A Local Index Formula for the Quantum Sphere

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Abstract

For the Dirac operator D on the standard quantum sphere we obtain an asymptotic expansion of the $SU_q(2)$-equivariant entire cyclic cocycle corresponding to $\varepsilon^{\frac{1}{2}}D$ when evaluated on the element $k^2 \in U_q(\mathfrak{su}_2)$. The constant term of this expansion is a twisted cyclic cocycle which up to a scalar coincides with the volume form and computes the quantum as well as the classical Fredholm indices.

Introduction

The geometry of q-deformed spaces, see e.g. [KS], and the theory of non-commutative geometry of Connes [C1] have developed considerably over the last twenty years. Crucial in establishing connections between these two areas is the construction of Dirac operators which give rise to reasonable differential calculi. Other aspects of the theory include computations of cyclic cohomology and analysis of the index theorem. Work in this direction has progressed furthest for spaces like $SU_q(2)$ and the quantum spheres, see e.g. [CP, DS, SW, MNW1, MNW2, C2].

The purpose of this paper is to show that, although the spectral triple associated with the Dirac operator [DS] on the homogeneous sphere S^2_q of Podleś has properties quite different from the classical one, the machine of non-commutative geometry by Connes, when gently tuned, works perfectly well also in this case. Specifically, we obtain an index formula for S^2_q.

Concerning the differences from the classical case, we mention that the ζ-function of the Dirac operator has infinitely many poles on vertical lines and the traces $\mathrm{Tr}(e^{-tD^2})$ of the heat operators tend to infinity slower than t^{-p} for any $p > 0$. More importantly, the spectral triple does not satisfy the regularity assumption, a condition which is often overshadowed by other assumptions such as boundedness of commutators, but which is crucial for a detailed analysis. This can mean that the Dirac operator is, in fact, not the right one, and one can try to construct another operator with the same Fredholm module. However, in this process one will most likely lose the $SU_q(2)$-equivariance [DS]. What one gains is not so clear: with the absence of Getzler's symbol calculus and of a Wodzicki-type geometric description of the Dixmier trace, the computation of the indices in terms of the Dixmier trace remains for the moment a non-trivial problem. So we will stick to the standard Dirac operator. The associated spectral triple is $SU_q(2)$-equivariant and thus, via the JLO-cocycle, defines the Chern character in the equivariant entire cyclic cohomology of S^2_q. We evaluate this cocycle on $k^2 \in U_q(\mathfrak{su}_2)$. The resulting object is an entire twisted cyclic cocycle, and its pairing with the equivariant K-theory computes the quantum Fredholm index, that is, the difference of quantum dimensions of the kernel and the cokernel [NT]. The possibility of detecting cocycles by evaluating them on k^2 was pointed out by Connes in [C2]. As was promised in [NT] such evaluations are much easier to compute. The philosophical reason for this is that the quantum dimension is intrinsically associated with the tensor category of finite dimensional corepresentations of a compact quantum group. On the technical side, the evaluation on k^2 gives

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an immediate connection to the Haar state, as was already remarked in [G, SW], and thus serves as a replacement for the trace theorem by Connes on equality of the Dixmier trace and the Wodzicki residue. Note also that for q-deformations the classical index can be recovered from the quantum one: if one can prove that the Fredholm operators depend continuously on q and has a polynomial formula in q and q^{-1} for the quantum index, the classical index is obtained by simply setting $q = 1$.

The evaluation on k^2 does not, however, solve all problems: the spectral triple is still non-regular, and while the quantum traces of the heat operators are better controlled than the classical ones, they possess a strange oscillating behavior near zero. The non-regularity of the spectral triple can be illuminated by saying that though the principal symbol of $|D|$ is scalar-valued, the operator $|D|^z T|D|^{-z}$ does not necessarily have the same symbol as T. The symbol is, however, computable, and this is the key property allowing to apply the ideas of the proof of the local index formula of Connes-Moscovici [CM2] to our situation. It seems that the development of a full-scaled pseudo-differential calculus for the Dirac operators on q-deformed spaces is the main prerequisite for the analysis of more general examples, such as the ones in [Kr].

The Dirac operator on the quantum sphere is defined using the standard differential calculus, and in the course of our analysis we use very little beyond basic properties of this calculus and some information on the spectrum of the Dirac operator. In this respect the fact that Connes' non-commutative geometry can be applied successfully to the study of a q-deformed space is more important than the final result for the quantum sphere.

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1 Cyclic Cohomology

In this section we formulate some results of non-commutative geometry in the Hopf algebra equivariant setting. Fortunately, as the Hopf algebra equivariant cyclic cohomology theory is now available [AK, NT], a simple argument involving crossed products allows to transfer such results from the non-equivariant case without having to prove each one of them from scratch.

We use the same notation and make the same assumptions as in [NT]. So let $(\mathcal{H}, \Delta)$ be a Hopf algebra over $\mathbb{C}$ with invertible antipode S and counit ε, and adapt the Sweedler notation $\Delta(\omega) = \omega_{(0)} \otimes \omega_{(1)}$. The results below are also true for the algebra of finitely supported functions on a discrete quantum group, which is of main interest for us. Let $\mathcal{B}$ be a unital right $\mathcal{H}$-module algebra with right action of $(\mathcal{H}, \Delta)$ denoted by $\triangleleft$. Consider the space $C^n_{\mathcal{H}}(\mathcal{B})$ of $\mathcal{H}$-invariant n-cochains, so $C^n_{\mathcal{H}}(\mathcal{B})$ consists of linear functionals on $\mathcal{H} \otimes \mathcal{B}^{\otimes(n+1)}$ such that

$$f(S^{-1}(\omega_{(0)})\eta\omega_{(1)}; b_0 \triangleleft \omega_{(2)}, \dots, b_n \triangleleft \omega_{(n+2)}) = \varepsilon(\omega) f(\eta; b_0, \dots, b_n)$$

for any $\omega, \eta \in \mathcal{H}$ and $b_0, \dots, b_n \in \mathcal{B}$. The coboundary operator $b_n \colon C^{n-1}_{\mathcal{H}}(\mathcal{B}) \to C^n_{\mathcal{H}}(\mathcal{B})$ and the cyclic operator $\lambda_n \colon C^n_{\mathcal{H}}(\mathcal{B}) \to C^n_{\mathcal{H}}(\mathcal{B})$ are given by

$$(b_n f)(\omega; b_0, \dots, b_n) = \sum_{i=0}^{n-1} (-1)^i f(\omega; b_0, \dots, b_{i-1}, b_i b_{i+1}, b_{i+2}, \dots, b_n)$$
$$+(-1)^n f(\omega_{(0)}; b_n \triangleleft \omega_{(1)} b_0, b_1, \dots, b_{n-1})$$

and

$$(\lambda_n f)(\omega; b_0, \dots, b_n) = (-1)^n f(\omega_{(0)}; b_n \triangleleft \omega_{(1)}, b_0, \dots, b_{n-1}).$$

The subcomplex $(\mathrm{Ker}(\iota-\lambda),b)$ of $(C^\bullet_{\mathcal{H}}(\mathcal{B}),b)$ is denoted by $(C^\bullet_{\mathcal{H},\lambda}(\mathcal{B}),b)$ and its cohomology is denoted by $HC^\bullet_{\mathcal{H}}(\mathcal{B})$. There is a pairing $\langle\cdot,\cdot\rangle\colon HC^{2n}_{\mathcal{H}}(\mathcal{B})\times K^{\mathcal{H}}_0(\mathcal{B})\to R(\mathcal{H})$ of the even cyclic cohomology with equivariant K-theory, where $R(\mathcal{H})$ is the space of $\mathcal{H}$-invariant linear functionals on $\mathcal{H}$ with $\mathcal{H}$ acting on itself by

$$\eta\triangleleft\omega = S^{-1}(\omega_{(0)})\eta\omega_{(1)}.$$

If $p\in\mathcal{B}$ is an $\mathcal{H}$-invariant idempotent and $f\in C^{2n}_{\mathcal{H},\lambda}(\mathcal{B})$ is a cyclic cocycle, then

$$\langle [f],[p]\rangle(\omega) = \frac{1}{n!}f(\omega;p,\dots,p). \tag{1.1}$$

More generally, if X is a finite dimensional (as a vector space) right $\mathcal{H}$-module, then $\mathrm{End}(X)\otimes\mathcal{B}$ is a right $\mathcal{H}$-module algebra with right action $\blacktriangleleft$ given by

$$(T\otimes b)\blacktriangleleft\omega = \pi_X(\omega_{(0)})T\pi_X S^{-1}(\omega_{(2)})\otimes b\triangleleft\omega_{(1)},$$

where $\pi_X\colon\mathcal{H}\to\mathrm{End}(X)$ is the anti-homomorphism defining the $\mathcal{H}$-module structure on X. Then we have a map $\Psi^n_X\colon C^n_{\mathcal{H}}(\mathcal{B})\to C^n_{\mathcal{H}}(\mathrm{End}(X)\otimes\mathcal{B})$ given by

$$(\Psi^n_X f)(\omega;T_0\otimes b_0,\dots,T_n\otimes b_n) = f(\omega_{(0)};b_0,\dots,b_n)\mathrm{Tr}(\pi_X S^{-1}(\omega_{(1)})T_0\dots T_n).$$

Now suppose $p\in\mathrm{End}(X)\otimes\mathcal{B}$ is an $\mathcal{H}$-invariant idempotent, so it defines an element of $K^{\mathcal{H}}_0(\mathcal{B})$. By definition we have

$$\langle [f],[p]\rangle = \langle[\Psi^{2n}_X f],[p]\rangle$$

for a cyclic cocycle $f\in C^{2n}_{\mathcal{H},\lambda}(\mathcal{B})$.

