Entropy of automorphisms of II_1-factors arising from the dynamical systems theory

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Abstract

Let a countable amenable group G act freely and ergodically on a Lebesgue space (X,μ), preserving the measure μ. If $T\in \mathrm{Aut}\,(X,\mu)$ is an automorphism of the equivalence relation defined by G then T can be extended to an automorphism α_T of the II_1-factor $M=L^\infty(X,\mu)\rtimes G$. We prove that if T commutes with the action of G then $H(\alpha_T)=h(T)$, where $H(\alpha_T)$ is the Connes-Størmer entropy of α_T, and $h(T)$ is the Kolmogorov–Sinai entropy of T. We prove also that for given s and t, $0\le s\le t\le\infty$, there exists a T such that $h(T)=s$ and $H(\alpha_T)=t$.

Introduction

Entropy is an important notion in classical statistical mechanics and information theory. Initially the conception of entropy for automorphism in the ergodic theory was introduced by Kolmogorov and Sinai in 1958. This invariant proved to be extremely useful in the classical dynamical systems theory and topological dynamics. The extension of this notion onto quantum dynamical systems was done by Connes, Narnhofer, Størmer and Thirring [CS, CNT]. At the present time there are several other promising approaches to entropy of C^*-dynamical systems [S, AF, V].

An important trend in dynamical entropy is its computation for various models. A lot of interesting results were obtained in this field in the recent years. We note several of them. Størmer, Voiculescu [SV], and the second author [N] computed the entropy of Bogoliubov automorphisms of CAR and CCR algebras (see also [BG, GN2]). Pimsner, Popa [PP], Choda [Ch1] computed the entropy of shifts of Temperley-Lieb algebras, Choda [Ch2], Hiai [H] and Størmer [St] computed the entropy of canonical shifts. The first author, Størmer [GS1, GS2], Price [Pr] computed entropy for a wide class of binary shifts.

In this paper we consider automorphisms of II_1 factors arising from the dynamical systems theory. Let a countable group G act freely and ergodically on a Lebesgue space (X,μ) and preserves μ. Then one can construct the crossed product $M=L^\infty(X,\mu)\rtimes G$, which, as is well known, is a II_1-factor. If $T\in\mathrm{Aut}\,(X,\mu)$ defines an automorphism of the ergodic equivalence relation induced by G then T can be extended to an automorphism α_T of M [FM]. It is a natural problem to compute the dynamical entropy $H(\alpha_T)$ in the sense of [CS] and to compare it with the Kolmogorov-Sinai entropy $h(T)$ of T. It should be noted that this last problem is a part of a more general problem. Namely, let M be a II_1-factor, $\alpha\in\mathrm{Aut}\;M$, A its α-invariant Cartan subalgebra, $\alpha(A)=A$, then it is natural to investigate when $H(\alpha)$ is equal to $H(\alpha|_A)$.

These problems are studied in our paper. In Section 1 we prove that if T commutes with the action of G then $H(\alpha_T) = h(T)$. More generally, we prove that this result is valid for crossed products of arbitrary algebras for the entropies of Voiculescu [V] and of Connes-Narnhofer-Thirring [CNT]. In Section 2 we consider two examples to illustrate this result. These examples give non-isomorphic ergodic automorphisms of the hyperfinite ergodic equivalence relation with the same entropy. In Section 3 we construct several examples showing that the entropies $h(T)$ and $H(\alpha_T)$ can be distinct. These systems are non-commutative analogues of dynamical systems of algebraic origin (see [A, Y, LSW, S]). In particular, some of our examples are automorphisms of non-commutative tori. In Section 4 we construct flows T_t such that $H(\alpha_{T_1}) > h(T_1)$. In particular, we show that the values $h(T)$ and $H(\alpha_T)$ can be arbitrary.

1 Computation of entropy of automorphisms of crossed products

Let (X, μ) be a Lebesgue space, G a countable amenable group of automorphisms S_g, $g \in G$, of (X, μ) preserving μ, and T an automorphism of (X, μ), $\mu \circ T = \mu$, such that

$$TS_g = S_gT, \quad g \in G.$$

Theorem 1.1 *Let (X, μ), G and T be as above. Suppose G acts freely and ergodically on (X, μ). Then $M = L^\infty(X, \mu) \rtimes_S G$ is the hyperfinite II_1-factor with the trace-state τ induced by μ. The automorphism T can be canonically extended to an automorphism α_T of M, and*

$$H(\alpha_T) = h(T)\,,$$

where $H(\alpha_T)$ is the Connes-Størmer entropy of α_T, and $h(T)$ is the Kolmogorov-Sinai entropy of T.

We will prove the following more general result.

Theorem 1.2 *Let M be an approximately finite-dimensional W^*-algebra, σ its normal state, T a σ-preserving automorphism. Suppose a discrete amenable group G acts on M by automorphisms S_g that commute with T and preserve σ. The automorphism T defines an automorphism α_T of $M \rtimes_S G$, and the state σ is extended to the dual state which we continue to denote by σ. Then*

(i) $hcpa_\sigma(\alpha_T) = hcpa_\sigma(T)$, where $hcpa_\sigma$ is the completely positive approximation entropy of Voiculescu [V];

(ii) $h_\sigma(\alpha_T) = h_\sigma(T)$, where h_σ is the dynamical entropy of Connes-Narnhofer-Thirring [CNT].

