

Hopf Algebra Equivariant Cyclic Cohomology, K-theory and Index Formulas

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Abstract

For an algebra \mathcal{B} with an action of a Hopf algebra \mathcal{H} we establish the pairing between equivariant cyclic cohomology and equivariant K -theory for \mathcal{B} . We then extend this formalism to compact quantum group actions and show that equivariant cyclic cohomology is a target space for the equivariant Chern character of equivariant summable Fredholm modules. We prove an analogue of Julg's theorem relating equivariant K -theory to ordinary K -theory of the C^* -algebra crossed product, and characterize equivariant vector bundles on quantum homogeneous spaces.

Introduction

This paper is part of a project which aims to extend the theory of noncommutative geometry of Connes [C] to q -homogeneous spaces in such a way that the various q -deformed geometric notions play as natural a role as possible. This is beneficial from the point of view of computations. In this paper we focus on the pairing between equivariant cyclic cohomology and equivariant K -theory, acknowledging the Chern character as the backbone of noncommutative geometry.

Equivariant cyclic cohomology for group actions was introduced in [KKL]. It is straightforward to generalize these results to the context of Hopf algebras with involutive antipode. In the general case the pertinent formulas are less obvious. Several approaches have been proposed in [AK1, AK2]. In the previous version of this paper we introduced equivariant cyclic cohomology in the case when the Hopf algebra admits a modular element, by which we mean a group-like element implementing the square of the antipode. However, this cyclic cohomology turns out to be isomorphic to the one introduced in the latest version of [AK1], so we prefer working with the latter definition.

In Section 1 we establish basic properties of equivariant cyclic cohomology such as Morita invariance and the construction of the trace map. In Section 2 we set up the pairing between equivariant cyclic cohomology and the equivariant K -groups. Restricting to the case when the Hopf algebra in question has a modular element, we discuss the relation of this pairing to twisted cyclic cohomology [KMT]. In Section 3 we extend these results to compact quantum groups in that we replace the Hopf algebra by the algebra of finitely supported functions on the dual discrete quantum group. Incorporating equivariant summable even Fredholm modules allows us to state an index formula. In the remaining part of the paper we study equivariant K -theory for compact quantum group actions. We establish its relation to ordinary K -theory of the crossed product algebra, thus generalizing a result of Julg [J]. Finally we obtain an analogue of the fact that equivariant vector bundles on homogeneous spaces are induced from representations of the stabilizer group.

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1 Cyclic cohomology

1.1 Cocyclic objects

Recall [L] that a cocyclic object in an abelian category \mathcal{C} consists of a sequence of objects C^n in \mathcal{C} , $n \geq 0$, together with morphisms

$$d_i^n: C^{n-1} \rightarrow C^n, \quad s_i^n: C^{n+1} \rightarrow C^n, \quad t_n: C^n \rightarrow C^n, \quad (0 \leq i \leq n)$$

that satisfy

$$\begin{aligned} d_j^m d_i^{m-1} &= d_i^m d_{j-1}^{m-1} \text{ for } i < j; \\ s_j^n s_i^{n+1} &= s_i^n s_{j+1}^{n+1} \text{ for } i \leq j; \\ s_j^n d_i^{m+1} &= d_i^m s_{j-1}^{n-1} \text{ for } i < j; \\ s_j^n d_i^{m+1} &= \iota \text{ for } i = j \text{ or } i = j + 1; \\ s_j^n d_i^{m+1} &= d_{i-1}^m s_j^{n-1} \text{ for } i > j + 1; \\ t_n d_i^m &= d_{i-1}^m t_{n-1} \text{ for } i \geq 1, \text{ and } t_n d_0^m = d_n^m; \\ t_n s_i^n &= s_{i-1}^n t_{n+1} \text{ for } i \geq 1, \text{ and } t_n s_0^n = s_n^n t_{n+1}^2; \\ t_n^{n+1} &= \iota. \end{aligned}$$

Every cocyclic object gives a bicomplex

$$\begin{array}{ccccccc} \vdots & & \vdots & & \vdots & & \vdots \\ b \uparrow & & -b' \uparrow & & b \uparrow & & -b' \uparrow \\ C^2 & \xrightarrow{1-\lambda} & C^2 & \xrightarrow{N} & C^2 & \xrightarrow{1-\lambda} & C^2 \xrightarrow{N} \dots \\ b \uparrow & & -b' \uparrow & & b \uparrow & & -b' \uparrow \\ C^1 & \xrightarrow{1-\lambda} & C^1 & \xrightarrow{N} & C^1 & \xrightarrow{1-\lambda} & C^1 \xrightarrow{N} \dots \\ b \uparrow & & -b' \uparrow & & b \uparrow & & -b' \uparrow \\ C^0 & \xrightarrow{1-\lambda} & C^0 & \xrightarrow{N} & C^0 & \xrightarrow{1-\lambda} & C^0 \xrightarrow{N} \dots \end{array}$$

with morphisms $b_n = \sum_{i=0}^n (-1)^i d_i^n$, $b'_n = \sum_{i=0}^{n-1} (-1)^i d_i^n$, $\lambda_n = (-1)^n t_n$ and $N^n = \sum_{i=0}^n \lambda_n^i$. The cyclic cohomology HC^\bullet of a cocyclic object is by definition the cohomology of the total complex associated to its bicomplex, whereas the Hochschild cohomology HH^\bullet is defined to be the cohomology of the complex (C^\bullet, b) . A cocyclic object yields a long exact sequence, referred to as the *IBS*-sequence of Connes, written

$$\dots \longrightarrow HC^n \xrightarrow{I} HH^n \xrightarrow{B} HC^{n-1} \xrightarrow{S} HC^{n+1} \xrightarrow{I} \dots,$$

where S is the periodicity operator and $B_n = N^n s_n^n t_{n+1} (\iota - \lambda_{n+1})$.

We denote the cohomology of the complex $(\text{Ker}(\iota - \lambda), b)$ by H_λ^\bullet . The equality $HC^\bullet = H_\lambda^\bullet$ holds whenever the objects C^n are vector spaces over a ground field that contains the rational numbers. In this case the periodicity operator gives the periodic cyclic cohomology HP^0 and HP^1 as direct limits of HC^{2n} and HC^{2n+1} , respectively.

1.2 Equivariant cyclic cohomology

By (\mathcal{H}, Δ) we mean a Hopf algebra (over \mathbb{C}) with comultiplication Δ , invertible antipode S and counit ε . We adapt the Sweedler notation

$$\Delta^n(\omega) = \omega_{(0)} \otimes \dots \otimes \omega_{(n)},$$

(with summation understood) where $\Delta^n = (\iota \otimes \Delta)\Delta^{n-1}$. Let \mathcal{B} denote a unital right \mathcal{H} -module algebra, that is, a unital algebra with a right action \triangleleft of \mathcal{H} on \mathcal{B} such that

$$(b_1 b_2) \triangleleft \omega = (b_1 \triangleleft \omega_{(0)})(b_2 \triangleleft \omega_{(1)}) \quad \text{and} \quad 1 \triangleleft \omega = \varepsilon(\omega)1.$$

Following [AK1] we introduce a cocyclic object in the category of vector spaces using the above data. The associated cyclic cohomology is called the *equivariant cyclic cohomology* of \mathcal{B} and denoted by $HC_{\mathcal{H}}^{\bullet}(\mathcal{B})$. Likewise we write $HH_{\mathcal{H}}^{\bullet}(\mathcal{B})$ and $HP_{\mathcal{H}}^{\bullet}(\mathcal{B})$ for the *equivariant Hochschild cohomology* and the *equivariant periodic cyclic cohomology*, respectively. Let $C^n = C_{\mathcal{H}}^n(\mathcal{B})$ be the space of linear functionals f on $\mathcal{H} \otimes \mathcal{B}^{\otimes(n+1)}$ that are \mathcal{H} -invariant in the sense that

$$\omega \triangleright f = \varepsilon(\omega)f \quad \text{for all } \omega \in \mathcal{H}.$$

Here the left action \triangleright of \mathcal{H} on the space of linear functionals on $\mathcal{H} \otimes \mathcal{B}^{\otimes(n+1)}$ is given by

$$(\omega \triangleright f)(\eta \otimes b_0 \otimes \dots \otimes b_n) = f(S^{-1}(\omega_{(0)})\eta\omega_{(1)} \otimes b_0 \triangleleft \omega_{(2)} \otimes \dots \otimes b_n \triangleleft \omega_{(n+2)}).$$

To see that this actually is a left action, recall that the tensor product $X_1 \otimes X_2$ of two right \mathcal{H} -modules X_1 and X_2 is again a right \mathcal{H} -module with action

$$(x_1 \otimes x_2) \triangleleft \omega = x_1 \triangleleft \omega_{(0)} \otimes x_2 \triangleleft \omega_{(1)}.$$

Thus, considering \mathcal{H} itself as a right \mathcal{H} -module with respect to the action

$$\eta \triangleleft \omega = S^{-1}(\omega_{(0)})\eta\omega_{(1)},$$

we regard $\mathcal{H} \otimes \mathcal{B}^{\otimes(n+1)}$ as a right \mathcal{H} -module, which in turn shows that the space of linear functionals on $\mathcal{H} \otimes \mathcal{B}^{\otimes(n+1)}$ is a left \mathcal{H} -module. Now define

$$\begin{aligned} (t_n f)(\omega \otimes b_0 \otimes \dots \otimes b_n) &= f(\omega_{(0)} \otimes b_n \triangleleft \omega_{(1)} \otimes b_0 \otimes \dots \otimes b_{n-1}), \\ (s_i^n f)(\omega \otimes b_0 \otimes \dots \otimes b_n) &= f(\omega \otimes b_0 \otimes \dots \otimes b_i \otimes 1 \otimes b_{i+1} \otimes \dots \otimes b_n), \\ (d_i^n f)(\omega \otimes b_0 \otimes \dots \otimes b_n) &= f(\omega \otimes b_0 \otimes \dots \otimes b_{i-1} \otimes b_i b_{i+1} \otimes b_{i+2} \otimes \dots \otimes b_n) \quad \text{for } i < n \end{aligned}$$

and put $d_n^n = t_n d_0^n$. These formulas give linear maps

$$t_n: C_{\mathcal{H}}^n(\mathcal{B}) \rightarrow C_{\mathcal{H}}^n(\mathcal{B}), \quad d_i^n: C_{\mathcal{H}}^{n-1}(\mathcal{B}) \rightarrow C_{\mathcal{H}}^n(\mathcal{B}), \quad s_i^n: C_{\mathcal{H}}^{n+1}(\mathcal{B}) \rightarrow C_{\mathcal{H}}^n(\mathcal{B})$$

which define a cocyclic object in the category of vector spaces. Indeed, if we consider a new Hopf algebra $(\mathcal{H}_1, \Delta_1)$ with $\mathcal{H}_1 = \mathcal{H}$ and $\Delta_1 = \Delta^{op}$, and define a left action of \mathcal{H}_1 on \mathcal{B} by $\omega \triangleright b = b \triangleleft S(\omega)$, then the correspondence $f \mapsto \tilde{f}$, where

$$\tilde{f}(b_0 \otimes \dots \otimes b_n)(\omega) = f(S^2(\omega) \otimes b_0 \otimes \dots \otimes b_n),$$

defines an isomorphism of the cocyclic object above with the cocyclic object associated with the left \mathcal{H}_1 -algebra \mathcal{B} as defined by [AK1]. The axioms of the cocyclic objects can also be checked directly. We include a proof of the most critical identity $t_n^{n+1} = \iota$:

