HECKE ALGEBRAS OF SEMIDIRECT PRODUCTS AND THE FINITE PART OF THE CONNES-MARCOLLI C*-ALGEBRA

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Abstract. We study a C*-dynamical system arising from the ring inclusion of the 2×2 integer matrices in the rational ones. The orientation preserving affine groups of these rings form a Hecke pair that is closely related to a recent construction of Connes and Marcolli; our dynamical system consists of the associated reduced Hecke C*-algebra endowed with a canonical dynamics defined in terms of the determinant function. We show that the Schlichting completion also consists of affine groups of matrices, over the finite adeles, and we obtain results about the structure and induced representations of the Hecke C*-algebra. In a somewhat unexpected parallel with the one dimensional case studied by Bost and Connes, there is a group of symmetries given by an action of the finite integral ideles, and the corresponding fixed point algebra decomposes as a tensor product over the primes. This decomposition allows us to obtain a complete description of a natural class of equilibrium states which conjecturally includes all KMS$_\beta$-states for $\beta \neq 0, 1$.

Introduction

Let $\mathbb{H}$ denote the upper halfplane and let $\mathbb{A}_f$ be the ring of finite adeles. The group $\mathrm{GL}_2^+(\mathbb{Q})$ of 2×2 rational matrices with positive determinant acts on $\mathbb{H}\times \mathrm{Mat}_2(\mathbb{A}_f)$, by Möbius transformations on $\mathbb{H}$ and by left multiplication on $\mathrm{Mat}_2(\mathbb{A}_f)$. Roughly speaking, the C*-algebra underlying the GL_2-system of Connes and Marcolli [3] can be constructed by effecting two modifications on the corresponding transformation groupoid $\mathrm{GL}_2^+(\mathbb{Q}) \times (\mathbb{H} \times \mathrm{Mat}_2(\mathbb{A}_f))$; the first one is to cut down from $\mathbb{A}_f$ to the compact open subring of finite integral adeles $\hat{\mathbb{Z}} = \prod_p \mathbb{Z}_p$ yielding a kind of semigroup crossed product by $\mathrm{Mat}_2^+(\mathbb{Z})$, and the second one is to eliminate the degeneracy due to the $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ symmetries by factoring out the action of $\Gamma\times\Gamma$ given by $(\gamma_1,\gamma_2)(g,x) = (\gamma_1 g\gamma_2^{-1}, \gamma_2 x)$. These modifications destroy the initial groupoid and semigroup crossed product structures, but the resulting C*-algebra

$$C^*_r(\Gamma\backslash \mathrm{GL}_2^+(\mathbb{Q}) \boxtimes_\Gamma (\mathbb{H}\times \mathrm{Mat}_2(\hat{\mathbb{Z}}))),$$

for which we use the notation of [12], retains enough of the original transformation groupoid flavor that it is possible to use slightly modified crossed product techniques in its study. The convolution formula that gives the product on the Connes-Marcolli C*-algebra is based on the convolution formula for the classical Hecke algebra. This connection was pursued early on by Tzanev [23], who pointed out that the Connes-Marcolli C*-algebra could also be described as $C^*_r(P_0\backslash P\times_{P_0}\mathbb{H})$, where

$$(P,P_0) = \left(\begin{pmatrix}1 & \mathrm{Mat}_2(\mathbb{Q})\\ 0 & \mathrm{GL}_2^+(\mathbb{Q})\end{pmatrix}, \begin{pmatrix}1 & \mathrm{Mat}_2(\mathbb{Z})\\ 0 & \mathrm{SL}_2(\mathbb{Z})\end{pmatrix}\right)$$

is a Hecke pair and the action of P on $\mathbb{H}$ is defined by Möbius transformations through the obvious homomorphism $P\to \mathrm{GL}_2^+(\mathbb{Q})$ (technically, the action of P_0 is not proper, but the construction of $C^*_r(P_0\backslash P\times_{P_0}\mathbb{H})$ still makes sense by [12, Remark 1.4]). Thus, the Connes-Marcolli C*-algebra can be

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thought of as a new type of crossed product: of the algebra $C_0(\Gamma\backslash\mathbb{H})$ by the Hecke pair (P, P_0), and, in particular, the reduced Hecke C*-algebra $C^*_r(P, P_0)$ is contained in the multiplier algebra of the Connes-Marcolli C*-algebra, see [12, Lemma 1.3]. Because of this, abusing slightly the terminology, we will refer to $C^*_r(P, P_0)$ as the *finite part of the Connes-Marcolli C*-algebra*, and we point out that this finite part corresponds to the quotient of the determinant part of the GL_2-system, cf. [3, Section 1.7], by the above action of $\Gamma \times \Gamma$.

The goal of the present work is to study the structure of $C^*_r(P, P_0)$ and the phase transition of the corresponding C*-dynamical system. We were initially motivated by our belief that it should be possible to exploit the crossed product structure observed by Tzanev in order to study the phase transition of the Connes-Marcolli system, and that in order to do this one would have to understand first the structure and the phase transition of the finite part. We were also motivated by the observation that the Hecke pair (P, P_0) consists of the orientation preserving affine transformations of the rings of 2×2 matrices over the rationals and over the integers, and hence the associated C*-dynamical system is a very natural (albeit somewhat naïve) higher dimensional version of the one studied by Bost and Connes [1], which certainly deserves consideration. In addition (P, P_0) is a very interesting example of a Hecke inclusion of semidirect product groups, a class that has received considerable attention in recent years, see e.g. [2, 11, 15, 6].

As it turned out, we were able to study the Connes-Marcolli phase transition and to prove the uniqueness of the KMS_β-states for β in the critical interval by a more direct method that does not require consideration of $C^*_r(P, P_0)$, although it does rely on it for insight, see [12]. Interestingly enough, the phase transition of the finite part of the Connes-Marcolli system seems to be a more resilient problem than for their full GL_2-system. The main reason for this is that the freeness resulting from the 'infinite part', that is to say, freeness of $\mathrm{GL}_2^+(\mathbb{Q})$ acting on $\mathbb{H} \times (\mathrm{Mat}_2(\mathbb{A}_f) \setminus \{0\})$, is a crucial ingredient in reducing KMS-states of the Connes-Marcolli GL_2-system to measures on $\mathbb{H} \times \mathrm{Mat}_2(\mathbb{A}_f)$. Because of this our classification of the KMS-states of the finite part relies on an extra hypothesis of regularity which allows us to use techniques similar to those of [1, 10, 16, 3, 12]. This regularity property seems natural and we believe it to be automatic, but we have not been able to prove this.

A brief summary of the contents of each section follows. In Section 1 we study Hecke pairs of semidirect products modeled on our main example (P, P_0), but general enough to be of independent interest. The results include necessary and sufficient conditions for an inclusion of semidirect products to be a Hecke pair, structural results that highlight the role of the Hecke algebras of the factors in that of the semidirect product, and the discussion of a (nonselfadjoint) Hecke algebra associated to the Hecke inclusion of a group in a semigroup determined by the inclusion of semidirect products. In Section 2 we study completions of a given Hecke pair of semidirect products to topological Hecke pairs. We show that a completion of a semidirect product $V \rtimes G$ can be chosen to be a semidirect product itself, which can be computed in terms of completions of V and G. We then use this fact to construct induced representations of the Hecke algebra from representations of the group V, and we use this in Theorem 2.14 to obtain certain faithful representations of $C^*_r(P, P_0)$ indexed by $\mathrm{GL}_2(\hat{\mathbb{Z}})$ in the case of our main motivating example. In Section 3 we show that the finite part of the GL_2-system carries an action of the group of finite integral ideles $\hat{\mathbb{Z}}^* = \prod_p \mathbb{Z}_p^*$ as symmetries, whose fixed point algebra decomposes as a tensor product over the primes, Theorem 3.2. Therefore the situation in this finite part is rather surprisingly quite similar to the one dimensional case of Bost and Connes [1]. We also characterize two important subalgebras of this fixed point algebra in terms of integer lattices. The results on representations, completions and symmetries come together in Section 4, where we obtain our classification result, Theorem 4.6, in which we show that the phase transition of regular KMS-states for $\beta > 2$ is indexed by the symmetry group $\hat{\mathbb{Z}}^*$, and that for $\beta \in (1, 2]$ there is a unique regular KMS_β-state.

1. Hecke pairs from semidirect products

Motivated by the inclusion of the orientation preserving affine group of $\mathrm{Mat}_2(\mathbb{Z})$ in that of $\mathrm{Mat}_2(\mathbb{Q})$, we are interested in the following general situation: let G be a group acting by automorphisms of another group V, and let Γ be a subgroup of G leaving a subgroup V_0 of V invariant, so the semidirect product $V_0 \rtimes \Gamma$ can be viewed as a subgroup of $V \rtimes G$. We aim to study the Hecke algebra of the inclusion $V_0 \rtimes \Gamma \subset V \rtimes G$, and start by recalling basic definitions.

1.1. **Hecke pairs and their $*$-algebras.** As customary, by a *Hecke pair* (N, N_0) we mean a group N with a subgroup N_0 such that

$$L_{N_0}(x) := [N_0 : N_0 \cap xN_0x^{-1}]$$

is finite for all x in N. More generally, when X is a right N_0-invariant subset of N we denote by $L_{N_0}(X)$ the number of left N_0-cosets in X. Then $L_{N_0}(x) = L_{N_0}(N_0xN_0)$, so the quantity $L_{N_0}(x)$ is the number of left cosets in the double coset N_0xN_0, and $R_{N_0}(x) := L_{N_0}(x^{-1})$ is the number of right cosets. We shall often drop the subindex from the notation $L_{N_0}(x)$ when there is no risk of confusion.

The formula $\Delta_{N_0}(x) := L_{N_0}(x)/L_{N_0}(x^{-1})$ defines a homomorphism $N \to \mathbb{Q}^+$, see for instance [7, Proposition I.3.6]. We refer to it as the modular function of (N, N_0).

The *Hecke algebra* $\mathcal{H}(N, N_0)$ of a Hecke pair (N, N_0) consists of the vector space of complex valued N_0-biinvariant functions supported on finitely many double cosets,

$$\{f \colon N \to \mathbb{C} \mid f(n_1xn_2) = f(x), \forall\, n_1, n_2 \in N_0,\ \mathrm{supp}(f) \text{ finite in } N_0\backslash N/N_0\},$$

endowed with the product

$$(f_1 * f_2)(x) = \sum_{y \in N_0\backslash N} f_1(xy^{-1})f_2(y) = \sum_{y \in N/N_0} f_1(y)f_2(y^{-1}x), \tag{1.1}$$

(where the first summation is over representatives of the right cosets, and the second one is over representatives of the left cosets) and the involution given by

$$f^*(x) = \overline{f(x^{-1})}, \tag{1.2}$$

for $x \in N$, cf. [7]. When $x \in N$ we denote the characteristic function of a double coset N_0xN_0 variously by $[N_0xN_0]$ or $[x]_{N_0}$ or simply $[x]$.

1.2. **The Hecke algebra of $(V \rtimes G, V_0 \rtimes \Gamma)$.** We aim to identify conditions which will ensure that $(V \rtimes G, V_0 \rtimes \Gamma)$ is a Hecke pair and to study the Hecke algebra of this new pair in terms of the Hecke algebras of (G, Γ) and (V, V_0). To gain insight, we begin with a simplified but important case.

Our convention is that for two groups V and G and a group homomorphism $\alpha : G \to \mathrm{Aut}(V)$, the semidirect product $V \rtimes_\alpha G$ is the cartesian product $V \times G$ endowed with the operations

$$(v, g)(w, h) = (v\alpha_g(w), gh) \ \text{ and } \ (v, g)^{-1} = (\alpha_{g^{-1}}(v^{-1}), g^{-1})$$

for $v, w \in V$ and $g, h \in G$. We shall omit α from the notation and write $g(v)$ for $\alpha_g(v)$ whenever convenient. We shall also identify V and G with their images in $V \rtimes G$ and hence write vg instead of (v, g) for a generic element of $V \rtimes G$.

Proposition 1.1. *Let V be a group with a subgroup V_0, and let Γ be a group acting by automorphisms of V in such a way that V_0 is Γ-invariant. The following conditions are equivalent:*

(i) *$(V \rtimes \Gamma, V_0 \rtimes \Gamma)$ is a Hecke pair;*

(ii) *(V, V_0) is a Hecke pair and the action of Γ on $V_0\backslash V/V_0$ has finite orbits.*

Let Γ_v denote the stabilizer of the double coset V_0vV_0 for $v \in V$. If (i) *and* (ii) *hold, then $R_{V_0 \rtimes \Gamma}(v) = |\Gamma/\Gamma_v| R_{V_0}(v)$, and the map*

$$[v]_{V_0 \rtimes \Gamma} \mapsto \sum_{\gamma \in \Gamma/\Gamma_v} [\gamma(v)]_{V_0} \tag{1.3}$$

implements an isomorphism from $\mathcal{H}(V \rtimes \Gamma, V_0 \rtimes \Gamma)$ *onto* $\mathcal{H}(V, V_0)^\Gamma$ *which respects the R-functions.*

Proof. Note first that any right coset of $V_0 \rtimes \Gamma$ is of the form $(V_0 \rtimes \Gamma)v$ with $v \in V$. Let $v \in V$ and suppose that

$$(V_0 \rtimes \Gamma)v(V_0 \rtimes \Gamma) = \bigsqcup_{j \in J} (V_0 \rtimes \Gamma)w_j \tag{1.4}$$

is a disjoint decomposition of right $(V_0 \rtimes \Gamma)$-cosets, with $w_j \in V$ for all j in J. Denote by $O_\Gamma(v)$ the orbit of v in V. Intersecting the left hand side of (1.4) with V gives, using the hypothesis $\Gamma(V_0) \subset V_0$, the set $V_0 O_\Gamma(v) V_0$. Since $y \in (V_0 \rtimes \Gamma)z$ is equivalent to $y \in V_0 z$ for all y, z in V, the right hand side of (1.4) intersects V in the disjoint decomposition $\bigsqcup_{j \in J} V_0 w_j$. Thus

$$V_0 O_\Gamma(v) V_0 = \bigsqcup_{j \in J} V_0 w_j, \tag{1.5}$$

and the equivalence of (i) and (ii) follows.

Since $R_{V_0}(\gamma(v)) = R_{V_0}(v)$ for every γ in Γ, the double V_0-cosets contained in $V_0 O_\Gamma(v) V_0$ contain the same number of right cosets each. Thus, decomposing $V_0 O_\Gamma(v) V_0$ first into a union of double cosets, and then decomposing each double coset into right cosets, we see that $R_{V_0 \rtimes \Gamma}(v) = |\Gamma/\Gamma_v| R_{V_0}(v)$.