The referee brought to our attention the interesting paper of Brown [B]. As one can see from our proof, Theorem 1.2 can be extended to the case of topological entropy considered by Brown.

Since CNT-entropy coincides with KS-entropy in the classical case, and with CS-entropy for tracial σ and approximately finite-dimensional M, Theorem 1.1 follows from Theorem 1.2.

To prove Theorem 1.2 we will generalize a construction of Voiculescu [V] (see also [B]).

Lemma 1.3 *Let B be a C^*-algebra, $x_1, \dots, x_n \in B$. Then the mapping $\Psi\colon \mathrm{Mat}_n(\mathbb{C}) \otimes B \to B$,*

$$\Psi(e_{ij} \otimes b) = x_i b x_j^*,$$

is completely positive.

Proof. Consider the element $V \in \mathrm{Mat}_n(B) = \mathrm{Mat}_n(\mathbb{C}) \otimes B$,

$$V = \begin{pmatrix} x_1 & \dots & x_n \\ 0 & \dots & 0 \\ & \dots & \\ 0 & \dots & 0 \end{pmatrix}.$$

Consider also the projection $p = e_{11} \otimes 1 \in \mathrm{Mat}_n(\mathbb{C}) \otimes B$. Then Ψ is the mapping $\mathrm{Mat}_n(B) \to p\mathrm{Mat}_n(B)p = B$, $x \mapsto VxV^*$.

■

Let λ be the canonical representation of G in $M \rtimes G$, so that $(\mathrm{Ad}\,\lambda(g))(a) = S_g(a)$ for $a \in M$.

Lemma 1.4 *For any finite subset F of G, there exist normal unital completely positive mappings $I_F\colon B(l^2(F)) \otimes M \to M \rtimes G$ and $J_F\colon M \rtimes G \to B(l^2(F)) \otimes M$ such that*

$$\begin{aligned}
I_F(e_{g,h} \otimes a) &= \frac{1}{|F|}\lambda(g)a\lambda(h)^* = \frac{1}{|F|}\lambda(gh^{-1})S_h(a), \\
J_F(\lambda(g)a) &= \sum_{h \in F \cap g^{-1}F} e_{gh,h} \otimes S_{h^{-1}}(a), \\
(I_F \circ J_F)(\lambda(g)a) &= \frac{|F \cap g^{-1}F|}{|F|}\lambda(g)a, \\
\sigma \circ I_F &= \mathrm{tr}_F \otimes \sigma, \quad \alpha_T \circ I_F = I_F \circ (\mathrm{id} \otimes T), \\
(\mathrm{tr}_F \otimes \sigma) \circ J_F &= \sigma, \quad (\mathrm{id} \otimes T) \circ J_F = J_F \circ \alpha_T,
\end{aligned}$$

where tr_F is the unique tracial state on $B(l^2(F))$.

Proof. The complete positivity of I_F follows from Lemma 1.3. Consider J_F. Suppose that $M \subset B(H)$, and consider the regular representation of $M \rtimes G$ on $l^2(G) \otimes H$:

$$\lambda(g)(\delta_h \otimes \xi) = \delta_{gh} \otimes \xi, \quad a(\delta_h \otimes \xi) = \delta_h \otimes S_{h^{-1}}(a)\xi \quad (a \in M).$$

Let P_F be the projection onto $l^2(F) \otimes H$. Then a direct computation shows that the mapping $J_F(x) = P_F x P_F$, $x \in M \rtimes G$, has the form written above. All other assertions follow immediately.

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Proof of Theorem 1.2.
(i) Since there exists a τ-preserving conditional expectation $M \rtimes G \to M$, we have $hcpa_\sigma(\alpha_T) \ge hcpa_\sigma(T)$. To prove the opposite inequality we have to show that $hcpa_\sigma(\alpha_T, \omega) \le hcpa_\sigma(T)$ for any finite subset ω of $M \rtimes G$. Fix $\varepsilon > 0$. We can find a finite subset F of G such that $||(I_F \circ J_F)(x) - x||_\sigma < \varepsilon$ for any $x \in \omega$. Let $(\psi, \phi, B) \in CPA(B(l^2(F)) \otimes M, \mathrm{tr}_F \otimes \sigma)$ (see [V, Section 3] for notations). Then $(I_F \circ \psi, \phi \circ J_F, B) \in CPA(M \rtimes G, \sigma)$. Suppose

$$||(\psi \circ \phi)(J_F(x)) - J_F(x)||_{\mathrm{tr}_F \otimes \sigma} < \delta$$

for some $x \in \alpha_T^k(\omega)$ and $k \in \mathbb{N}$. Then

$$||(I_F \circ \psi \circ \phi \circ J_F)(x) - x||_\sigma \le ||(\psi \circ \phi)(J_F(x)) - J_F(x)||_{\sigma \circ I_F} + ||(I_F \circ J_F)(x) - x||_\sigma < \delta + \varepsilon,$$

where we have used the facts that $\sigma \circ I_F = \mathrm{tr}_F \otimes \sigma$ and that α_T commutes with $I_F \circ J_F$. Since $J_F \circ \alpha_T = (\mathrm{id} \otimes T) \circ J_F$, we infer that