We will also need the (b,B)-bicomplex description of the periodic cyclic cohomology. So consider the operator $B=NB_0$, where $N_n\colon C^n_{\mathcal{H}}(\mathcal{B})\to C^n_{\mathcal{H}}(\mathcal{B})$, $N_n=\sum_{i=0}^n\lambda^i_n$, $B^n_0\colon C^{n+1}_{\mathcal{H}}(\mathcal{B})\to C^n_{\mathcal{H}}(\mathcal{B})$, $B^n_0=(-1)^n s^n_n(\iota-\lambda_n)$, so

$$(B^n_0 f)(\omega;b_0,\dots,b_n) = f(\omega;1,b_0,\dots,b_n)-(-1)^{n+1}f(\omega;b_0,\dots,b_n,1).$$

Set $\mathcal{C}^0_{\mathcal{H}}(\mathcal{B})=\oplus_{n=0}^\infty C^{2n}_{\mathcal{H}}(\mathcal{B})$ and $\mathcal{C}^1_{\mathcal{H}}(\mathcal{B})=\oplus_{n=0}^\infty C^{2n+1}_{\mathcal{H}}(\mathcal{B})$. Then we have a well-defined complex

$$\mathcal{C}^1_{\mathcal{H}}(\mathcal{B})\xrightarrow{b+B}\mathcal{C}^0_{\mathcal{H}}(\mathcal{B})\xrightarrow{b+B}\mathcal{C}^1_{\mathcal{H}}(\mathcal{B})\xrightarrow{b+B}\mathcal{C}^0_{\mathcal{H}}(\mathcal{B}).$$

The cohomology of this small complex is denoted by $HP^n_{\mathcal{H}}(\mathcal{B})$, $n=0,1$. Then the formula

$$\langle[(f_{2n})_n],[p]\rangle(\omega) = \sum_{n=0}^\infty(-1)^n\frac{(2n)!}{n!}(\Psi^{2n}_X f_{2n})(\omega;p-\frac{1}{2},p,\dots,p) \tag{1.2}$$

defines a pairing $HP^0_{\mathcal{H}}(\mathcal{B})\times K^{\mathcal{H}}_0(\mathcal{B})\to R(\mathcal{H})$ (this can be deduced from the non-equivariant case in the same way as in [NT]). The map

$$C^{2n}_{\mathcal{H}}(\mathcal{B})\to\mathcal{C}^0_{\mathcal{H}}(\mathcal{B}),\quad f\mapsto\frac{(-1)^n}{(2n)!}f, \tag{1.3}$$

induces a homomorphism $HC^{2n}_{\mathcal{H}}(\mathcal{B})\to HP^0_{\mathcal{H}}(\mathcal{B})$ compatible with pairings (1.1) and (1.2) and which respects the periodicity operator in the sense that the images of f and Sf in HP^0 are equal.

Consider the subspaces $CE^k_{\mathcal{H}}(\mathcal{B})\subset\prod_{n=0}^\infty C^{2n+k}_{\mathcal{H}}(\mathcal{B})$, $k=0,1$, consisting of all cochains $(f_{2n+k})_n$ such that for any $\omega\in\mathcal{H}$ and any finite subset $F\subset\mathcal{B}$ there exists $C>0$ such that

$$|f_{2n+k}(\omega_{(0)};b_0\triangleleft\omega_{(1)},\dots,b_{j-1}\triangleleft\omega_{(j)},b_j,\dots,b_{2n+k})|\le\frac{C}{n!}$$

for any $n \geq 0$, $0 \leq j \leq 2n+k+1$ and $b_i \in F$. Denote by $HE^k_{\mathcal{H}}(\mathcal{B})$ the cohomology of the complex

$$CE^1_{\mathcal{H}}(\mathcal{B}) \xrightarrow{b+B} CE^0_{\mathcal{H}}(\mathcal{B}) \xrightarrow{b+B} CE^1_{\mathcal{H}}(\mathcal{B}) \xrightarrow{b+B} CE^0_{\mathcal{H}}(\mathcal{B}).$$

The formula (1.2) still defines a pairing of $HE^0_{\mathcal{H}}(\mathcal{B})$ with $K^{\mathcal{H}}_0(\mathcal{B})$.

Suppose we are given an equivariant even spectral triple. By this we mean a Hilbert space H with grading operator γ, an odd selfadjoint operator D with compact resolvent, and even representations of $\mathcal{B}$ and $\mathcal{H}$ on H such that D commutes with $\mathcal{H}$, $[D,b]$ is bounded for any $b \in \mathcal{B}$ and

$$b\omega = \omega_{(0)} b \triangleleft \omega_{(1)} \quad \forall b \in \mathcal{B},\ \forall \omega \in \mathcal{H}. \tag{1.4}$$

If the spectral triple is θ-summable in the sense that e^{-tD^2} is trace-class for all $t>0$, set

$$\mathrm{Ch}^{2n}(D)(\omega; b_0,\dots,b_{2n}) = \int_{\Delta_{2n}} dt \mathrm{Tr}(\gamma\omega b_0 e^{-t_0D^2}[D,b_1]e^{-t_1D^2}\dots[D,b_{2n}]e^{-t_{2n}D^2}),$$

where $\int_{\Delta_{2n}} dt$ means integration over the simplex $\Delta_{2n} = \{(t_0,\dots,t_{2n}) \mid t_i \geq 0, \sum_i t_i = 1\}$. Then $(\mathrm{Ch}^{2n}(D))_n$ is a cocycle in $CE^0_{\mathcal{H}}(\mathcal{B})$ having all the usual properties of a JLO-cocycle [C1, CM1, GBVF]:

(i) $(\mathrm{Ch}^{2n}(tD))_n$ and $(\mathrm{Ch}^{2n}(sD))_n$ are cohomologous for any $s,t>0$;

(ii) the pairing of $K^{\mathcal{H}}_0(\mathcal{B})$ with $(\mathrm{Ch}^{2n}(tD))_n$ computes the index map defined by D (so e.g. for an $\mathcal{H}$-invariant idempotent $p \in B$ we get $\langle[(\mathrm{Ch}^{2n}(D))_n],[p]\rangle(\omega) = \mathrm{Tr}(\omega|_{\mathrm{Ker}\, p_-Dp_+}) - \mathrm{Tr}(\omega|_{\mathrm{Ker}\, p_+Dp_-})$);

(iii) if the spectral triple is p-summable in the sense that $|D|^{-p}$ is a trace-class operator, $F = D|D|^{-1}$, and τ^{2m}_F denotes the Chern character of the Fredholm module (H,F,γ) in the equivariant cyclic cohomology for some fixed m, $2m \geq p$, so

$$\tau^{2m}_F(\omega; b_0,\dots,b_{2m}) = \frac{(-1)^m}{2} m! \mathrm{Tr}(\gamma\omega F[F,b_0]\dots[F,b_{2m}]),$$

then $(\mathrm{Ch}^{2n}(D))_n$ is cohomologous to the image of τ^{2m}_F under the map $C^{2m}_{\mathcal{H}}(\mathcal{B}) \to CE^0_{\mathcal{H}}(\mathcal{B})$ given by (1.3).

As in the non-equivariant case, all these properties are consequences of the homotopy invariance of the cohomology class of $(\mathrm{Ch}^{2n}(D))_n$ meaning that if we have an $\mathcal{H}$-invariant homotopy D_t such that e.g. $\dot{D}_t$ is bounded, then $(\mathrm{Ch}^{2n}(D_0))_n$ and $(\mathrm{Ch}^{2n}(D_1))_n$ are cohomologous (concerning (ii) see also [KL]). This in turn can be deduced from the non-equivariant case as follows, see the discussion at the end of [C2]. Consider the crossed product algebra $\mathcal{B} \rtimes \mathcal{H}$, i.e. the vector space $\mathcal{B} \otimes \mathcal{H}$ with product

$$(b \otimes \omega)(c \otimes \eta) = bc \triangleleft S^{-1}(\omega_{(1)}) \otimes \omega_{(0)}\eta.$$

In the sequel we write $b\omega$ instead of $b \otimes \omega$. Then we have a map $\Phi^n\colon C^n_{\mathcal{H}}(\mathcal{B}) \to C^n(\mathcal{B} \rtimes \mathcal{H})$ given by

$$(\Phi^n f)(b_0\omega^0,\dots,b_n\omega^n) = f(\omega^0_{(0)}\dots\omega^n_{(0)}; b_0 \triangleleft (\omega^0_{(1)}\dots\omega^n_{(1)}), b_1 \triangleleft (\omega^1_{(2)}\dots\omega^n_{(2)}),\dots, b_n \triangleleft \omega^n_{(n+1)}).$$

The definition of Φ^n is motivated by the following easily proved lemma.

Lemma 1.1 *Consider a covariant representation of $\mathcal{B}$ and $\mathcal{H}$ on H (that is, identity (1.4) holds). Suppose $c_0,\dots,c_n \in B(H)$ commute with $\mathcal{H}$. Then for any $b_0,\dots,b_n \in \mathcal{B}$ and $\omega^0,\dots,\omega^n \in \mathcal{H}$ we have*

$$c_0b_0\omega^0\dots c_nb_n\omega^n = \omega^0_{(0)}\dots\omega^n_{(0)} c_0b_0 \triangleleft (\omega^0_{(1)}\dots\omega^n_{(1)}) c_1b_1 \triangleleft (\omega^1_{(2)}\dots\omega^n_{(2)})\dots c_nb_n \triangleleft \omega^n_{(n+1)}.$$

■

It is not difficult to check that the maps Φ^n constitute a morphism of cocyclic objects, so they induce maps for the various cyclic cohomology theories. Note also that if $\mathcal{H}$ is unital, or $\mathcal{H}$ has an approximate unit in an appropriate sense, then Φ^n is injective. So to prove a property for a cochain in an equivariant theory it is enough to establish the analogous property for the image of the cochain under Φ. Now note that any equivariant spectral triple can be considered as a spectral triple for $\mathcal{B} \rtimes \mathcal{H}$. It follows immediately from Lemma 1.1 that $(\Phi^{2n}(\mathrm{Ch}^{2n}(D))_n$ is the JLO-cocycle for this spectral triple. This allows to deduce the properties of $(\mathrm{Ch}^{2n}(D))_n$ from the non-equivariant case.