Note that

$$(t_n^{n+1} f)(\omega \otimes b_0 \otimes \dots \otimes b_n) = f(\omega_{(0)} \otimes b_0 \triangleleft \omega_{(1)} \otimes \dots \otimes b_n \triangleleft \omega_{(n+1)}).$$

Hence it suffices to show that if X is a right \mathcal{H} -module and f is a linear functional on $\mathcal{H} \otimes X$ such that $\eta \triangleright f = \varepsilon(\eta)f$, then

$$f(\omega \otimes x) = f(\omega_{(0)} \otimes x \triangleleft \omega_{(1)}).$$

Due to the identity $\omega = \omega_{(0)} \triangleleft \omega_{(1)}$ we indeed obtain

$$f(\omega_{(0)} \otimes x \triangleleft \omega_{(1)}) = f(\omega_{(0)} \triangleleft \omega_{(1)} \otimes x \triangleleft \omega_{(2)}) = \varepsilon(\omega_{(1)})f(\omega_{(0)} \otimes x) = f(\omega \otimes x).$$

1.3 Morita invariance

Let X be a right \mathcal{H} -module and let $\text{End}(X)$ denote the algebra of all complex linear maps on X . Write π_X for the antihomomorphism $\mathcal{H} \rightarrow \text{End}(X)$ given by the right \mathcal{H} -action. Then $\text{End}(X)$ is clearly a right \mathcal{H} -module algebra with respect to the adjoint action

$$T \blacktriangleleft \omega = \pi_X(\omega_{(0)})T\pi_X S^{-1}(\omega_{(1)}).$$

If in addition X is a unital algebra, we identify X with the subalgebra of $\text{End}(X)$ consisting of those endomorphisms which are given by multiplication from the left. In this case the adjoint action on $\text{End}(X)$ restricts to the original action of \mathcal{H} on X if and only if X is a right \mathcal{H} -module algebra.

Lemma 1.1 *For any right \mathcal{H} -module X , the subalgebra $\text{End}(X) \otimes \mathcal{B}$ of $\text{End}(X \otimes \mathcal{B})$ is an \mathcal{H} -submodule with respect to the adjoint action of \mathcal{H} on $\text{End}(X \otimes \mathcal{B})$, where the tensor product $X \otimes \mathcal{B}$ is regarded as a right \mathcal{H} -module. Moreover, the adjoint action on $\text{End}(X) \otimes \mathcal{B}$ is given by*

$$(T \otimes b) \blacktriangleleft \omega = \pi_X(\omega_{(0)})T\pi_X S^{-1}(\omega_{(2)}) \otimes b \blacktriangleleft \omega_{(1)},$$

for which $\text{End}(X) \otimes \mathcal{B}$ is a right \mathcal{H} -module algebra.

Proof. First observe that $\pi_{X \otimes \mathcal{B}}(\omega) = \pi_X(\omega_{(0)}) \otimes \pi_{\mathcal{B}}(\omega_{(1)})$. To any $b \in \mathcal{B}$ define L_b to be the endomorphism on \mathcal{B} given by left multiplication with b . Then for $T \in \text{End}(X)$, we derive

$$\begin{aligned} (T \otimes L_b) \blacktriangleleft \omega &= \pi_{X \otimes \mathcal{B}}(\omega_{(0)})(T \otimes L_b)\pi_{X \otimes \mathcal{B}} S^{-1}(\omega_{(1)}) \\ &= \pi_X(\omega_{(0)})T\pi_X S^{-1}(\omega_{(3)}) \otimes \pi_{\mathcal{B}}(\omega_{(1)})L_b\pi_{\mathcal{B}} S^{-1}(\omega_{(2)}) \\ &= \pi_X(\omega_{(0)})T\pi_X S^{-1}(\omega_{(2)}) \otimes L_b \blacktriangleleft \omega_{(1)}. \end{aligned}$$

It is now immediate that $\text{End}(X) \otimes \mathcal{B}$ is an \mathcal{H} -submodule subalgebra of $\text{End}(X \otimes \mathcal{B})$. ■

Note that \mathcal{B} is in general not an \mathcal{H} -submodule subalgebra of $\text{End}(X) \otimes \mathcal{B}$, while $\text{End}(X)$ is. Remark also that since $\text{End}(X)$ and \mathcal{B} are right \mathcal{H} -modules, one can consider the \mathcal{H} -module tensor product action

$$(T \otimes b) \blacktriangleleft \omega = T \blacktriangleleft \omega_{(0)} \otimes b \blacktriangleleft \omega_{(1)}.$$

When the Hopf algebra \mathcal{H} is cocommutative this action coincides with the action above. In general, however, these two actions are different, and $\text{End}(X) \otimes \mathcal{B}$ equipped with the tensor product action need not even be an \mathcal{H} -module algebra.

Suppose now that X is finite dimensional. For each $n \geq 0$ and $f \in C_{\mathcal{H}}^n(\mathcal{B})$, define

$$(\Psi^n f)(\omega \otimes (T_0 \otimes b_0) \otimes \dots \otimes (T_n \otimes b_n)) = f(\omega_{(0)} \otimes b_0 \otimes \dots \otimes b_n) \text{Tr}(\pi_X S^{-1}(\omega_{(1)})T_0 \dots T_n),$$

where Tr is the canonical non-normalized trace on $\text{End}(X)$.

Lemma 1.2 *The formula above gives maps*

$$\Psi^n: C_{\mathcal{H}}^n(\mathcal{B}) \rightarrow C_{\mathcal{H}}^n(\text{End}(X) \otimes \mathcal{B}),$$

which constitute a morphism of cocyclic objects, and thus induces homomorphisms

$$[\Psi^n]: HC_{\mathcal{H}}^n(\mathcal{B}) \rightarrow HC_{\mathcal{H}}^n(\text{End}(X) \otimes \mathcal{B}).$$

Proof. Let $f \in C_{\mathcal{H}}^n(\mathcal{B})$. To check \mathcal{H} -invariance of $\Psi^n f$, compute

$$\begin{aligned} & (\Psi^n f)((\eta \otimes (T_0 \otimes b_0) \otimes \dots \otimes (T_n \otimes b_n)) \triangleleft \omega) \\ &= (\Psi^n f)(S^{-1}(\omega_{(0)})\eta\omega_{(1)} \otimes (\pi_X(\omega_{(2)})T_0\pi_X S^{-1}(\omega_{(4)}) \otimes b_0 \triangleleft \omega_{(3)}) \otimes \dots \\ & \quad \dots \otimes (\pi_X(\omega_{(3n+2)})T_n\pi_X S^{-1}(\omega_{(3n+4)}) \otimes b_n \triangleleft \omega_{(3n+3)}) \\ &= f(S^{-1}(\omega_{(1)})\eta_{(0)}\omega_{(2)} \otimes b_0 \triangleleft \omega_{(5)} \otimes \dots \otimes b_n \triangleleft \omega_{(3n+5)}) \text{Tr}(\pi_X S^{-2}(\omega_{(0)})\pi_X S^{-1}(\eta_{(1)}) \times \\ & \quad \times \pi_X S^{-1}(\omega_{(3)})\pi_X(\omega_{(4)})T_0\pi_X S^{-1}(\omega_{(6)}) \dots \pi_X(\omega_{(3n+4)})T_n\pi_X S^{-1}(\omega_{(3n+6)})). \end{aligned}$$

Using the identity $\pi_X S^{-1}(\omega_{(0)})\pi_X(\omega_{(1)}) = \varepsilon(\omega)$ repeatedly, this expression simplifies to

$$\begin{aligned} & f(S^{-1}(\omega_{(1)})\eta_{(0)}\omega_{(2)} \otimes b_0 \triangleleft \omega_{(3)} \otimes \dots \otimes b_n \triangleleft \omega_{(n+3)}) \text{Tr}(\pi_X S^{-2}(\omega_{(0)})\pi_X S^{-1}(\eta_{(1)})T_0 \dots T_n \pi_X S^{-1}(\omega_{(n+4)})) \\ &= \varepsilon(\omega_{(1)})f(\eta_{(0)} \otimes b_0 \otimes \dots \otimes b_n) \text{Tr}(\pi_X S^{-2}(\omega_{(0)})\pi_X S^{-1}(\eta_{(1)})T_0 \dots T_n \pi_X S^{-1}(\omega_{(2)})) \\ &= f(\eta_{(0)} \otimes b_0 \otimes \dots \otimes b_n) \text{Tr}(\pi_X S^{-2}(\omega_{(0)})S(\omega_{(1)})\pi_X S^{-1}(\eta_{(1)})T_0 \dots T_n) \\ &= \varepsilon(\omega)(\Psi^n f)(\eta \otimes (T_0 \otimes b_0) \otimes \dots \otimes (T_n \otimes b_n)). \end{aligned}$$

Thus Ψ^n is a well-defined map $C_{\mathcal{H}}^n(\mathcal{B}) \rightarrow C_{\mathcal{H}}^n(\text{End}(X) \otimes \mathcal{B})$. It is readily verified that Ψ^\bullet commutes with the maps t_n , d_i^n and s_i^n . ■

Suppose $p \in \text{End}(X)$ is an idempotent such that $p \triangleleft \omega = \varepsilon(\omega)p$ and pX is a submodule with trivial \mathcal{H} -action. In other words, $\pi_X(\omega)p = p\pi_X(\omega) = \varepsilon(\omega)p$. Then the map $\Phi_p: \mathcal{B} \rightarrow \text{End}(X) \otimes \mathcal{B}$ given by $\Phi_p(b) = p \otimes b$ is a homomorphism of algebras which respects the right actions of \mathcal{H} . Hence it induces homomorphisms $[\Phi_p^n]: HC_{\mathcal{H}}^n(\text{End}(X) \otimes \mathcal{B}) \rightarrow HC_{\mathcal{H}}^n(\mathcal{B})$.