$$rcp_\sigma(\omega \cup \alpha_T(\omega) \cup \ldots \cup \alpha_T^{n-1}(\omega); \delta + \varepsilon) \le rcp_{\mathrm{tr}_F \otimes \sigma}(J_F(\omega) \cup \ldots \cup (\mathrm{id} \otimes T)^{n-1}(J_F(\omega)); \delta),$$

so that (for $\delta < \varepsilon$)

$$\begin{aligned} hcpa_\sigma(\alpha_T, \omega; 2\varepsilon) &\le hcpa_\sigma(\alpha_T, \omega; \varepsilon + \delta) \le hcpa_{\mathrm{tr}_F \otimes \sigma}(\mathrm{id} \otimes T, J_F(\omega); \delta) \\ &\le hcpa_{\mathrm{tr}_F \otimes \sigma}(\mathrm{id} \otimes T) = hcpa_\sigma(T), \end{aligned}$$

where the last equality follows from the subadditivity of the entropy [V]. Since $\varepsilon > 0$ was arbitrary, the proof of the inequality $hcpa_\sigma(\alpha_T, \omega) \le hcpa_\sigma(T)$ is complete.

(ii) We always have $h_\sigma(\alpha_T) \ge h_\sigma(T)$. To prove the opposite inequality consider a channel $\gamma: B \to M \rtimes G$, i. e., a unital completely positive mapping of a finite-dimensional C*-algebra B. We have to prove that $h_\sigma(\alpha_T; \gamma) \le h_\sigma(T)$. Fix $\varepsilon > 0$. We can choose F such that

$$||(I_F \circ J_F \circ \gamma - \gamma)(x)||_\sigma \le \varepsilon ||x|| \quad \text{for any } x \in B.$$

By [CNT, Theorem IV.3],

$$\frac{1}{n} H_\sigma(\gamma, \alpha_T \circ \gamma, \ldots, \alpha_T^{n-1} \circ \gamma) \le \delta + \frac{1}{n} H_\sigma(I_F \circ J_F \circ \gamma, \alpha_T \circ I_F \circ J_F \circ \gamma, \ldots, \alpha_T^{n-1} \circ I_F \circ J_F \circ \gamma), \quad (1.1)$$

where $\delta = \delta(\varepsilon, \mathrm{rank}\, B) \to 0$ as $\varepsilon \to 0$. Since $\sigma \circ I_F = \mathrm{tr}_F \otimes \sigma$, it is easy to see from the definition of mutual entropy H_σ [CNT] that

$$H_\sigma(I_F \circ J_F \circ \gamma, I_F \circ J_F \circ \alpha_T \circ \gamma, \ldots, I_F \circ J_F \circ \alpha_T^{n-1} \circ \gamma) \le H_{\mathrm{tr}_F \otimes \sigma}(J_F \circ \gamma, J_F \circ \alpha_T \circ \gamma, \ldots, J_F \circ \alpha_T^{n-1} \circ \gamma) \quad (1.2)$$

Since $I_F \circ J_F$ commutes with α_T, and $J_F \circ \alpha_T = (\mathrm{id} \otimes T) \circ J_F$, we infer from (1.1) and (1.2) that

$$h_\sigma(\alpha_T; \gamma) \le \delta + h_{\mathrm{tr}_F \otimes \sigma}(\mathrm{id} \otimes T; J_F \circ \gamma) \le \delta + h_{\mathrm{tr}_F \otimes \sigma}(\mathrm{id} \otimes T).$$

Since we could choose F such that δ was arbitrary small, we see that it suffices to prove that $h_{\mathrm{tr}_F \otimes \sigma}(\mathrm{id} \otimes T) = h_\sigma(T)$. For abelian M this is proved by standard arguments, using [CNT, Corollary VIII.8]. To handle the general case we need the following lemma.

Lemma 1.5 *For any finite-dimensional C*-algebra B, any state ϕ of B, and any positive linear functional ψ on* $\mathrm{Mat}_n(\mathbb{C}) \otimes B$*, we have*

$$S(\mathrm{tr}_n \otimes \phi, \psi) \le S(\phi, \psi|_B) + 2\psi(1) \log n.$$

Proof. By [OP, Theorem 1.13],

$$S(\mathrm{tr}_n \otimes \phi, \psi) = S(\phi, \psi|_B) + S(\psi \circ E, \psi),$$

where $E = \mathrm{tr}_n \otimes \mathrm{id}: \mathrm{Mat}_n(\mathbb{C}) \otimes B \to B$ is the $(\mathrm{tr}_n \otimes \phi)$-preserving conditional expectation (note that we adopt the notations of [CNT], so we denote by $S(\omega_1, \omega_2)$ the quantity which is denoted by $S(\omega_2, \omega_1)$ in [OP]). By the Pimsner-Popa inequality [PP, Theorem 2.2], we have

$$E(x) \ge \frac{1}{n^2} x \quad \text{for any} \quad x \in \mathrm{Mat}_n(\mathbb{C}) \otimes B, \ x \ge 0.$$

In particular, $\psi \circ E \geq \frac{1}{n^2}\psi$, whence $S(\psi \circ E, \psi) \leq 2\psi(1)\log n$.