From now onwards we assume that $(\mathcal{H}, \Delta)$ is the algebra of finitely supported functions on a discrete quantum group $(\hat{A}, \hat{\Delta})$, and as in [NT] we write $(\hat{\mathcal{A}}, \hat{\Delta})$ instead of $(\mathcal{H}, \Delta)$. Denote by $\rho \in M(\hat{\mathcal{A}})$ the Woronowicz character f_{-1}. We will be interested in evaluating equivariant cocycles on $\omega = \rho$. The following known lemma explains why such evaluations are easier to deal with.

Lemma 1.2 *Let $U \in M(A \otimes K(H))$ be a unitary corepresentation of the dual compact group (A, Δ), and $\alpha_U\colon B(H) \to M \otimes B(H)$ the coaction of the von Neumann closure (M, Δ) of (A, Δ) on $B(H)$,*

$$\alpha_U(x) = U^*(1 \otimes x)U.$$

Consider also the corresponding representation of $\hat{A}$ on H, so $\omega\xi = (\omega \otimes \iota)(U)\xi$. Then the map $a \mapsto \mathrm{Tr}(\cdot a\rho)$ defines a one-to-one correspondence between positive elements $a \in B(H)^{\alpha_U}$ such that $\mathrm{Tr}(a\rho) = 1$ and normal α_U-invariant states on $B(H)$.

In particular, if $B \subset B(H)$ is an α_U-invariant C-subalgebra with a unique α_U-invariant state φ, then*

$$\mathrm{Tr}(ba\rho) = \varphi(b)\mathrm{Tr}(a\rho)$$

for any $b \in B$ and any $a \in B(H)^{\alpha_U}_+$ with $\mathrm{Tr}(a\rho) < \infty$. ■

Note also that the fixed point algebra $B(H)^{\alpha_U}$ is precisely the commutant of $\hat{A}$.

Since in general $\rho \notin \hat{\mathcal{A}}$, one needs some care in dealing with this element. From now on we assume as in [NT] that we have a left coaction α of (A, Δ) on a C*-algebra B, and $\triangleleft$ is the corresponding action of $\hat{\mathcal{A}}$,

$$b\triangleleft\omega = (\omega \otimes \iota)\alpha(b),$$

while $\mathcal{B} = B\triangleleft\hat{\mathcal{A}}$. Then $\sigma_z(b) = b\triangleleft\rho^z$ is a well-defined element of $\mathcal{B}$ for any $b \in \mathcal{B}$ and $z \in \mathbb{C}$. The automorphism $\sigma = \sigma_1$ will be called the twist. Now if we have an equivariant θ-summable spectral triple, for the expression $\mathrm{Ch}^{2n}(D)(\rho; b_0, \dots, b_{2n})$ to make sense it is enough to require that

$$\mathrm{Tr}(\rho e^{-sD^2}) < \infty \quad \forall s > 0.$$

Indeed, since $z \mapsto \rho^{-z}[D, b]\rho^z = [D, b\triangleleft\rho^z] \in B(H)$ is an analytic function, the Hölder inequality and the identity

$$\begin{aligned} &e^{-t_0D^2}[D, b_1]e^{-t_1D^2} \dots [D, b_{2n}]e^{-t_{2n}D^2}\rho \\ &\quad = e^{-t_0D^2}\rho^{t_0}[D, b_1\triangleleft\rho^{t_0}]e^{-t_1D^2}\rho^{t_1}[D, b_2\triangleleft\rho^{t_0+t_1}]e^{-t_2D^2}\rho^{t_2} \dots [D, b_{2n}\triangleleft\rho^{1-t_{2n}}]e^{-t_{2n}D^2}\rho^{t_{2n}} \end{aligned} \tag{1.5}$$

show that $\mathrm{Ch}^{2n}(D)(\rho; b_0, \dots, b_{2n})$ is well-defined. In fact, these elements form a cocycle in an appropriate cyclic cohomology theory, namely, in the entire twisted cyclic cohomology $HE^0_\sigma(\mathcal{B})$, cf [KMT, G]. This is immediate as such twisted theories are obtained by setting $\omega = \rho$ in the formulas above. Thus the space of n-cochains $C^n_\sigma(\mathcal{B})$ becomes the space of linear functionals f on $\mathcal{B}^{\otimes(n+1)}$ such that $f(\sigma(b_0), \dots, \sigma(b_n)) = f(b_0, \dots, b_n)$, the cyclic operator is given by $(\lambda_n f)(b_0, \dots, b_n) = (-1)^n f(\sigma(b_n), b_0, \dots, b_{n-1})$, and so on. In particular, we still have pairings of $K^{\mathcal{H}}_0(\mathcal{B})$ with $HC^{2n}_\sigma(\mathcal{B})$, $HP^0_\sigma(\mathcal{B})$ and $HE^0_\sigma(\mathcal{B})$.

2 Differential Calculus on the Quantum Sphere

From now on (A, Δ) will denote the compact quantum group $SU_q(2)$ of Woronowicz [W], $q \in (0, 1)$. So A is the universal unital C*-algebra with generators α and γ satisfying the relations

$$\alpha^*\alpha + \gamma^*\gamma = 1, \quad \alpha\alpha^* + q^2\gamma^*\gamma = 1, \quad \gamma^*\gamma = \gamma\gamma^*, \quad \alpha\gamma = q\gamma\alpha, \quad \alpha\gamma^* = q\gamma^*\alpha.$$

The comultiplication Δ is determined by the formulas

$$\Delta(\alpha) = \alpha \otimes \alpha - q\gamma^* \otimes \gamma, \quad \Delta(\gamma) = \gamma \otimes \alpha + \alpha^* \otimes \gamma.$$

Consider the quantized universal enveloping algebra $U_q(\mathfrak{su}_2) \subset M(\hat{\mathcal{A}})$. Recall that it is the universal unital $*$-algebra generated by elements e, f, k, k^{-1} satisfying the relations

$$kk^{-1} = k^{-1}k = 1, \quad ke = qek, \quad kf = q^{-1}fk, \quad ef - fe = \frac{k^2 - k^{-2}}{q - q^{-1}},$$

$$k^* = k, \quad e^* = f.$$

The algebra $U_q(\mathfrak{su}_2)$ is a Hopf subalgebra of $M(\hat{\mathcal{A}})$ with

$$\hat{\Delta}(k) = k \otimes k, \quad \hat{\Delta}(e) = e \otimes k^{-1} + k \otimes e, \quad \hat{\Delta}(f) = f \otimes k^{-1} + k \otimes f,$$

$$\hat{S}(k) = k^{-1}, \quad \hat{S}(e) = -q^{-1}e, \quad \hat{S}(f) = -qf,$$

$$\hat{\varepsilon}(k) = 1, \quad \hat{\varepsilon}(e) = \hat{\varepsilon}(f) = 0.$$

The set I of equivalence classes of irreducible corepresentations of (A, Δ) is identified with the set $\frac{1}{2}\mathbb{Z}_+$ of non-negative half-integers. The fundamental corepresentation ($s = \frac{1}{2}$) is defined by

$$U^{\frac{1}{2}} = (u^{\frac{1}{2}}_{ij})_{ij} = \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix}, \tag{2.1}$$

The corresponding representation of $U_q(\mathfrak{su}_2)$ is given by

$$k \mapsto \begin{pmatrix} q^{\frac{1}{2}} & 0 \\ 0 & q^{-\frac{1}{2}} \end{pmatrix}, \quad e \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{2.2}$$

The formulas (2.1–2.2) completely determine the pairing between $(\mathcal{A}, \Delta)$ and $(U_q(\mathfrak{su}_2), \hat{\Delta})$. Note also that

$$\rho = f_{-1} = k^2. \tag{2.3}$$

For $a \in \mathcal{A}$ and $\omega \in M(\hat{\mathcal{A}})$ define elements in $\mathcal{A}$ by

$$\partial_\omega(a) = \omega * a = (\iota \otimes \omega)\Delta(a) \quad \text{and} \quad a \triangleleft \omega = a * \omega = (\omega \otimes \iota)\Delta(a).$$

Then the modular property of the Haar state h can be expressed as

$$h(a_1 a_2) = h(a_2 f_1 * a_1 * f_1) = h(a_2 \partial_{k^{-2}}(a_1 \triangleleft k^{-2})). \tag{2.4}$$

For $n \in \mathbb{Z}$ set

$$\mathcal{A}_n = \{a \in \mathcal{A} \mid \partial_k(a) = q^{\frac{n}{2}} a\}.$$

The norm closure $B = C(S^2_q) \subset A$ of $\mathcal{A}_0$ with the left coaction $\Delta|_B$ of (A, Δ) is the quantum homogeneous sphere of Podleś [P1]. Note that $\mathcal{B} = \mathcal{A}_0$. The norm closure A_n of $\mathcal{A}_n$ is an analogue of the space of continuous sections of the line bundle over the sphere with winding number n.