The algebra $\mathcal{H}(V \rtimes \Gamma, V_0 \rtimes \Gamma)$ is spanned by the characteristic functions $[v]$ of double cosets $(V_0 \rtimes \Gamma)v(V_0 \rtimes \Gamma)$ for $v \in V$. Equation (1.5) implies that $[v]|_V$ depends only on the Γ-orbit of $V_0 v V_0$ in $V_0 \backslash V / V_0$. Hence the map in (1.3) is just $\mathcal{H}(V \rtimes \Gamma, V_0 \rtimes \Gamma) \ni f \mapsto f|_V \in \mathcal{H}(V, V_0)^\Gamma$, and is a linear isomorphism. To see that it is multiplicative we compute $[v]\,[w](x)$ as

$$R_{V_0 \rtimes \Gamma}((V_0 \rtimes \Gamma)v^{-1}(V_0 \rtimes \Gamma)x \cap (V_0 \rtimes \Gamma)w(V_0 \rtimes \Gamma)),$$

and we claim that this equals

$$\sum_{\alpha \in \Gamma/\Gamma_v, \beta \in \Gamma/\Gamma_w} R_{V_0}(V_0 \alpha(v^{-1}) V_0 x \cap V_0 \beta(w) V_0), \tag{1.6}$$

for every $v, w, x \in V$. The claim follows, upon invoking (1.4) and (1.5), from the decompositions

$$(V_0 \rtimes \Gamma)v^{-1}(V_0 \rtimes \Gamma)x = \bigcup_{\alpha \in \Gamma/\Gamma_v} (V_0 \rtimes \Gamma) V_0 \alpha(v^{-1}) V_0 x$$

and $(V_0 \rtimes \Gamma)w(V_0 \rtimes \Gamma) = \bigcup_{\beta \in \Gamma/\Gamma_w} (V_0 \rtimes \Gamma) V_0 \beta(w) V_0$. ☐

As a corollary, we obtain the following generalization of [14, Proposition 1.7 (II.3)].

Corollary 1.2. *With the hypotheses of Proposition 1.1, assume that* V_0 *is normal in* V*. Then there is an isomorphism*

$$\mathcal{H}(V \rtimes \Gamma, V_0 \rtimes \Gamma) \cong \mathbb{C}[V/V_0]^\Gamma,$$

given by the restriction map $f \mapsto f|_V$*. Furthermore the product in* $\mathcal{H}(V \rtimes \Gamma, V_0 \rtimes \Gamma)$ *is given by*

$$[v][w] = \sum_{\alpha \in \Gamma/\Gamma_v, \beta \in \Gamma/\Gamma_w} |\Gamma/\Gamma_{\alpha(v)\beta(w)}|^{-1} [\alpha(v)\beta(w)] \quad \textit{for} \quad v, w \in V.$$

Proof. Since (1.3) is multiplicative and $V_0 \trianglelefteq V$, we obtain from (1.6) that

$$[v][w](x) = \#\{(\alpha, \beta) \in \Gamma/\Gamma_v \times \Gamma/\Gamma_w \,|\, x = \alpha(v)\beta(w) \text{ in } V/V_0\}. \tag{1.7}$$

Note that $[\alpha(v)\beta(w)](x) = 1$ exactly when $\alpha(v)\beta(w) \in O_\Gamma(x) V_0$, and in this case

$$|\Gamma/\Gamma_{\alpha(v)\beta(w)}| = |\Gamma/\Gamma_x|.$$

We claim that

$$\#\{(\alpha, \beta) \,|\, \alpha(v)\beta(w) = x \text{ in } V/V_0\} = \frac{\#\{(\alpha, \beta) \,|\, \alpha(v)\beta(w) \in O_\Gamma(x) V_0\}}{|\Gamma/\Gamma_x|}.$$

To prove the claim, note that $(y,z)\mapsto yz$ is a Γ-equivariant map from $O_\Gamma(v)V_0\times O_\Gamma(w)V_0$ into V/V_0, and then the number of elements in the preimage of $O_\Gamma(x)V_0$ under this map is the product of the number of preimages of $xV_0\in V/V_0$ multiplied by the size of the orbit, as needed. The result follows from the claim and (1.7). □

We consider next the more general situation in which there is a group G containing Γ and acting on V by an extension of the action of Γ. To state the next result, we need to recall from e.g. [7, Chapter I] that two subgroups K_1 and K_2 of a group L are commensurable if $K_1\cap K_2$ has finite index in both K_1 and K_2.

Proposition 1.3. *Let G be a group acting by automorphisms of a group V, and let Γ be a subgroup of G leaving a subgroup V_0 of V invariant. Then $(V\rtimes G, V_0\rtimes\Gamma)$ is a Hecke pair if and only if the following conditions are satisfied:*

(i) *(V,V_0) is a Hecke pair such that the action of Γ on $V_0\backslash V/V_0$ has finite orbits and*

(ii) *(G,Γ) is a Hecke pair such that V_0 and $g(V_0)$ are commensurable subgroups of V for every $g\in G$.*

If (i) *and* (ii) *hold, then we have an embedding $\mathcal{H}(V,V_0)^\Gamma\hookrightarrow\mathcal{H}(V\rtimes G,V_0\rtimes\Gamma)$, and*

$$R_{V_0\rtimes\Gamma}(vg) = R_\Gamma(g)\,|V_0\backslash V_0O_{\Gamma_g}(v)g(V_0)|, \tag{1.8}$$
$$L_{V_0\rtimes\Gamma}(vg) = L_\Gamma(g)\,|V_0O_{\Gamma_g}(v)g(V_0)/g(V_0)|, \tag{1.9}$$

where $\Gamma_g=g\Gamma g^{-1}\cap\Gamma$ and $O_{\Gamma_g}(v)$ denotes the Γ_g-orbit of $v\in V$. Furthermore,

$$\Delta_{V_0\rtimes\Gamma}(vg)=\Delta_{V_0}(v)\Delta_\Gamma(g)\frac{|V_0/(V_0\cap g(V_0))|}{|V_0/(V_0\cap g^{-1}(V_0))|}.$$

Proof. We begin by proving (1.9) without assuming (i) and (ii). To simplify the notation we let $P_0:=V_0\rtimes\Gamma$. Let $\{\gamma_i\}_i$ be representatives of left Γ_g-cosets in Γ. Put $g_i=\gamma_i g$. Then, since Γ_g is exactly the set of elements γ such that $\gamma g\Gamma=g\Gamma$, $\{g_i\}_i$ are representatives of left Γ-cosets in $\Gamma g\Gamma$. We claim that P_0vgP_0 is the union of the sets

$$V_0\gamma_i(O_{\Gamma_g}(v))g_i(V_0)g_i\Gamma.$$

Indeed, P_0vgP_0 is the union of the sets $V_0\gamma_i\Gamma_g vgV_0\Gamma=V_0\gamma_i\Gamma_g vg(V_0)g\Gamma$. Observe that $\gamma(g(V_0))=g(V_0)$ for $\gamma\in\Gamma_g$. Hence $\Gamma_g vg(V_0)=O_{\Gamma_g}(v)g(V_0)\Gamma_g$, so that

$$V_0\gamma_i\Gamma_g vg(V_0)g\Gamma=V_0\gamma_iO_{\Gamma_g}(v)g(V_0)g\Gamma=V_0\gamma_i(O_{\Gamma_g}(v))\gamma_i(g(V_0))\gamma_i g\Gamma,$$

and since $\gamma_i g=g_i$, our claim is proved.

Since left P_0-cosets have the form $whP_0=wh(V_0)h\Gamma$, the sets $V_0\gamma_i(O_{\Gamma_g}(v))g_i(V_0)g_i\Gamma$ do not intersect, and

$$L_{P_0}(vg)=\sum_i|V_0\gamma_i(O_{\Gamma_g}(v))g_i(V_0)/g_i(V_0)|.$$

By applying the automorphisms γ_i^{-1} we see that all summands in the formula above are equal to $|V_0O_{\Gamma_g}(v))g(V_0)/g(V_0)|$, and we therefore get (1.9). Then

$$R_{P_0}(vg)=L_{P_0}(g^{-1}v^{-1})=L_{P_0}(g^{-1}(v^{-1})g^{-1})=L_\Gamma(g^{-1})\,|V_0O_{\Gamma_{g^{-1}}}(g^{-1}(v^{-1}))g^{-1}(V_0)/g^{-1}(V_0)|.$$

Applying the automorphism g we get that the last expression equals

$$L_\Gamma(g^{-1})\,|g(V_0)O_{g\Gamma_{g^{-1}}g^{-1}}(v^{-1})V_0/V_0|=R_\Gamma(g)|g(V_0)O_{\Gamma_g}(v^{-1})V_0/V_0|.$$

Finally, by applying the anti-automorphism $V\ni w\mapsto w^{-1}$, we get

$$R_{P_0}(vg)=R_\Gamma(g)|V_0\backslash V_0O_{\Gamma_g}(v)g(V_0)|,$$

and (1.8) is also proved.

It is now immediate that $(V\rtimes G,V_0\rtimes\Gamma)$ is a Hecke pair if and only if (i) and (ii) hold. That we have an embedding $\mathcal{H}(V,V_0)^\Gamma\hookrightarrow\mathcal{H}(V\rtimes G,V_0\rtimes\Gamma)$ follows from Proposition 1.1.

It remains to prove the formula for the modular function (note that it will also follow from Section 2.3 below). We have

$$\Delta_{P_0}(vg) = \Delta_\Gamma(g)\frac{|V_0 O_{\Gamma_g}(v) g(V_0)/g(V_0)|}{|V_0\backslash V_0 O_{\Gamma_g}(v) g(V_0)|}.$$

Since the numbers $|V_0 w g(V_0)/g(V_0)|$ and $|V_0\backslash V_0 w g(V_0)|$ are independent of $w \in O_{\Gamma_g}(v)$, we conclude that $\Delta_{P_0}(vg)$ is equal to

$$\Delta_\Gamma(g)\frac{|V_0 v g(V_0)/g(V_0)|}{|V_0\backslash V_0 v g(V_0)|} = \Delta_\Gamma(g)\frac{|v^{-1}V_0 v g(V_0)/g(V_0)|}{|v^{-1}V_0 v\backslash v^{-1}V_0 v g(V_0)|} = \Delta_\Gamma(g)\frac{|v^{-1}V_0 v/(v^{-1}V_0 v\cap g(V_0))|}{|g(V_0)/(v^{-1}V_0 v\cap g(V_0))|}.$$

If K_1, K_2 and K_3 are commensurable groups then

$$\frac{|K_1/(K_1\cap K_2)|}{|K_2/(K_1\cap K_2)|} = \frac{|K_1/(K_1\cap K_3)|}{|K_3/(K_1\cap K_3)|}\,\frac{|K_3/(K_2\cap K_3)|}{|K_2/(K_2\cap K_3)|}.$$

Applying this to $K_1 = v^{-1}V_0 v$, $K_2 = g(V_0)$ and $K_3 = V_0$ we get

$$\begin{aligned}\Delta_{P_0}(vg) &= \Delta_\Gamma(g)\frac{|v^{-1}V_0 v/(v^{-1}V_0 v\cap V_0)|}{|V_0/(v^{-1}V_0 v\cap V_0)|}\,\frac{|V_0/(g(V_0)\cap V_0)|}{|g(V_0)/(g(V_0)\cap V_0)|}\\ &= \Delta_\Gamma(g)\Delta_{V_0}(v)\frac{|V_0/(g(V_0)\cap V_0)|}{|g(V_0)/(g(V_0)\cap V_0)|},\end{aligned}$$

which completes the proof of the proposition. □

1.3. **A Hecke pair from the inclusion** $\mathrm{Mat}_2(\mathbb{Z}) \subset \mathrm{Mat}_2(\mathbb{Q})$**.** We now explain our motivating example, which is the 2×2 matrix analogue of the Hecke pair $(P^+_{\mathbb{Q}}, P^+_{\mathbb{Z}})$ of Bost and Connes [1], and arises from the following setup: let $V := \mathrm{Mat}_2(\mathbb{Q})$ and $V_0 := \mathrm{Mat}_2(\mathbb{Z})$ be the additive groups in the ring inclusion $\mathrm{Mat}_2(\mathbb{Q}) \supset \mathrm{Mat}_2(\mathbb{Z})$, and let G and Γ be their orientation preserving groups of invertible elements acting by multiplication on the right. Specifically,

$$\begin{aligned} G &= \mathrm{GL}_2^+(\mathbb{Q}) := \{g \in \mathrm{Mat}_2(\mathbb{Q}) \mid \det(g) > 0\},\\ \Gamma &= \mathrm{SL}_2(\mathbb{Z}) := \{m \in \mathrm{Mat}_2(\mathbb{Z}) \mid \det(m) = 1\},\end{aligned}$$

and $\alpha_g(m) = mg^{-1}$ for $g \in \mathrm{GL}_2^+(\mathbb{Q})$ and $m \in \mathrm{Mat}_2(\mathbb{Q})$. Since $(0,g)(m,1)(0,g^{-1}) = (\alpha_g(m),1)$ is compatible with the matrix multiplication

$$\begin{pmatrix}1 & 0\\ 0 & g\end{pmatrix}\begin{pmatrix}1 & m\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & g^{-1}\end{pmatrix} = \begin{pmatrix}1 & mg^{-1}\\ 0 & 1\end{pmatrix},$$

we can view the element $(\alpha_g(m), g)$ of $V \rtimes G$ as the 4×4 matrix $\begin{pmatrix}1 & m\\ 0 & g\end{pmatrix}$.

To argue that $(V\rtimes G, V_0\rtimes\Gamma)$ is a Hecke pair does not require any computations, because it can be deduced from the following general result: if $\mathcal{H}$ is a linear algebraic group defined over $\mathbb{Q}$, and $H = \mathcal{H}(\mathbb{Q})$ and $H_0 = \mathcal{H}(\mathbb{Z})$ are the groups of rational and integral points of $\mathcal{H}$, then (H, H_0) is a Hecke pair, see e.g. [18, Corollary 1 of Proposition 4.1]. However, to compute the R-function we need the elementary considerations of Proposition 1.3 anyway, so at this point we prefer to rely on that proposition.

It is well known that (G,Γ) is a Hecke pair, in fact $\mathcal{H}(G,\Gamma)$ is the classical Hecke algebra generated by the Hecke operators, see e.g. [7, Chapter IV] for details. The remaining assumptions of Proposition 1.3 are easily verified. E.g. to show that Γ-orbits in V/V_0 are finite, take $m \in \mathrm{Mat}_2(\mathbb{Q})$ and choose $d \in \mathbb{N}^*$ such that $dm \in \mathrm{Mat}_2(\mathbb{Z})$. Then $dm\mathrm{SL}_2(\mathbb{Z}) \subset \mathrm{Mat}_2(\mathbb{Z})$, and thus the $\mathrm{SL}_2(\mathbb{Z})$-orbit of $m + \mathrm{Mat}_2(\mathbb{Z})$ in $\mathrm{Mat}_2(\mathbb{Q})/\mathrm{Mat}_2(\mathbb{Z})$ is finite, and in fact has at most $|\mathrm{Mat}_2(\mathbb{Z})d^{-1}/\mathrm{Mat}_2(\mathbb{Z})| = d^4$ elements. Thus $(V\rtimes G, V_0\rtimes\Gamma)$ is a Hecke pair, and since (G,Γ) is unimodular, we have

$$\Delta_{V_0\rtimes\Gamma}\begin{pmatrix}1 & m\\ 0 & g\end{pmatrix} = \frac{|\mathrm{Mat}_2(\mathbb{Z})/(\mathrm{Mat}_2(\mathbb{Z})\cap\mathrm{Mat}_2(\mathbb{Z})g^{-1})|}{|\mathrm{Mat}_2(\mathbb{Z})/(\mathrm{Mat}_2(\mathbb{Z})\cap\mathrm{Mat}_2(\mathbb{Z})g)|} = \det(g)^{-2}.$$

We have therefore obtained the following.