■

Since M is an AFD-algebra, to compute the entropy of $\mathrm{id} \otimes T$ it suffices to consider subalgebras of the form $B(l^2(F)) \otimes B$, where $B \subset M$. From Lemma 1.5 and the definitions [CNT] we immediately get

$$h_{\mathrm{tr}_F \otimes \sigma}(\mathrm{id} \otimes T; B(l^2(F)) \otimes B) \leq h_\sigma(T; B) + 2\log|F|.$$

Hence $h_{\mathrm{tr}_F \otimes \sigma}(\mathrm{id} \otimes T) \leq h_\sigma(T) + 2\log|F|$. Applying this inequality to T^m, we obtain

$$h_{\mathrm{tr}_F \otimes \sigma}((\mathrm{id} \otimes T)^m) \leq h_\sigma(T^m) + 2\log|F| \quad \forall m \in \mathbb{N}.$$

But since M is an AFD-algebra, we have $h_{\mathrm{tr}_F \otimes \sigma}((\mathrm{id} \otimes T)^m) = m \cdot h_{\mathrm{tr}_F \otimes \sigma}(\mathrm{id} \otimes T)$ and $h_\sigma(T^m) = m \cdot h_\sigma(T)$. So dividing the above inequality by m, and letting $m \to \infty$, we obtain $h_{\mathrm{tr}_F \otimes \sigma}(\mathrm{id} \otimes T) \leq h_\sigma(T)$, and the proof of Theorem is complete.

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Remarks.
(i) For any AFD-algebra N and any normal state ω of N, we have $h_{\omega \otimes \sigma}(\mathrm{id} \otimes T) = h_\sigma(T)$. Indeed, we may suppose that N is finite-dimensional and ω is faithful (because if p is the support of ω, then $h_{\omega \otimes \sigma}(\mathrm{id} \otimes T) = h_{\omega \otimes \sigma}((\mathrm{id} \otimes T)|_{pNp \otimes M})$). Now the only thing we need is a generalization of the Pimsner-Popa inequality. Let $p_1, \ldots, p_m$ be the atoms of a maximal abelian subalgebra of the centralizer of the state ω. Then

$$(\omega \otimes \mathrm{id})(x) \geq \left(\sum_{i=1}^{m} \frac{1}{\omega(p_i)} \right)^{-1} x \quad \text{for any} \quad x \in N \otimes M,\ x \geq 0,$$

by [L, Theorem 4.1 and Proposition 5.4].
(ii) By Corollary 3.8 in [V], $hcpa_\mu(T) = h(T)$ for ergodic T. For non-ergodic T, the entropies can be distinct. Indeed, let X_1 be a T-invariant measurable subset of X, $\lambda = \mu(X)$, $0 < \lambda < 1$. Set $\mu_1 = \lambda^{-1}\mu|_{X_1}$, $T_1 = T|_{X_1}$, $X_2 = X \backslash X_1$, $\mu_2 = (1-\lambda)^{-1}\mu|_{X_2}$, $T_2 = T|_{X_2}$. It is easy to see that $h(T) = \lambda h(T_1) + (1-\lambda)h(T_2)$. On the other hand, it can be proved that

$$hcpa_\mu(T) = \max\{hcpa_{\mu_1}(T_1), hcpa_{\mu_2}(T_2)\}.$$

So if $h(T_1), h(T_2) < \infty$, $h(T_1) \neq h(T_2)$, then $h(T) < hcpa_\mu(T)$.

To obtain an invariant which coincides with KS-entropy in the classical case, one can modify Voiculescu's definition replacing $\operatorname{rank} B$ with $\exp S(\sigma \circ \psi)$ in [V, Definition 3.1]. Theorem 1.2 remains true for this modified entropy.

2 Examples

We present two examples to illustrate Theorem 1.1. These examples give non-isomorphic ergodic automorphisms of amenable equivalence relations with the same KS-entropy.

Let us first describe a general construction.

Proposition 2.1 *Let S_0, S_1, S_2 be ergodic automorphisms of (X, μ) such that S_0 commutes with S_1 and S_2, and S_1 is conjugate with neither S_2, nor S_2^{-1} by an automorphism commuting with S_0. Set $M_i = L^\infty(X, \mu) \rtimes_{S_i} \mathbb{Z}$, $i = 1, 2$, and let α_i be the automorphism of M_i induced by S_0. Then there is no isomorphism ϕ of M_1 onto M_2 such that $\phi \circ \alpha_1 = \alpha_2 \circ \phi$ and $\phi(L^\infty(X, \mu)) = L^\infty(X, \mu)$.*

Proof. Suppose such a ϕ exists. Let $U_i \in M_i$ be the unitary corresponding to S_i, $i = 1, 2$, $A = L^\infty(X, \mu) \subset M_1$. Set $U = \phi^{-1}(U_2)$. Since U is a unitary operator from M_1 such that $(\operatorname{Ad} U)(A) = A$, it is easy to check that U has the form

$$U = \sum_{i \in \mathbb{Z}} a_i U_1^i,$$

where $\{a_i\}_i$ is a family of elements of A, $a_i a_j = 0$ for $i \neq j$. Since $\alpha_1(U) = U$, we have $\alpha_1(a_i) = a_i$, $i \in \mathbb{Z}$. But S_0 is ergodic, therefore a_i are constants. Hence $U = a_i U_1^i$ for some $i \in \mathbb{Z}$ and $a_i \in \mathbb{T}$. Since ϕ is an isomorphism, we have either $i = -1$, or $i = 1$. We see that $\phi|_{L^\infty(X,\mu)}$ is an automorphism that commutes with S_0 and conjugates S_2 with either S_1^{-1}, or S_1.