Theorem 1.3 *Suppose X is a finite dimensional vector space endowed with the trivial right action $x \triangleleft \omega = \varepsilon(\omega)x$ of \mathcal{H} . Then $[\Psi^n]: HC_{\mathcal{H}}^n(\mathcal{B}) \rightarrow HC_{\mathcal{H}}^n(\text{End}(X) \otimes \mathcal{B})$ from Lemma 1.2 is an isomorphism with inverse $[\Phi_p^n]$, where p is any one-dimensional idempotent in $\text{End}(X)$.*

Proof of Theorem 1.3. The demonstration is similar to the proof in the non-equivariant case: Recall [L, Corollary 2.2.3] that the *IBS*-sequence of Connes implies that if a morphism of cocyclic objects induces an isomorphism for Hochschild cohomology then it also does so for cyclic cohomology. Since $\Phi_p \circ \Psi = \iota$ already on the level of complexes, it is therefore enough to show that $\Psi \circ \Phi_p$ induces the identity map on Hochschild cohomology. Thus it suffices to show that $\Psi \circ \Phi_p$ is homotopic to the identity regarded as a morphism of $(C_{\mathcal{H}}^\bullet(\text{End}(X) \otimes \mathcal{B}), b)$. Let x_1, \dots, x_m be a basis in X with corresponding matrix units $m_{ij} \in \text{End}(X)$. We may assume that $p = m_{11}$. Then [L, Theorem 1.2.4] the required homotopy $h^n: C_{\mathcal{H}}^{n+1}(\text{End}(X) \otimes \mathcal{B}) \rightarrow C_{\mathcal{H}}^n(\text{End}(X) \otimes \mathcal{B})$ is given by $h^n = \sum_{j=0}^n (-1)^j h_j^n$, where

$$\begin{aligned} & (h_j^n f)(\omega \otimes (T_0 \otimes b_0) \otimes \dots \otimes (T_n \otimes b_n)) = \sum_{k_0, \dots, k_{j+1}} (T_0)_{k_0 k_1} (T_1)_{k_1 k_2} \dots (T_j)_{k_j k_{j+1}} \times \\ & \times f(\omega \otimes (m_{k_0 1} \otimes b_0) \otimes (m_{11} \otimes b_1) \otimes \dots \otimes (m_{11} \otimes b_j) \otimes (m_{1 k_{j+1}} \otimes 1) \otimes (T_{j+1} \otimes b_{j+1}) \otimes \dots \otimes (T_n \otimes b_n)) \end{aligned}$$

and T_{ij} stand for matrix coefficients of T . ■

The following corollary is a standard consequence of Morita invariance. It is obtained by considering two embeddings $b \mapsto m_{11} \otimes b$, $b \mapsto m_{22} \otimes b$ of \mathcal{B} into $\text{Mat}_2(\mathbb{C}) \otimes \mathcal{B} = \text{Mat}_2(\mathcal{B})$ and the automorphism $\text{Ad} \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}$ of $\text{Mat}_2(\mathcal{B})$.

Corollary 1.4 *Let b be an invertible \mathcal{H} -invariant element of \mathcal{B} . Then the automorphism $\text{Ad } b$ induces the identity map on $HC_{\mathcal{H}}^\bullet(\mathcal{B})$.* ■

2 K-theory

We make the same assumptions on (\mathcal{H}, Δ) and \mathcal{B} as in the previous section.

2.1 Equivariant K-theory

By an \mathcal{H} -equivariant \mathcal{B} -module we mean a vector space with \mathcal{H} -module and \mathcal{B} -module structures compatible in the sense that

$$(xb)\triangleleft\omega = (x\triangleleft\omega_{(0)})(b\triangleleft\omega_{(1)}).$$

Equivalently, equivariant modules can be considered as $(\mathcal{B} \rtimes \mathcal{H})$ -modules, where the crossed product $\mathcal{B} \rtimes \mathcal{H}$ is the algebra with underlying space $\mathcal{B} \otimes \mathcal{H}$ and product

$$(b \otimes \omega)(c \otimes \eta) = b(c\triangleleft S^{-1}(\omega_{(1)})) \otimes \omega_{(0)}\eta.$$

In the sequel we will always write $b\omega$ instead of $b \otimes \omega$ whenever we mean an element of $\mathcal{B} \rtimes \mathcal{H}$. The identities

$$ab = (b\triangleleft S^{-1}(\omega_{(1)}))\omega_{(0)} \quad \text{and} \quad b\omega = \omega_{(0)}(b\triangleleft\omega_{(1)})$$

in $\mathcal{B} \rtimes \mathcal{H}$ are easily verified.

Let $\mathcal{C}_{\mathcal{H}}(\mathcal{B})$ be the full subcategory of the category of $(\mathcal{B} \rtimes \mathcal{H})$ -modules consisting of those modules which are isomorphic to $X \otimes \mathcal{B}$ for finite dimensional (as vector spaces) right \mathcal{H} -modules X . As before, we consider $X \otimes \mathcal{B}$ as a right \mathcal{H} -module with tensor product action, while the \mathcal{B} -module structure is defined by the right action of \mathcal{B} on itself. Furthermore, let $\tilde{\mathcal{C}}_{\mathcal{H}}(\mathcal{B})$ denote the associated pseudo-abelian category, that is, the category of modules isomorphic to $X_p = p(X \otimes \mathcal{B})$ for idempotents $p \in \text{End}_{\mathcal{B} \rtimes \mathcal{H}}(X \otimes \mathcal{B})$. Note that for $T \in \text{End}_{\mathcal{B}}(X \otimes \mathcal{B}) = \text{End}(X) \otimes \mathcal{B}$ we have $T \in \text{End}_{\mathcal{H}}(X \otimes \mathcal{B})$ if and only if T is \mathcal{H} -invariant. Thus $\text{End}_{\mathcal{B} \rtimes \mathcal{H}}(X \otimes \mathcal{B})$ is the algebra of \mathcal{H} -invariant elements in $\text{End}(X) \otimes \mathcal{B}$.

Let $K_0^{\mathcal{H}}(\mathcal{B})$ be the Grothendieck group of the category $\tilde{\mathcal{C}}_{\mathcal{H}}(\mathcal{B})$. More concretely, consider all possible \mathcal{H} -invariant idempotents $p \in \text{End}(X) \otimes \mathcal{B}$. Say that two such idempotents $p \in \text{End}(X) \otimes \mathcal{B}$ and $p' \in \text{End}(X') \otimes \mathcal{B}$ are equivalent if there exist \mathcal{H} -invariant elements $\gamma \in \text{Hom}_{\mathbb{C}}(X, X') \otimes \mathcal{B}$ and $\gamma' \in \text{Hom}_{\mathbb{C}}(X', X) \otimes \mathcal{B}$ such that $\gamma\gamma' = p'$ and $\gamma'\gamma = p$. Note that $\text{Hom}_{\mathbb{C}}(X, X') \otimes \mathcal{B}$ can be considered as a subspace of $\text{End}(X \oplus X') \otimes \mathcal{B}$ and as such is endowed with a canonical \mathcal{H} -module structure. The set of equivalence classes of \mathcal{H} -invariant idempotents is an abelian semigroup in a standard fashion. Then $K_0^{\mathcal{H}}(\mathcal{B})$ coincides with the associated Grothendieck group of this semigroup.

Recall also that if for two \mathcal{H} -invariant idempotents $p \in \text{End}(X) \otimes \mathcal{B}$ and $p' \in \text{End}(X') \otimes \mathcal{B}$ there exist \mathcal{H} -invariant elements $\gamma \in \text{Hom}_{\mathbb{C}}(X, X') \otimes \mathcal{B}$ and $\gamma' \in \text{Hom}_{\mathbb{C}}(X', X) \otimes \mathcal{B}$ such that $\gamma\gamma' = p'$ and $\gamma'\gamma = p$, then, if we consider p and p' as elements of $\text{End}(X \oplus X') \otimes \mathcal{B}$, there exists an \mathcal{H} -invariant invertible element $\gamma_0 \in \text{End}(X \oplus X') \otimes \mathcal{B}$ such that $\gamma_0 p \gamma_0^{-1} = p'$. Namely, put

$$\gamma_0 = \begin{pmatrix} 1 - p & p\gamma'p' \\ p'\gamma p & 1 - p' \end{pmatrix}.$$

Hence we can equivalently define $K_0^{\mathcal{H}}(\mathcal{B})$ as the Grothendieck group of the semigroup of equivalence classes of \mathcal{H} -invariant idempotents with the equivalence relation generated by similarity (two idempotents p and p' are called similar if there exists an invertible \mathcal{H} -invariant element $\gamma \in \text{Hom}_{\mathbb{C}}(X, X') \otimes \mathcal{B}$ such that $\gamma p \gamma^{-1} = p'$) and the condition $p \sim p \oplus 0$, where we think of $p \oplus 0$ as living in $\text{End}(X \oplus X') \otimes \mathcal{B}$.

To define $K_1^{\mathcal{H}}(\mathcal{B})$, consider invertible elements $u \in \text{End}_{\mathcal{B} \rtimes \mathcal{H}}(Y)$, where Y is an object in the category $\mathcal{C}_{\mathcal{H}}(\mathcal{B})$. Then $K_1^{\mathcal{H}}(\mathcal{B})$ is by definition the abelian group generated by isomorphism classes of such u satisfying the following relations:

- (i) $[u] + [v] = [uv]$ for $u, v \in \text{End}_{\mathcal{B} \rtimes \mathcal{H}}(Y)$;
(ii) $[u_1] + [u_2] = [u]$ whenever there exists a *split* exact sequence

$$0 \longrightarrow Y_1 \xrightarrow{i} Y \xrightarrow{j} Y_2 \longrightarrow 0$$

with $u_1 \in \text{End}_{\mathcal{B} \rtimes \mathcal{H}}(Y_1)$, $u_2 \in \text{End}_{\mathcal{B} \rtimes \mathcal{H}}(Y_2)$ and $u \in \text{End}_{\mathcal{B} \rtimes \mathcal{H}}(Y)$ such that $ui = iu_1$ and $u_2j = ju$.

The relation (ii) can be rewritten as follows. Given invertible \mathcal{H} -invariant elements $u_i \in \text{End}(X_i) \otimes \mathcal{B}$, $i = 1, 2$, and an \mathcal{H} -invariant element $T \in \text{Hom}_{\mathbb{C}}(X_2, X_1) \otimes \mathcal{B}$, we have

$$[u_1] + [u_2] = \left[\begin{pmatrix} u_1 & T \\ 0 & u_2 \end{pmatrix} \right].$$

Note also that if we use $\tilde{\mathcal{C}}_{\mathcal{H}}(\mathcal{B})$ instead of $\mathcal{C}_{\mathcal{H}}(\mathcal{B})$ in the definition of $K_1^{\mathcal{H}}(\mathcal{B})$, we get exactly the same group.

If \mathcal{H} is semisimple, which in particular implies that it is finite dimensional, we can equivalently describe the category $\tilde{\mathcal{C}}_{\mathcal{H}}(\mathcal{B})$ as the category of f.g. (finitely generated) projective $(\mathcal{B} \rtimes \mathcal{H})$ -modules.