It is well-known that the standard differential calculus on the quantum sphere [P2] can be obtained from various covariant differential calculi on $SU_q(2)$, see e.g. [S]. Consider the $3D$-calculus of Woronowicz [W, KS], that is, the left-covariant first order differential calculus $(\mathcal{A}\omega_{-1} \oplus \mathcal{A}\omega_0 \oplus \mathcal{A}\omega_1, d)$ of $(\mathcal{A}, \Delta)$ with differential

$$da = \partial_{fk^{-1}}(a)\omega_{-1} + \frac{a - \partial_{k^{-4}}(a)}{1 - q^{-2}}\omega_0 + \partial_{ek^{-1}}(a)\omega_1,$$

and right $\mathcal{A}$-module action on the left-invariant forms $\omega_{-1}, \omega_0, \omega_1$ given by $\omega_0 a = \partial_{k^{-4}}(a)\omega_0$ and $\omega_i a = \partial_{k^{-2}}(a)\omega_i$ for $i = \pm 1$ and $a \in \mathcal{A}$. The associated exterior differential algebra spanned by $a_0 da_1 \dots da_n$, $a_i \in \mathcal{A}$, is completely described by the following rules:

$$\omega_{-1}^2 = \omega_0^2 = \omega_1^2 = 0, \quad \omega_{-1}\omega_1 = -q^2\omega_1\omega_{-1}, \quad \omega_0\omega_1 = -q^4\omega_1\omega_0, \quad \omega_{-1}\omega_0 = -q^4\omega_0\omega_{-1},$$

$$d\omega_{-1} = (q^2 + q^4)\omega_0\omega_{-1}, \quad d\omega_0 = -\omega_1\omega_{-1}, \quad d\omega_1 = (q^2 + q^4)\omega_1\omega_0.$$

The standard differential calculus on S_q^2 is then obtained by restricting the differential d to $\mathcal{B}$, so the exterior algebra $(\Gamma_{\mathcal{B}}^{\wedge}, d)$ for S_q^2 is the projective $\mathcal{B}$-module given by $\Gamma_{\mathcal{B}}^{\wedge} = \operatorname{span}\{b_0 db_1 \cdots db_n \mid b_i \in \mathcal{B}\}$, and we have the following concrete description.

Theorem 2.1 *Set* $\Omega^{0,1}(\mathcal{B}) = \mathcal{A}_{-2}$, $\Omega^{1,0}(\mathcal{B}) = \mathcal{A}_2$ *and* $\Omega^{1,1}(\mathcal{B}) = \mathcal{B}$. *Then*
(i) *the de Rham complex on* S_q^2 *is the graded differential algebra*

$$\Gamma_{\mathcal{B}}^{\wedge} = \Gamma_{\mathcal{B}}^{\wedge 0} \oplus \Gamma_{\mathcal{B}}^{\wedge 1} \oplus \Gamma_{\mathcal{B}}^{\wedge 2} = \mathcal{B} \oplus (\Omega^{0,1}(\mathcal{B}) \oplus \Omega^{1,0}(\mathcal{B})) \oplus \Omega^{1,1}(\mathcal{B})$$

with multiplication $\wedge$ *given by*
$(a_{0,0},\, a_{0,1},\, a_{1,0},\, a_{1,1}) \wedge (b_{0,0},\, b_{0,1},\, b_{1,0},\, b_{1,1})$

$$= (a_{0,0}b_{0,0},\, a_{0,0}b_{0,1} + a_{0,1}b_{0,0},\, a_{1,0}b_{0,0} + a_{0,0}b_{1,0},\, a_{0,0}b_{1,1} - a_{0,1}b_{1,0} + q^2 a_{1,0}b_{0,1} + a_{1,1}b_{0,0})$$

and differential $d = \partial + \bar{\partial}$ *given by*

$$\partial(a_{0,0},\, a_{0,1},\, a_{1,0},\, a_{1,1}) = (0,\, 0,\, \partial_e(a_{0,0}),\, q\partial_e(a_{0,1})),$$

$$\bar{\partial}(a_{0,0},\, a_{0,1},\, a_{1,0},\, a_{1,1}) = (0,\, \partial_f(a_{0,0}),\, 0,\, -q\partial_f(a_{1,0}));$$

(ii) *if we define* $\int \omega = h(\omega)$ *for* $\omega \in \Gamma_{\mathcal{B}}^{\wedge 2}$ *by identifying* $\Gamma_{\mathcal{B}}^{\wedge 2}$ *with* $\mathcal{B}$, *then* $\int$ *is closed in the sense that* $\int d\omega = 0$ *for any* $\omega \in \Gamma_{\mathcal{B}}^{\wedge 1}$;
(iii) *the twist* σ *of* $\mathcal{B}$ *extends uniquely to an automorphism of* $\Gamma_{\mathcal{B}}^{\wedge}$ *commuting with d, which we again denote by* σ, *and then*

$$\int \omega \wedge \omega' = (-1)^{\#\omega\#\omega'} \int \sigma(\omega') \wedge \omega$$

for any $\omega, \omega' \in \Gamma_{\mathcal{B}}^{\wedge}$ *with* $\#\omega + \#\omega' = 2$.

Proof. Part (i) follows from the equality $\Gamma_{\mathcal{B}}^{\wedge} = \mathcal{B} + \mathcal{A}_{-2}\omega_{-1} + \mathcal{A}_2\omega_1 + \mathcal{B}\omega_1\omega_{-1}$, which is easily checked. Since $\Gamma_{\mathcal{B}}^{\wedge 1} = \mathcal{B}d\mathcal{B}$, to prove (ii) it is enough to show that $\int db_1 \wedge db_2 = 0$, that is

$$q^2 h(\partial_e(b_1)\partial_f(b_2)) = h(\partial_f(b_1)\partial_e(b_2)). \tag{2.5}$$

We have

$$h(\partial_e(b_1)\partial_f(b_2)) = h(b_1\partial_{\hat{S}(e)f}(b_2)) = -q^{-1}h(b_1\partial_{ef}(b_2)).$$

Similarly $h(\partial_f(b_1)\partial_e(b_2)) = -qh(b_1\partial_{fe}(b_2))$. Since $\partial_{ef} = \partial_{fe}$ on $\mathcal{B}$, assertion (ii) is proved. The existence of an extension of σ is obvious: it comes from the automorphism $a \mapsto a\triangleleft k^2$ of $\mathcal{A}$. Concerning the equality in part (iii), the only interesting case is when $\omega \in \Omega^{1,0}(\mathcal{B})$ and $\omega' \in \Omega^{0,1}(\mathcal{B})$. In this case we have to prove that for $a \in \mathcal{A}_2$ and $a' \in \mathcal{A}_{-2}$ we have

$$q^2 h(aa') = h(\sigma(a')a). \tag{2.6}$$

But this is true, since $h(\sigma(a')a) = h(a\partial_{k^{-2}}(a')) = q^2 h(aa')$ by (2.4).

■

Define the volume form by

$$\tau(b_0, b_1, b_2) = \int b_0 db_1 \wedge db_2 = h(b_0(q^2\partial_e(b_1)\partial_f(b_2) - \partial_f(b_1)\partial_e(b_2))).$$

Set also

$$\tau_1(b_0, b_1, b_2) = h(b_0\partial_e(b_1)\partial_f(b_2)) \quad \text{and} \quad \tau_2(b_0, b_1, b_2) = h(b_0\partial_f(b_1)\partial_e(b_2)).$$

These forms were introduced in [SW], where part of the following proposition was proved.

Proposition 2.2 *We have that*
(i) *τ is a cocycle in the twisted cyclic complex $(C^\bullet_{\sigma,\lambda}(\mathcal{B}), b)$;*
(ii) *τ_1 and τ_2 are cocycles in the complex $(\mathcal{C}^\bullet_\sigma(\mathcal{B}), b + B)$;*
(iii) *the cocycle $q^2\tau_1 + \tau_2 \in \mathcal{C}^0_\sigma(\mathcal{B})$ is a coboundary; in particular, the cocycles τ, $2q^2\tau_1$ and $-2\tau_2$ are cohomologous.*

Proof. Part (i) is a standard consequence of the properties of the integral, cf [C1, KMT].
To prove (iii), set $\tilde\tau(b_0, b_1) = h(\partial_f(b_0)\partial_e(b_1))$. Then

$$(B_0\tilde\tau)(b_0) = (B_0^0\tilde\tau)(b_0) = \tilde\tau(b_0, 1) + \tilde\tau(1, \sigma(b_0)) = 0.$$

On the other hand, using that ∂_e and ∂_f are derivations on $\mathcal{B}$ and that $h(\sigma(b)a) = h(ab)$ for any $a, b \in \mathcal{B}$, we get

$$(b_2\tilde\tau)(b_0, b_1, b_2) = h(b_0\partial_f(b_1)\partial_e(b_2)) + h(\partial_f(\sigma(b_2))b_0\partial_e(b_1)) = \tau_2(b_0, b_1, b_2) + q^2\tau_1(b_0, b_1, b_2),$$

where we have used (2.6) with $a = b_0\partial_e(b_1)$ and $a' = \partial_f(a_2)$. Thus $q^2\tau_1 + \tau_2$ is the coboundary of $\tilde\tau$.
Since $\tau = \tau_1 + \tau_2$ and $q^2 \neq 1$, statement (ii) follows immediately from (i) and (iii). It can also be checked directly.

■

Note that although τ is cyclic, this is not the case for τ_1 and τ_2.

3 The Dirac Operator

Dirac operators on the quantum sphere have appeared in several papers, see e.g. [O, PS, DS, SW, M, Kr] and references therein. However, they all coincide by an important result of [DS] stating that the Dirac operator is essentially uniquely determined by the requirements of $SU_q(2)$-invariance, boundedness of commutators and the first order condition. In the next paragraph and in Proposition 3.1 below we summarize the properties of the Dirac operator which we shall need.