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Remark. It follows from Proposition 2.1 that S_0 defines non-isomorphic automorphisms of the ergodic equivalence relations induced by S_1 and S_2 on X correspondingly, despite of $H(\alpha_1) = H(\alpha_2) = h(S_0)$.

Example 2.2 Let $X = [0, 1]$ be the unit interval, μ the Lebesgue measure on X, t_0, t_1 and t_2 irrational numbers from $[0, 1]$ such that $t_2 \neq t_1, 1 - t_1$. Consider the shifts $S_i x = x + t_i$ (mod 1), $x \in [0, 1]$. Any automorphism of X commuting with S_0 commutes with S_1 and S_2. Since $S_1 \neq S_2^{\pm 1}$, Proposition 2.1 is applicable. Note that $h(S_0) = 0$.

Example 2.3 Let (X, μ) be a Lebesgue space, T_t a Bernoulli flow on (X, μ) with $h(T_1) = \log 2$ [O]. Choose $t_i \in \mathbb{R}$, $t_i \neq 0$ $(i = 0, 1, 2)$, $t_1 \neq \pm t_2$, and set $S_i = T_{t_i}$. Then $h(S_1) \neq h(S_2)$, and we can apply Proposition 2.1.

3 Entropy of automorphisms and their restrictions to a Cartan subalgebra

Let M be a II_1-factor, A its Cartan subalgebra, $\alpha \in \operatorname{Aut} M$ such that $\alpha(A) = A$. We consider cases when $H(\alpha) > H(\alpha|_A)$.

Suppose a discrete abelian group G acts freely and ergodically by automorphisms S_g on (X, μ), β an automorphism of G, and S an automorphism of (X, μ) such that $TS_g = S_{\beta(g)} T$. Then T induces an automorphism α_T of $M = L^\infty(X, \mu) \rtimes_S G$. Explicitly,

$$\alpha_T(f)(x) = f(T^{-1}x) \text{ for } f \in L^\infty(X, \mu),\ \alpha_T(\lambda(g)) = \lambda(\beta(g)).$$

The algebra $A = L^\infty(X, \mu)$ is a Cartan subalgebra of M. On the other hand, the operators $\lambda(g)$ generate a maximal abelian subalgebra $B \cong L^\infty(\hat{G})$ of M, and $\alpha_T|_B = \hat{\beta}$, the dual automorphism of $\hat{G}$. We have

$$H(\alpha_T) \geq \max\{h(T), h(\hat{\beta})\},$$

so if $h(\hat{\beta}) > h(T)$, then $H(\alpha_T) > H(\alpha_T|_A)$.

To construct such examples we consider systems of algebraic origin.

Let G_1 and G_2 be discrete abelian groups, and T_1 an automorphism of G_1. Suppose there exists an embedding $l \colon G_2 \hookrightarrow \hat{G}_1$ such that $l(G_2)$ is a dense $\hat{T}_1$-invariant subgroup. Set $T_2 =$

$\hat{T}_1|_{G_2}$. The group G_2 acts by translations on $\hat{G}_1$ ($g_2 \cdot \chi_1 = \chi_1 + l(g_2)$), and we fall into the situation described above (with $X = \hat{G}_1$, $G = G_2$, $T = \hat{T}_1$ and $\beta = T_2$).

The roles of G_1 and G_2 above are almost symmetric. Indeed, to be given an embedding $G_2 \hookrightarrow \hat{G}_1$ with dense range is just the same as to be given a non-degenerate pairing $\langle\cdot\,,\,\cdot\rangle\colon G_1 \times G_2 \to \mathbb{T}$, then the equality $T_2 = \hat{T}_1|_{G_2}$ means that this pairing is $T_1 \times T_2$-invariant. The pairing gives rise to an embedding $r\colon G_1 \hookrightarrow \hat{G}_2$. Then G_1 acts on $\hat{G}_2$ by translations $g_1 \cdot \chi_2 = \chi_2 - r(g_1)$, and $L^\infty(\hat{G}_1) \rtimes G_2 \cong G_1 \ltimes L^\infty(\hat{G}_2)$. In fact, both algebras are canonically isomorphic to the twisted group W*-algebra $W^*(G_1 \times G_2, \omega)$, where ω is the bicharacter defined by

$$\omega((g_1', g_2'), (g_1'', g_2'')) = \langle g_1'', g_2'\rangle.$$

Then α_T is nothing else than the automorphism induced by the ω-preserving automorphism $T_1 \times T_2$.

