar V_\mu\otimes V_{\lambda+\mu}\ \text{ such that }\ \bar\xi_{\mu+\eta}\otimes\xi_{\lambda+\mu+\eta}\mapsto\bar\xi_\mu\otimes\xi_{\lambda+\mu}.$$

Using these morphisms define an inverse limit $U_q\mathfrak{g}$-module

$$M_\lambda=\varprojlim_\mu \bar V_\mu\otimes V_{\lambda+\mu}.$$

Denote by $\mathrm{tr}_{\mu,\lambda+\mu}$ the canonical map $M_\lambda\to\bar V_\mu\otimes V_{\lambda+\mu}$. The module M_λ is considered as a topological $U_q\mathfrak{g}$-module with a base of neighborhoods of zero formed by the kernels of the maps $\mathrm{tr}_{\mu,\lambda+\mu}$, while all modules in our category $\mathcal{C}_q(\mathfrak{g})$ are considered with discrete topology. Then $\mathrm{Hom}_{U_q\mathfrak{g}}(M_\lambda,V)$ is the inductive limit of the spaces $\mathrm{Hom}_{U_q\mathfrak{g}}(\bar V_\mu\otimes V_{\lambda+\mu},V)$. The vectors $\bar\xi_\mu\otimes\xi_{\lambda+\mu}$ define a topologically cyclic vector $\Omega_\lambda\in M_\lambda$. For any finite dimensional admissible $U_q\mathfrak{g}$-module V the map

$$\eta_V\colon\ \mathrm{Hom}_{U_q\mathfrak{g}}(\oplus_\lambda M_\lambda,V)\to V,\quad \eta_V(f)=\sum_\lambda f(\Omega_\lambda),$$

is an isomorphism, so the topological $U_q\mathfrak{g}$-module $M=\oplus_\lambda M_\lambda$ represents the forgetful functor. Furthermore, the representation of $U_q\mathfrak{g}$ in the endomorphism ring of the forgetful functor is implemented by the antihomomorphism $\pi\colon U_q\mathfrak{g}\to\mathrm{End}_{U_q\mathfrak{g}}(M)$ defined by $\pi(E_i)\Omega_\lambda=E_i\Omega_{\lambda-\alpha_i}$, $\pi(F_i)\Omega_i=F_i\Omega_{\lambda+\alpha_i}$ and $\pi(K_i)\Omega_\lambda=q_i^{\lambda(i)}\Omega_\lambda$. In other words, M is a $U_q\mathfrak{g}$-bimodule.

It was shown in [5, Section 4], see the arguments up to (but not including) Lemma 4.3 there, that condition (2) is exactly what is needed to define an action of any invariant cocycle $\mathcal{E}$ satisfying (2) on the $U_q\mathfrak{g}$-bimodule M. More precisely, we showed that there exist a character χ of P/Q, an invertible morphism $\mathcal{E}_0$ of $M=\oplus_\lambda M_\lambda$ onto itself preserving the direct sum decomposition, and an invertible element c in the center of $\mathcal{U}(G_q)$ such that

$$\mathrm{tr}_{\mu,\lambda+\mu}\mathcal{E}_0=\chi(\mu)^{-1}\mathcal{E}\,\mathrm{tr}_{\mu,\lambda+\mu}\quad\text{and}\quad \eta_V(f\mathcal{E}_0)=c\,\eta_V(f) \tag{3}$$

for all $\mu\in P_+$, $\lambda\in P$ such that $\lambda+\mu\in P_+$, and for all finite dimensional admissible $U_q\mathfrak{g}$-modules V and $f\in\mathrm{Hom}_{U_q\mathfrak{g}}(M_\lambda,V)$. We will show now that this is already enough to conclude that $\mathcal{E}$ is, in fact, symmetric.

Proof of Proposition 3. We want to show that $\mathcal{R}_\hbar\mathcal{E}=\mathcal{E}_{21}\mathcal{R}_\hbar$ for some $\hbar$ such that $q=e^{\pi i\hbar}$. We will prove a stronger statement: $\sigma\mathcal{E}=\mathcal{E}\sigma$ for any braiding σ on $\mathcal{C}_q(\mathfrak{g})$.