One of the easiest ways to construct the Dirac operator is to recall [F] that the spinor bundle on a classical Kähler manifold M of complex dimension m is $\oplus_{i=0}^m \wedge^{0,i} \otimes \mathcal{S}$, where $\mathcal{S}$ is a square root of the canonical line bundle $\wedge^{m,0}$, and then the Dirac operator D is $\sqrt{2}(\bar\partial + \bar\partial^*)$. In view of the previous section these notions have rather straightforward analogues for the quantum sphere. So consider the space $H = L^2(A, h)$ of the GNS-representation of $A = C(SU_q(2))$ corresponding to the Haar state h. Then

$$H = \oplus_{n\in\mathbb{Z}} L^2(A_n, h).$$

The left actions $a \mapsto \partial_\omega(a)$ and $a \mapsto a\triangleleft\hat{S}^{-1}(\omega)$ of $\hat{\mathcal{A}}$ on A extend to $*$-representations of $\hat{\mathcal{A}}$ on H. These are, in fact, the left and the right regular representations of $\hat{\mathcal{A}}$. In the sequel we will write ∂_ω for the operators of the first representation and simply ω for the operators of the second one. Then the space of L^2-spinors and the Dirac operator are defined by

$$H_+ \oplus H_- = L^2(A_1, h) \oplus L^2(A_{-1}, h) \quad \text{and} \quad D = \begin{pmatrix} 0 & \partial_e \\ \partial_f & 0 \end{pmatrix},$$

respectively. Note that $\partial_f(\mathcal{A}_n) \subset \mathcal{A}_{n-2}$ and $\partial_e(\mathcal{A}_n) \subset \mathcal{A}_{n+2}$. This shows that D is indeed an operator on $H_+ \oplus H_-$.

Proposition 3.1 *We have*

(i) $D^2 = C$, *where* $C = fe + \left(\dfrac{q^{\frac{1}{2}}k - q^{-\frac{1}{2}}k^{-1}}{q - q^{-1}}\right)^2$ *is the Casimir;*

(ii) $[D, b] = \begin{pmatrix} 0 & q^{\frac{1}{2}}\partial_e(b) \\ q^{-\frac{1}{2}}\partial_f(b) & 0 \end{pmatrix}$ *for any* $b \in \mathcal{B}$.

Proof. We have $D^2 = \begin{pmatrix} \partial_{ef} & 0 \\ 0 & \partial_{fe} \end{pmatrix}$. Since $\partial_k|_{L^2(A_n,h)} = q^{\frac{n}{2}}$ and $\partial_{ef} - \partial_{fe} = (q-q^{-1})^{-1}(\partial_{k^2} - \partial_{k^{-2}})$, we see that $D^2 = \partial_C$. To prove (i) it remains to note that $\partial_C = C$ on H as C is in the center of $M(\hat{\mathcal{A}})$ and $\hat{S}(C) = C$.

Part (ii) follows from the identity

$$[\partial_e, a] = \partial_e(a)\partial_{k^{-1}} + \partial_k(a)\partial_e - a\partial_e,$$

which is valid for any $a \in \mathcal{A}$, and from a similar identity for ∂_f.

■

We now want to develop a simple pseudo-differential calculus for our Dirac operator. More precisely, we want to compute the principal symbol of the operator of the form $|D|^{2z}db|D|^{-2z}$, where $db = [D, b]$. For this it is more convenient to consider a more general problem. The Casimir C defines a scale of Hilbert spaces $\mathcal{H}_t$, $t \in \mathbb{R}$, so that for $t \ge 0$

$$\mathcal{H}_t = \mathrm{Domain}(C^{t/2}), \quad \|\xi\|_t = \|C^{t/2}\xi\|.$$

Set $\mathcal{H}_\infty = \cap_t \mathcal{H}_t$. Note that $\mathcal{H}_\infty$ is a dense subspace of $\mathcal{H}_t$ for every t. We say that an operator $T\colon \mathcal{H}_\infty \to \mathcal{H}_\infty$ is of order $r \in \mathbb{R}$ if it extends by continuity to a bounded operator $\mathcal{H}_t \to \mathcal{H}_{t-r}$ for every t. In this case we write $T \in op^r$. If $T \in op^0$, we denote by $\|T\|_t$ the norm of the operator $T\colon \mathcal{H}_t \to \mathcal{H}_t$. Recall that $\|T\|_t$ is a convex function of t. Note that $C^z \in op^{2\mathrm{Re}\, z}$.

For $F \subset \mathbb{Z}$, denote by p_F the projection onto $\oplus_{n\in F} L^2(A_n, h)$ with respect to the decomposition $H = \oplus_{n\in\mathbb{Z}} L^2(A_n, h)$.

Proposition 3.2 *We have*

(i) $\mathcal{A} \subset op^0$ *and* $p_F \in op^0$ *for any* $F \subset \mathbb{Z}$;

(ii) *for* $x = ap_F$ *with* $a \in \mathcal{A}_n$ *and* $F \subset \mathbb{Z}$ *finite, the function* $\mathbb{C} \ni z \mapsto x(z) = (C^z x C^{-z} - q^{nz}x)C^{\frac{1}{2}}$ *takes values in* op^0 *and has at most linear growth on vertical strips; more precisely, for any finite interval* $I \subset \mathbb{R}$, *there exists* $\lambda > 0$ *such that* $\|x(z)\|_t \le \lambda|z|$ *for* $t \in I$ *and* $\mathrm{Re}\, z \in I$.

Proof. Since p_F is a projection commuting with C, it is obvious that $p_F \in op^0$. It is also clear that to prove that $\mathcal{A} \subset op^0$ it is enough to consider generators of $\mathcal{A}$. Similarly, if (ii) is proved for $x_i = a_i p_{F_i}$, $a_i \in \mathcal{A}_{n_i}$, $i = 1, 2$, and it is proved that $a_1, a_2 \in op^0$, then $x_1 x_2 = a_1 a_2 p_{F_1 - n_2} p_{F_2} = a_1 a_2 p_{(F_1-n_2)\cap F_2}$ and

$(C^z x_1 x_2 C^{-z} - q^{(n_1+n_2)z} x_1 x_2) C^{\frac{1}{2}}$

$$= (C^z x_1 C^{-z} - q^{n_1 z} x_1) C^{\frac{1}{2}} (C^{-\frac{1}{2}+z} x_2 C^{\frac{1}{2}-z}) + q^{n_1 z} x_1 (C^z x_2 C^{-z} - q^{n_2 z} x_2) C^{\frac{1}{2}},$$

so (ii) is also true for $a_1 a_2 p_{(F_1-n_2)\cap F_2}$. Analogously, if (ii) is true for $a p_F$ with $a \in \mathcal{A}_n$, then it is also true for $a^* p_{F+n}$. It follows that to prove (i) and (ii) it is enough to consider generators of $\mathcal{A}$ and arbitrary finite $F \subset \mathbb{Z}$. We will only consider the generator α, the proof for γ is similar. Consider the orthonormal basis $\{\xi^s_{ij} \mid s \in \frac{1}{2}\mathbb{Z}_+,\ i, j = -s, \dots, s\}$ given by normalized matrix coefficients, $\xi^s_{ij} = q^i d_s^{\frac{1}{2}} u^s_{ij} \xi_h$, where $d_s = [2s+1]_q$. Then α can be written as the sum of two operators α^+ and α^- such that

$$\alpha^+ \xi^s_{ij} = \alpha^{s+}_{ij} \xi^{s+\frac{1}{2}}_{i-\frac{1}{2}, j-\frac{1}{2}} \quad \text{and} \quad \alpha^- \xi^s_{ij} = \alpha^{s-}_{ij} \xi^{s-\frac{1}{2}}_{i-\frac{1}{2}, j-\frac{1}{2}}.$$

What we have to know about the numbers α^{s+}_{ij} and α^{s-}_{ij} is that they are of modulus ≤ 1 and if $|j| \le m$, then $|\alpha^{s+}_{ij}| \le \lambda_m q^s$ for some λ_m depending only on m and q (in fact, we have $\alpha^{s+}_{ij} = q^{\frac{1}{2}} (d_s d_{s+\frac{1}{2}})^{-\frac{1}{2}} q^{\frac{2s+i+j}{2}} ([s-i+1]_q [s-j+1]_q)^{\frac{1}{2}}$, cf [V]). In our basis we have also $C\xi^s_{ij} = [s+\frac{1}{2}]^2_q \xi^s_{ij}$. It follows immediately that $\alpha^+, \alpha^- \in op^0$. Then

$(C^{z+t} \alpha^- C^{-z-t} - q^z C^t \alpha^- C^{-t}) C \xi^s_{ij}$

$$\begin{aligned} &= [s+\frac{1}{2}]^2_q \left(\left(\frac{[s]_q}{[s+\frac{1}{2}]_q} \right)^{2z+2t} - q^z \left(\frac{[s]_q}{[s+\frac{1}{2}]_q} \right)^{2t} \right) \alpha^{s-}_{ij} \xi^{s-\frac{1}{2}}_{i-\frac{1}{2}, j-\frac{1}{2}} \\ &= q^{z+t} [s+\frac{1}{2}]^2_q \left(\frac{1-q^{2s}}{1-q^{2s+1}} \right)^{2t} \left(\left(\frac{1-q^{2s}}{1-q^{2s+1}} \right)^{2z} - 1 \right) \alpha^{s-}_{ij} \xi^{s-\frac{1}{2}}_{i-\frac{1}{2}, j-\frac{1}{2}}. \end{aligned}$$