Proposition 1.4. *The inclusion* $\mathrm{Mat}_2(\mathbb{Z}) \rtimes \mathrm{SL}_2(\mathbb{Z}) \subset \mathrm{Mat}_2(\mathbb{Q}) \rtimes \mathrm{GL}_2^+(\mathbb{Q})$ *is a Hecke pair. Furthermore the associated modular function is given by* $\Delta \begin{pmatrix} 1 & m \\ 0 & g \end{pmatrix} = \det(g)^{-2}$.

1.4. **Hecke algebras from subsemigroups.** Suppose that (N, N_0) is a Hecke pair and S is a subsemigroup of N containing N_0. The set $\mathcal{H}(S, N_0)$ of functions in $\mathcal{H}(N, N_0)$ that are supported on the double cosets of elements of S forms an algebra, with convolution formula (1.1) restricted to a summation over y in S/N_0 but without adjoints. It is easy to see that $\mathcal{H}(S, N_0)$ is a subalgebra of $\mathcal{H}(N, N_0)$, cf. [7, Lemma I.4.9].

Example 1.5. The main example of this situation is again classical: for the Hecke pair $(G, \Gamma) = (\mathrm{GL}_2^+(\mathbb{Q}), \mathrm{SL}_2(\mathbb{Z}))$ of Section 1.3, we take S to be the subsemigroup of integer matrices with positive determinant, $\mathrm{Mat}_2^+(\mathbb{Z}) := \{g \in \mathrm{Mat}_2(\mathbb{Z}) \mid \det(g) > 0\}$; the Hecke algebra $\mathcal{H}(S, \Gamma)$ is described in [7, Chapter IV].

We have seen in Proposition 1.3 that $\mathcal{H}(V \rtimes G, V_0 \rtimes \Gamma)$ contains a copy of the fixed point algebra of $\mathcal{H}(V, V_0)$ under the action of Γ, so it is natural to wonder whether $\mathcal{H}(G, \Gamma)$ plays any role in $\mathcal{H}(V \rtimes G, V_0 \rtimes \Gamma)$. The following proposition shows that there is an actual algebraic embedding, but only of the nonselfadjoint Hecke subalgebra of $\mathcal{H}(G, \Gamma)$ corresponding to a convenient subsemigroup of G.

Proposition 1.6. *Suppose that* $(V \rtimes G, V_0 \rtimes \Gamma)$ *is a Hecke pair as in Proposition 1.3 and let* $S = \{s \in G \mid V_0 \subset s(V_0)\}$. *Then* S *is a semigroup containing* Γ *and the map*

$$\iota\colon [s]_\Gamma \in \mathcal{H}(S, \Gamma) \mapsto [s]_{V_0 \rtimes \Gamma} \in \mathcal{H}(V \rtimes G, V_0 \rtimes \Gamma)$$

extends by linearity to an injective homomorphism.

Proof. Clearly S is a semigroup. To prove the second assertion it is convenient to use the notation $P = V \rtimes G$ and $P_0 = V_0 \rtimes \Gamma$. The set SV_0 is a subsemigroup of $V \rtimes G$ containing $V_0 \rtimes \Gamma$, so $\mathcal{H}(SV_0, P_0)$ is a subalgebra of $\mathcal{H}(P, P_0)$. Furthermore, the identity map on S gives a bijection between $\Gamma \backslash S / \Gamma$ and $P_0 \backslash SV_0 / P_0$. It follows that ι extends to a bijective linear map from $\mathcal{H}(S, \Gamma)$ onto $\mathcal{H}(SV_0, P_0)$. To see that this is an isomorphism of algebras, we have to show that

$$R_{P_0}(P_0 t^{-1} P_0 r \cap P_0 s P_0) = R_\Gamma(\Gamma t^{-1} \Gamma r \cap \Gamma s \Gamma)$$

for $r, s, t \in S$. Since $P_0 t^{-1} P_0 r = V_0 \Gamma t^{-1} \Gamma r$, we have $P_0 t^{-1} P_0 r \cap P_0 s P_0 = V_0(\Gamma t^{-1} \Gamma r \cap \Gamma s \Gamma)$, as required. ☐

It is easy to see that the embedding ι of Proposition 1.6 restricts to the Hecke algebra of any subsemigroup of S containing Γ. But the embedding does not extend in general to a $*$-preserving homomorphism from $\mathcal{H}(G, \Gamma)$ to $\mathcal{H}(V \rtimes G, V_0 \rtimes \Gamma)$; in particular, this happens in our main example, essentially because the classical Hecke algebra $\mathcal{H}(\mathrm{GL}_2^+(\mathbb{Q}), \mathrm{SL}_2(\mathbb{Z}))$ is commutative, but double cosets of $\mathrm{Mat}_2(\mathbb{Z}) \rtimes \mathrm{SL}_2(\mathbb{Z})$ do not $*$-commute in general.

Remark 1.7. Under the assumptions of Proposition 1.6, if $\phi\colon S \to \mathbb{R}^*$ is a homomorphism whose kernel contains Γ, then by essentially the same proof, $[s]_\Gamma \mapsto \phi(s)[s]_{V_0 \rtimes \Gamma}$ is also a homomorphism from $\mathcal{H}(S, \Gamma)$ into $\mathcal{H}(V \rtimes G, V_0 \rtimes \Gamma)$. We shall see in Section 2.4 that the choice $\phi(s) := [s(V_0) : V_0]^{-1/2}$ instead of $\phi(s)$=1 gives rise to a more natural homomorphism of Hecke algebras.

2. Completions of Hecke pairs

If N is a locally compact group and N_0 is a compact open subgroup then (N, N_0) forms a Hecke pair. It is known that any Hecke pair gives rise to a pair of this form with isomorphic Hecke algebra, and we shall review the known facts, but also describe slightly more general topological Hecke pairs that turn out to be useful in the context of semidirect product groups.

2.1. **Completions of Hecke pairs and their C*-algebras.** Let (N, N_0) be a Hecke pair.

Definition 2.1. A pair $(\bar N, \bar N_0)$ and a group homomorphism $\rho\colon N \to \bar N$ form a *completion* of a Hecke pair (N, N_0) if $\bar N$ is a locally compact group, $\bar N_0$ is a compact open subgroup of $\bar N$, $\rho(N)$ is dense in $\bar N$, $\rho(N_0)$ is dense in $\bar N_0$, and $\rho^{-1}(\bar N_0) = N_0$.

Completions always exist, and in fact one can produce a canonical one, called the *Schlichting completion* [22, 6], see also [4, 15] for slightly different approaches. We recall from [22, 6] that a Hecke pair (N, N_0) is called *reduced* if $\bigcap_{x\in N} xN_0x^{-1} = \{e\}$ or, equivalently, if N_0 contains no nontrivial normal subgroup of N. The *Hecke topology* on N is the topology determined by a neighbourhood subbase at e consisting of the sets xN_0x^{-1} for $x \in N$. Then the Schlichting completion of (N, N_0) is a Hecke pair $(\bar N, \bar N_0)$ together with a homomorphism $\phi\colon N \to \bar N$ characterized uniquely by the properties that $\bar N$ is a locally compact totally disconnected group, $\bar N_0$ is a compact open subgroup, $(\bar N, \bar N_0)$ is reduced, $\phi(N)$ is dense in $\bar N$, and $\phi^{-1}(\bar N_0) = N_0$. Furthermore, if (N, N_0) is reduced then ϕ is a homeomorphism of N with its Hecke topology onto its image inside $\bar N$. The Schlichting completion is universal in the following sense.

Proposition 2.2. *Let (N, N_0) be a Hecke pair, and suppose that $(\tilde N, \tilde N_0)$ and $\rho\colon N \to \tilde N$ form a completion of (N, N_0). Assume also that $(\bar N, \bar N_0)$ together with a homomorphism $\phi\colon N \to \bar N$ is the Schlichting completion of (N, N_0). Then there exists a unique continuous homomorphism $\tilde\phi\colon \tilde N \to \bar N$ such that $\tilde\phi \circ \rho = \phi$. The homomorphism $\tilde\phi$ is onto, and it is an isomorphism if and only if the pair $(\tilde N, \tilde N_0)$ is reduced.*

Proof. For the proof note that we can identify N/N_0 with $\tilde N/\tilde N_0$, and then $\bar N$ is the Schlichting completion of $\tilde N$ by construction, see e.g. [6]. □

Let now $(\bar N, \bar N_0)$ together with $\rho\colon N \to \bar N$ be a completion of (N, N_0). Denote by μ the left Haar measure on $\bar N$ such that $\mu(\bar N_0) = 1$, and by $\Delta = \Delta_{\bar N}$ the modular function of $\bar N$. Note that by [19, Lemma 1] the composition of $\Delta_{\bar N}$ with ρ coincides with Δ_{N_0} from Section 1.1.

The space $C_c(\bar N)$ of compactly supported continuous functions on $\bar N$ is a $*$-algebra with the convolution product

$$(f_1 * f_2)(x) = \int_{\bar N} f_1(y)f_2(y^{-1}x)d\mu(y) = \int_{\bar N} \Delta(y^{-1})f_1(xy^{-1})f_2(y)d\mu(y) \tag{2.1}$$

and involution

$$f^*(x) = \Delta(x^{-1})\overline{f(x^{-1})}. \tag{2.2}$$

Denote by p the characteristic function $\mathbb{1}_{\bar N_0}$ of $\bar N_0$, which is a self-adjoint projection in $C_c(\bar N)$. Then we have the following result similar to the situation of Schlichting completions in [22, 6] (note that in [6] the definition of the involution has an extra factor $\Delta_{N_0}(x^{-1})$).

Lemma 2.3. *The homomorphism $\Psi\colon \mathcal{H}(N, N_0) \to C_c(\bar N)$, $\Psi([N_0xN_0]) := \Delta_{N_0}^{-1/2}(x)[\bar N_0\rho(x)\bar N_0]$, is a $*$-isomorphism from $\mathcal{H}(N, N_0)$ onto the $*$-algebra $pC_c(\bar N)p$ with operations* (2.1)–(2.2).

Suppose further that $\pi\colon \bar N \to U(H)$ is a unitary representation of $\bar N$ on a Hilbert space H. The integrated form π_* of π carries p into the projection in $B(H)$ onto the space $\{\xi \in H \mid \pi(n)\xi = \xi, \forall\, n \in \bar N_0\}$ of $\bar N_0$-fixed vectors. Hence by Lemma 2.3 we obtain a $*$-representation

$$\pi_* \circ \Psi\colon \mathcal{H}(N, N_0) \to B(\pi_*(p)H).$$

In particular, when π is the regular representation of $\bar N$, we get a representation $\pi_* \circ \Psi$ of $\mathcal{H}(N, N_0)$ on the space of $\bar N_0$-fixed vectors in $L^2(\bar N)$. The map $U\colon \ell^2(N_0\backslash N) \to L^2(\bar N)$ defined by

$$(U\xi)(n) := \Delta_{\bar N}(n)^{-1/2}\xi(x),$$

where x is such that $n \in \bar{N}_0\rho(x)$, is an isometry of $\ell^2(N_0\backslash N)$ onto the space of $\bar{N}_0$-fixed vectors in $L^2(\bar{N})$. It follows that $U^*(\pi_* \circ \Psi)(\cdot)U$ is a $*$-representation of $\mathcal{H}(N, N_0)$ on $\ell^2(N_0\backslash N)$. We claim that this is the familiar *regular representation* λ of $\mathcal{H}(N, N_0)$ from [1, Proposition 3], which satisfies

$$(\lambda(f)\xi)(x) = \sum_{y \in N_0\backslash N} f(xy^{-1})\xi(y) = \sum_{y \in N/N_0} f(y)\xi(y^{-1}x). \tag{2.3}$$

Indeed, denoting by $s\colon \bar{N} \to N$ a map such that $n \in \bar{N}_0\rho(s(n))$ for $n \in \bar{N}$, we compute

$$\begin{aligned} ((\pi_* \circ \Psi)(f)U\xi)(m) &= \int_{\bar{N}} \Psi(f)(n)(U\xi)(n^{-1}m)d\mu(n) \\ &= \Delta(m)^{-1/2} \sum_{y \in N/N_0} \int_{\rho(y)\bar{N}_0} f(s(n))\xi(s(n^{-1}m))d\mu(n) \\ &= \Delta(m)^{-1/2} \sum_{y \in N/N_0} \int_{\bar{N}_0} f(y)\xi(y^{-1}s(m))d\mu(n) \\ &= \Delta(m)^{-1/2} \sum_{y \in N/N_0} f(y)\xi(y^{-1}s(m)). \end{aligned} \tag{2.4}$$

The *reduced Hecke C^*-algebra* $C^*_r(N, N_0)$ is the C*-algebra generated by the image of $\mathcal{H}(N, N_0)$ in $B(\ell^2(N_0\backslash N))$ under the representation λ. The isomorphism $\Psi\colon \mathcal{H}(N, N_0) \to pC_c(\bar{N})p$ extends to an isomorphism of $C^*_r(N, N_0)$ onto $pC^*_r(\bar{N})p$, which we shall still denote by Ψ.

Notice that if S is a subsemigroup of N containing N_0 then the formula (2.3) makes sense when $f \in \mathcal{H}(S, N_0)$, $\xi \in \ell^2(N_0\backslash S)$ and the summation is over $y \in N_0\backslash S$. Hence we can define the regular representation of $\mathcal{H}(S, N_0)$ on $\ell^2(N_0\backslash S)$ by

$$(\lambda(f)\xi)(s) = \sum_{t \in N_0\backslash S} f(st^{-1})\xi(t) \text{ for } s \in S. \tag{2.5}$$

2.2. **The Schlichting completion of** $(\mathrm{Mat}_2(\mathbb{Q}) \rtimes \mathrm{GL}_2^+(\mathbb{Q}), \mathrm{Mat}_2(\mathbb{Z}) \rtimes \mathrm{SL}_2(\mathbb{Z}))$**.** We first recall from [18, Section 5.1] a standard way of getting completions of Hecke pairs arising from algebraic groups (see also [22, Section 4]).

Let $\hat{\mathbb{Z}}$ denote the ring $\prod_{p \in \mathcal{P}} \mathbb{Z}_p$, where $\mathcal{P}$ is the set of prime numbers. The additive group of $\hat{\mathbb{Z}}$ is a compact group. The ring of finite adeles is, by definition,

$$\mathbb{A}_f = \prod_{p \in \mathcal{P}} (\mathbb{Q}_p : \mathbb{Z}_p),$$

and as an additive group has the locally compact totally disconnected topology of a restricted product in which $\hat{\mathbb{Z}}$ is a compact open subgroup. Then $\mathbb{Q}$ embeds diagonally into $\mathbb{A}_f$, and $\mathbb{Q} \cap \hat{\mathbb{Z}} = \mathbb{Z}$. Consider now the group $\mathrm{GL}_n(\mathbb{A}_f)$ where the topology is defined by the embedding $\mathrm{GL}_n(\mathbb{A}_f) \ni g \mapsto (g, \det(g)^{-1}) \in \mathrm{Mat}_n(\mathbb{A}_f) \times \mathbb{A}_f$. Equivalently, $\mathrm{GL}_n(\mathbb{A}_f)$ is the restricted topological product of the groups $\mathrm{GL}_n(\mathbb{Q}_p)$ with respect to the subgroups $\mathrm{GL}_n(\mathbb{Z}_p)$, so that $\mathrm{GL}_n(\hat{\mathbb{Z}})$ is a compact open subgroup of $\mathrm{GL}_n(\mathbb{A}_f)$.