By (3), since $\mathrm{tr}_{\mu,\lambda+\mu}(\Omega_\lambda)=\bar\xi_\mu\otimes\xi_{\lambda+\mu}$, for any $\mu,\eta,\nu\in P_+$ and $f\in\mathrm{Hom}_{U_q\mathfrak{g}}(\bar V_\mu\otimes V_\eta,V_\nu)$ we have

$$\chi(\mu)^{-1}f\mathcal{E}(\bar\xi_\mu\otimes\xi_\eta)=c(\nu)f(\bar\xi_\mu\otimes\xi_\eta).$$

As the vector $\bar\xi_\mu\otimes\xi_\eta$ is cyclic, this means that $f\mathcal{E}=\chi(\mu)c(\nu)f$. Since this is true for all f, we conclude that $\mathcal{E}$ acts on the isotypic component of $\bar V_\mu\otimes V_\eta$ with highest weight ν as multiplication by $\chi(\mu)c(\nu)$. In other words, $\mathcal{E}$ acts on the isotypic component of $V_\mu\otimes V_\eta$ with highest weight ν as multiplication by $\chi(\bar\mu)c(\nu)$. It follows that

$$\sigma\mathcal{E}=\chi(\bar\mu-\bar\eta)\mathcal{E}\sigma\ \text{ on }\ V_\mu\otimes V_\eta.$$

But by assumption (2) the element $\mathcal{E}$ is the identity on the isotypic component of $V_\mu \otimes V_\eta$ with highest weight $\mu+\eta$, so by considering the above identity on this isotypic component we conclude that $\chi(\bar\mu-\bar\eta)=1$. Thus χ is the trivial character and $\sigma\mathcal{E}=\mathcal{E}\sigma$. By [5, Corollary 4.4] we then get that $\mathcal{E}=1$. This completes the proof of Proposition 3 and hence of Theorem 1. □

As our first application we will classify Drinfeld twists, relating the coproducts on $U_q\mathfrak{g}$ and $U\mathfrak{g}$, that do not necessarily respect braiding.

Corollary 4. *Let $\varphi\colon \mathcal{U}(G_q)\to\mathcal{U}(G)$ be an isomorphism extending the canonical identifications of the centers of these algebras with the algebra of functions on P_+, and let $\hbar$ be such that $q=e^{\pi i\hbar}$. Suppose $\mathcal{F}\in\mathcal{U}(G\times G)$ is an invertible element such that*

(i) $(\varphi\otimes\varphi)\hat\Delta_q=\mathcal{F}\hat\Delta\varphi(\cdot)\mathcal{F}^{-1}$;

(ii) *the element $(\iota\otimes\hat\Delta)(\mathcal{F}^{-1})(1\otimes\mathcal{F}^{-1})(\mathcal{F}\otimes 1)(\hat\Delta\otimes\iota)(\mathcal{F})$ coincides with Drinfeld's KZ-associator $\Phi_{KZ}(\hbar t_{12},\hbar t_{23})$, where $t\in\mathfrak{g}\otimes\mathfrak{g}$ is the element defined by our fixed* ad*-invariant form on $\mathfrak{g}$.*

Assume $\mathcal{F}'\in\mathcal{U}(G\times G)$ is another element with the same properties. Then there exist a $\mathbb{T}$-valued 2-cocycle c on P/Q and an invertible central element $a\in\mathcal{U}(G)$ such that $\mathcal{F}'=\mathcal{E}_c\mathcal{F}(a\otimes a)\hat\Delta(a)^{-1}$.