Theorem 2.1 *Suppose \mathcal{H} is semisimple. Then there exist bijective correspondences between*

- (i) *the equivalence classes of \mathcal{H} -invariant idempotents in $\text{End}(X) \otimes \mathcal{B}$, where X ranges over all finite dimensional \mathcal{H} -modules;*
(ii) *the isomorphism classes of \mathcal{H} -equivariant f.g. projective right \mathcal{B} -modules;*
(iii) *the isomorphism classes of f.g. projective right $(\mathcal{B} \rtimes \mathcal{H})$ -modules.*

In particular, $K_0^{\mathcal{H}}(\mathcal{B}) \cong K_0(\mathcal{B} \rtimes \mathcal{H})$ and $K_1^{\mathcal{H}}(\mathcal{B}) \cong K_1(\mathcal{B} \rtimes \mathcal{H})$.

Proof. For any \mathcal{H} -invariant idempotent $p \in \text{End}(X) \otimes \mathcal{B}$ we have already seen that $X_p = p(X \otimes \mathcal{B})$ is an \mathcal{H} -equivariant f.g. projective right \mathcal{B} -module. It is furthermore clear that two idempotents $p \in \text{End}(X) \otimes \mathcal{B}$ and $p' \in \text{End}(X') \otimes \mathcal{B}$ are equivalent if and only if X_p and $X_{p'}$ are isomorphic as \mathcal{H} -equivariant \mathcal{B} -modules, or as $(\mathcal{B} \rtimes \mathcal{H})$ -modules. So to prove the theorem we just have to show that

- (a) X_p is a projective $(\mathcal{B} \rtimes \mathcal{H})$ -module;
(b) any \mathcal{H} -equivariant f.g. projective right \mathcal{B} -module is isomorphic to X_p for some X and p ;
(c) any f.g. projective right $(\mathcal{B} \rtimes \mathcal{H})$ -module is isomorphic to X_p for some X and p .

Since X_p is a direct summand of $X \otimes \mathcal{B}$, in proving (a) we may assume $p = 1$. By semisimplicity X is isomorphic to a finite direct sum of modules $q_1\mathcal{H}, \dots, q_n\mathcal{H}$ for some idempotents $q_1, \dots, q_n \in \mathcal{H}$. Thus we may assume $X = q\mathcal{H}$ for some idempotent $q \in \mathcal{H}$. The map $q\mathcal{H} \otimes \mathcal{B} \rightarrow q(\mathcal{B} \rtimes \mathcal{H})$, $\omega \otimes b \mapsto \omega b$, is an isomorphism with inverse obtained by restricting the map $\mathcal{B} \rtimes \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{B}$, $b\omega \mapsto \omega_{(0)} \otimes b \triangleleft \omega_{(1)}$, to $q(\mathcal{B} \rtimes \mathcal{H})$. Thus X_p is indeed a f.g. projective $(\mathcal{B} \rtimes \mathcal{H})$ -module.

To establish (b) consider an \mathcal{H} -equivariant f.g. projective right \mathcal{B} -module Y with generators y_1, \dots, y_n . Set $X = y_1 \triangleleft \mathcal{H} + \dots + y_n \triangleleft \mathcal{H}$. Then X is an \mathcal{H} -module which is finite dimensional as a vector space. The map $T: X \otimes \mathcal{B} \rightarrow Y$, $x \otimes b \mapsto xb$, is a surjective homomorphism of $(\mathcal{B} \rtimes \mathcal{H})$ -modules. Since Y is a projective \mathcal{B} -module, there exists a \mathcal{B} -module map $T': Y \rightarrow X \otimes \mathcal{B}$ such that $TT' = \iota$. By semisimplicity there exists a right integral η in \mathcal{H} (that is, an element such that $\eta\omega = \varepsilon(\omega)\eta$ for any $\omega \in \mathcal{H}$) such that $\varepsilon(\eta) = 1$. Replacing T' by $T' \triangleleft \eta = \pi_{X \otimes \mathcal{B}}(\eta_{(0)})T'\pi_Y S^{-1}(\eta_{(1)})$, it is clear that we can assume that T' is also an \mathcal{H} -module map. Then $Y \cong X_p$ with $p = T'T$. The proof of (c) is similar, but shorter since T' from the beginning can be chosen to be a $(\mathcal{B} \rtimes \mathcal{H})$ -module map.

The established bijective correspondences immediately imply $K_0^{\mathcal{H}}(\mathcal{B}) \cong K_0(\mathcal{B} \rtimes \mathcal{H})$. Since every exact sequence of projective modules is split, the result for K_1 also follows. ■

2.2 Pairing with cyclic cohomology

Let $R(\mathcal{H})$ be the space of \mathcal{H} -invariant linear functionals on \mathcal{H} . As before, we consider the action $\eta \triangleleft \omega = S^{-1}(\omega_{(0)})\eta\omega_{(1)}$ of \mathcal{H} on itself.

Theorem 2.2 *There exists pairings*

$$\langle \cdot, \cdot \rangle: HC_{\mathcal{H}}^{2n}(\mathcal{B}) \times K_0^{\mathcal{H}}(\mathcal{B}) \rightarrow R(\mathcal{H}) \quad \text{and} \quad \langle \cdot, \cdot \rangle: HC_{\mathcal{H}}^{2n+1}(\mathcal{B}) \times K_1^{\mathcal{H}}(\mathcal{B}) \rightarrow R(\mathcal{H}),$$

such that for $f \in C_{\mathcal{H}}^{2n}(\mathcal{B})$ and $p \in \text{End}(X) \otimes \mathcal{B}$ we have

$$\langle [f], [p] \rangle(\omega) = (\Psi^{2n} f)(\omega \otimes p \otimes \dots \otimes p),$$

and such that for $f \in C_{\mathcal{H}}^{2n+1}(\mathcal{B})$ and $u \in \text{End}(X) \otimes \mathcal{B}$ we have

$$\langle [f], [u] \rangle(\omega) = (\Psi^{2n+1} f)(\omega \otimes (u^{-1} - 1) \otimes (u - 1) \otimes \dots \otimes (u^{-1} - 1) \otimes (u - 1)),$$

where Ψ^\bullet is the map defined before Lemma 1.2.

Proof. The proof can be reduced to the non-equivariant case as follows. First observe that the quantities $\langle f, p \rangle(\omega) = (\Psi^{2n} f)(\omega \otimes p \otimes \dots \otimes p)$ and

$$\langle f, u \rangle(\omega) = (\Psi^{2n+1} f)(\omega \otimes (u^{-1} - 1) \otimes (u - 1) \otimes \dots \otimes (u^{-1} - 1) \otimes (u - 1))$$

remain unchanged if we replace p by $p \oplus 0$ and u by $u \oplus 1$. Secondly note that the relation

$$\langle f, u_1 \rangle(\omega) + \langle f, u_2 \rangle(\omega) = \left\langle f, \begin{pmatrix} u_1 & T \\ 0 & u_2 \end{pmatrix} \right\rangle(\omega)$$

follows from $\langle f, uv \rangle(\omega) = \langle f, u \rangle(\omega) + \langle f, v \rangle(\omega)$, because $\begin{pmatrix} u_1 & T \\ 0 & u_2 \end{pmatrix} = \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix} \begin{pmatrix} 1 & u_1^{-1}T \\ 0 & 1 \end{pmatrix}$ and $\langle f, u \rangle(\omega) = 0$, whenever $(u - 1)^2 = 0$, due to the condition $t_{2n+1}f = -f$.

By substituting \mathcal{B} with $\text{End}(X) \otimes \mathcal{B}$ it is thus enough to show that $\langle f, p \rangle(\omega)$ and $\langle f, u \rangle(\omega)$ depend only on the cohomology class of f , the similarity class of $p \in \mathcal{B}^{\mathcal{H}}$ and the isomorphism class of $u \in \mathcal{B}^{\mathcal{H}}$, and that $\langle f, uv \rangle(\omega) = \langle f, u \rangle(\omega) + \langle f, v \rangle(\omega)$ for invertible elements u and v in $\mathcal{B}^{\mathcal{H}}$. Now for a fixed $\omega \in \mathcal{H}$ consider the morphism $\omega_*: C_{\mathcal{H}}^\bullet(\mathcal{B}) \rightarrow C^\bullet(\mathcal{B}^{\mathcal{H}})$ of cocyclic objects given by

$$(\omega_* f)(b_0 \otimes \dots \otimes b_n) = f(\omega \otimes b_0 \otimes \dots \otimes b_n)$$

for $f \in C_{\mathcal{H}}^n(\mathcal{B})$. Then $\langle f, p \rangle(\omega) = \langle \omega_* f, p \rangle$ and $\langle f, u \rangle(\omega) = \langle \omega_* f, u \rangle$, so the result follows from the analogous result in the non-equivariant case [C]. ■

Note that the proof in [C] implies even a stronger result that $HC_{\mathcal{H}}^{2n+1}(\mathcal{B})$ pairs with $K_1^{\mathcal{H}}(\mathcal{B}) / \sim_h$, where \sim_h is the equivalence relation given by polynomial homotopy.

Suppose now that we have a group-like element ρ of \mathcal{H} . Consider the twisted cyclic cohomology $HC_{\theta_\rho}^\bullet(\mathcal{B})$ of \mathcal{B} introduced in [KMT], where θ_ρ is the twist automorphism of \mathcal{B} given by $\theta_\rho(b) = b \triangleleft \rho$. More generally, for any finite dimensional right \mathcal{H} -module X consider the twist automorphism $\theta_{X,\rho} = \text{Ad } \pi_X(\rho) \otimes \theta_\rho$ on $\text{End}(X) \otimes \mathcal{B}$. As in Subsection 1.3 we can construct a map

$$\Psi_\rho^n: HC_{\theta_\rho}^n(\mathcal{B}) \rightarrow HC_{\theta_{X,\rho}}^n(\text{End}(X) \otimes \mathcal{B})$$

given by

$$(\Psi_\rho^n f)((T_0 \otimes b_0) \otimes \dots \otimes (T_n \otimes b_n)) = f(b_0 \otimes \dots \otimes b_n) \text{Tr}(\pi_X(\rho^{-1})T_0 \dots T_n).$$

Using this map we get a pairing

$$\langle \cdot, \cdot \rangle_\rho: HC_{\theta_\rho}^{2n}(\mathcal{B}) \times K_0^{\mathcal{H}}(\mathcal{B}) \rightarrow \mathbb{C}$$

given by $\langle [f], [p] \rangle_\rho = (\Psi_\rho^{2n} f)(p^{\otimes(2n+1)})$. On the other hand, there exists a map $\rho_*: HC_{\mathcal{H}}^n(\mathcal{B}) \rightarrow HC_{\theta_\rho}^n(\mathcal{B})$ determined by

$$(\rho_*\phi)(b_0 \otimes \dots \otimes b_n) = \phi(\rho \otimes b_0 \otimes \dots \otimes b_n),$$

and such that the following diagram

$$\begin{array}{ccc} HC_{\mathcal{H}}^{2n}(\mathcal{B}) \times K_0^{\mathcal{H}}(\mathcal{B}) & \xrightarrow{\langle \cdot, \cdot \rangle} & R(\mathcal{H}) \\ \rho_* \downarrow & & \rho_* \downarrow \\ HC_{\theta_\rho}^{2n}(\mathcal{B}) \times K_0^{\mathcal{H}}(\mathcal{B}) & \xrightarrow{\langle \cdot, \cdot \rangle_\rho} & \mathbb{C} \end{array}$$

commutes, where $\rho_*: R(\mathcal{H}) \rightarrow \mathbb{C}$ is the evaluation map $\rho_*(f) = f(\rho)$. This diagram is particularly interesting for Hopf algebras which admit a modular element ρ . We shall return to it in Subsection 3.2.