Let $R = \mathbb{Z}[t, t^{-1}]$ be the ring of Laurent polynomials over $\mathbb{Z}$, $f \in \mathbb{Z}[t]$, $f \neq 1$, a polynomial whose irreducible factors are not cyclotomic, equivalently, f has no roots of modulus 1. Fix $n \in \{2, 3, \dots, \infty\}$. Set $G_1 = R/(f^\sim)$ and $G_2 = \overset{n}{\underset{k=1}{\oplus}} R/(f)$, where $f^\sim(t) = f(t^{-1})$. Let T_i be the automorphism of G_i of multiplication by t. Let χ be a character of G_2. Then the mapping $R \ni f_1 \mapsto f_1(\hat{T}_2)\chi \in \hat{G}_2$ defines an equivariant homomorphism $G_1 \to \hat{G}_2$. In other words, if $\chi = (\chi_1, \dots, \chi_n) \in \hat{G}_2 \subset \hat{R}^n$, then the pairing is given by

$$\langle f_1, (g_1, \dots, g_n)\rangle = \prod_{k=1}^{n} \chi_k(f_1^\sim \cdot g_k),$$

where $(f_1^\sim \cdot g_k)(t) = f_1(t^{-1})g_k(t)$. This pairing is non-degenerate iff the orbit of χ under the action of $\hat{T}_2$ generates a dense subgroup of $\hat{G}_2$. Since T_2 is aperiodic, the dual automorphism is ergodic. Hence the orbit is dense for almost every choice of χ.

Now let us estimate entropy. First, by Yuzvinskii's formula [Y, LW], $h(\hat{T}_1) = m(f)$, $h(\hat{T}_2) = n \cdot m(f)$, where $m(f)$ is the logarithmic Mahler measure of f,

$$m(f) = \int_0^1 \log|f(e^{2\pi i s})|ds = \log|a_m| + \sum_{j : |\lambda_j| > 1} \log|\lambda_j|,$$

where a_m is the leading coefficient of f, and $\{\lambda_j\}_j$ are the roots of f. Now suppose that the coefficients of the leading and the lowest terms of f are equal to 1. Then $G_1 \times G_2$ is a free abelian group of rank $(n+1)\deg f$, and by a result of Voiculescu [V] we have $H(\alpha_T) \le h(\hat{T}_1 \times \hat{T}_2) = (n+1)m(f)$.

Note also that since the automorphism $T_1 \times T_2$ is aperiodic, the automorphism α_T is mixing.

Let us summarize what we have proved.

Theorem 3.1 *For given $n \in \{2, 3, \dots, \infty\}$ and a polynomial $f \in \mathbb{Z}[t]$, $f \neq 1$, whose coefficients of the leading and the lowest terms are equal to 1 and which has no roots of modulus 1, there exist a mixing automorphism α of the hyperfinite II_1-factor and an α-invariant Cartan subalgebra A such that*

$$H(\alpha|_A) = m(f), \quad n \cdot m(f) \le H(\alpha) \le (n+1)m(f).$$

The possibility of constructing in this way systems with arbitrary values $H(\alpha|_A) < H(\alpha)$ is closely related to the question, whether 0 is a cluster point of the set $\{m(f) \mid f \in \mathbb{Z}[t]\}$ (note that it suffices to consider irreducible polynomials whose leading coefficients and constant terms are equal to 1). This question is known as Lehmer's problem, and there is evidence that the answer is *negative* (see [LSW] for a discussion).

In estimating the entropy above we used the result of Voiculescu stating that the entropy of an automorphism of a non-commutative torus is not greater than the entropy of its abelian counterpart. It is clear that this result should be true for a wider class of systems. Consider the most simple case where the polynomial f is a constant.

Example 3.2 Let $f = 2$ and $n = 2$. Then $G_1 = R/(2) \cong \underset{k\in\mathbb{Z}}{\oplus} \mathbb{Z}/2\mathbb{Z}$, $G_2 = G_1 \oplus G_1$, T_1 is the shift to the right, $T_2 = T_1 \oplus T_1$. Let $G_1(0) = \mathbb{Z}/2\mathbb{Z} \subset G_1$ and $G_2(0) = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \subset G_2$ be the subgroups sitting at the 0th place. Set

$$G_i^{(n)} = G_i(0) \oplus T_iG_i(0) \oplus \ldots \oplus T_i^nG_i(0).$$

Then $H(\alpha_T) \le hcpa_\tau(\alpha_T) \le \lim\limits_{n\to\infty} \dfrac{1}{n} \log \operatorname{rank} C^*(G_1^{(n)} \times G_2^{(n)}, \omega) \le 3\log 2$, so (for $A = L^\infty(\hat{G}_1)$)

$$H(\alpha_T|_A) = \log 2 \quad\text{and}\quad 2\log 2 \le H(\alpha_T) \le 3\log 2.$$

The actual value of $H(\alpha_T)$ is probably depends on the choice of the character $\chi \in \hat{G}_2$. We want to show that $H(\alpha_T) = 2\log 2$ for some special choice of χ. For this it suffices to require the pairing $\langle\cdot\,,\,\cdot\rangle|_{G_1^{(n)}\times G_2^{(n)}}$ be non-degenerate in the first variable for any $n \ge 0$ (so that $C^*(G_2^{(n)})$ is a maximal abelian subalgebra of $C^*(G_1^{(n)} \times G_2^{(n)}, \omega)$, and the rank of the latter algebra is equal to $4(n+1)$). The embedding $G_1 \hookrightarrow \hat{G}_2$ is given by

$$g_1 \mapsto \prod_{n\in\mathbb{Z}: g_1(n)\ne 0} \hat{T}_2^n\chi, \quad g_1 = (g_1(n))_n \in \underset{n\in\mathbb{Z}}{\oplus} \mathbb{Z}/2\mathbb{Z}.$$