Now if $\mathcal{H} \subset \mathrm{GL}_n$ is a linear algebraic group defined over $\mathbb{Q}$, and $H = \mathcal{H}(\mathbb{Q})$, $H_0 = \mathcal{H}(\mathbb{Z})$, we have an embedding $H \hookrightarrow \mathrm{GL}_n(\mathbb{A}_f)$, which together with the groups $\tilde{H} := \overline{\mathcal{H}(\mathbb{Q})}$ and $\tilde{H}_0 := \tilde{H} \cap \mathrm{GL}_n(\hat{\mathbb{Z}}) = \overline{\mathcal{H}(\mathbb{Z})}$ forms a completion of (H, H_0).

Returning to our Hecke pair from Section 1.3, we have an algebraic subgroup $\mathcal{H} := \mathrm{Mat}_2 \rtimes \mathrm{GL}_2$ in GL_4, and then $\mathrm{Mat}_2(\mathbb{Q}) \rtimes \mathrm{GL}_2^+(\mathbb{Q})$ is a subgroup of index 2 in $\mathcal{H}(\mathbb{Q})$. Since its closure in $\overline{\mathcal{H}(\mathbb{Q})} \subset \mathrm{GL}_4(\mathbb{A}_f)$ will then be open, to find a completion it suffices to find the closure of

$$\begin{pmatrix} 1 & \mathrm{Mat}_2(\mathbb{Q}) \\ 0 & \mathrm{GL}_2^+(\mathbb{Q}) \end{pmatrix}$$

in $\mathrm{GL}_4(\mathbb{A}_f)$. Since the group $\mathrm{Mat}_2(\mathbb{Q})$ is dense in $\mathrm{Mat}_2(\mathbb{A}_f)$, it remains to find the closure of $\mathrm{GL}_2^+(\mathbb{Q})$ in $\mathrm{GL}_2(\mathbb{A}_f)$. This requires more care than the one dimensional case, in which $\mathbb{Q}^*_+$ is discrete in $\mathrm{GL}_1(\mathbb{A}_f)$ because $\mathbb{Q}^*_+ \cap \hat{\mathbb{Z}}^* = \{1\}$; now by the strong approximation theorem [18, Theorem 7.12], the closure of $\mathrm{SL}_2(\mathbb{Z})$ inside $\mathrm{GL}_2(\mathbb{A}_f)$ is $\mathrm{SL}_2(\hat{\mathbb{Z}})$, the compact subgroup of matrices in $\mathrm{GL}_2(\hat{\mathbb{Z}})$ which have determinant 1, see also [20, Lemma 1.38] for an elementary proof of our particular case.

Lemma 2.4. *The closure of* $\mathrm{GL}_2^+(\mathbb{Q})$ *in* $\mathrm{GL}_2(\mathbb{A}_f)$ *is the set*

$$\mathrm{GL}_2^+(\mathbb{A}_f) := \{g \in \mathrm{GL}_2(\mathbb{A}_f) \mid \det(g) \in \mathbb{Q}^*_+\}.$$

Moreover,

$$\mathrm{GL}_2^+(\mathbb{A}_f) = \mathrm{GL}_2^+(\mathbb{Q})\mathrm{SL}_2(\hat{\mathbb{Z}}). \tag{2.6}$$

Note that the factorization in (2.6) is not unique. The factors are only determined up to an element of $\mathrm{SL}_2(\mathbb{Z}) = \mathrm{GL}_2^+(\mathbb{Q}) \cap \mathrm{SL}_2(\hat{\mathbb{Z}})$.

Proof. Since the determinant function is continuous on $\mathrm{GL}_2(\mathbb{A}_f)$ and $\mathbb{Q}^*$ is closed in $\mathrm{GL}_1(\mathbb{A}_f)$, the subgroup $\mathrm{GL}_2^+(\mathbb{A}_f)$ is closed, and hence is a totally disconnected, locally compact group in its own right. In particular, the closure of $\mathrm{GL}_2^+(\mathbb{Q})$ is contained in $\mathrm{GL}_2^+(\mathbb{A}_f)$.

Clearly $\mathrm{GL}_2^+(\mathbb{Q})\mathrm{SL}_2(\hat{\mathbb{Z}}) \subset \mathrm{GL}_2^+(\mathbb{A}_f)$. In order to prove the reverse inclusion, let $g \in \mathrm{GL}_2^+(\mathbb{A}_f)$. We need to show that $g = g_0 r$ for matrices $g_0 \in \mathrm{GL}_2^+(\mathbb{Q})$ and $r \in \mathrm{SL}_2(\hat{\mathbb{Z}})$. Since $g\hat{\mathbb{Z}}^2$ is a lattice in $\mathbb{A}_f^2$, that is, a compact open $\hat{\mathbb{Z}}$-submodule, by [24, Theorem V.2] there exists $g_0 \in \mathrm{GL}_2^+(\mathbb{Q})$ such that $g\hat{\mathbb{Z}}^2 \cap \mathbb{Q}^2 = g_0\mathbb{Z}^2$. Thus $g\hat{\mathbb{Z}}^2 = g_0\hat{\mathbb{Z}}^2$, which implies that $g_0^{-1}g \in \mathrm{GL}_2(\hat{\mathbb{Z}})$. Since g has positive determinant, $\det(g_0^{-1}g) \in \mathbb{Q}^*_+ \cap \hat{\mathbb{Z}}^* = \{1\}$, and hence $g = g_0(g_0^{-1}g) \in \mathrm{GL}_2^+(\mathbb{Q})\mathrm{SL}_2(\hat{\mathbb{Z}})$, as required. As $\mathrm{SL}_2(\mathbb{Z})$ is dense in $\mathrm{SL}_2(\hat{\mathbb{Z}})$, this in particular implies that $\mathrm{GL}_2^+(\mathbb{Q})$ is dense in $\mathrm{GL}_2^+(\mathbb{A}_f)$. □

We therefore obtain the following result.

Proposition 2.5. *The pair* $(\mathrm{Mat}_2(\mathbb{A}_f) \rtimes \mathrm{GL}_2^+(\mathbb{A}_f), \mathrm{Mat}_2(\hat{\mathbb{Z}}) \rtimes \mathrm{SL}_2(\hat{\mathbb{Z}}))$ *is the Schlichting completion of* $(\mathrm{Mat}_2(\mathbb{Q}) \rtimes \mathrm{GL}_2^+(\mathbb{Q}), \mathrm{Mat}_2(\mathbb{Z}) \rtimes \mathrm{SL}_2(\mathbb{Z}))$.

Proof. That $(\mathrm{Mat}_2(\mathbb{A}_f) \rtimes \mathrm{GL}_2^+(\mathbb{A}_f), \mathrm{Mat}_2(\hat{\mathbb{Z}}) \rtimes \mathrm{SL}_2(\hat{\mathbb{Z}}))$, with the obvious embedding of $\mathrm{Mat}_2(\mathbb{Q}) \rtimes \mathrm{GL}_2^+(\mathbb{Q})$ into $\mathrm{Mat}_2(\mathbb{A}_f) \rtimes \mathrm{GL}_2^+(\mathbb{A}_f)$, form a completion follows from the preceding lemma and the discussion about algebraic groups. To see that it is in fact the Schlichting completion, notice that by Proposition 2.2 we just have to check that $(\mathrm{Mat}_2(\mathbb{A}_f) \rtimes \mathrm{GL}_2^+(\mathbb{A}_f), \mathrm{Mat}_2(\hat{\mathbb{Z}}) \rtimes \mathrm{SL}_2(\hat{\mathbb{Z}}))$ is a reduced pair, which can be done using the following simple sufficient condition. □

Lemma 2.6. *A Hecke pair* $(V \rtimes G, V_0 \rtimes \Gamma)$ *is reduced if the following two conditions are satisfied:*
(i) $\bigcap_{g\in G} g(V_0) = \{e\}$;
(ii) *if* $\gamma(v) = v$ *for some* $\gamma \in \bigcap_{g\in G} g\Gamma g^{-1}$ *and all* $v \in V$ *then* $\gamma = e$.

Proof. Assume $v\gamma$ lies in a normal subgroup of $V \rtimes G$ which is contained in $V_0 \rtimes \Gamma$. Then $v\gamma$ belongs to $gV_0\Gamma g^{-1} = g(V_0)g\Gamma g^{-1}$ for all $g \in G$, which shows that $v \in \bigcap_{g\in G} g(V_0)$ and $\gamma \in \bigcap_{g\in G} g\Gamma g^{-1}$. Hence $v = e$ by condition (i). But then $\gamma wg = \gamma(w)\gamma g$ belongs to $wgV_0\Gamma = wg(V_0)g\Gamma$ for all $w \in V$, so that in particular $\gamma(w) \in \bigcap_{g\in G} wg(V_0) = \{w\}$. Hence $\gamma = e$ by condition (ii). □

2.3. Completions of $(V \rtimes G, V_0 \rtimes \Gamma)$.

If $(V \rtimes G, V_0 \rtimes \Gamma)$ is a Hecke pair, there is no apparent reason for its Schlichting completion to be again a semidirect product, although this is the case in our main example. However, it turns out that if we do not require a reduced pair as a completion, we can get one that is a semidirect product.

Remark 2.7. Let $\bar{V}$ be a locally compact group. Let $\mathrm{HAut}(\bar{V})$ be the group of homeomorphic group automorphisms of $\bar{V}$. It is a topological group for the topology that has as a neighbourhood base at the identity the sets

$$\{\beta \in \mathrm{HAut}(\bar{V}) \mid \beta(u) \in Uu \text{ and } \beta^{-1}(u) \in Uu \text{ for all } u \in K\},$$

for all compact subsets K of $\bar V$ and all neighbourhoods U of e in $\bar V$, see [5, (26.5)]

If $\bar G$ is another locally compact group with an action α on $\bar V$, then this action is continuous, in the sense that the map $\bar G \times \bar V \to \bar V$ is continuous, if and only if the homomorphism $\alpha\colon \bar G \to \mathrm{HAut}(\bar V)$ is continuous. In this case $\bar V \rtimes_\alpha \bar G$ is a locally compact group with the topology inherited from the product topology on $\bar V \times \bar G$.

Proposition 2.8. *Suppose that* $(\overline{V \rtimes G}, \overline{V_0 \rtimes \Gamma})$ *and* $\rho\colon V \rtimes G \to \overline{V \rtimes G}$ *form a completion of* $(V \rtimes G, V_0 \rtimes \Gamma)$*, and for each subset* $X \subset V \rtimes G$ *let* $X^\sharp$ *denote the closure of* $\rho(X)$ *in* $\overline{V \rtimes G}$*. Then there is a continuous action of* $G^\sharp$ *by homeomorphic automorphisms of* $V^\sharp$ *such that* $(V^\sharp \rtimes G^\sharp, {V_0}^\sharp \rtimes \Gamma^\sharp)$ *and the homomorphism* $\iota\colon (v,g) \mapsto (\rho(v), \rho(g))$ *from* $V \rtimes G$ *into* $V^\sharp \rtimes G^\sharp$ *form a completion of* $(V \rtimes G, V_0 \rtimes \Gamma)$*.*

Proof. Since $\rho(G)$ normalizes $\rho(V)$, it normalizes $V^\sharp$. Hence $G^\sharp$ normalizes $V^\sharp$. We therefore have a continuous action of $G^\sharp$ on $V^\sharp$ by conjugation.

Since both $V_0^\sharp$ and $\Gamma^\sharp$ are compact, $V_0^\sharp\Gamma^\sharp$ is a closed subgroup of $\overline{V_0 \rtimes \Gamma}$, and hence $V_0^\sharp\Gamma^\sharp = \overline{V_0 \rtimes \Gamma}$. We claim that $V^\sharp \cap (V_0^\sharp\Gamma^\sharp) = V_0^\sharp$. Indeed, take v in $V^\sharp \cap (V_0^\sharp\Gamma^\sharp)$, and write $v = \lim_i \rho(v_i)$ for $v_i \in V$. The set $V_0^\sharp\Gamma^\sharp$ being open, it eventually contains $\rho(v_i)$, and so $v_i \in V \cap \rho^{-1}(\overline{V_0 \rtimes \Gamma}) = V \cap (V_0 \rtimes \Gamma) = V_0$, proving that $v \in V_0^\sharp$. The claim implies that $V_0^\sharp$ is open in $V^\sharp$, and that

$$V \cap \rho^{-1}(V_0^\sharp) \subset V \cap \rho^{-1}(\overline{V_0 \rtimes \Gamma}) = V \cap (V_0 \rtimes \Gamma) = V_0.$$

Hence $V \cap \rho^{-1}(V_0^\sharp) = V_0$, and thus the pair $(V^\sharp, V_0^\sharp)$ and $\rho|_V$ form a completion of (V, V_0). Similarly, $(G^\sharp, \Gamma^\sharp)$ and $\rho|_G$ form a completion of (G, Γ), and then the map $\iota = (\rho|_V, \rho|_G)$ satisfies the claim of the proposition. □

The next theorem shows that it is possible to produce completions of semidirect product pairs from completions of the component pairs. To simplify the notation we shall only consider reduced pairs and identify a group with its image in the Schlichting completion, suppressing the corresponding injective homomorphism.

Theorem 2.9. *Let* $(V \rtimes G, V_0 \rtimes \Gamma)$ *be a Hecke pair as in Proposition 1.3, and suppose that it is reduced. Denote by* $\bar V$ *and* $\bar V_0$ *the closures of* V *and* V_0 *in the Schlichting completion of* $(V \rtimes G, V_0)$*, and suppose that* $(\tilde G, \tilde \Gamma)$ *and* $\tilde\rho\colon G \to \tilde G$ *form a completion of* (G, Γ)*.*

(i) *The map* $\rho(g) := (\mathrm{Ad}\, g, \tilde\rho(g))$ *is a homomorphism from* G *into* $\mathrm{HAut}(\bar V) \rtimes \tilde G$*, and the closures* $\bar G := \overline{\rho(G)}$ *and* $\bar\Gamma := \overline{\rho(\Gamma)}$ *satisfy that* $(\bar V \rtimes \bar G, \bar V_0 \rtimes \bar \Gamma)$ *is a completion of* $(V \rtimes G, V_0 \rtimes \Gamma)$ *together with the map* $\iota\colon V \rtimes G \to \bar V \rtimes \bar G$ *given by* $\iota\colon (v,g) \mapsto (v, \rho(g))$*.*

(ii) *Let* $(\overline{V \rtimes G}, \overline{V_0 \rtimes \Gamma})$ *be the Schlichting completion of* $(V \rtimes G, V_0 \rtimes \Gamma)$*. If* $(\tilde G, \tilde\Gamma)$ *is the Schlichting completion of* (G, Γ) *and for every* v *in* V *there is a finite set* $\{g_1, \ldots, g_n\}$ *in* G *such that*

$$\bigcap_{i=1}^n g_i(V_0) \subset v V_0 v^{-1}, \tag{2.7}$$

then ι *extends to a topological isomorphism of* $\overline{V \rtimes G}$ *onto* $\bar V \rtimes \bar G$ *and of* $\overline{V_0 \rtimes \Gamma}$ *onto* $\bar V_0 \rtimes \bar\Gamma$*.*

Proof. We claim that the closure of the image of Γ in $\mathrm{HAut}(\bar V)$ under the map Ad is compact. To see this note first that $(V \rtimes G, V_0 \rtimes \Gamma)$ being reduced implies on one hand that $(V \rtimes G, V_0)$ is reduced, and on the other that $\overline{V \rtimes G}$ has the Hecke topology, given by a neighbourhood subbase at e consisting of sets of the form $(v,g)(V_0 \rtimes \Gamma)(v,g)^{-1}$ for $(v,g) \in V \rtimes G$. The relative topology on V has the sets of the form $vg(V_0)v^{-1}$ as elements of the subbase, and these are precisely the sets defining the Hecke topology on $V \rtimes G$ for the pair $(V \rtimes G, V_0)$. Hence $\bar V$ can be considered as a closed normal subgroup of $\overline{V \rtimes G}$.