3 Compact quantum group actions

3.1 Extending the formalism

So far we have only considered Hopf algebra actions. The theory does however apply to more general contexts, notably to the case when \mathcal{H} is the algebra of finitely supported functions on a discrete quantum group. As explained in Introduction this is in fact the main motivation for the present work, and the purpose of this section is to extend our results to this setting.

Throughout this section (A, Δ) denotes a compact quantum group in the sense of Woronowicz, and $\alpha: B \rightarrow A \otimes B$ denotes a left coaction of (A, Δ) on a unital C^* -algebra B . We shall follow notation and conventions in [NT] with the only exception that we replace ρ by ρ^{-1} . In particular, we assume that (A, Δ) is a reduced compact quantum group, so the Haar state φ on A is faithful. The coaction α is assumed to be non-degenerate in the sense that $(A \otimes 1)\alpha(B)$ is dense in $A \otimes B$.

The dual discrete quantum group $(\hat{A}, \hat{\Delta})$ has a canonical dense $*$ -subalgebra $\hat{\mathcal{A}}$, which can be considered as the algebra of finitely supported functions on the discrete quantum group. The elements of $\hat{\mathcal{A}}$ are bounded linear functionals on A . Thus the coaction α defines a right action of $\hat{\mathcal{A}}$ on B given by

$$b \triangleleft \omega = (\omega \otimes \iota)\alpha(b).$$

Set $\mathcal{B} = B \triangleleft \hat{\mathcal{A}}$. The assumption on non-degeneracy ensures that \mathcal{B} is a dense $*$ -subalgebra of B such that $\alpha(\mathcal{B}) \subset \mathcal{A} \otimes \mathcal{B}$ and $\mathcal{B} \triangleleft \hat{\mathcal{A}} = \mathcal{B}$, where $\mathcal{A} \subset A$ is the Hopf $*$ -algebra of matrix coefficients of irreducible unitary corepresentations of (A, Δ) .

The $*$ -algebra $\hat{\mathcal{A}}$ is isomorphic to the algebraic direct sum $\bigoplus_{s \in I} B(H_s)$ of full matrix algebras $B(H_s)$, where I denotes the set of equivalence classes of irreducible unitary corepresentations of (A, Δ) . The comultiplication $\hat{\Delta}$ restricts to a $*$ -homomorphism

$$\hat{\Delta}: \hat{\mathcal{A}} \rightarrow M(\hat{\mathcal{A}} \otimes \hat{\mathcal{A}}) = \prod_{s,t \in I} B(H_s) \otimes B(H_t).$$

If $\hat{\Delta}(\omega) = (\omega_{st})_{s,t \in I}$, then the matrix $(\omega_{st})_{s,t}$ has only finitely many non-zero components in each column and row. This together with the property $\mathcal{B} \triangleleft \hat{\mathcal{A}} = \mathcal{B}$ shows that expressions like $b_1 \triangleleft \omega_{(0)} \otimes$

$b_2 \triangleleft \omega_{(1)}$ make sense as elements of $\mathcal{B} \otimes \mathcal{B}$. It is then straightforward to check that all the results and notions from the previous sections remain true for \mathcal{B} whenever (\mathcal{H}, Δ) is replaced by $(\hat{\mathcal{A}}, \hat{\Delta})$. Even Theorem 2.1, which is not directly applicable since $\hat{\mathcal{A}}$ is not semisimple, has an analogue in this context, as will be shown in Subsection 3.3 below.

Remark that if (A, Δ) is a q -deformation of a compact connected and simply connected semisimple Lie group G , then $\mathcal{A} = M(\hat{\mathcal{A}})$ has a dense Hopf subalgebra, namely the q -deformed universal enveloping algebra of the Lie algebra of the group G . One can work with this algebra instead of $\hat{\mathcal{A}}$. Then our previous results do not require any modifications. Since this algebra consists of unbounded elements, in many cases it is however more convenient to work with $(\hat{\mathcal{A}}, \hat{\Delta})$.

3.2 The Chern character and index formulas

Suppose (π, H, F, γ, U) is an α -equivariant even Fredholm module over \mathcal{B} . By this we mean that π is a bounded $*$ -representation of \mathcal{B} on a graded Hilbert space $H = H_- \oplus H_+$, that F is a symmetry on H ($F = F^*$, $F^2 = 1$), that $\gamma = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ is the grading operator for which F is odd and $\pi(b)$, $b \in \mathcal{B}$, is even, and that the commutator $[F, \pi(b)]$ is a compact operator on H for each $b \in \mathcal{B}$. Furthermore, the element $U \in M(A \otimes K(H))$ is a unitary corepresentation of (A, Δ) , so $(\Delta \otimes \iota)(U) = U_{13}U_{23}$, which commutes with $1 \otimes F$ and with $1 \otimes \gamma$, and

$$U(\iota \otimes \pi)\alpha(b) = (1 \otimes \pi(b))U.$$

The corepresentation U defines a bounded $*$ -representation of $\hat{\mathcal{A}}$ on H by $\pi_U(\omega) = (\omega \otimes \iota)(U)$. Recall that a Fredholm module (π, H, F) over \mathcal{B} is called $(2n + 1)$ -summable if $[F, \pi(b)] \in \mathcal{L}^{2n+1}(H)$ for any $b \in \mathcal{B}$, where $\mathcal{L}^{2n+1}(H)$ is the Schatten $(2n + 1)$ -class. Then we can define

$$\phi_F(\omega \otimes b_0 \otimes \dots \otimes b_{2n}) = \frac{(-1)^n}{2} \text{Tr}(\gamma \pi_U(\omega) F [F, \pi(b_0)] \dots [F, \pi(b_{2n})]).$$

The identity $U(\iota \otimes \pi)\alpha(b) = (1 \otimes \pi(b))U$ implies $\pi_U(\omega_{(0)})\pi(b \triangleleft \omega_{(1)}) = \pi(b)\pi_U(\omega)$. It is now easy to see that ϕ_F is a cyclic cocycle and thus defines an element of $HC_{\hat{\mathcal{A}}}^{2n}(\mathcal{B})$.

Suppose $p \in \mathcal{B}$ is a $\hat{\mathcal{A}}$ -invariant projection, and define projections $p_+ = \frac{1}{2}(1 + \gamma)\pi(p)$ and $p_- = \frac{1}{2}(1 - \gamma)\pi(p)$. Then $p_- F p_+ : p_+ H_+ \rightarrow p_- H_-$ is a Fredholm operator. Since F , γ and $\pi(p)$ all commute with the algebra $\pi_U(\hat{\mathcal{A}})$, the finite dimensional spaces $\text{Ker}(p_- F p_+)$ and $\text{Im}(p_- F p_+)^\perp$ are $\hat{\mathcal{A}}$ -invariant. Denote by $\tilde{\pi}_+$ and $\tilde{\pi}_-$ the finite dimensional representations of $\hat{\mathcal{A}}$ on $\text{Ker}(p_- F p_+)$ and $\text{Im}(p_- F p_+)^\perp$, respectively, given by restriction. Define $\phi_\pm(\omega) = \text{Tr}(\tilde{\pi}_\pm(\omega))$ and $(\text{Ind}_F p)(\omega) = \phi_+(\omega) - \phi_-(\omega)$. Note that if ω is a sufficiently large projection in $\hat{\mathcal{A}}$, then

$$(\text{Ind}_F p)(\omega) = \dim \tilde{\pi}_+ - \dim \tilde{\pi}_-.$$

Theorem 3.1 *We have $(\text{Ind}_F p)(\omega) = \langle [\phi_F], [p] \rangle(\omega)$.*

Proof. The proof is analogous to the one in [C], and is based on the fact that if τ is trace on an algebra, then $\tau((p - p')^{2n+1}) = \tau(p - p')$ for any two idempotents p and p' and any $n \in \mathbb{N}$. ■

Consider now a finite dimensional unitary corepresentation $V \in A \otimes B(H_V)$ of (A, Δ) . Set $\tilde{U} = U_{13}V_{12} \in M(A \otimes K(H_V \otimes H))$ and consider the coaction $\tilde{\alpha}_V$ of (A, Δ) on $B(H_V) \otimes B$ given by

$$\tilde{\alpha}_V(T \otimes b) = (V^* \otimes 1)(1 \otimes T \otimes 1)\alpha(b)_{13}(V \otimes 1).$$

Then $(\iota \otimes \pi, H_V \otimes H, 1 \otimes F, 1 \otimes \gamma, \tilde{U})$ is an $\tilde{\alpha}_V$ -equivariant even Fredholm module over $B(H_V) \otimes \mathcal{B}$. For any $\tilde{\alpha}_V$ -invariant projection $p \in B(H_V) \otimes \mathcal{B}$, we define $\text{Ind}_F p$ as above. Note that the right

action of $\hat{\mathcal{A}}$ on $B(H_V) \otimes \mathcal{B}$ corresponding to $\tilde{\alpha}_V$ in the sense that $x \triangleleft \omega = (\omega \otimes \iota) \tilde{\alpha}_V(x)$ is precisely the action introduced in Lemma 1.1, where H_V is the right $\hat{\mathcal{A}}$ -module with action $\xi \triangleleft \omega = \pi_V \hat{S}(\omega) \xi$. Indeed,

$$\begin{aligned} (\omega \otimes \iota) \tilde{\alpha}_V(T \otimes b) &= (\omega \otimes \iota \otimes \iota) ((V^* \otimes 1)(1 \otimes T \otimes 1) \alpha(b)_{13} (V \otimes 1)) \\ &= ((\omega_{(0)} \otimes \iota)(V^*) \otimes 1)(T \otimes b \triangleleft \omega_{(1)}) ((\omega_{(2)} \otimes \iota)(V) \otimes 1) \\ &= \pi_V \hat{S}(\omega_{(0)}) T \pi_V(\omega_{(2)}) \otimes b \triangleleft \omega_{(1)}. \end{aligned}$$

Thus p defines an element of $K_0^{\hat{\mathcal{A}}}(\mathcal{B})$ and we get a homomorphism $\text{Ind}_F: K_0^{\hat{\mathcal{A}}}(\mathcal{B}) \rightarrow R(\hat{\mathcal{A}})$. The cocycle corresponding to the Fredholm module $(\iota \otimes \pi, H_V \otimes H, 1 \otimes F, 1 \otimes \gamma, \tilde{U})$ is exactly $\Psi^{2n} \phi_F$. Hence $\text{Ind}_F = \langle [\phi_F], \cdot \rangle$ on $K_0^{\hat{\mathcal{A}}}(\mathcal{B})$.