So we must choose χ in a way such that the character $\prod_{k=1}^m \hat{T}_2^{n_k}\chi$ is non-trivial on $G_2^{(n)}$ for any $0 \le n_1 < \ldots < n_m \le n$. Identify $\hat{G}_2$ with $\prod_{n\in\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z})$. Then $\hat{T}_2$ is the shift to the right, and we may take any $\chi = (\chi_n)_n$ such that

(i) $\chi_n = 0$ for $n < 0$, $\chi_0 \ne 0$;

(ii) the group generated by $\hat{T}_2^n\chi$ is dense in $\hat{G}_2$.

Finally, we will show that it is possible to construct systems with positive entropy, which have zero entropy on a Cartan subalgebra.

Example 3.3 Let p be a prime number, $p \ne 2$, $\hat{G}_1 = \mathbb{Z}_p$ (the group of p-adic integers), $G_2 = \cup_{n\in\mathbb{N}} 2^{-n}\mathbb{Z} \subset \hat{G}_1$, $\hat{T}_1$ and T_2 the automorphisms of multiplication by 2. The group G_1 is the inductive limit of the groups $\mathbb{Z}/p^n\mathbb{Z}$, and T_1 acts on them as the automorphism of division by 2. Hence

$$H(\alpha_T|_A) = \lim_{n\to\infty} H(\alpha_T|_{C^*(\mathbb{Z}/p^n\mathbb{Z})}) = 0.$$

Since $G_2 = R/(t-2)$, we have $h(\hat{T}_2) = \log 2$, so $H(\alpha_T) \ge \log 2$. We state that

$$H(\alpha_T) = hcpa_\tau(\alpha_T) = \log 2.$$

The automorphism $T_1^{p^{n-1}(p-1)}$ is identical on $\mathbb{Z}/p^n\mathbb{Z}$. Since

$$W^*(\mathbb{Z}/p^n\mathbb{Z} \times G_2, \omega) = \mathbb{Z}/p^n\mathbb{Z} \ltimes L^\infty(\hat{G}_2),$$

by Theorem 1.2 we infer

$$hcpa_\tau(\alpha_T^{p^{n-1}(p-1)}|_{W^*(\mathbb{Z}/p^n\mathbb{Z}\times G_2,\omega)}) = h(\hat{T}_2^{p^{n-1}(p-1)}),$$

whence $hcpa_\tau(\alpha_T|_{W^*(\mathbb{Z}/p^n\mathbb{Z}\times G_2,\omega)}) = \log 2$, and

$$hcpa_\tau(\alpha_T) = \lim_{n\to\infty} hcpa_\tau(\alpha_T|_{W^*(\mathbb{Z}/p^n\mathbb{Z}\times G_2,\omega)}) = \log 2.$$

4 Flows on II_1-factors with invariant Cartan subalgebras

Using examples of previous sections and the construction of associated flow we will construct systems with arbitrary values of $H(\alpha|_A)$ and $H(\alpha)$ $(0 \le H(\alpha|_A) \le H(\alpha) \le \infty)$.

Suppose a discrete amenable group G acts freely and ergodically by measure-preserving transformations S_g on (X,μ), T an automorphism of (X,μ) and β an automorphism of G such that $TS_g = S_{\beta(g)}T$. Consider the flow F_t associated with T. So $Y = \mathbb{R}/\mathbb{Z}\times X$, $d\nu = dt\times d\mu$,

$$F_t(\dot r, x) = (\dot r + \dot t, T^{[r+t]}x) \ \text{ for } \ r\in[0,1),\ x\in X,$$

where $t\mapsto \dot t$ is the factorization mapping $\mathbb{R}\to\mathbb{R}/\mathbb{Z}$. The semidirect product group $G_0 = G\times_\beta\mathbb{Z}$ acts on (X,μ). This action is ergodic. It is also free if

$$\text{there exist no } g\in G \text{ and no } n\in\mathbb{N} \text{ such that } S_g = T^n \text{ on a set of positive measure.} \tag{4.1}$$

Let Γ be a countable dense subgroup of $\mathbb{R}/\mathbb{Z}$, it acts by translations on $\mathbb{R}/\mathbb{Z}$. Set $\mathcal{G} = \Gamma\times G_0$. The group $\mathcal{G}$ is amenable. It acts freely and ergodically on (Y,ν). The corresponding equivalence relation is invariant under the flow, so we obtain a flow α_t on $L^\infty(Y,\nu)\rtimes\mathcal{G}$. We next compute its entropy. Let α_T be the automorphism of $L^\infty(X,\mu)\rtimes G$ defined by T. We state that

$$H(\alpha_t) = |t|H(\alpha_T),\quad hcpa_\tau(\alpha_t) = |t|hcpa_\tau(\alpha_T),\quad\text{and}\quad H(\alpha_t|_{L^\infty(Y,\nu)}) = |t|h(T). \tag{4.2}$$