The closure of $V \rtimes G$ in the Hecke topology coming from $(V \rtimes G, V_0)$ acts on $\bar V$ by conjugation, and this action drops to an action of G on $\bar V$. Since the closure of Γ in $\overline{V \rtimes G}$ is compact, the closure of Γ in $\mathrm{HAut}(\bar V)$ is compact, so our claim is proved.

We claim next that $(\bar{G}, \bar{\Gamma})$ and the map $\rho\colon g \mapsto (\operatorname{Ad} g, \tilde{\rho}(g))$ form a completion of (G, Γ). That $\bar{\Gamma}$ is compact in the product $\operatorname{HAut}(\bar{V}) \times \tilde{G}$ follows because this is true in each component. We have $\rho^{-1}(\bar{\Gamma}) \subset \tilde{\rho}^{-1}(\tilde{\Gamma}) = \Gamma$, hence $\rho^{-1}(\bar{\Gamma}) = \Gamma$, as required. To see that $\bar{\Gamma}$ is open, let $\pi_2\colon \bar{G} \to \tilde{G}$ be the projection map, and take x in $\pi_2^{-1}(\tilde{\Gamma})$. We can approximate x by an element of the form $\rho(g)$ for $g \in G$, and then $\tilde{\rho}(g) = \pi_2(\rho(g)) \in \tilde{\Gamma}$, showing that $g \in \tilde{\rho}^{-1}(\tilde{\Gamma}) = \Gamma$. Hence $x \in \bar{\Gamma}$, and so $\bar{\Gamma}$ is $\pi_2^{-1}(\tilde{\Gamma})$, and is therefore open, proving the claim.

Since $\bar{G}$ has a continuous action on $\bar{V}$ by construction, and since this action gives by restriction to $\bar{\Gamma}$ an action on $\bar{V}_0$, the pair $(\bar{V} \rtimes \bar{G}, \bar{V}_0 \rtimes \bar{\Gamma})$ and the map ι satisfy the conditions of Definition 2.1, and we have proved (i).

To prove (ii) it suffices by Proposition 2.2 to ensure that $(\bar{V} \rtimes \bar{G}, \bar{V}_0 \rtimes \bar{\Gamma})$ is reduced. We shall check that the conditions of Lemma 2.6 are satisfied.

Since the Schlichting completion of $(V \rtimes G, V_0)$ is by definition reduced, we have

$$\bigcap_{g\in G, v\in V} g(v\bar{V}_0 v^{-1}) = \bigcap_{g\in G, v\in V} gv\bar{V}_0 v^{-1} g^{-1} = \{e\}.$$

By (2.7) the left hand side coincides with $\bigcap_{g\in G} g(\bar{V}_0)$, and thus assumption (i) of Lemma 2.6 is satisfied.

Assume now that $\gamma \in \bigcap_{g\in G} \rho(g)\bar{\Gamma}\rho(g)^{-1}$ acts trivially on $\bar{V}$. Since the action is defined using the projection $\pi_1\colon \operatorname{HAut}(\bar{V}) \times \tilde{G} \to \operatorname{HAut}(\bar{V})$, we have $\pi_1(\gamma) = \mathrm{id}$. On the other hand, $\pi_2(\gamma) \in \bigcap_{g\in G} \tilde{\rho}(g)\tilde{\Gamma}\tilde{\rho}(g)^{-1}$. Since $(\tilde{G}, \tilde{\Gamma})$ is reduced, we get $\pi_2(\gamma) = e$. Thus $\gamma = e$, and assumption (ii) of Lemma 2.6 is also satisfied. □

2.4. **Induced representations of $C_r^*(V \rtimes G, V_0 \rtimes \Gamma)$.** Suppose that $(V \rtimes G, V_0 \rtimes \Gamma)$ is a Hecke pair. We denote by ρ the dense embedding of $(V \rtimes G, V_0 \rtimes \Gamma)$ in a completion of the form $(\bar{V} \rtimes \bar{G}, \bar{V}_0 \rtimes \bar{\Gamma})$, which exists by Proposition 2.8.

Choose left Haar measures $\mu_{\bar{V}}$ and $\mu_{\bar{G}}$ on $\bar{V}$ and $\bar{G}$ normalized by

$$\mu_{\bar{V}}(\bar{V}_0) = 1 \quad\text{and}\quad \mu_{\bar{G}}(\bar{\Gamma}) = 1, \tag{2.8}$$

and for each $g \in \bar{G}$, let $\delta(g)$ be defined by the formula $\mu_{\bar{V}}(\alpha_{g^{-1}}(\bar{V}_0)) = \delta(g)\mu_{\bar{V}}(\bar{V}_0)$, see for example [5, (15.29)]. Thus, a left Haar measure and the modular function for $\bar{V} \rtimes \bar{G}$ are given by

$$d\mu_{\bar{V}\rtimes\bar{G}}(vg) := \delta(g) d\mu_{\bar{V}}(v) d\mu_{\bar{G}}(g)$$

and

$$\Delta_{\bar{V}\rtimes\bar{G}}(vg) = \delta(g)\Delta_{\bar{V}}(v)\Delta_{\bar{G}}(g),$$

for $v \in \bar{V}$ and $g \in \bar{G}$. Recall also that $\Delta_{\bar{V}\rtimes\bar{G}} \circ \rho$ is the modular function $\Delta_{V_0\rtimes\Gamma}$ associated to the given Hecke pair. Likewise for $\Delta_{\bar{V}}$ and $\Delta_{\bar{G}}$.

Suppose now that $\chi\colon V \to \mathbb{T}$ is a character such that $\chi(V_0) = \{1\}$. We can extend χ to a unique continuous character on $\bar{V}$, which we continue to denote by χ. The induced representation $\operatorname{Ind}_{\bar{V}}^{\bar{V}\rtimes\bar{G}} \chi$ acts, by definition, on the Hilbert space of functions $\tilde{\xi}\colon \bar{V} \rtimes \bar{G} \to \mathbb{C}$ such that

$$\tilde{\xi}(xv) = \overline{\chi(v)}\tilde{\xi}(x) \text{ for } v \in \bar{V} \text{ and } x \in \bar{V} \rtimes \bar{G},$$

and $|\tilde{\xi}| \in L^2((\bar{V} \rtimes \bar{G})/\bar{V})$. The representation is simply given by left translations on this space

$$\left(\left(\operatorname{Ind}_{\bar{V}}^{\bar{V}\rtimes\bar{G}} \chi\right)(x)\tilde{\xi}\right)(y) = \tilde{\xi}(x^{-1}y) \text{ for } y \in \bar{V} \rtimes \bar{G}.$$

The map $L^2(\bar{G}) \ni \xi \mapsto \tilde{\xi}$ defined by $\tilde{\xi}(vg) := \xi(g)\overline{\chi(g^{-1}v)}$ is an isomorphism of $L^2(\bar{G})$ onto the space of such functions, transforming $\operatorname{Ind}_{\bar{V}}^{\bar{V}\rtimes\bar{G}} \chi$ into the representation $\pi\colon \bar{V} \rtimes \bar{G} \to B(L^2(\bar{G}))$ given by

$$\pi(vg)\xi(h) = \chi(\alpha_{h^{-1}}(v))\xi(g^{-1}h) \tag{2.9}$$

for $v \in \bar{V}$ and $g, h \in \bar{G}$.

Lemma 2.10. *Let χ be a character of V whose kernel contains V_0, and let S be the subset of G defined by*

$$S := \{g \in G \mid (\chi \circ \alpha_{g^{-1}})|_{V_0} \equiv 1\}. \tag{2.10}$$

Then the space of $\bar{V}_0 \rtimes \bar{\Gamma}$-invariant vectors for the representation π defined by (2.9) *coincides with the space of $\bar{\Gamma}$-invariant functions $f \in L^2(\bar{G})$ with support in the closure $\bar{S}$ of $\rho(S)$ in $\bar{G}$.*

Proof. Since Γ leaves V_0 invariant, we have $\Gamma S = S$, and since $\bar{\Gamma}$ is an open subgroup of $\bar{G}$, this implies that $\bar{S}$ is $\bar{\Gamma}\rho(S)$. But $\bar{G} = \bar{\Gamma}\rho(G)$, and thus

$$\bar{S} = \{g \in \bar{G} \mid (\chi \circ \alpha_{g^{-1}})|_{\bar{V}_0} \equiv 1\}.$$

Denote by π_* the integrated form of the representation π from (2.9), and let $p \in C_c(\bar{V} \rtimes \bar{G})$ be the characteristic function of the compact open subgroup $\bar{V}_0 \rtimes \bar{\Gamma}$, which is a self-adjoint projection because of our normalization (2.8); then

$$(\pi_*(p)\xi)(h) = \int_{\bar{V}_0} \chi(\alpha_{h^{-1}}(v))d\mu_{\bar{V}}(v) \int_{\bar{\Gamma}} \xi(g^{-1}h)d\mu_{\bar{G}}(g) \text{ for } \xi \in L^2(\bar{G}).$$

Since the first factor is zero or one, depending on whether h belongs to $\bar{S}$ or not, we get the result. □

We thus obtain a $*$-representation, which we continue to denote by π_*, of $pC_c(\bar{V} \rtimes \bar{G})p$ on the subspace of $\bar{\Gamma}$-invariant functions in $L^2(\bar{G})$ with support in $\bar{S}$. By composing π_* with the isomorphism $\Psi\colon \mathcal{H}(V \rtimes G, V_0 \rtimes \Gamma) \to pC_c(\bar{V} \rtimes \bar{G})p$ of Lemma 2.3 we get a representation of the Hecke algebra. A computation similar to (2.4) yields

$$((\pi_* \circ \Psi)(f)\xi)(h) = \sum_{y=vg \in V \rtimes G/V_0 \rtimes \Gamma} \Delta_{V_0 \rtimes \Gamma}(y)^{-1/2} f(y)\chi(\alpha_{h^{-1}}(\rho(v)))\xi(\rho(g)^{-1}h)$$

for $f \in \mathcal{H}(V \rtimes G, V_0 \rtimes \Gamma)$, $\xi \in L^2(\bar{G})$ a $\bar{\Gamma}$-invariant function with support in $\bar{S}$, and $h \in \bar{S}$. The unitary isomorphism $(U^*\xi)(s) := \Delta_{\bar{G}}^{1/2}(\rho(s))\xi(\rho(s))$ from the subspace of $\bar{\Gamma}$-invariant functions in $L^2(\bar{G})$ with support in $\bar{S}$ onto $\ell^2(\Gamma\backslash S)$ conjugates $\pi_* \circ \Psi$ into a representation π_χ of $\mathcal{H}(V \rtimes G, V_0 \rtimes \Gamma)$ on $\ell^2(\Gamma\backslash S)$. Since

$$\Delta_{V_0 \rtimes \Gamma}(vg)\Delta_\Gamma(g)^{-1} = \delta(g)\Delta_{V_0}(v)$$

for $vg \in V \rtimes G$, and since $(V_0 \rtimes \Gamma)gv = \Gamma g \alpha_{g^{-1}}(V_0)v$, we get that

$$\begin{aligned}
(\pi_\chi(f)\xi)(t) &= \sum_{vg \in V \rtimes G/V_0 \rtimes \Gamma} \delta(g)^{-1/2}\Delta_{V_0}(v)^{-1/2}\chi(\alpha_{t^{-1}}(v))f(vg)\xi(g^{-1}t) \\
&= \sum_{gv \in V_0 \rtimes \Gamma\backslash V \rtimes G} \delta(g)^{1/2}\Delta_{V_0}(v)^{1/2}\overline{\chi(\alpha_{t^{-1}}(v))}f(v^{-1}g^{-1})\xi(gt) \\
&= \sum_{g \in \Gamma\backslash G} \sum_{v \in \alpha_{g^{-1}}(V_0)\backslash V} \delta(g)^{1/2}\Delta_{V_0}(v)^{1/2}\overline{\chi(\alpha_{t^{-1}}(v))}f(g^{-1}\alpha_g(v)^{-1})\xi(gt) \\
&= \sum_{g \in \Gamma\backslash G} \sum_{v \in V_0\backslash V} \delta(g)^{1/2}\Delta_{V_0}(v)^{1/2}\overline{\chi(\alpha_{(gt)^{-1}}(v))}f(g^{-1}v^{-1})\xi(gt) \\
&= \sum_{g \in \Gamma\backslash G} \sum_{v \in V_0\backslash V} \delta(gt^{-1})^{1/2}\Delta_{V_0}(v)^{1/2}\overline{\chi(\alpha_{g^{-1}}(v))}f(tg^{-1}v^{-1})\xi(g),
\end{aligned}$$

for $t \in S$. Since $\xi \in \ell^2(\Gamma\backslash S)$, the last summation is actually over $g \in \Gamma\backslash S$. We have thus established the first claim of the next proposition.

Proposition 2.11. *Let χ be a character of V whose kernel contains V_0, and let $S\subset G$ be the set defined by (2.10); then the formula*

$$(\pi_\chi(f)\xi)(t)=\sum_{s\in\Gamma\backslash S}\sum_{v\in V_0\backslash V}\delta(ts^{-1})^{-1/2}\Delta_{V_0}(v)^{1/2}\overline{\chi(\alpha_{s^{-1}}(v))}f(ts^{-1}v^{-1})\xi(s)$$

defines a $*$-representation $\pi_\chi\colon\mathcal{H}(V\rtimes G,V_0\rtimes\Gamma)\to B(\ell^2(\Gamma\backslash S))$.

If χ is weakly contained in the regular representation of $\bar V$, then π_χ extends to a representation of $C^*_r(V\rtimes G,V_0\rtimes\Gamma)$ on $\ell^2(\Gamma\backslash S)$.

Proof. The second assertion holds since if χ is weakly contained in the regular representation of $\bar V$ then π_* descends to a representation of $C^*_r(\bar V\rtimes\bar G)$. ☐

Lemma 2.12. *Under the assumptions of Proposition 2.11, suppose that $V_0\subset\alpha_s(V_0)$ for every $s\in S$. Then S is a semigroup and*

$$\pi_\chi([s]_{V_0\rtimes\Gamma})=\delta(s)^{-1/2}\lambda([s]_\Gamma)\ \textit{for}\ s\in S,$$

where $\lambda\colon\mathcal{H}(S,\Gamma)\to B(\ell^2(\Gamma\backslash S))$ is the regular representation of $\mathcal{H}(S,\Gamma)$ given by (2.5).

Proof. Since $\ker\chi$ contains V_0, the identity of G is in S, and since $(s_1s_2)^{-1}(V_0)=s_2^{-1}(s_1^{-1}(V_0))\subset V_0$ for $s_1,s_2\in S$, the set S is multiplicatively closed.