Using the simple estimate $|(1-\beta)^z - 1| = |\beta z \int_0^1 (1-\beta\tau)^{z-1} d\tau| \le \beta |z| (1-\beta_0)^{\operatorname{Re} z - 1}$ for $0 \le \beta \le \beta_0 < 1$, we conclude that the operator $(C^{z+t} \alpha^- C^{-z-t} - q^z C^t \alpha^- C^{-t}) C$ is bounded, and moreover, if t and $\operatorname{Re} z$ lie in some finite interval then the operator norm can be estimated by some multiple of $|z|$. Thus the function $z \mapsto (C^z \alpha^- C^{-z} - q^z \alpha^-) C$ takes values in op^0 and has at most linear growth on vertical strips. Then surely the function $z \mapsto (C^z \alpha^- C^{-z} - q^z \alpha^-) C^{\frac{1}{2}}$ has the same properties. Similarly the function $z \mapsto (C^z \alpha^+ C^{-z} - q^{-z} \alpha^+) C^{\frac{1}{2}}$ takes values in op^0 and has at most linear growth on vertical strips. Then

$(C^z \alpha p_F C^{-z} - q^z \alpha p_F) C^{\frac{1}{2}}$

$$= (C^z \alpha^- C^{-z} - q^z \alpha^-) C^{\frac{1}{2}} p_F + (C^z \alpha^+ C^{-z} - q^{-z} \alpha^+) C^{\frac{1}{2}} p_F + (q^{-z} - q^z) \alpha^+ p_F C^{\frac{1}{2}}.$$

It remains to note that p_F is the projection onto the space spanned by the vectors ξ^s_{ij} such that $-2j \in F$. Since $|\alpha^{s+}_{ij}| \le \lambda_F q^s$ if $-2j \in F$ for some λ_F, we conclude that $\alpha^+ p_F C^{\frac{1}{2}} \in op^0$. ∎

It is worth noting that by analyzing the proof of Theorem B.1 in [CM2] the slightly weaker result that $x(z)$ has polynomial growth on vertical strip can be deduced using only the fact that $Cx - q^n x C \in op^1$. This, in turn, follows from the identity

$$\partial_{fe} a - q^n a \partial_{fe} = \partial_{fe}(a) \partial_{k^{-2}} + \partial_{fk}(a) \partial_{k^{-1}e} + \partial_{ke}(a) \partial_{fk^{-1}}$$

for $a \in \mathcal{A}_n$. Thus a variant of Proposition 3.2 with polynomial growth (which would, in fact, be sufficient for our purposes) can be proved without any knowledge about the Clebsch-Gordan coefficients.

Corollary 3.3 *Consider the operator* $\chi = \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}$ *on* $H_+ \oplus H_-$. *Then for any* $b \in \mathcal{B}$ *there exists an analytic function* $z \mapsto b(z) \in B(H_+ \oplus H_-)$ *with at most linear growth on vertical strips such that*

$$|D|^{-2z} db = db\chi^{2z}|D|^{-2z} + b(z)|D|^{-2z-1} = \chi^{-2z} db|D|^{-2z} + b(z)|D|^{-2z-1}.$$

Proof. Since $|D|^2 = C$ and $\partial_f(b) \in \mathcal{A}_{-2}$, $\partial_e(b) \in \mathcal{A}_2$ for $b \in \mathcal{B} = \mathcal{A}_0$, this is an immediate consequence of Propositions 3.1 and 3.2.

■

Next we study the behavior of the heat operators. Consider the direct sum of irreducible corepresentations of (A, Δ) with spins $s \in \frac{1}{2} + \mathbb{Z}_+$. Denote by $\tilde{H}$ the space of this corepresentation, and consider the corresponding representation of $\hat{A}$ on $\tilde{H}$. Note that both $H_+ = L^2(A_1, h)$ and $H_- = L^2(A_{-1}, h)$ can be identified with $\tilde{H}$ (since in the notation of the proof of Proposition 3.2 the spaces H_+ and H_- are spanned by the vectors $\xi^s_{i,-\frac{1}{2}}$ and $\xi^s_{i,\frac{1}{2}}$, respectively).

Lemma 3.4 *On* $\tilde{H}$ *we have that*

(i) *the function* $\varepsilon \mapsto \varepsilon \mathrm{Tr}(e^{-\varepsilon C}\rho)$ *is continuous and bounded on* $(0, \infty)$;

(ii) $\lim_{\varepsilon \to 0+} \varepsilon(\mathrm{Tr}(e^{-\varepsilon C}\rho) - q^2 \mathrm{Tr}(e^{-\varepsilon q^2 C}\rho)) = 0$;

(iii) $\lim_{\varepsilon \to 0+} \varepsilon \int_{q^2}^1 \mathrm{Tr}(e^{-\varepsilon t C}\rho)dt = \mu = q^{-1} - q.$

Using the Karamata theorem (see e.g. [BGV]) it can be shown that the limit $\lim_{\varepsilon \to 0+} \varepsilon \mathrm{Tr}(e^{-\varepsilon C}\rho)$ does not exist.

Proof of Lemma 3.4. If f is a positive function then $\mathrm{Tr}(f(C)\rho) = \sum_{n=1}^{\infty} [2n]_q f([n]_q^2)$. Since it is obvious that the function $\varepsilon \mapsto \varepsilon \mathrm{Tr}(e^{-\varepsilon C}\rho)$ is continuous and vanishes at infinity, to prove (i) it is thus enough to show that the function $\varepsilon \mapsto \sum_{n=1}^{\infty} \varepsilon q^{-2n} e^{-\varepsilon [n]_q^2}$ is bounded near zero. As $[n]_q^2 \geq (q^{-2n} - 2)\mu^{-2}$, for $q^{2m} \leq \varepsilon \leq q^{2(m-1)}$ we get

$$\sum_{n=1}^{\infty} \varepsilon q^{-2n} e^{-\varepsilon [n]_q^2} \leq \sum_{n=1}^{\infty} q^{-2(n-m+1)} e^{-(q^{-2(n-m)} - 2q^{2m})\mu^{-2}} \leq q^{-2} e^{2\mu^{-2}} \sum_{n=-\infty}^{\infty} q^{-2n} e^{-q^{-2n}\mu^{-2}}.$$

Thus (i) is proved. Since $q^{-2n}\mu^{-2} \geq [n]_q^2 \geq (q^{-2n} - 2)\mu^{-2}$, we see also that

$$\lim_{\varepsilon \to 0+} \varepsilon(\mathrm{Tr}(e^{-\varepsilon C}\rho) - \sum_{n=1}^{\infty} q^{-2n}\mu^{-1} e^{-\varepsilon q^{-2n}\mu^{-2}}) = 0.$$

Replacing ε by $\varepsilon\mu^2$, to prove (ii) and (iii) we thus have to show that

$$\lim_{\varepsilon \to 0+} \varepsilon(\sum_{n=1}^{\infty} q^{-2n} e^{-\varepsilon q^{-2n}} - q^2 \sum_{n=1}^{\infty} q^{-2n} e^{-\varepsilon q^2 q^{-2n}}) = 0$$

and

$$\lim_{\varepsilon \to 0+} \varepsilon \int_{q^2}^1 dt \sum_{n=1}^{\infty} q^{-2n} e^{-\varepsilon t q^{-2n}} = 1.$$

Both statements are obvious.

■

4 Local Index Formula

Let us first briefly recall the proof of the local index formula of Connes and Moscovici [CM2], which is a far-reaching generalization of the local index theorem of Gilkey-Atiyah-Bott-Patodi [BGV]. Suppose we are given a p-summable even spectral triple for an algebra $\mathcal{B}$. For each $\varepsilon > 0$ consider the JLO-cocycle corresponding to the Dirac operator $\varepsilon^{\frac{1}{2}}D$, so

$$\psi^{\varepsilon}_{2n}(b_0,\dots,b_{2n}) = \varepsilon^n \int_{\Delta_{2n}} dt \operatorname{Tr}(\gamma b_0 e^{-\varepsilon t_0 D^2}[D,b_1]e^{-\varepsilon t_1 D^2}\dots[D,b_{2n}]e^{-\varepsilon t_{2n} D^2}),$$

and note that we always have the following estimate

$$|\psi^{\varepsilon}_{2n}(b_0,\dots,b_{2n})| \le C_p \frac{\varepsilon^{n-\frac{p}{2}}}{(2n)!}\operatorname{Tr}((1+D^2)^{-\frac{p}{2}})\|b_0\|\prod_{i=1}^{2n}\|[D,b_i]\|,$$

where C_p is a universal constant. In the course of proving the local index formula one provides a finite decomposition of ψ^{ε}_{2n}, $0 \le 2n \le p$, of the form

$$\psi^{\varepsilon}_{2n} = \sum_{k,l} \alpha^{(2n)}_{k,l}\varepsilon^{-p_k(n)}(\log\varepsilon)^l + o(1), \tag{4.1}$$

where $\alpha^{(2n)}_{k,l}$ are $2n$-cochains expressed in terms of the residues of certain ζ-functions. Since the pairing with K-theory is independent of ε, one concludes that it is given by the pairing with the cocycle $(\alpha^{(2n)}_{0,0})_{0\le 2n\le p}$. The explicit form of this cocycle and the fact that it is cohomologous to $(\operatorname{Ch}^{2n}(D))_n$ is the main content of the local index formula.