Proof. The proof is similar to that of [5, Theorem 5.2]. Define $\mathcal{E}=(\varphi^{-1}\otimes\varphi^{-1})(\mathcal{F}'\mathcal{F}^{-1})\in\mathcal{U}(G_q\times G_q)$. It is easy to check that $\mathcal{E}$ is an invariant 2-cocycle on $\hat G_q$. By Theorem 1, $\mathcal{E}=\mathcal{E}_c(b\otimes b)\hat\Delta_q(b)^{-1}$ for a 2-cocycle c on P/Q and a central element $b\in\mathcal{U}(G_q)$. Letting $a=\varphi(b)$, we obtain $\mathcal{F}'=\mathcal{E}_c(a\otimes a)(\varphi\otimes\varphi)(\hat\Delta_q(b)^{-1})\mathcal{F}=\mathcal{E}_c\mathcal{F}(a\otimes a)\hat\Delta(a)^{-1}$. □

Note that this corollary implies that the Dirac operator defined as in [4] is the same (for fixed φ) for any choice of a unitary element $\mathcal{F}$ satisfying properties (i) and (ii). This extends [5, Theorem 6.1].

We now turn to our main application, the computation of the group of $\mathbb{C}$-linear monoidal autoequivalences of $\mathcal{C}_q(\mathfrak{g})$ identified up to monoidal natural isomorphisms.

Any automorphism α of the based root datum $\Psi_\mathfrak{g}$ of $\mathfrak{g}$ defines an automorphism of the Hopf algebra $U_q\mathfrak{g}$, hence an autoequivalence $\tilde\alpha$ of $\mathcal{C}_q(\mathfrak{g})$. On the other hand, for any 2-cocycle c on P/Q we can define an autoequivalence β_c which acts trivially on objects and morphisms, while the tensor structure is given by the action of $\mathcal{E}_c^{-1}$. It turns out that any autoequivalence of $\mathcal{C}_q(\mathfrak{g})$ is monodially naturally isomorphic to a composition of two autoequivalences defined either by an automorphism of $\Psi_\mathfrak{g}$ or by a cocycle on P/Q.

Theorem 5. *The group of $\mathbb{C}$-linear monoidal autoequivalences of the tensor category $\mathcal{C}_q(\mathfrak{g})$ is canonically isomorphic to $H^2(P/Q;\mathbb{T})\rtimes\mathrm{Aut}(\Psi_\mathfrak{g})$.*

Proof. The proof is essentially identical to [7, Theorem 2.5]. Briefly, by a result of McMullen [3] any automorphism of the fusion ring of $\mathcal{C}_q(\mathfrak{g})$, mapping irreducibles into irreducibles, is implemented by an automorphism of $\Psi_\mathfrak{g}$. Hence for any autoequivalence γ of $\mathcal{C}_q(\mathfrak{g})$ there exists a unique automorphism α of $\Psi_\mathfrak{g}$ such that $\tilde\alpha\circ\gamma$ maps every object into an isomorphic one; that is, ignoring the tensor structure, $\tilde\alpha\circ\gamma$ is naturally isomorphic to the identity functor. Possible tensor structures on the identity functor are, in turn, described by invariant 2-cocycles on $\hat G_q$. □

We next consider $q=1$ and extend the above results to compact connected groups.

The group P/Q is canonically identified with the dual of the center $Z(G)$ of the group G, and so, for $q=1$, Theorem 1 can be formulated as $H^2_G(\hat G;\mathbb{C}^*)\cong H^2(\widehat{Z(G)};\mathbb{C}^*)$.

Theorem 6. *For any compact connected separable group G we have a canonical isomorphism*

$$H^2_G(\hat G;\mathbb{C}^*)\cong H^2(\widehat{Z(G)};\mathbb{C}^*).$$

Proof. For Lie groups the proof is essentially the same as above, with P replaced by the weight lattice of a maximal torus of G. In the general case we have a homomorphism $H^2(\widehat{Z(G)};\mathbb{C}^*)\to H^2_G(\hat G;\mathbb{C}^*)$

obtained by considering $\mathcal{U}(Z(G))$ as a subring of $\mathcal{U}(G)$. To construct the inverse homomorphism, for every quotient H of G which is a Lie group consider the composition

$$H^2_G(\hat{G};\mathbb{C}^*)\to H^2_H(\hat{H};\mathbb{C}^*)\to H^2(\widehat{Z(H)};\mathbb{C}^*),$$