By Proposition 1.6 the map $[s]_\Gamma\mapsto[s]_{V_0\rtimes\Gamma}$ defines a homomorphism $\mathcal{H}(S,\Gamma)\to\mathcal{H}(V\rtimes G,V_0\rtimes\Gamma)$. By composing π_χ with it we get a representation of $\mathcal{H}(S,\Gamma)$ on $\ell^2(\Gamma\backslash S)$. To see that this representation matches with λ as claimed, we let $s,t\in S$, $\xi\in\ell^2(\Gamma\backslash S)$. Then

$$(\pi_\chi([s]_{V_0\rtimes\Gamma})\xi)(t)=\sum_{y\in\Gamma\backslash S}\sum_{v\in V_0\backslash V}\delta(ty^{-1})^{-1/2}\Delta_{V_0}(v)^{1/2}\overline{\chi(\alpha_{y^{-1}}(v))}[s]_{V_0\rtimes\Gamma}(ty^{-1}v^{-1})\xi(y).$$

The value of $[s]_{V_0\rtimes\Gamma}$ at $ty^{-1}v^{-1}$ is zero unless $ty^{-1}v^{-1}\in V_0\Gamma sV_0\Gamma=\Gamma s\Gamma V_0$, in which case $ty^{-1}\in\Gamma s\Gamma$ and $v\in V_0$. Since then $\delta(ty^{-1})=\delta(s)$ and $\chi(\alpha_{y^{-1}}(v))=1$, we can compute further that

$$(\pi_\chi([s]_{V_0\rtimes\Gamma})\xi)(t)=\sum_{y\in\Gamma\backslash S}\delta(s)^{-1/2}[s]_\Gamma(ty^{-1})\xi(y).$$

But the last expression is precisely $\delta(s)^{-1/2}(\lambda([s]_\Gamma)\xi)(t)$, and the lemma follows. ☐

In other words, the map $[s]_\Gamma\mapsto\delta(s)^{1/2}[s]_{V_0\rtimes\Gamma}$ from Remark 1.7 gives an embedding of $\mathcal{H}(S,\Gamma)$ into $\mathcal{H}(V\rtimes G,V_0\rtimes\Gamma)$ such that the restriction of the representation π_χ to $\mathcal{H}(S,\Gamma)$ coincides with the regular representation λ of $\mathcal{H}(S,\Gamma)$.

Our next immediate goal is to apply the preceding considerations to the Hecke pair (P,P_0) obtained by setting $V=\mathrm{Mat}_2(\mathbb{Q})$, $V_0=\mathrm{Mat}_2(\mathbb{Z})$, $G=\mathrm{GL}_2^+(\mathbb{Q})$ and $\Gamma=\mathrm{SL}_2(\mathbb{Z})$. Since the Hecke pairs $(\mathrm{Mat}_2(\mathbb{Q}),\mathrm{Mat}_2(\mathbb{Z}))$ and $(\mathrm{GL}_2^+(\mathbb{Q}),\mathrm{SL}_2(\mathbb{Z}))$ are unimodular, it follows from Proposition 1.3 and Proposition 1.4 that

$$\delta(g)=\Delta_{P_0}(g)=\det(g)^{-2}\ \text{for}\ g\in G.$$

We now fix a character χ on $\mathbb{A}_f$ such that the corresponding pairing $(x,y)\mapsto\chi^y(x):=\chi(xy)$ for $x,y\in\mathbb{A}_f$ gives a self-duality isomorphism $y\mapsto\chi^y$ of $\mathbb{A}_f$ to $\widehat{\mathbb{A}_f}$ in which $\hat{\mathbb{Z}}$ corresponds to $\hat{\mathbb{Z}}^\perp$. Thus in particular $\chi(\hat{\mathbb{Z}})=\{1\}$. Denote by Tr the usual trace on 2×2 matrices; then a similar pairing given by

$$(a,b)\mapsto\chi^b(a):=\chi(\mathrm{Tr}(ab)),\ a,b\in\mathrm{Mat}_2(\mathbb{A}_f),\tag{2.11}$$

implements a self-duality isomorphism $a\mapsto\chi^a$ of $\mathrm{Mat}_2(\mathbb{A}_f)$ onto $\widehat{\mathrm{Mat}_2(\mathbb{A}_f)}$ in which $\mathrm{Mat}_2(\hat{\mathbb{Z}})$ corresponds to $\mathrm{Mat}_2(\hat{\mathbb{Z}})^\perp$. This pairing is noncanonical, but for any choice of χ as above we have that

$$\chi^n(\alpha_g(m))=\chi(\mathrm{Tr}(mg^{-1}n))=\chi^{g^{-1}n}(m)$$

for $g \in \mathrm{GL}_2^+(\mathbb{A}_f)$, and this says that the action by right multiplication by g on $\mathrm{Mat}_2(\mathbb{A}_f)$ is transformed by the self-duality into left multiplication.

An element $g \in \mathrm{GL}_2^+(\mathbb{Q})$ belongs to $S := \{h \in \mathrm{GL}_2^+(\mathbb{Q}) \mid (\chi \circ \alpha_{h^{-1}})|_{\mathrm{Mat}_2(\mathbb{Z})} \equiv 1\}$ precisely when $gw \in \mathrm{Mat}_2(\hat{\mathbb{Z}})^\perp = \mathrm{Mat}_2(\hat{\mathbb{Z}})$ for all $w \in \mathrm{Mat}_2(\hat{\mathbb{Z}})$, that is, when $g \in \mathrm{GL}_2^+(\mathbb{Q}) \cap \mathrm{Mat}_2(\hat{\mathbb{Z}}) = \mathrm{Mat}_2^+(\mathbb{Z})$. We thus get a representation of our Hecke algebra on the space $\ell^2(\Gamma \backslash S)$. This representation is faithful by the following known result.

Lemma 2.13. *Let G be a locally compact group acting on a locally compact space X, and suppose that x is a point in X with dense orbit. Let χ_x denote evaluation at x on $C_0(X)$. Then $\mathrm{Ind}_{C_0(X)}^{C_0(X)\rtimes_r G} \chi_x$ is faithful.*

Proof. Denote by π the representation $\oplus_{g\in G} \chi_{gx}$ of $C_0(X)$. The representations $\mathrm{Ind}_{C_0(X)}^{C_0(X)\rtimes_r G} \chi_{gx}$ and $\mathrm{Ind}_{C_0(X)}^{C_0(X)\rtimes_r G} \chi_x$ are equivalent for every g in G, and therefore $\mathrm{Ind}_{C_0(X)}^{C_0(X)\rtimes_r G} \chi_x$ is quasi-equivalent to the representation $\mathrm{Ind}_{C_0(X)}^{C_0(X)\rtimes_r G} \pi$, which is faithful because π is faithful. The lemma follows. □

We want to apply Lemma 2.13 to the group $G = \mathrm{GL}_2^+(\mathbb{A}_f)$ acting on $\hat{V} = \mathrm{Mat}_2(\mathbb{A}_f)$ by multiplication on the left. The density of $\mathrm{Mat}_2^+(\mathbb{Z})$ in $\mathrm{Mat}_2(\hat{\mathbb{Z}})$ implies that $\mathrm{GL}_2^+(\mathbb{A}_f)w$ is dense in $\mathrm{Mat}_2(\mathbb{A}_f)$ for every $w \in \mathrm{GL}_2(\hat{\mathbb{Z}})$. Therefore the lemma applies, and we get the following.

Theorem 2.14. *For the Hecke pair $(P, P_0) = (\mathrm{Mat}_2(\mathbb{Q}) \rtimes \mathrm{GL}_2^+(\mathbb{Q}), \mathrm{Mat}_2(\mathbb{Z}) \rtimes \mathrm{SL}_2(\mathbb{Z}))$, the semigroup $S = \mathrm{Mat}_2^+(\mathbb{Z})$, and each $w \in \mathrm{GL}_2(\hat{\mathbb{Z}})$, the representation $\pi_w := \pi_{\chi^w}$ constructed as in Proposition 2.11 from the character χ^w defined by* (2.11) *is a faithful representation of $C_r^*(P, P_0)$ on $\ell^2(\Gamma\backslash S)$. The restriction of this representation to the subalgebra $\mathcal{H}(\mathrm{Mat}_2^+(\mathbb{Z}), \mathrm{SL}_2(\mathbb{Z}))$ is simply the left regular representation rescaled by a factor given by the determinant:*

$$\pi_w([s]_{P_0}) = \det(s)\lambda([s]_\Gamma) \text{ for } s \in \mathrm{Mat}_2^+(\mathbb{Z}).$$

3. The symmetry group and the symmetric part of the algebra

In this section we continue our study of the reduced Hecke C*-algebra of the pair

$$(P, P_0) = \left(\begin{pmatrix} 1 & \mathrm{Mat}_2(\mathbb{Q}) \\ 0 & \mathrm{GL}_2^+(\mathbb{Q}) \end{pmatrix}, \begin{pmatrix} 1 & \mathrm{Mat}_2(\mathbb{Z}) \\ 0 & \mathrm{SL}_2(\mathbb{Z}) \end{pmatrix} \right),$$

with the aim of showing that there exists an action of the compact group $\hat{\mathbb{Z}}^*$ of invertible elements in $\hat{\mathbb{Z}}$ such that the fixed point algebra decomposes into a tensor product of algebras corresponding to different primes.

3.1. The crossed product picture of $C_r^*(P, P_0)$ and symmetries.

Note that if we have a Hecke pair (N, N_0), then every automorphism θ of N leaving N_0 invariant defines an automorphism α_θ of the Hecke algebra by $\alpha_\theta(f) = f \circ \theta^{-1}$. Moreover, since any completion of (N, N_0) defines an isomorphic Hecke algebra, we can consider automorphisms of completions. As showed in Section 2.2, the Schlichting completion of our pair (P, P_0) is

$$(\bar{P}, \bar{P}_0) = \left(\begin{pmatrix} 1 & \mathrm{Mat}_2(\mathbb{A}_f) \\ 0 & \mathrm{GL}_2^+(\mathbb{A}_f) \end{pmatrix}, \begin{pmatrix} 1 & \mathrm{Mat}_2(\hat{\mathbb{Z}}) \\ 0 & \mathrm{SL}_2(\hat{\mathbb{Z}}) \end{pmatrix} \right).$$

For $r \in \mathrm{GL}_2(\hat{\mathbb{Z}})$ the automorphism of $\bar{P}$ defined by

$$\bar{P} \ni x \mapsto \begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix} x \begin{pmatrix} 1 & 0 \\ 0 & r^{-1} \end{pmatrix}$$

leaves $\bar{P}_0$ invariant, and hence defines an automorphism α_r of $\mathcal{H}(\bar{P}, \bar{P}_0)$. Since $\mathcal{H}(\bar{P}, \bar{P}_0)$ consists by definition of $\bar{P}_0$-biinvariant functions, α_r is trivial for $r \in \mathrm{SL}_2(\hat{\mathbb{Z}})$. Thus we get an action of the group $\mathrm{GL}_2(\hat{\mathbb{Z}})/\mathrm{SL}_2(\hat{\mathbb{Z}}) \cong \hat{\mathbb{Z}}^*$ on $\mathcal{H}(\bar{P}, \bar{P}_0)$ and on $\mathcal{H}(P, P_0)$.

Recall from Section 2.1 that the reduced Hecke C*-algebra $C_r^*(P,P_0)$ is isomorphic to the corner in $C_r^*(\bar P)$ defined by the projection corresponding to the compact open subgroup $\bar P_0$. It is standard that the C*-algebra of a semidirect product can be written as a crossed product, so $C_r^*(\bar P)\cong C_r^*(\mathrm{Mat}_2(\mathbb{A}_f))\rtimes \mathrm{GL}_2^+(\mathbb{A}_f)$, where the action of $\mathrm{GL}_2^+(\mathbb{A}_f)$ on $C_r^*(\mathrm{Mat}_2(\mathbb{A}_f))$ is obtained from right multiplication of $\mathrm{GL}_2^+(\mathbb{A}_f)$ on $\mathrm{Mat}_2(\mathbb{A}_f)$. But the self-duality of $\mathrm{Mat}_2(\mathbb{A}_f)$ discussed before Lemma 2.13 allows us to transpose the action of $\mathrm{GL}_2^+(\mathbb{A}_f)$ into left multiplication on $\mathrm{Mat}_2(\mathbb{A}_f)$. Hence using the Fourier transform and the transposed action we may write

$$C_r^*(\bar P)\cong C_r^*(\mathrm{Mat}_2(\mathbb{A}_f))\rtimes \mathrm{GL}_2^+(\mathbb{A}_f)\cong C_0(\mathrm{Mat}_2(\mathbb{A}_f))\rtimes_r \mathrm{GL}_2^+(\mathbb{A}_f)$$

where the action of $\mathrm{GL}_2^+(\mathbb{A}_f)$ on $C_0(\mathrm{Mat}_2(\mathbb{A}_f))$ is given by $(g\cdot f)(m)=f(g^{-1}m)$ for $g\in \mathrm{GL}_2^+(\mathbb{A}_f)$ and $m\in \mathrm{Mat}_2(\mathbb{A}_f)$. Moreover, since $\mathrm{Mat}_2(\hat{\mathbb{Z}})^\perp=\mathrm{Mat}_2(\hat{\mathbb{Z}})$, this isomorphism carries the projection corresponding to $\bar P_0$ into

$$p_0=\mathbb{1}_{\mathrm{Mat}_2(\hat{\mathbb{Z}})}\int_{\mathrm{SL}_2(\hat{\mathbb{Z}})}\lambda_g dg,$$

where $\mathbb{1}_{\mathrm{Mat}_2(\hat{\mathbb{Z}})}$ is the characteristic function of the set $\mathrm{Mat}_2(\hat{\mathbb{Z}})$, and λ_g the element corresponding to $g\in\mathrm{SL}_2(\hat{\mathbb{Z}})$ in the multiplier algebra of the second crossed product. Thus, with Ψ from Lemma 2.3 we have

$$\Psi: C_r^*(P,P_0)\xrightarrow{\cong} p_0(C_0(\mathrm{Mat}_2(\mathbb{A}_f))\rtimes_r \mathrm{GL}_2^+(\mathbb{A}_f))p_0. \tag{3.1}$$

We can regard $C_0(\mathrm{Mat}_2(\mathbb{A}_f))\rtimes_r \mathrm{GL}_2^+(\mathbb{A}_f)$ as a completion of the algebra of continuous complex-valued functions with compact support on $\mathrm{GL}_2^+(\mathbb{A}_f)\times\mathrm{Mat}_2(\mathbb{A}_f)$, endowed with the convolution product

$$(f_1*f_2)(g,m)=\int_{\mathrm{GL}_2^+(\mathbb{A}_f)}f_1(gh^{-1},hm)f_2(h,m)dh \tag{3.2}$$

and involution

$$f^*(g,m)=f(g^{-1},gm). \tag{3.3}$$

Cutting down to the corner determined by p_0 has two effects on functions f on the space $\mathrm{GL}_2^+(\mathbb{A}_f)\times\mathrm{Mat}_2(\mathbb{A}_f)$. The first one is that it reduces the support to pairs $(g,m)\in\mathrm{GL}_2^+(\mathbb{A}_f)\times\mathrm{Mat}_2(\mathbb{A}_f)$ such that both m and gm are in $\mathrm{Mat}_2(\hat{\mathbb{Z}})$ and second, it forces the invariance under the action $\mathrm{SL}_2(\hat{\mathbb{Z}})\times\mathrm{SL}_2(\hat{\mathbb{Z}})$ on $\mathrm{GL}_2^+(\mathbb{A}_f)\times\mathrm{Mat}_2(\mathbb{A}_f)$ given by

$$(g_1,g_2)(g,m)=(g_1gg_2^{-1},g_2m)\text{ for }g_1,g_2\in\mathrm{SL}_2(\hat{\mathbb{Z}}). \tag{3.4}$$