To get (4.1) one needs to commute $e^{-\varepsilon t D^2}$ through db. For this one first uses the identity

$$e^{-\varepsilon t D^2} = \frac{1}{2\pi i}\int_{C_\lambda} dz \Gamma(z)|D|^{-2z}(\varepsilon t)^{-z},$$

where $C_\lambda = \lambda + i\mathbb{R}$, $\lambda > 0$. Then one invokes the expression

$$b_0|D|^{-2z_0}db_1|D|^{-2z_1}\dots db_{2n}|D|^{-2z_n} = \sum_{k=0}^{p-2n+1} f_k(b_0,\dots,b_{2n};z_0,\dots,z_{2n})|D|^{-2(z_0+\dots+z_{2n})-k} \tag{4.2}$$

given by the pseudo-differential calculus for D, where f_k are analytic operator-valued functions with at most polynomial growth on vertical strips. Since the last term ($k = p-2n+1$) is of trace-class for $\operatorname{Re} z_i \ge \frac{2n-1}{2(2n+1)}$, it contributes to ψ^{ε}_{2n} as $O(\varepsilon^{\frac{1}{2}})$. On the other hand, assuming that $\operatorname{Tr}(f_k(b_0,\dots,b_{2n};z_0,\dots,z_{2n})|D|^{-2(z_0+\dots+z_{2n})-k})$ extends to a nicely behaved meromorphic function (note that a priori it is defined for $\operatorname{Re} z_i \ge \frac{p-k}{2(2n+1)}$), the contribution of the kth term can be estimated by counting the residues of this function in the region $\frac{n}{2n+1} - \delta < \operatorname{Re} z_i < \frac{p-k}{2(2n+1)}$ for some $\delta > 0$.

We now want to apply this method to our situation. Although the decomposition (4.2) will be different from the one obtained in [CM2], the method itself works perfectly well. So consider the twisted entire cocycle $(\psi^{\varepsilon}_{2n})_n$ given by

$$\psi^{\varepsilon}_{2n}(b_0,\dots,b_{2n}) = \operatorname{Ch}^{2n}(\varepsilon^{\frac{1}{2}}D)(\rho;b_0,\dots,b_{2n}).$$

Using Lemma 1.2 we get

$$\psi^{\varepsilon}_0(b_0) = \operatorname{Tr}(\gamma b_0 e^{-\varepsilon D^2}\rho) = h(b_0)\operatorname{Tr}(e^{-\varepsilon C}\rho) - h(b_0)\operatorname{Tr}(e^{-\varepsilon C}\rho) = 0,$$

where we think of the Casimir C as the operator acting on $\tilde{H}$, see the end of Section 3. Consider ψ_2^ε. Fix $b_0, b_1, b_2 \in \mathcal{B}$ and set

$$\zeta(z_0, z_1, z_2) = \mathrm{Tr}(\gamma b_0 |D|^{-2z_0} db_1 |D|^{-2z_1} db_2 |D|^{-2z_2} \rho).$$

This function is well-defined and analytic in the region $\mathrm{Re}\,(z_0 + z_1 + z_2) > 1$. By Corollary 3.3,

$$\zeta(b_0, b_1, b_2) = \mathrm{Tr}(\gamma b_0 db_1 db_2 \chi^{2z_1} |D|^{-2(z_0+z_1+z_2)} \rho) + \tilde{\zeta}(z_0, z_1, z_2),$$

where $\tilde{\zeta}(z_0, z_1, z_2) =$

$$= \mathrm{Tr}(\gamma b_0 db_1 \chi^{2z_0} b_2(z_0 + z_1) |D|^{-2(z_0+z_1+z_2)-1} \rho) + \mathrm{Tr}(\gamma b_0 b_1(z_0) |D|^{-2(z_0+z_1)-1} db_2 |D|^{-2z_2} \rho).$$

The function $\tilde{\zeta}$ is holomorphic in the region $\mathrm{Re}\,(z_0 + z_1 + z_2) > \frac{1}{2}$. Moreover, if we fix $\lambda, \lambda_0 > \frac{1}{6}$, $\lambda > \lambda_0$, then

$$\tilde{\zeta}(z_0, z_1, z_2)(1 + |z_0|)^{-1}(1 + |z_1|)^{-1}$$

is bounded in the region $\lambda_0 < \mathrm{Re}\, z_i < \lambda$. Fix $\lambda > \frac{1}{3}$ and $\frac{1}{6} < \lambda_0 < \frac{1}{3}$. Then

$$\psi_2^\varepsilon(b_0, b_1, b_2) = \frac{\varepsilon}{(2\pi i)^3} \int_{\Delta_2} dt \int_{C_\lambda^3} dz \Gamma(z_0)\Gamma(z_1)\Gamma(z_2)(\varepsilon t_0)^{-z_0}(\varepsilon t_1)^{-z_1}(\varepsilon t_2)^{-z_2} \zeta(z_0, z_1, z_2),$$

where $C_\lambda = \lambda + i\mathbb{R}$. If we consider the same expression for $\tilde{\zeta}$, we can replace integration over C_λ^3 by integration over $C_{\lambda_0}^3$, since the integrand is holomorphic and vanishes at infinity. It follows that such an expression is $O(\varepsilon^{1-3\lambda_0})$. Thus, as $\varepsilon \to 0$,

$$\begin{aligned}
\psi_2^\varepsilon(b_0, b_1, b_2) &= \frac{\varepsilon}{(2\pi i)^3} \int_{\Delta_2} dt \int_{C_\lambda^3} dz \Gamma(z_0)\Gamma(z_1)\Gamma(z_2)(\varepsilon t_0)^{-z_0}(\varepsilon t_1)^{-z_1}(\varepsilon t_2)^{-z_2} \\
&\qquad \times \mathrm{Tr}(\gamma b_0 db_1 db_2 \chi^{2z_1} |D|^{-2(z_0+z_1+z_2)} \rho) + o(1) \\
&= \varepsilon \int_{\Delta_2} dt \mathrm{Tr}(\gamma b_0 db_1 db_2 e^{-\varepsilon(t_0 + \chi^{-2} t_1 + t_2) D^2} \rho) + o(1) \\
&= \varepsilon \int_{\Delta_2} dt \mathrm{Tr}(\gamma b_0 db_1 db_2 e^{-\varepsilon(1 + (\chi^{-2} - 1)t_1) D^2} \rho) + o(1) \\
&= \varepsilon \int_0^1 (1 - t) \mathrm{Tr}(\gamma b_0 db_1 db_2 e^{-\varepsilon(1 + (\chi^{-2} - 1)t) D^2} \rho) dt + o(1).
\end{aligned}$$

By Proposition 3.1(ii), we have

$$b_0 db_1 db_2 = \begin{pmatrix} b_0 \partial_e(b_1) \partial_f(b_2) & 0 \\ 0 & b_0 \partial_f(b_1) \partial_e(b_2) \end{pmatrix}.$$

Hence by Lemma 1.2 we get

$$\begin{aligned}
\psi_2^\varepsilon(b_0, b_1, b_2) &= h(b_0 \partial_e(b_1) \partial_f(b_2)) \varepsilon \int_0^1 (1 - t) \mathrm{Tr}(e^{-\varepsilon(1 + (q^{-2} - 1)t) C} \rho) dt \\
&\qquad - h(b_0 \partial_f(b_1) \partial_e(b_2)) \varepsilon \int_0^1 (1 - t) \mathrm{Tr}(e^{-\varepsilon(1 + (q^2 - 1)t) C} \rho) dt + o(1). \qquad (4.3)
\end{aligned}$$

We have

$$\varepsilon \int_0^1 (1 - t) \mathrm{Tr}(e^{-\varepsilon(1 + (q^2 - 1)t) C} \rho) dt = -\frac{\varepsilon}{(1 - q^2)^2} \int_{q^2}^1 (q^2 - t) \mathrm{Tr}(e^{-\varepsilon t C} \rho) dt. \qquad (4.4)$$

On the other hand,

$$\begin{aligned}\varepsilon \int_0^1 (1-t)\mathrm{Tr}(e^{-\varepsilon(1+(q^{-2}-1)t)C}\rho)dt &= \frac{\varepsilon}{(1-q^2)^2}\int_{q^2}^1 (1-t)\mathrm{Tr}(e^{-\varepsilon q^{-2}tC}\rho)dt \\ &= \frac{\varepsilon q^2}{(1-q^2)^2}\int_{q^2}^1 (1-t)\mathrm{Tr}(e^{-\varepsilon tC}\rho)dt + o(1) \qquad (4.5)\end{aligned}$$

by Lemma 3.4(ii). By Lemma 3.4(iii), $\varepsilon\int_{q^2}^1 \mathrm{Tr}(e^{-\varepsilon tC}\rho)dt \to q^{-1}(1-q^2)$. Thus putting (4.3-4.5) together we get

$$\psi_2^\varepsilon = \frac{q}{1-q^2}(\tau_1+\tau_2) - (q^2\tau_1+\tau_2)\frac{\varepsilon}{(1-q^2)^2}\int_{q^2}^1 \mathrm{Tr}(e^{-\varepsilon tC}\rho)tdt + o(1).$$

Finally, consider ψ_{2n}^ε with $n>1$. Combining (1.5) with Lemmas 10.8 and 10.11 in [GBVF] yields the following standard estimate

$$|\psi_{2n}^\varepsilon(b_0,\dots,b_{2n})| \le C_p\frac{\varepsilon^{n-\frac{p}{2}}}{(2n)!}\mathrm{Tr}(C^{-\frac{p}{2}}\rho)\|b_0\|\prod_{i=1}^{2n}\max_{0\le t\le 1}\{\|[D, b_i\triangleleft\rho^t]\|\},$$

where p is any number larger than 2. Thus we get the first part of the following theorem, which is the main result of the paper.