where the first homomorphism is defined using the quotient map $\mathcal{U}(G)\to\mathcal{U}(H)$. The map $Z(G)\to Z(H)$ is surjective (since this is true for Lie groups), so $Z(G)$ is the inverse limit of the groups $Z(H)$. Then $H^2(\widehat{Z(G)};\mathbb{C}^*)$ is the inverse limit of the groups $H^2(\widehat{Z(H)};\mathbb{C}^*)$. Therefore the above maps $H^2_G(\hat{G};\mathbb{C}^*)\to H^2(\widehat{Z(H)};\mathbb{C}^*)$ define a homomorphism $H^2_G(\hat{G};\mathbb{C}^*)\to H^2(\widehat{Z(G)};\mathbb{C}^*)$. It is clearly a left inverse of the map $H^2(\widehat{Z(G)};\mathbb{C}^*)\to H^2_G(\hat{G};\mathbb{C}^*)$, so it remains to show that it is injective.

In other words, we have to check that if $\mathcal{E}$ is an invariant cocycle on $\hat{G}$ such that its image in $\mathcal{U}(H\times H)$ is a coboundary for every Lie group quotient H of G, then $\mathcal{E}$ itself is a coboundary. If $\mathcal{E}$ were unitary, this could be easily shown by taking a weak operator limit point of cochains, see the proof of [7, Theorem 2.2], and would not require the separability of G. In the non-unitary case we can argue as follows.

Since G is separable, there exists a decreasing sequence of closed normal subgroups N_n of G such that $\cap_{n\ge1}N_n=\{e\}$ and the quotients $H_n=G/N_n$ are Lie groups. Let $\mathcal{E}_n$ be the image of $\mathcal{E}$ in $\mathcal{U}(H_n\times H_n)$. By assumption there exist invertible central elements $c_n\in\mathcal{U}(H_n)$ such that $\mathcal{E}_n=(c_n\otimes c_n)\hat{\Delta}(c_n)^{-1}$. For a fixed n consider the image a of c_{n+1} in $\mathcal{U}(H_n)$. Then c_na^{-1} is a central group-like element in $\mathcal{U}(H_n)$. By [5, Theorem A.1] it is therefore defined by an element of the center of the complexification $(H_n)_\mathbb{C}$ of H_n. Since the homomorphism $(H_{n+1})_\mathbb{C}\to(H_n)_\mathbb{C}$ is surjective, we conclude that there exists a central group-like element b in $\mathcal{U}(H_{n+1})$ such that its image in $\mathcal{U}(H_n)$ is c_na^{-1}. Replacing c_{n+1} by $c_{n+1}b$ we get an element such that $\mathcal{E}_{n+1}=(c_{n+1}\otimes c_{n+1})\hat{\Delta}(c_{n+1})^{-1}$ and the image of c_{n+1} in $\mathcal{U}(H_n)$ is c_n. Applying this procedure inductively we can therefore assume that the image of c_{n+1} in $\mathcal{U}(H_n)$ is c_n for all $n\ge1$. Then the elements c_n define a central element $c\in\mathcal{U}(G)$ such that $\mathcal{E}=(c\otimes c)\hat{\Delta}(c)^{-1}$. ☐

In [7, Theorem 2.5] we computed the group of autoequivalences of the C*-tensor category of finite dimensional unitary representations of G. The above theorem and the same arguments as in the proof of Theorem 5 allow us to get a similar result ignoring the C*-structure.

Theorem 7. *For any compact connected separable group G, the group of $\mathbb{C}$-linear monoidal autoequivalences of the category of finite dimensional representations of G is canonically isomorphic to $H^2(\widehat{Z(G)};\mathbb{C}^*)\rtimes\mathrm{Out}(G)$.*

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Department of Mathematics, University of Oslo, P.O. Box 1053 Blindern, NO-0316 Oslo, Norway
E-mail address: sergeyn@math.uio.no

Faculty of Engineering, Oslo University College, P.O. Box 4 St. Olavs plass, NO-0130 Oslo, Norway
E-mail address: Lars.Tuset@iu.hio.no