Therefore, if we denote by $\mathrm{SL}_2(\hat{\mathbb{Z}})\backslash\mathrm{GL}_2^+(\mathbb{A}_f)\boxtimes_{\mathrm{SL}_2(\hat{\mathbb{Z}})}\mathrm{Mat}_2(\hat{\mathbb{Z}})$ the quotient of the space

$$\{(g,m)\in\mathrm{GL}_2^+(\mathbb{A}_f)\times\mathrm{Mat}_2(\hat{\mathbb{Z}})\mid gm\in\mathrm{Mat}_2(\hat{\mathbb{Z}})\}$$

by the above action of $\mathrm{SL}_2(\hat{\mathbb{Z}})\times\mathrm{SL}_2(\hat{\mathbb{Z}})$, then $p_0(C_0(\mathrm{Mat}_2(\mathbb{A}_f))\rtimes_r\mathrm{GL}_2^+(\mathbb{A}_f))p_0$ is a completion of the algebra $C_c(\mathrm{SL}_2(\hat{\mathbb{Z}})\backslash\mathrm{GL}_2^+(\mathbb{A}_f)\boxtimes_{\mathrm{SL}_2(\hat{\mathbb{Z}})}\mathrm{Mat}_2(\hat{\mathbb{Z}}))$ of compactly supported continuous functions on $\mathrm{SL}_2(\hat{\mathbb{Z}})\backslash\mathrm{GL}_2^+(\mathbb{A}_f)\boxtimes_{\mathrm{SL}_2(\hat{\mathbb{Z}})}\mathrm{Mat}_2(\hat{\mathbb{Z}})$ with convolution and involution given by (3.2) and (3.3). Since $\mathrm{SL}_2(\hat{\mathbb{Z}})\backslash\mathrm{GL}_2^+(\mathbb{A}_f)=\Gamma\backslash\mathrm{GL}_2^+(\mathbb{Q})$ by Lemma 2.4, we can refine formula (3.2) for the convolution of two such functions to

$$(f_1*f_2)(g,m)=\sum_{h\in\Gamma\backslash\mathrm{GL}_2^+(\mathbb{Q}):\,hm\in\mathrm{Mat}_2(\hat{\mathbb{Z}})}f_1(gh^{-1},hm)f_2(h,m). \tag{3.5}$$

If we regard elements of $\mathcal{H}(P,P_0)$ as functions defined on $\mathrm{GL}_2^+(\mathbb{Q})\times\mathrm{Mat}_2(\mathbb{Q})$ (it is convenient now to use this order), the isomorphism Ψ from (3.1) obtained by applying the Fourier transform in the second variable and multiplying by $\Delta_{P_0}^{-1/2}$ (due to Lemma 2.3) is then explicitly given by

$$\Psi(f)(g,m)=\det(g)\sum_{n\in\mathrm{Mat}_2(\mathbb{Q})/\mathrm{Mat}_2(\mathbb{Z})}f(g,n)\chi(\mathrm{Tr}(nm)). \tag{3.6}$$

For the rest of the paper we shall identify $C_r^*(P,P_0)$ with its image under Ψ. So with this convention, the automorphism α_r is now given by

$$\alpha_r(f)(g,m)=f(r^{-1}gr,r^{-1}m), \tag{3.7}$$

for $f\in C_c(\mathrm{SL}_2(\hat{\mathbb{Z}})\backslash\mathrm{GL}_2^+(\mathbb{A}_f)\boxtimes_{\mathrm{SL}_2(\hat{\mathbb{Z}})}\mathrm{Mat}_2(\hat{\mathbb{Z}}))$.

3.2. **Tensor product decomposition of the fixed point algebra.** To understand the fixed point algebra of the action (3.7), for each prime p define a subgroup G_p of $\mathrm{GL}_2^+(\mathbb{Q})$ by

$$G_p:=\mathrm{GL}_2^+(\mathbb{Z}[p^{-1}])=\mathrm{GL}_2(\mathbb{Z}[p^{-1}])\cap\mathrm{GL}_2^+(\mathbb{Q}).$$

It is not difficult to show, see [12, Section 3], that G_p is the subgroup of $\mathrm{GL}_2^+(\mathbb{Q})$ generated by Γ and $\begin{pmatrix}1&0\\0&p\end{pmatrix}$, and also that it is the group of elements $g\in\mathrm{GL}_2^+(\mathbb{Q})$ such that the image of g in $\mathrm{GL}_2(\mathbb{Q}_q)$ lies in $\mathrm{GL}_2(\mathbb{Z}_q)$ for $q\ne p$. Consider now the Hecke pair

$$(\mathrm{Mat}_2(\mathbb{Z}[p^{-1}])\rtimes G_p,\mathrm{Mat}_2(\mathbb{Z})\rtimes\Gamma)=\left(\begin{pmatrix}1&\mathrm{Mat}_2(\mathbb{Z}[p^{-1}])\\0&\mathrm{GL}_2^+(\mathbb{Z}[p^{-1}])\end{pmatrix},\begin{pmatrix}1&\mathrm{Mat}_2(\mathbb{Z})\\0&\mathrm{SL}_2(\mathbb{Z})\end{pmatrix}\right). \tag{3.8}$$

Similarly to Proposition 2.5, the Schlichting completion of (3.8) is the pair

$$(\mathrm{Mat}_2(\mathbb{Q}_p)\rtimes\mathrm{GL}_2^+(\mathbb{Q}_p),\mathrm{Mat}_2(\mathbb{Z}_p)\rtimes\mathrm{SL}_2(\mathbb{Z}_p)),$$

where $\mathrm{GL}_2^+(\mathbb{Q}_p)$ is the group of elements $g\in\mathrm{GL}_2(\mathbb{Q}_p)$ such that $\det(g)$ is an integral power of p. The reduced Hecke C*-algebra of this pair can be regarded as a subalgebra of $C_r^*(P,P_0)$. If we consider $C_r^*(P,P_0)$ as a completion of $C_c(\mathrm{SL}_2(\hat{\mathbb{Z}})\backslash\mathrm{GL}_2^+(\mathbb{A}_f)\boxtimes_{\mathrm{SL}_2(\hat{\mathbb{Z}})}\mathrm{Mat}_2(\hat{\mathbb{Z}}))$ as explained above, then this subalgebra has the following description.

Lemma 3.1. *The reduced Hecke C*-algebra* $\mathfrak{A}_p:=C_r^*(\mathrm{Mat}_2(\mathbb{Z}[p^{-1}])\rtimes G_p,\mathrm{Mat}_2(\mathbb{Z})\rtimes\Gamma)$ *of the pair* (3.8) *is the closure of the space of* $(\mathrm{SL}_2(\hat{\mathbb{Z}})\times\mathrm{SL}_2(\hat{\mathbb{Z}}))$*-invariant compactly supported continuous functions* f *on*

$$\{(g,m)\in\mathrm{GL}_2^+(\mathbb{A}_f)\times\mathrm{Mat}_2(\hat{\mathbb{Z}})\mid gm\in\mathrm{Mat}_2(\hat{\mathbb{Z}})\}$$

such that $f(g,m)=0$ *if* $g\notin G_p\mathrm{SL}_2(\hat{\mathbb{Z}})$*, and* $f(g,m)=f(g,m')$ *if* $m_p=m'_p$.

Proof. The closure of G_p in $\mathrm{GL}_2^+(\mathbb{A}_f)$ is the group $G_p\mathrm{SL}_2(\hat{\mathbb{Z}})$, and the closure of $\mathrm{Mat}_2(\mathbb{Z}[p^{-1}])$ in $\mathrm{Mat}_2(\mathbb{A}_f)$ is $\mathrm{Mat}_2(\mathbb{Q}_p\times\prod_{q\ne p}\mathbb{Z}_q)$. The annihilator of $\mathrm{Mat}_2(\mathbb{Q}_p\times\prod_{q\ne p}\mathbb{Z}_q)$ is the group of elements $m\in\mathrm{Mat}_2(\mathbb{A}_f)$ such that $m_p=0$ and $m_q\in\mathrm{Mat}_2(\mathbb{Z}_q)$ for $q\ne p$. Since a function on $\mathrm{Mat}_2(\mathbb{A}_f)/\mathrm{Mat}_2(\hat{\mathbb{Z}})$ is supported on $\mathrm{Mat}_2(\mathbb{Q}_p\times\prod_{q\ne p}\mathbb{Z}_q)$ if and only if its Fourier transform is invariant under the translations by elements of the annihilator, and the latter means that the value of the Fourier transform at $m\in\mathrm{Mat}_2(\hat{\mathbb{Z}})$ depends only on m_p, the result follows. □

Note that the action of α_r on $\mathfrak{A}_p$ depends only on r_p. Indeed, assume $r_p=1$ in $\mathrm{GL}_2(\mathbb{Z}_p)$. Since every double coset of Γ in $\mathrm{GL}_2^+(\mathbb{Q})$ has a diagonal representative, see e.g. [7, Chapter IV], it suffices to compute $\alpha_r(f)(g,m)$ for $g\in G_p$ diagonal. Since α_r depends only on $\det(r)$, we may assume that r is also diagonal, and since $(r^{-1}m)_p=m_p$, we get

$$\alpha_r(f)(g,m)=f(g,r^{-1}m)=f(g,m).$$

Therefore the action of $\hat{\mathbb{Z}}^*$, when restricted to $\mathfrak{A}_p$, defines an action of $\mathbb{Z}_p^*$. Alternatively, this action can also be obtained from conjugation of elements in $\begin{pmatrix}1&\mathrm{Mat}_2(\mathbb{Q}_p)\\0&\mathrm{GL}_2^+(\mathbb{Q}_p)\end{pmatrix}$ by matrices $\begin{pmatrix}1&0\\0&r\end{pmatrix}$ with $r\in\mathrm{GL}_2(\mathbb{Z}_p)$.

We can now formulate the main result of the section.

Theorem 3.2. *The subalgebras* $\mathfrak{A}_p^{\mathbb{Z}_p^*} = C_r^*(\mathrm{Mat}_2(\mathbb{Z}[p^{-1}]) \rtimes G_p, \mathrm{Mat}_2(\mathbb{Z}) \rtimes \Gamma)^{\mathbb{Z}_p^*}$ *of* $C_r^*(P, P_0)$ *mutually commute for different primes, and*

$$C_r^*(P, P_0)^{\hat{\mathbb{Z}}^*} \cong \bigotimes_{p \in \mathcal{P}} \mathfrak{A}_p^{\mathbb{Z}_p^*}.$$

In the proof of this theorem we rely on lattices and their properties, and since we will also use these later we collect the data we need in the next remark.

Remark 3.3. A lattice L in $\mathbb{R}^2$ is commensurable with $\mathbb{Z}^2$ if it is contained in $\mathbb{Q}^2$. We identify $\Gamma\backslash S$, where $S = \mathrm{Mat}_2^+(\mathbb{Z})$, with the set of lattices in $\mathbb{R}^2$ containing $\mathbb{Z}^2$: namely, for $s \in S$ we let $L = s^{-1}\mathbb{Z}^2$. Equivalently, we can consider lattices in $\mathbb{A}_f^2$ containing $\hat{\mathbb{Z}}^2$ as follows: given a lattice $L \subset \mathbb{Q}^2$, the closure $\bar{L}$ of L in $\mathbb{A}_f^2$ is a lattice, and by [24, Theorem V.2], the map $L \mapsto \bar{L}$ is a bijection between lattices in $\mathbb{Q}^2$ and lattices in $\mathbb{A}_f^2$ with inverse given by $\bar{L} \mapsto \mathbb{Q}^2 \cap \bar{L}$. The group $\mathrm{GL}_2(\hat{\mathbb{Z}})$ acts on the space of lattices by

$$rL := r\bar{L} \cap \mathbb{Q}^2 \text{ for } r \in \mathrm{GL}_2(\hat{\mathbb{Z}}).$$

Given a lattice L commensurable with $\mathbb{Z}^2$, we denote by L_p the lattice $\mathbb{Q}^2 \cap (\mathbb{Z}_p \otimes_{\mathbb{Z}} L)$: in other words, L_p is the unique lattice in $\mathbb{Q}^2$ such that the closure of L_p in $\mathbb{Q}_p^2$ coincides with that of L, and the closure of L_p in $\mathbb{Q}_q^2$ is $\mathbb{Z}_q^2$ for $q \neq p$. If L contains $\mathbb{Z}^2$ and we write $L_p = s^{-1}\mathbb{Z}^2$ and use that the closure of L_p in $\mathbb{Q}_q^2$ is $\mathbb{Z}_q^2$ for $q \neq p$, we deduce that $s \in \mathrm{GL}_2(\mathbb{Z}_q)$ for $q \neq p$. Hence $s \in S_p := G_p \cap S$. Note that S_p is the semigroup of matrices $m \in \mathrm{Mat}_2^+(\mathbb{Z})$ such that $\det(m)$ is a nonnegative power of p. Thus to every $L \in \Gamma\backslash S$ corresponds a family $(L_p)_{p \in \mathcal{P}}$ with $L_p \in \Gamma\backslash S_p$. An equivalent description of $\Gamma\backslash S_p$ is as the set of lattices L such that $\mathbb{Z}^2 \subset L$ and $L/\mathbb{Z}^2$ is a p-group, i.e. its order is a power of p. The map $\delta_L \mapsto \otimes_{p \in \mathcal{P}} \delta_{L_p}$ defines a unitary isomorphism

$$(\ell^2(\Gamma\backslash S), \delta_{\mathbb{Z}^2}) \cong \otimes_{p \in \mathcal{P}} (\ell^2(\Gamma\backslash S_p), \delta_{\mathbb{Z}^2}). \tag{3.9}$$

Finally, since $\mathrm{SL}_2(\hat{\mathbb{Z}})\backslash \mathrm{GL}_2^+(\mathbb{A}_f) = \Gamma\backslash \mathrm{GL}_2^+(\mathbb{Q})$ can be identified with the set of lattices commensurable with $\mathbb{Z}^2$, a function on $\mathrm{SL}_2(\hat{\mathbb{Z}})\backslash \mathrm{GL}_2^+(\mathbb{A}_f) \boxtimes_{\mathrm{SL}_2(\hat{\mathbb{Z}})} \mathrm{Mat}_2(\hat{\mathbb{Z}})$ can be thought of as a function f on the set of pairs (L, m), where L is a lattice commensurable with $\mathbb{Z}^2$ and $m \in \mathrm{Mat}_2(\hat{\mathbb{Z}})$ is a matrix whose columns belong to $\bar{L} \subset \mathbb{A}_f^2$, such that f is invariant under the action of $\mathrm{SL}_2(\hat{\mathbb{Z}})$ given by $\gamma(L, m) = (\gamma L, \gamma m)$. In this picture the action of $\mathrm{GL}_2(\hat{\mathbb{Z}})$ on $C_r^*(P, P_0)$ is given by $\alpha_r(f)(L, m) = f(r^{-1}L, r^{-1}m)$.