Recall now that there exists a canonical element $\rho \in M(\hat{\mathcal{A}})$ such that $\hat{\Delta}(\rho) = \rho \otimes \rho$ and $\hat{S}^2(\omega) = \rho^{-1} \omega \rho$. Then in the above notation $(\text{Ind}_F p)(\rho) = \phi_+(\rho) - \phi_-(\rho)$ is nothing but the difference between the quantum dimensions of $\tilde{\pi}_+$ and $\tilde{\pi}_-$. We denote this quantity by $q\text{-Ind}_F p$ and think of it as a quantum Fredholm index of $p\text{-}Fp_+$.

We say that an α -equivariant Fredholm module (π, H, F, γ, U) over \mathcal{B} is $(2n+1, \rho)$ -summable if

$$\pi_U(\rho)^{\frac{1}{2(2n+1)}} |[F, \pi(b)]| \pi_U(\rho)^{\frac{1}{2(2n+1)}} \in \mathcal{L}^{2n+1}(H)$$

for any $b \in \mathcal{B}$. Under this condition $q\text{-Ind}_F = \langle \tilde{\phi}_F, \cdot \rangle_\rho$, where $\tilde{\phi}_F \in HC_{\theta_\rho}^{2n}(\mathcal{B})$ is the twisted cocycle given by

$$\tilde{\phi}_F(b_0 \otimes \dots \otimes b_{2n}) = \frac{(-1)^n}{2} \text{Tr}(\gamma F [F, \pi(b_0)] \dots [F, \pi(b_{2n})] \pi_U(\rho)).$$

As will be shown elsewhere, the quantum index is much easier to compute than the usual one and the equivariant index Ind_F .

3.3 Julg's theorem

We have shown how an α -equivariant Fredholm module gives rise to an index map on equivariant K -theory. In the rest of the paper we address the question of how to compute the K -theory. First we shall obtain an analogue of Theorem 2.1 and thus extend the result of [J] to compact quantum groups.

Equivariant K -theory for coactions of arbitrary locally compact quantum groups was defined in [BS1]. While the general case requires KK -theory technique, the case of compact quantum groups can be handled using the simple-minded definition in Subsection 2.1. Thus we define $K_0^{\hat{\mathcal{A}}}(B)$ as the Grothendieck group associated with the semigroup of equivalence classes of all $\hat{\mathcal{A}}$ -invariant projections in $B(H_V) \otimes B$ for all finite dimensional unitary corepresentations of V of (A, Δ) (in view of [BS1] this group should rather be denoted by $K_0^{\hat{\mathcal{A}}}(B)$). Since $(B(H_V) \otimes B) \triangleleft \hat{\mathcal{A}} = B(H_V) \otimes \mathcal{B}$, by definition $K_0^{\hat{\mathcal{A}}}(B) = K_0^{\hat{\mathcal{A}}}(\mathcal{B})$. We shall now describe $K_0^{\hat{\mathcal{A}}}(B)$ in terms of α -equivariant Hilbert B -modules.

By an α -equivariant Hilbert B -module we mean a right Hilbert B -module X together with a non-degenerate continuous linear map $\delta: X \rightarrow A \otimes X$ (here we consider $A \otimes X$ as a right Hilbert $(A \otimes B)$ -module, and non-degeneracy means that $\delta(X)(A \otimes 1)$ is dense in $A \otimes X$) such that $(\Delta \otimes \iota) \delta = (\iota \otimes \delta) \delta$ and

- (i) $\delta(\xi b) = \delta(\xi) \alpha(b)$ for $\xi \in X$ and $b \in B$;
- (ii) $\langle \delta(\xi_1), \delta(\xi_2) \rangle_{A \otimes B} = \alpha(\langle \xi_1, \xi_2 \rangle_B)$ for $\xi_1, \xi_2 \in X$.

If $V \in A \otimes B(H_V)$ is a finite dimensional unitary corepresentation of (A, Δ) and $p \in B(H_V) \otimes B$ is a $\hat{\mathcal{A}}$ -invariant projection, then $H_{V,p} = p(H_V \otimes B)$ is an α -equivariant Hilbert B -module with

$$\delta(\xi \otimes b) = (V^* \otimes 1)(1 \otimes \xi \otimes 1) \alpha(b)_{13} \quad \text{and} \quad \langle \xi_1 \otimes b_1, \xi_2 \otimes b_2 \rangle = \langle \xi_2, \xi_1 \rangle b_1^* b_2.$$

Lemma 3.2 Any α -equivariant f.g. Hilbert B -module X is isomorphic to $H_{V,p}$ for some V and p .

Proof. The coaction δ defines a right action of \hat{A} on X , $\xi \triangleleft \omega = (\omega \otimes \iota)\delta(\xi)$. By non-degeneracy $X \triangleleft \hat{A}$ is dense in X . Since X is f.g. as a B -module, we can find a finite number of generators in $X \triangleleft \hat{A}$, cf [J]. Since $\xi \triangleleft \hat{A}$ is a finite dimensional space for any $\xi \in X \triangleleft \hat{A}$, we conclude that there exists a finite dimensional non-degenerate \hat{A} -submodule X_0 of X such that $X_0 B = X$. There exists a finite dimensional unitary corepresentation $V \in A \otimes B(H_V)$ such that the right \hat{A} -modules H_V and X_0 are isomorphic. Fix an isomorphism $T_0: H_V \rightarrow X_0$ and define $T: H_V \otimes B \rightarrow X$ by $T(\xi \otimes b) = (T_0 \xi)b$. The mapping T is a surjective morphism of B -modules. Since $H_V \otimes B$ is a f.g. Hilbert B -module, it makes sense to consider the polar decomposition of T^* , $T^* = u|T^*|$. Then $|T^*|$ is an invertible morphism of the B -module X , and $u: X \rightarrow H_V \otimes B$ is a B -module mapping such that $u^*u = \iota$. Property (ii) in the definition of α -equivariant Hilbert modules together with non-degeneracy ensure that T^* is a morphism of α -equivariant modules, hence so are $|T^*|$, $u = T^*|T^*|^{-1}$ and u^* . Thus u is an isomorphism of the α -equivariant B -module X onto $H_{V,p}$ with $p = uu^*$. ■

Thus $K_0^{\hat{A}}(B)$ can equivalently be described as the Grothendieck group of the semigroup of isomorphism classes of α -equivariant f.g. Hilbert B -modules. Note that $K_1^{\hat{A}}(B)$ as defined in [BS1] is identical to K_1 of the Banach category of α -equivariant f.g. right Hilbert B -modules [K]. To get the pairing with cyclic cohomology, one needs a continuous version of equivariant cyclic cohomology, but we shall not discuss this any further here.

Consider the C^* -algebra crossed product $B \rtimes \hat{A}$, that is, the C^* -algebra generated by $\alpha(B)(\hat{A} \otimes 1)$, where we consider A and \hat{A} as subalgebras of $B(L^2(A, \varphi))$. Note that as \hat{A} is non-unital, the algebra B should be considered as a subalgebra of $M(B \rtimes \hat{A})$. As before, we write just $b\omega$ instead of $\alpha(b)(\omega \otimes 1)$. This notation agrees with the notation introduced earlier in the sense that $\mathcal{B} \rtimes \hat{A}$ becomes a dense $*$ -subalgebra of $B \rtimes \hat{A}$. In the proof below we denote by \mathcal{B}_s the spectral subspace of B corresponding to an equivalence class s of irreducible unitary corepresentations of (A, Δ) , so

$$\mathcal{B}_s = B \triangleleft B(H_s) = B \triangleleft I_s,$$

where we identify \hat{A} with $\bigoplus_{t \in I} B(H_t)$ and where I_s is the unit in $B(H_s)$.

Lemma 3.3 We have $(\mathcal{B} \rtimes \hat{A})(B \rtimes \hat{A})(\mathcal{B} \rtimes \hat{A}) \subset \mathcal{B} \rtimes \hat{A}$.

Proof. Let p be a central projection in \hat{A} . It is enough to prove that $p(B \rtimes \hat{A})p \subset \mathcal{B} \rtimes \hat{A}$. Suppose a net $\{x_n\}_n$ in $\mathcal{B} \rtimes \hat{A}$ with $px_n p = x_n$ converges to an element $x \in B \rtimes \hat{A}$. Fix a basis $\omega_1, \dots, \omega_m$ in $\hat{A}p$. Then $x_n = \sum_i b_i(n)\omega_i$ for some $b_i(n) \in \mathcal{B}$. Since

$$x_n = px_n = \sum_i (b_i(n) \triangleleft \hat{S}^{-1}(p_{(1)})) p_{(0)} \omega_i,$$

we may assume that there exists a finite set F of equivalence classes of irreducible unitary corepresentations of (A, Δ) such that $b_i(n) \in \bigoplus_{s \in F} \mathcal{B}_s$ for all n and i . Namely, if $p = \sum_{t \in F_0} I_t$, then for F we can take the set of irreducible components of $\bar{t}_1 \times t_2$ for $t_1, t_2 \in F_0$. Fix matrix units m_{jk}^s in $B(H_s)$, and let $u_{jk}^s \in \mathcal{A}$ be the corresponding matrix coefficients. Then there exists uniquely defined elements $b_i^s(n), b_{ijk}^s(n) \in \mathcal{B}_s$ such that $b_i(n) = \sum_{s \in F} b_i^s(n)$ and

$$\alpha(b_i^s(n)) = \sum_{j,k} u_{jk}^s \otimes b_{ijk}^s(n).$$

By assumption the net

$$\sum_i \alpha(b_i(n))(\omega_i \otimes 1) = \sum_{s \in F} \sum_{i=1}^m \sum_{j,k} u_{jk}^s \omega_i \otimes b_{ijk}^s(n)$$

converges in $K(L^2(A, \varphi)) \otimes B$. Since the map $\mathcal{A} \otimes \hat{\mathcal{A}} \rightarrow K(L^2(A, \varphi))$, $a \otimes \omega \mapsto a\omega$, is injective, we conclude that the nets $\{b_{ijk}^s(n)\}_n$ converge in B for all s, i, j, k . Then $b_i^s(n) = \sum_{j,k} \varepsilon(u_{jk}^s) b_{ijk}^s(n)$ converge to elements $b_i^s \in \mathcal{B}_s$, so $x = \sum_s \sum_i b_i^s \omega_i$ is an element of $\mathcal{B} \rtimes \hat{\mathcal{A}}$. \blacksquare