Since $h(F_t) = |t|h(F_1) = |t|h(\mathrm{id}\times T)$, the last equality in (4.2) is evident. To prove the first two note that

$$H(\alpha_t) = |t|H(\alpha_1) \ \text{ and } \ hcpa_\tau(\alpha_t) = |t|hcpa_\tau(\alpha_1)$$

(see [OP, Proposition 10.16] for the first equality, the second is proved analogously). We have

$$L^\infty(Y,\nu)\rtimes\mathcal{G} = (L^\infty(\mathbb{R}/\mathbb{Z})\rtimes\Gamma)\otimes(L^\infty(X,\mu)\rtimes G_0),\quad \alpha_1 = \mathrm{id}\otimes\tilde\alpha_T,$$

where $\tilde\alpha_T$ is the automorphism of $L^\infty(X,\mu)\rtimes G_0$ defined by T. Since completely positive approximation entropy is subadditive and monotone [V], we have $hcpa_\tau(\mathrm{id}\otimes\tilde\alpha_T) = hcpa_\tau(\tilde\alpha_T)$. We have also $H(\mathrm{id}\otimes\tilde\alpha_T) = H(\tilde\alpha_T)$ by Remark following the proof of Theorem 1.2. Since

$$L^\infty(X,\mu)\rtimes G_0 = (L^\infty(X,\mu)\rtimes_S G)\rtimes_{\alpha_T}\mathbb{Z},$$

we obtain $hcpa_\tau(\tilde\alpha_T) = hcpa_\tau(\alpha_T)$ and $H(\tilde\alpha_T) = H(\alpha_T)$ by virtue of Theorem 1.2. So $hcpa_\tau(\alpha_1) = hcpa_\tau(\alpha_T)$ and $H(\alpha_1) = H(\alpha_T)$, and the proof of the equalities (4.2) is complete.

Theorem 4.1 *For any s and t, $0 \leq s < t \leq \infty$, there exist an automorphism α of the hyperfinite II_1-factor and an α-invariant Cartan subalgebra A such that*

$$H(\alpha|_A) = s \quad \textit{and} \quad H(\alpha) = t.$$

Proof. Consider a system from Example 3.3. Then the condition (4.1) is satisfied, so the construction above leads to a flow α_t and an α_t-invariant Cartan subalgebra A_1 such that

$$H(\alpha_t|_{A_1}) = 0 \quad \text{and} \quad H(\alpha_t) = hcpa_\tau(\alpha_t) = |t| \log 2.$$

As in Example 2.3, consider a Bernoulli flow S_t on (X, μ) with $h(S_1) = \log 2$. Then for the corresponding flow β_t on $L^\infty(X, \mu) \rtimes_{S_1} \mathbb{Z}$ we have (with $A_2 = L^\infty(X, \mu)$)

$$H(\beta_t|_{A_2}) = H(\beta_t) = hcpa_\tau(\beta_t) = |t| \log 2.$$

Since Connes-Størmer' entropy is superadditive [SV] and Voiculescu's entropies are subadditive, we conclude that

$$H((\alpha_t \otimes \beta_s)|_{A_1 \otimes A_2}) = |s| \log 2, \quad H(\alpha_t \otimes \beta_s) = H(\alpha_t) + H(\beta_s) = (|t| + |s|) \log 2.$$

Finally, consider an infinite tensor product of systems from Example 3.3. Thus we obtain an automorphism γ and an α-invariant Cartan subalgebra A_3 such that

$$H(\gamma|_{A_3}) = 0 \quad \text{and} \quad H(\gamma) = \infty.$$

Then $H(\beta_s \otimes \gamma)|_{A_2 \otimes A_3}) = |s| \log 2$, $H(\beta_s \otimes \gamma) = \infty$.

■

5 Final remarks

5.1. Let p_1 and p_2 be prime numbers, $p_i \geq 3$, $i = 1, 2$. Construct automorphisms α_1 and α_2 as in Example 3.3.

Proposition 5.1 *If $p_1 \neq p_2$, then α_1 and α_2 are not conjugate as automorphisms of the hyperfinite II_1-factor, though $H(\alpha_1) = H(\alpha_2) = \log 2$.*

Proof. Indeed, the automorphisms define unitary operators U_i on $L^2(M, \tau)$. As we can see, the point part S_i of the spectrum of U_i is non-trivial. If $p_1 \neq p_2$, then $S_1 \neq S_2$, so α_1 and α_2 are not conjugate.

■

5.2. The automorphisms of Theorem 3.1 and Example 3.2 are ergodic. On the other hand, the automorphisms of Example 3.3 are not ergodic, even on the Cartan subalgebra. Moreover, any ergodic automorphism of compact abelian group has positive entropy (it is even Bernoullian), so with the methods of Section 3 we can not construct ergodic automorphisms with positive entropy and zero entropy restriction to a Cartan subalgebra (however, for actions of $\mathbb{Z}^d$, $d \geq 2$, we are able to construct such examples).

The construction of Section 4 leads to non-ergodic automorphisms also, even if we start with an ergodic automorphism (such as in Example 3.2).

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