Theorem 4.1 *For $\varepsilon>0$, consider the cocycle $(\psi_{2n}^\varepsilon)_n$ in the twisted entire cyclic cohomology of $\mathcal{B}$ given by*

$$\psi_{2n}^\varepsilon(b_0,\dots,b_{2n}) = \mathrm{Ch}^{2n}(\varepsilon^{\frac12}D)(\rho; b_0,\dots,b_{2n}).$$

Then

$$\begin{aligned}\psi_0^\varepsilon &= 0;\\ \psi_2^\varepsilon &= \frac{q}{1-q^2}(\tau_1+\tau_2) - (q^2\tau_1+\tau_2)\frac{\varepsilon}{(1-q^2)^2}\int_{q^2}^1 \mathrm{Tr}(e^{-\varepsilon tC}\rho)tdt + o(1) \quad \text{as} \quad \varepsilon\to 0;\\ \|\psi_{2n}^\varepsilon\| &\le C_\delta\frac{\varepsilon^{n-1-\delta}}{(2n)!} \quad \text{for any} \quad \delta>0 \quad \text{and} \quad n>1,\end{aligned}$$

where the norm of a multi-linear form on $\mathcal{B}$ is defined using the norm

$$\max_{0\le t\le 1}\{\|b\triangleleft\rho^t\| + \|[D, b\triangleleft\rho^t]\|\}$$

on $\mathcal{B}$. In particular, the map $q\text{−}\mathrm{Ind}_D$ on $K_0^{SU_q(2)}(C(S_q^2))$ is given by the pairing with the twisted cyclic cocycle $-q^{-1}\tau$, where $\tau(b_0,b_1,b_2) = h(b_0(q^2\partial_e(b_1)\partial_f(b_2) - \partial_f(b_1)\partial_e(b_2)))$.

Proof. The pairing of $(\psi_{2n}^\varepsilon)_n$ with K-theory does not depend on ε. Using the Karamata theorem it can be shown that the limit $\lim_{\varepsilon\to 0+}\varepsilon\int_{q^2}^1 \mathrm{Tr}(e^{-\varepsilon tC}\rho)tdt$ does not exist. It follows that the cocycle $q^2\tau_1+\tau_2$ pairs trivially with K-theory. Thanks to Proposition 2.2(iii) we know even more, $q^2\tau_1+\tau_2$ is a coboundary. We also conclude that the pairing is given by the cocycle $\frac{q}{1-q^2}(\tau_1+\tau_2)\in\mathcal{C}_\sigma^0(\mathcal{B})$, which is cohomologous to the cocycle $\frac{1}{2q}\tau$. Recalling definition (1.3) of the map $C_{\sigma,\lambda}^2(\mathcal{B})\to\mathcal{C}_\sigma^0(\mathcal{B})$, we see that the pairing is defined by the twisted cyclic cocycle $-q^{-1}\tau$.

■

We end the paper with an actual computation of indices. Observe first that the spaces A_n can be considered as equivariant Hilbert B-modules, and thus they define elements of $K_0^{\hat{A}}(B)$

which we denote by $[A_n]$. As in the classical case, the group $K_0^{\hat{A}}(B)$ is a free abelian group with generators $[A_n]$, see [NT]. Consider now the module A_1 and note that $\mathcal{A}_1 = \alpha\mathcal{B} + \gamma\mathcal{B}$. The map $T\colon H_{\frac{1}{2}} \otimes B \to A_1$ given by

$$T(\xi_{-\frac{1}{2}} \otimes b) = q\gamma b \quad \text{and} \quad T(\xi_{\frac{1}{2}} \otimes b) = -\alpha b,$$

where $H_{\frac{1}{2}}$ is the space of the spin $\frac{1}{2}$ corepresentation of (A, Δ), see Section 2, is an equivariant partial isometry. Thus A_1 is isomorphic to the equivariant Hilbert B-module $p(H_{\frac{1}{2}} \otimes B)$ with projection

$$p = T^*T = \begin{pmatrix} q^2\gamma^*\gamma & -\alpha\gamma^* \\ -\gamma\alpha^* & \alpha^*\alpha \end{pmatrix}.$$

The explicit form of a projection corresponding to A_n for arbitrary n can be found in [HM]. Let Tr_s be the trace on $\hat{\mathcal{A}}$ defined by the spin s representation of $\hat{\mathcal{A}}$, and let ϕ_s be the corresponding q-trace, $\phi_s = \mathrm{Tr}_s(\cdot\rho)$. By definition we have, for any twisted cyclic cocycle $\varphi \in C^2_\sigma(\mathcal{B})$, that

$$\langle[\varphi], [p]\rangle = \sum_{\substack{i_0,i_1,i_2 \\ j_0,j_1,j_2}} \phi_{\frac{1}{2}}(m_{i_0j_0}m_{i_1j_1}m_{i_2j_2})\varphi(p_{i_0j_0}, p_{i_1j_1}, p_{i_2j_2}) = \sum_{i_0,i_1,i_2} q^{-2i_0}\varphi(p_{i_0i_1}, p_{i_1i_2}, p_{i_2i_0}).$$

Using the formula $h((\gamma^*\gamma)^n) = (1-q^2)(1-q^{2(n+1)})^{-1}$ for the Haar state, a lengthy but straightforward computation yields

$$q\text{–}\mathrm{Ind}_D([A_1]) = q\text{–}\mathrm{Ind}_F([A_1]) = \langle[-q^{-1}\tau], [p]\rangle = -1.$$

This is enough to conclude that the equivariant index $q\text{–}\mathrm{Ind}_F([A_1])$ equals $-\mathrm{Tr}_0$. To see this we shall use a continuity argument for $q \in (0,1)$. Write $\alpha(q)$, $\gamma(q)$, and so on, to distinguish operators for different q. The spaces $L^2(C(SU_q(2)), h)$ can be identified for all q. We also identify the spaces $H_+ \oplus H_-$ of L^2-spinors. Note that $F = D(q)|D(q)|^{-1}$ is independent of q (in the notation of the proof of Proposition 3.2 we have $F\xi^s_{i,-\frac{1}{2}} = \xi^s_{i,\frac{1}{2}}$). The functions $q \mapsto \alpha(q)$ and $q \mapsto \gamma(q)$ are norm-continuous as can easily be verified by looking at the Clebsch-Gordan coefficients. It follows that our quantum Bott projections $p(q) \in B(H_{\frac{1}{2}} \otimes (H_+ \oplus H_-))$ depend continuously on q. Let $I_s(q) \in \hat{\mathcal{A}}(q)$ be the support of the spin s representation. Considered as operators on $H_{\frac{1}{2}} \otimes (H_+ \oplus H_-)$ the projections $I_s(q)$ depend continuously on q (and are, in fact, finite-rank operators). As the functions

$$m_s(q) = (2s+1)^{-1}\mathrm{Ind}(p(q)_- I_s(q)(1 \otimes F)p(q)_+ I_s(q))$$

are continuous and integer-valued, they are constant. We have by definition

$$\mathrm{Ind}_F([p(q)]) = \sum_s m_s \mathrm{Tr}_s.$$

Since

$$-1 = \mathrm{Ind}_F([p(q)])(\rho) = \sum_s m_s[2s+1]_q,$$

and the functions $q \mapsto [n]_q$, $n \in \mathbb{N}$, are linearly independent on any infinite set, we conclude that $m_0 = -1$ and $m_s = 0$ for $s > 0$. Thus $\mathrm{Ind}_F([p]) = -\mathrm{Tr}_0 = -\hat{\varepsilon}$. This, in turn, is sufficient in order to find the non-equivariant Chern character.

Proposition 4.2 *The image of the non-equivariant Chern character of our Fredholm module in $HP^0(\mathcal{B})$ coincides with the class of the cyclic 0-cocycle τ' given by*

$$\tau'(\alpha^{n-m}\gamma^m\gamma^{*n}) = \begin{cases} (1-q^{2n})^{-1} & \text{for } n = m > 0, \\ 0 & \text{otherwise,} \end{cases}$$

where we used the convention $\alpha^k = (\alpha^*)^{-k}$ for $k < 0$. In particular, $\mathrm{Ind}_F([A_n])(1) = -n$.

Proof. The cocycle τ' was found in [MNW2], and is one of the two generators of $HP^0(\mathcal{B}) \cong \mathbb{C}^2$. Since the class of a cocycle in $HP^0(\mathcal{B})$ is completely determined by its pairing with [1] and [p], we conclude that the Chern character is cohomologous to τ'. The equality $\langle[\tau'],[A_n]\rangle = -n$ was established in [H].

■

The fact that the non-equivariant Chern character is cohomologous to a 0-cocycle is natural as our spectral triple is ε-summable for any $\varepsilon > 0$. On the other hand, the spectral triple is $(2+\varepsilon,\rho)$-summable in the sense of [NT], so twisted cyclic cohomology does not see the dimension drop and captures the volume form.

We finally remark that $\mathrm{Ind}_F([A_n]) = -\mathrm{sign}(n)\mathrm{Tr}_{\frac{|n|-1}{2}}$ for $n \neq 0$. To prove this it suffices to check that $\langle[-q^{-1}\tau],[A_n]\rangle = -[n]_q$. Another possibility is to use the classical theory. To this end one just has to show that there are projections $p_n(q)$ representing $[A_n]$ with the property that $I_s(q)p_n(q)$ depend continuously on $q \in (0,1]$ (the projections $p_n(q)$ themselves can be discontinuous at $q=1$). In the classical case the operator $p_n(1\otimes D)p_n$ is homotopic to the operator $D_n = \begin{pmatrix} 0 & \partial_e \\ \partial_f & 0 \end{pmatrix}$ which acts on the Hilbert space $L^2(A_{n+1},h) \oplus L^2(A_{n-1},h)$. Both operators are differential operators of order 1 with the same principal symbol, and the index of $\partial_f\colon L^2(A_{n+1},h) \to L^2(A_{n-1},h)$ is given by the Borel-Weil-Bott theorem and can also easily be found by direct computations.

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