Let V be a finite dimensional unitary corepresentation of (A, Δ) and $p \in B(H_V) \otimes \mathcal{B}$ a $\hat{\mathcal{A}}$ -invariant projection. As in Subsection 2.1 we can consider $X_p = p(H_V \otimes \mathcal{B})$ as a $(\mathcal{B} \rtimes \hat{\mathcal{A}})$ -module. The same argument as we used there shows that it is isomorphic to a f.g. projective $(\mathcal{B} \rtimes \hat{\mathcal{A}})$ -module $q(\mathcal{B} \rtimes \hat{\mathcal{A}})^n$ for some idempotent $q \in \text{Mat}_n(\mathcal{B} \rtimes \hat{\mathcal{A}})$. More precisely, if $V = V^s$ is irreducible and $p = 1$, then for q we can take any one-dimensional idempotent in $B(H_{\bar{s}}) \subset \hat{\mathcal{A}}$. Conversely, if $q \in \text{Mat}_n(\mathcal{B} \rtimes \hat{\mathcal{A}})$, then there exists an idempotent e in $\hat{\mathcal{A}}$ such that $q(1 \otimes e) = q$ in $\text{Mat}_n(\mathbb{C}) \otimes (\mathcal{B} \rtimes \hat{\mathcal{A}})$. Then the $(\mathcal{B} \rtimes \hat{\mathcal{A}})$ -module $q(\mathcal{B} \rtimes \hat{\mathcal{A}})^n$ is a direct summand of the module $(e(\mathcal{B} \rtimes \hat{\mathcal{A}}))^n \cong (e\hat{\mathcal{A}} \otimes \mathcal{B})^n$, so it is of the form X_p for some V and p . We can now consider q as an idempotent in $\text{Mat}_n(B \rtimes \hat{\mathcal{A}})$, so the corresponding projective $(B \rtimes \hat{\mathcal{A}})$ -module is just $X_p \otimes_{B \rtimes \hat{\mathcal{A}}} (B \rtimes \hat{\mathcal{A}})$. Given two $\hat{\mathcal{A}}$ -invariant projections $p \in B(H_V) \otimes \mathcal{B}$ and $p' \in B(H_{V'}) \otimes \mathcal{B}$, it is clear that they are equivalent if and only if the α -equivariant modules $H_{V,p}$ and $H_{V',p'}$ are isomorphic, if and only if the $(\mathcal{B} \rtimes \hat{\mathcal{A}})$ -modules X_p and $X_{p'}$ are isomorphic, if and only if the corresponding idempotents q and q' in $\cup_{n \in \mathbb{N}} \text{Mat}_n(\mathcal{B} \rtimes \hat{\mathcal{A}})$ are equivalent. But according to Lemma 3.3 equivalence of idempotents in $\cup_{n \in \mathbb{N}} \text{Mat}_n(\mathcal{B} \rtimes \hat{\mathcal{A}})$ is the same as equivalence of idempotents in $\cup_{n \in \mathbb{N}} \text{Mat}_n(B \rtimes \hat{\mathcal{A}})$. We summarize this discussion in the following theorem.

Theorem 3.4 *There are bijective correspondences between*

- (i) *the isomorphism classes of α -equivariant f.g. right Hilbert B -modules;*
- (ii) *the equivalence classes of $\hat{\mathcal{A}}$ -invariant projections in $B(H_V) \otimes B$ for finite dimensional unitary corepresentations V of (A, Δ) ;*
- (iii) *the equivalence classes of idempotents in $\cup_{n \in \mathbb{N}} \text{Mat}_n(\mathcal{B} \rtimes \hat{\mathcal{A}})$;*
- (iv) *the equivalence classes of projections in $K(l^2(\mathbb{N})) \otimes (B \rtimes \hat{\mathcal{A}})$.*

In particular, $K_0^{\hat{\mathcal{A}}}(B) \cong K_0^{\hat{\mathcal{A}}}(\mathcal{B}) \cong K_0(\mathcal{B} \rtimes \hat{\mathcal{A}}) \cong K_0(B \rtimes \hat{\mathcal{A}})$.

Proof. The bijective correspondences are already explained. It remains to note that though the algebras $\mathcal{B} \rtimes \hat{\mathcal{A}}$ and $B \rtimes \hat{\mathcal{A}}$ are in general non-unital, we can choose an approximate unit $\{e_n\}_n$ in $B \rtimes \hat{\mathcal{A}}$ consisting of central projections in $\hat{\mathcal{A}}$, so that $K_0(B \rtimes \hat{\mathcal{A}})$ and $K_0(\mathcal{B} \rtimes \hat{\mathcal{A}})$ can be described in terms of idempotents in $\cup_{m \in \mathbb{N}} \cup_n \text{Mat}_m(e_n(B \rtimes \hat{\mathcal{A}})e_n) = \cup_{m \in \mathbb{N}} \cup_n \text{Mat}_m(e_n(\mathcal{B} \rtimes \hat{\mathcal{A}})e_n) = \cup_{m \in \mathbb{N}} \text{Mat}_m(\mathcal{B} \rtimes \hat{\mathcal{A}})$. \blacksquare

3.4 Equivariant vector bundles on homogeneous spaces

In this section we consider quantum homogeneous spaces and prove an analogue of the geometric fact that an equivariant vector bundle on a homogeneous space is completely determined by the representation of the stabilizer of some point in the fiber over this point. It seems that in the quantum case such a result is less obvious than in the classical case, and we deduce it from Theorem 3.4 combined with the Takesaki-Takai duality.

We consider a closed subgroup of (A, Δ) , that is, a compact quantum group (A_0, Δ_0) together with a surjective $*$ -homomorphism $P: A \rightarrow A_0$ such that $(P \otimes P)\Delta = \Delta_0 P$. Consider the right coaction $\Delta_R: A \rightarrow A \otimes A_0$ of (A_0, Δ_0) on A given by $\Delta_R(a) = (\iota \otimes P)\Delta(a)$, and the corresponding quotient space $B = A^{\Delta_R} = \{a \in A \mid \Delta_R(a) = a \otimes 1\}$ together with the left coaction $\alpha = \Delta|_B: B \rightarrow A \otimes B$ of (A, Δ) on B . Denote by $\text{Irr}(A_0, \Delta_0)$ the set of equivalence classes of irreducible unitary corepresentations of (A_0, Δ_0) . For each $t \in \text{Irr}(A_0, \Delta_0)$ fix a representative $V^t \in A_0 \otimes B(H_t)$ of this class. On $H_t \otimes A$ define a right A_0 -comodule structure by

$$\delta_t: H_t \otimes A \rightarrow H_t \otimes A \otimes A_0, \quad \delta_t(\xi \otimes a) = V_{31}^t(\xi \otimes 1 \otimes 1)(1 \otimes \Delta_R(a)).$$

The comultiplication Δ induces also a left A -comodule structure on $H_t \otimes A$ by

$$\delta: H_t \otimes A \rightarrow A \otimes H_t \otimes A, \quad \delta(\xi \otimes a) = (1 \otimes \xi \otimes 1)\Delta(a)_{13}.$$

Set

$$X_t = (H_t \otimes A)^{\delta_t} = \{x \in H_t \otimes A \mid \delta_t(x) = x \otimes 1\}.$$

Since $(\delta \otimes \iota)\delta_t = (\iota \otimes \delta_t)\delta$, the projection $E_t: H_t \otimes A \rightarrow X_t$, $E_t = (\iota \otimes \varphi_0)\delta_t$, has the property $\delta E_t = (\iota \otimes E_t)\delta$, so δ induces a structure of an α -equivariant right B -module on X_t . Here φ_0 denotes the Haar state on A_0 . To introduce a B -valued inner product on it, consider the conditional expectation $E: A \rightarrow B$ given by $E = (\iota \otimes \varphi_0)\Delta_R$. Then set

$$\langle \xi_1 \otimes a_1, \xi_2 \otimes a_2 \rangle = (\xi_2, \xi_1)E(a_1^* a_2).$$

Lemma 3.5 *The module X_t is an α -equivariant f.g. right Hilbert B -module.*

Proof. Consider the left action of \hat{A}_0 on A given by $\omega \triangleright a = (\iota \otimes \omega)\Delta_R(a)$. Let $A_t = B(H_{\bar{t}}) \triangleright A$ be the spectral subspace of A corresponding to t . Equivalently A_t can be described as the linear span of elements $(\iota \otimes \varphi_0(u \cdot))\Delta_R(a)$, where $a \in A$ and u is a matrix coefficient of V^t . Since $X_t = E_t(H_t \otimes A)$, it follows that $X_t \subset H_t \otimes A_t$. Thus it is enough to prove that A_t considered as a B -module with inner product $\langle a_1, a_2 \rangle = E(a_1^* a_2)$ is a f.g. right Hilbert B -module.

To check that A_t is complete, choose matrix units e_{ij} in $B(H_{\bar{t}})$ and consider the corresponding matrix coefficients v_{ij} for $V^{\bar{t}}$. Then define B -module mappings $E_{ij}: A_t \rightarrow A_t$ by letting $E_{ij}(a) = e_{ij} \triangleright a$, so

$$\Delta_R(a) = \sum_{i,j} E_{ij}(a) \otimes v_{ij}.$$

The orthogonality relations imply that the B -valued inner product $\langle a_1, a_2 \rangle' = \sum_{i,j} E_{ij}(a_1^*) E_{ij}(a_2)$ defines an equivalent norm. Since $a = \sum_{i,j} \varepsilon_0(v_{ij}) E_{ij}(a)$ for $a \in A_t$, where ε_0 is the counit on \mathcal{A}_0 , it follows that the topology on A_t is the usual norm topology inherited from A . Thus A_t is complete.

It remains to show that A_t is finitely generated. This is, in fact, known (see e.g. [D] for a more general statement), but we include a proof for the sake of completeness. Since $A_t^* = A_{\bar{t}}$, it is enough to show that A_t is finitely generated as a left B -module. Let $U \in A \otimes B(H)$ be a finite dimensional unitary corepresentation of (A, Δ) . Then $(P \otimes \iota)(U)$ is a unitary corepresentation of (A_0, Δ_0) . We can decompose it into irreducible ones, and then the image of the space of matrix coefficients of U under P is precisely the space of matrix coefficients of those irreducible unitary corepresentations. Since $P(\mathcal{A})$ is dense in A_0 , using orthogonality relations, we conclude that there exists an irreducible unitary corepresentation U such that $V^{\bar{t}}$ is a subcorepresentation of $(P \otimes \iota)(U)$. Thus we can identify $H_{\bar{t}}$ with a subspace of H and complete the orthonormal basis ξ_1, \dots, ξ_m in $H_{\bar{t}}$ to an orthonormal basis ξ_1, \dots, ξ_n in H . Consider the corresponding matrix coefficients u_{ij} , $1 \leq i, j \leq n$, of U . Then $P(u_{ij}) = v_{ij}$ for $1 \leq i, j \leq m$, so the elements $b_{ij} = u_{ij}$, $1 \leq i \leq n$, $1 \leq j \leq m$, lie in A_t . We claim that they generate A_t as a left B -module. Indeed, for $a \in A_t$ set

$$a_{ij} = \sum_{k=1}^m E_{kj}(a) b_{ik}^*, \quad 1 \leq i \leq n, \quad 1 \leq j \leq m.$$

Since $\Delta_R(E_{kj}(a)) = \sum_{p=1}^m E_{pj}(a) \otimes v_{pk}$, we get

$$\Delta_R(a_{ij}) = \sum_{k,p,q=1}^m (E_{pj}(a) \otimes v_{pk})(u_{iq}^* \otimes v_{qk}^*) = \sum_{p=1}^m E_{pj}(a) u_{ip}^* \otimes 1 = a_{ij} \otimes 1,$$

so $a_{ij} \in B$. Then we compute

$$\sum_{i=1}^n \sum_{j=1}^m a_{ij} b_{ij} = \sum_{i=1}^n \sum_{j,k=1}^m E_{kj}(a) u_{ik}^* u_{ij} = \sum_{j=1}^m E_{jj}(a) = I_{\hat{t}} a = a.$$

Now we can formulate the main result of this subsection. ■

Theorem 3.6 *Any α -equivariant f.g. right Hilbert B -module is isomorphic to a unique finite direct sum of modules X_t , $t \in \text{Irr}(A_0, \Delta_0)$.*

In particular, $K_0^{\hat{A}}(B)$ is a free abelian group with basis $[X_t]$, $t \in \text{Irr}(A_0, \Delta_0)$.

To prove Theorem we first realize $B \rtimes \hat{A}$ as a subalgebra of $B(L^2(A, \varphi))$. Since $B \subset A$, we can consider $B \rtimes \hat{A}$ as a subalgebra of $A \rtimes \hat{A}$. Then we can apply the following particular case of the Takesaki-Takai duality, see e.g. [BS2].

Lemma 3.7 *The representations of A and of \hat{A} on $L^2(A, \varphi)$ define an isomorphism of $A \rtimes \hat{A}$ onto $K(L^2(A, \varphi))$.*

Specifically, the homomorphism $B(L^2(A, \varphi)) \rightarrow B(L^2(A, \varphi) \otimes L^2(A, \varphi))$, $x \mapsto V(x \otimes 1)V^$, sends $\omega \in \hat{A}$ to $\omega \otimes 1$ and $a \in A$ to $\Delta(a)$, where $V = (J\hat{J} \otimes 1)\Sigma W\Sigma(J\hat{J} \otimes 1)$, J is the modular involution corresponding to φ , \hat{J} is the modular involution corresponding to the right invariant Haar weight on \hat{A} , Σ is the flip, and W is the multiplicative unitary for (A, Δ) . ■*

Thus there is no ambiguity in writing $a\omega$ irrespectively of whether we consider it as an element of $A \rtimes \hat{A}$ or as an operator on $L^2(A, \varphi)$.

Define a unitary U_0 on $L^2(A, \varphi) \otimes L^2(A_0, \varphi_0)$ by

$$U_0(a\xi_\varphi \otimes \xi) = \Delta_R(a)(\xi_\varphi \otimes \xi).$$

Then define a right coaction $\tilde{\Delta}_R: K(L^2(A, \varphi)) \rightarrow M(K(L^2(A, \varphi)) \otimes A_0)$ of (A_0, Δ_0) on $K(L^2(A, \varphi))$ by setting $\tilde{\Delta}_R(x) = U_0(x \otimes 1)U_0^*$. We obviously have $\tilde{\Delta}_R(a) = \Delta_R(a)$ for $a \in A$. Since the representation of \hat{A} on $L^2(A, \varphi)$ by definition has the property $\omega a \xi_\varphi = (\hat{S}^{-1}(\omega) \otimes 1)\Delta(a)\xi_\varphi$, we see also that $\tilde{\Delta}_R(\omega) = \omega \otimes 1$. It follows that we can identify $B \rtimes \hat{A}$ with $(A \rtimes \hat{A})^{\tilde{\Delta}_R}$.

Choose an irreducible unitary corepresentation $U \in A \otimes B(H)$ such that V^t is a subcorepresentation of $(P \otimes \iota)(U)$, and choose an orthonormal basis ξ_1, \dots, ξ_n in H such that ξ_1, \dots, ξ_m is an orthonormal basis in H_t . Let u_{ij} , $1 \leq i, j \leq n$, be the corresponding matrix coefficients for U , and v_{ij} , $1 \leq i, j \leq m$, the matrix coefficients for V^t . Fix i , $1 \leq i \leq n$.

Lemma 3.8 *The map $H_t \otimes \mathcal{A} \rightarrow K(L^2(A, \varphi))$, $\xi_j \otimes a \mapsto u_{ij} I_0 a$, defines a right $(\mathcal{A} \rtimes \hat{A})$ -module and right \mathcal{A}_0 -comodule isomorphism of $H_t \otimes \mathcal{A}$ onto $p_t(\mathcal{A} \rtimes \hat{A})$, where $p_t \in K(L^2(A, \varphi))$ is the projection onto the space spanned by $u_{ij} \xi_\varphi$, $1 \leq j \leq m$.*

Proof. We have already used the isomorphisms $\mathcal{A} \cong I_0 \hat{A} \otimes \mathcal{A} \cong I_0(\mathcal{A} \rtimes \hat{A})$, where $I_0 \in \hat{A}$ is the projection such that $I_0 \omega = \hat{\varepsilon}(\omega) I_0$. In other words, the map $\mathcal{A} \rightarrow I_0(\mathcal{A} \rtimes \hat{A})$, $a \mapsto I_0 a$, is an $(\mathcal{A} \rtimes \hat{A})$ -module isomorphism. Note also that since I_0 is the projection onto $\mathbb{C}\xi_\varphi$, the operator $u_{ij} I_0$ is, up to a scalar, a partial isometry with initial space $\mathbb{C}\xi_\varphi$ and range space $\mathbb{C}u_{ij} \xi_\varphi$. It follows that the map in the formulation of Lemma is an injective morphism of $(\mathcal{A} \rtimes \hat{A})$ -modules. Moreover, since p_t is a linear combination of $u_{ij} I_0 u_{ij}^*$, we have $u_{ij} I_0(\mathcal{A} \rtimes \hat{A}) \subset p_t(\mathcal{A} \rtimes \hat{A})$ and $p_t(\mathcal{A} \rtimes \hat{A}) \subset \sum_j u_{ij} I_0(\mathcal{A} \rtimes \hat{A})$. Thus the map is an isomorphism of the $(\mathcal{A} \rtimes \hat{A})$ -modules $H_t \otimes \mathcal{A}$ and $p_t(\mathcal{A} \rtimes \hat{A})$. It remains to check that it is an \mathcal{A}_0 -comodule morphism. By definition we have

$$\delta_t(\xi_j \otimes a) = V_{31}^t(\xi_j \otimes 1 \otimes 1)(1 \otimes \Delta_R(a)) = \sum_k (\xi_k \otimes 1 \otimes v_{kj})(1 \otimes \Delta_R(a)).$$

On the other hand,

$$\tilde{\Delta}_R(u_{ij}I_0a) = \sum_k (u_{ik} \otimes v_{kj})(I_0 \otimes 1)\Delta_R(a).$$

Thus the map is indeed an \mathcal{A}_0 -comodule morphism. ■

Proof of Theorem 3.6. Consider the $(\mathcal{B} \rtimes \hat{\mathcal{A}})$ -module $\chi_t = (H_t \otimes \mathcal{A})^{\delta_t}$. As an immediate consequence of Lemma 3.8 we have $\chi_t \cong p_t(\mathcal{B} \rtimes \hat{\mathcal{A}})$. By Theorem 3.4 we get a f.g. α -equivariant right Hilbert B -module $\tilde{X}_t = \chi_t \otimes_{\mathcal{B}} B$. The inclusion $\chi_t \hookrightarrow X_t$ induces a morphism $\tilde{X}_t \rightarrow X_t$ of α -equivariant f.g. right Hilbert B -modules. This map is surjective, since by the proof of Lemma 3.5, we know that $\chi_t B = X_t$. The map is also injective, since it is equivariant and injective on χ_t , and since any equivariant B -module map on \tilde{X}_t maps χ_t into itself. Thus $\tilde{X}_t \cong X_t$. By Theorem 3.4 it is thus enough to show that any projection in $K(l^2(\mathbb{N})) \otimes (B \rtimes \hat{\mathcal{A}})$ is equivalent to a unique direct sum of projections p_t , $t \in \text{Irr}(A_0, \Delta_0)$.

Note that $(U_0)_{21} \in M(A_0 \otimes K(L^2(A, \varphi)))$ is a unitary corepresentation of (A_0, Δ_0) . Thus we can decompose the space $L^2(A, \varphi) = \oplus_t (H_t \otimes L_t)$ so that $(U_0)_{21} = \oplus_t (V_t \otimes 1)$. Then $B \rtimes \hat{\mathcal{A}} \cong \oplus_t K(L_t)$, so to prove Theorem, it is enough to show that under this isomorphism p_t becomes a minimal projection in $K(L_t)$. Equivalently, we must show that the restriction of $(U_0)_{21}$ to $p_t L^2(A, \varphi)$ is isomorphic to V^t . This is indeed the case since in the notation of the proof of Lemma 3.8 we have

$$(U_0)_{21}(\xi \otimes u_{ij}\xi_\varphi) = \sum_k v_{kj}\xi \otimes u_{ik}\xi_\varphi.$$
■

References

- [AK1] Akbarpour R., Khalkhali M., *Equivariant cyclic cohomology of Hopf module algebras*, preprint math.KT/0009236.
- [AK2] Akbarpour R., Khalkhali M., *Hopf algebra equivariant cyclic homology and cyclic homology of crossed product algebras*, preprint math.KT/0011248.
- [BS1] Baaq S., Skandalis G., *C*-algèbres de Hopf et théorie de Kasparov équivariante*, *K-Theory* **2** (1989), 683–721.
- [BS2] Baaq S., Skandalis G., *Unitaires multiplicatifs et dualité pour les produits croisés de C*-algèbres*, *Ann. Sci. cole Norm. Sup. (4)* **26** (1993), 425–488.
- [C] Connes A. *Noncommutative geometry*. Academic Press, Inc., San Diego, CA, 1994.
- [D] Durdević M., *Geometry of quantum principal bundles. II*, *Rev. Math. Phys.* **9** (1997), 531–607.
- [J] Julg P., *K-théorie équivariante et produits croisés*, *C. R. Acad. Sci. Paris Sér. I Math.* **292** (1981), 629–632.
- [K] Karoubi M. *K-theory. An introduction*. Grundlehren der Mathematischen Wissenschaften, Band 226. Springer-Verlag, Berlin-New York, 1978.
- [KKL] Klimek S., Kondracki W., Lesniewski A., *Equivariant entire cyclic cohomology. I. Finite groups*, *K-Theory* **4** (1991), 201–218.

- [L] Loday J.-L. Cyclic homology. Springer-Verlag, Berlin, 1998.
- [KMT] Kustermans J., Murphy G. J., Tuset L., *Differential calculi over quantum groups and twisted cyclic cocycles*, J. Geom. Phys. **44** (2003), 570–594.
- [NT] Neshveyev S., Tuset L., *The Martin boundary of a discrete quantum group*, preprint math.OA/0209270.

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