Proof of Theorem 3.2. Denote by π the representation of $C_r^*(P, P_0)$ on $\ell^2(\Gamma\backslash S)$ defined by $w = 1$ as described in Theorem 2.14. Using the identification of the reduced Hecke C*-algebra with the corner of the crossed product via (3.1), for $f \in C_c(\mathrm{SL}_2(\hat{\mathbb{Z}})\backslash \mathrm{GL}_2^+(\mathbb{A}_f) \boxtimes_{\mathrm{SL}_2(\hat{\mathbb{Z}})} \mathrm{Mat}_2(\hat{\mathbb{Z}}))$ and $s \in S$ we get

$$\pi(f)\delta_{\Gamma s} = \sum_{t \in \Gamma\backslash S} f(ts^{-1}, s)\delta_{\Gamma t}.$$

If we view f as a function on a pair (L, m) as explained in Remark 3.3, for $L = s^{-1}\mathbb{Z}^2$ we obtain

$$\pi(f)\delta_L = \sum_{L' \supset \mathbb{Z}^2} f(sL', s)\delta_{L'}. \tag{3.10}$$

For each prime p we have a similar representation π_p of $\mathfrak{A}_p$ on $\ell^2(\Gamma\backslash S_p)$. Let $f \in \mathfrak{A}_p$ and view it, by Lemma 3.1 and Remark 3.3, as an $\mathrm{SL}_2(\hat{\mathbb{Z}})$-invariant function on (L, m) such that $f(L, m)$ is nonzero only if L is defined by an element in G_p, and $f(L, m) = f(L, m')$ when $m_p = m'_p$. Then for $L'' = t^{-1}\mathbb{Z}^2$ with $t \in S_p$, we have an analogue of (3.10) for π_p:

$$\pi_p(f)\delta_{L''} = \sum_{\substack{L' \supset \mathbb{Z}^2: \\ L'/\mathbb{Z}^2 \text{ is a } p\text{-group}}} f(tL', t)\delta_{L'}. \tag{3.11}$$

We claim that for every $f \in \mathfrak{A}_p{}^{\mathbb{Z}_p^*}$ we have $\pi(f) = \pi_p(f) \otimes 1$ with respect to the decomposition of $\ell^2(\Gamma\backslash S)$ into the tensor product of $\ell^2(\Gamma\backslash S_p)$ and $\otimes_{q\ne p}(\ell^2(\Gamma\backslash S_q), \delta_{\mathbb{Z}^2})$ given by (3.9). Indeed, fix such f, and let $L = s^{-1}\mathbb{Z}^2$ for $s \in S$ be a lattice containing $\mathbb{Z}^2$. In the right hand side of (3.10), the value $f(sL', s)$ is nonzero only if sL' is defined by an element in S_p, that is, $(sL')_q = \mathbb{Z}^2$ for $q \ne p$, or equivalently, $L'_q = (s^{-1}\mathbb{Z}^2)_q = L_q$. Then the summation is over lattices L' for which we have a decomposition $\delta_{L'} = \delta_{L'_p} \otimes \bigotimes_{q\ne p} \delta_{L_q}$, and (3.10) becomes

$$\pi(f)\delta_L = \sum_{\substack{L'\supset\mathbb{Z}^2:\\ L'_q=L_q \text{ for } q\ne p}} f(sL', s)\delta_{L'} = \left(\sum_{\substack{L'\supset\mathbb{Z}^2:\\ L'_q=L_q \text{ for } q\ne p}} f(sL', s)\delta_{L'_p} \right) \otimes \bigotimes_{q\ne p} \delta_{L_q}.$$

Comparing the last sum with (3.11) (with $L'' = L_p$) and using that when L' runs through all lattices containing $\mathbb{Z}^2$ such that $L'_q = L_q$ for $q \ne p$ then L'_p runs through all lattices such that $L'_p/\mathbb{Z}^2$ is a p-group, we see that to prove the claim it suffices to check that

$$f(sL', s) = f(tL'_p, t) \ \text{ if } \ L = s^{-1}\mathbb{Z}^2, \ L_p = t^{-1}\mathbb{Z}^2 \text{ and } L'_q = L_q \text{ for } q \ne p. \tag{3.12}$$

Since f is $\mathrm{GL}_2(\hat{\mathbb{Z}})$-invariant and $f(sL', s) = f(sL', m)$ if $s = m_p$ in $\mathrm{Mat}_2(\mathbb{Z}_p)$, equality (3.12) will hold if we find an element $r \in \mathrm{GL}_2(\hat{\mathbb{Z}})$ such that $s = r_p t$ in $\mathrm{GL}_2(\mathbb{Q}_p)$ and $sL' = rtL'_p$.

We take $r_p = st^{-1}$ and $r_q = 1$ for $q \ne p$. Since the closures of L and L_p in $\mathbb{Q}_p^2$ coincide by definition, we have $s^{-1}\mathbb{Z}_p^2 = t^{-1}\mathbb{Z}_p^2$, so that $r_p = st^{-1} \in \mathrm{GL}_2(\mathbb{Z}_p)$. Since the closures of L' and L'_p in $\mathbb{Q}_p^2$ coincide, it is also clear that the closures of sL' and rtL'_p in $\mathbb{Q}_p^2$ coincide. On the other hand, for $q \ne p$ we have $(sL')_q = \mathbb{Z}^2 = (tL'_p)_q = (rtL'_p)_q$, so that the closures of sL' and rtL'_p in $\mathbb{Q}_q^2$ coincide. Hence $sL' = rtL'_p$, and the claim is proved.

Since the representation π is faithful, the claim implies that the algebras $\mathfrak{A}_p^{\mathbb{Z}_p^*}$ mutually commute for different primes, and the C*-subalgebra they generate is isomorphic to their tensor product. It remains to show that this subalgebra coincides with the whole fixed point algebra.

Towards this end we first prove that if $f_p \in \mathfrak{A}_p^{\mathbb{Z}_p^*}$ for p in a finite set F of prime numbers, then

$$(*_{p\in F} f_p)(L, m) = \begin{cases} \prod_{p\in F} f_p(L_p, m), & \text{if } L_q = \mathbb{Z}^2 \text{ for } q \notin F, \\ 0, & \text{otherwise.} \end{cases} \tag{3.13}$$

This can be deduced from the equality $\pi(f) = \pi_p(f) \otimes 1$ above, but we can give a direct argument as follows. To simplify notation we only consider the case of a two-point set. So assume $F = \{p_1, p_2\}$. For f_1 and f_2 as in (3.13) we have

$$(f_{p_1} * f_{p_2})(L, m) = \sum_{h\in\Gamma\backslash\mathrm{GL}_2^+(\mathbb{Q}):\, hm\in\mathrm{Mat}_2(\hat{\mathbb{Z}})} f_{p_1}(hL, hm) f_{p_2}(h^{-1}\mathbb{Z}^2, m).$$

The second factor in the right hand side can be nonzero only if $h \in G_{p_2}$. But for the first factor to be nonzero we need $(hL)_q = \mathbb{Z}^2$ for $q \ne p_1$. In particular, $L_q = \mathbb{Z}^2$ for $q \ne p_1, p_2$, and $L_{p_2} = h^{-1}\mathbb{Z}^2$, which shows that there is at most one nonzero summand. The element r defined by $r_{p_1} = h$ and $r_q = 1$ for $q \ne p_1$ lies in $\mathrm{GL}_2(\hat{\mathbb{Z}})$, and $hL = rL_{p_1}$ because both lattices have the same closure in $\mathbb{Q}_{p_1}^2$ by the choice of r_{p_1}, and the same closure $\mathbb{Z}_q^2$ in $\mathbb{Q}_q^2$ for $q \ne p_1$. This implies that $f_{p_1}(hL, hm) = f_{p_1}(rL_{p_1}, rm)$, and this is $f_{p_1}(L_{p_1}, m)$ by $\mathrm{GL}_2(\hat{\mathbb{Z}})$-invariance, proving (3.13).

Now let $L \supset \mathbb{Z}^2$ be a lattice, and $F \subset \mathcal{P}$ a finite subset such that $L_q = \mathbb{Z}^2$ for $q \notin F$. Let U be an open compact subset of $\mathrm{Mat}_2(\hat{\mathbb{Z}})$ of the form

$$\prod_{p\in F} U_p \times \prod_{q\notin F} \mathrm{Mat}_2(\mathbb{Z}_q)$$

such that the column vectors of any element $m\in U$ belong to $\bar L$. Let f be the characteristic function of the set $\mathrm{GL}_2(\hat{\mathbb{Z}})(L,U)=\{(rL,rm)\mid r\in \mathrm{GL}_2(\hat{\mathbb{Z}}),\ m\in U\}$. The linear span of such functions is dense in $C^*_r(P,P_0)^{\hat{\mathbb{Z}}^*}$. On the other hand, denoting by f_p the characteristic function of the set $\mathrm{GL}_2(\hat{\mathbb{Z}})(L_p,U_p\times\prod_{q\ne p}\mathrm{Mat}_2(\mathbb{Z}_q))$ we have $f_p\in\mathfrak{A}_p^{\mathbb{Z}_p^*}$, and $f=*_{p\in F}f_p$ by (3.13). This completes the proof of the theorem. □

Remark 3.4. For each $w\in\mathrm{GL}_2(\hat{\mathbb{Z}})$ consider the representation π_w of $C^*_r(P,P_0)$ on $\ell^2(\Gamma\backslash S)$ defined in Theorem 2.14. As opposed to the one dimensional case, the restrictions of these representations to $C^*_r(P,P_0)^{\hat{\mathbb{Z}}^*}$ still depend on w. Let us show that nevertheless, they are equivalent.

Similarly to (3.10), for $L=s^{-1}\mathbb{Z}^2$ and $f\in C_c(\mathrm{SL}_2(\hat{\mathbb{Z}})\backslash\mathrm{GL}_2^+(\mathbb{A}_f)\boxtimes_{\mathrm{SL}_2(\hat{\mathbb{Z}})}\mathrm{Mat}_2(\hat{\mathbb{Z}}))$ we have

$$\pi_w(f)\delta_L=\sum_{L'\supset\mathbb{Z}^2}f(sL',sw)\delta_{L'}. \tag{3.14}$$

Define a unitary U_w on $\ell^2(\Gamma\backslash S)$ by $U_w\delta_L=\delta_{wL}$, and let $s_w\in S$ be such that $w^{-1}L=s_w^{-1}\mathbb{Z}^2$. Then

$$U_w\pi(f)U_w^*\delta_L=U_w\pi(f)\delta_{w^{-1}L}=\sum_{L'\supset\mathbb{Z}^2}f(s_wL',s_w)\delta_{wL'}=\sum_{L'\supset\mathbb{Z}^2}f(s_ww^{-1}L',s_w)\delta_{L'}$$

Since $w^{-1}s^{-1}\mathbb{Z}^2=w^{-1}L=s_w^{-1}\mathbb{Z}^2$, we have $sw=rs_w$ for some $r\in\mathrm{GL}_2(\hat{\mathbb{Z}})$. Therefore for $f\in C^*_r(P,P_0)^{\hat{\mathbb{Z}}^*}$ we get

$$f(s_ww^{-1}L',s_w)=f(r^{-1}sL',r^{-1}sw)=f(sL',sw).$$

Thus $U_w\pi(f)U_w^*=\pi_w(f)$ for $f\in C^*_r(P,P_0)^{\hat{\mathbb{Z}}^*}$. Note that if $\det(w)=1$ then $r\in\mathrm{SL}_2(\hat{\mathbb{Z}})$, and hence $U_w\pi(\cdot)U_w=\pi_w$ as representations of $C^*_r(P,P_0)$.

3.3. **The structure of the fixed point algebra.** The fixed point algebra $C^*_r(P,P_0)^{\hat{\mathbb{Z}}^*}$ contains two important subalgebras: the diagonal subalgebra and the semigroup Hecke algebra. We start by describing the diagonal subalgebra. Due to the equality

$$p_0C_0(\mathrm{Mat}_2(\mathbb{A}_f))p_0=C(\mathrm{SL}_2(\hat{\mathbb{Z}})\backslash\mathrm{Mat}_2(\hat{\mathbb{Z}}))p_0,$$

the algebra $C(\mathrm{SL}_2(\hat{\mathbb{Z}})\backslash\mathrm{Mat}_2(\hat{\mathbb{Z}}))$ can be viewed as a subalgebra of $p_0(C_0(\mathrm{Mat}_2(\mathbb{A}_f)\rtimes\mathrm{GL}_2^+(\mathbb{A}_f))p_0$. With our equivalent picture of functions on $\mathrm{SL}_2(\hat{\mathbb{Z}})\backslash\mathrm{GL}_2^+(\mathbb{A}_f)\boxtimes_{\mathrm{SL}_2(\hat{\mathbb{Z}})}\mathrm{Mat}_2(\hat{\mathbb{Z}})$, we are looking at $(\mathrm{SL}_2(\hat{\mathbb{Z}})\times\mathrm{SL}_2(\hat{\mathbb{Z}}))$-invariant functions on $\mathrm{SL}_2(\hat{\mathbb{Z}})\times\mathrm{Mat}_2(\hat{\mathbb{Z}})$, which depend only on the second coordinate. Then $C(\mathrm{GL}_2(\hat{\mathbb{Z}})\backslash\mathrm{Mat}_2(\hat{\mathbb{Z}}))=C(\mathrm{SL}_2(\hat{\mathbb{Z}})\backslash\mathrm{Mat}_2(\hat{\mathbb{Z}}))^{\hat{\mathbb{Z}}^*}$ is a subalgebra of $C^*_r(P,P_0)^{\hat{\mathbb{Z}}^*}$. Likewise, $C(\mathrm{GL}_2(\mathbb{Z}_p)\backslash\mathrm{Mat}_2(\mathbb{Z}_p))$ is a subalgebra of $\mathfrak{A}_p^{\mathbb{Z}_p^*}$ for each prime p.

The algebra $C(\mathrm{GL}_2(\hat{\mathbb{Z}})\backslash\mathrm{Mat}_2(\hat{\mathbb{Z}}))$ can be described in terms of lattices as follows. For $L=s^{-1}\mathbb{Z}^2$ with $s\in S$, we denote by π_L the characteristic function of the set $\mathrm{Mat}_2(\hat{\mathbb{Z}})s$. Then π_L belongs to $C(\mathrm{GL}_2(\hat{\mathbb{Z}})\backslash\mathrm{Mat}_2(\hat{\mathbb{Z}}))$.

Proposition 3.5. *The algebra $C(\mathrm{GL}_2(\hat{\mathbb{Z}})\backslash\mathrm{Mat}_2(\hat{\mathbb{Z}}))$ is the universal C*-algebra generated by projections π_L, $L\supset\mathbb{Z}^2$, satisfying the relations $\pi_L\pi_{L'}=\pi_{L+L'}$.*

Proof. We first check that $\pi_L\pi_{L'}=\pi_{L+L'}$ in $C(\mathrm{GL}_2(\hat{\mathbb{Z}})\backslash\mathrm{Mat}_2(\hat{\mathbb{Z}}))$. Note that if $L=s^{-1}\mathbb{Z}^2$ then $m\in\mathrm{Mat}_2(\hat{\mathbb{Z}})s$ if and only if $ms^{-1}\in\mathrm{Mat}_2(\hat{\mathbb{Z}})$, or equivalently, $ms^{-1}\hat{\mathbb{Z}}^2\subset\hat{\mathbb{Z}}^2$, that is, $m\bar L\subset\hat{\mathbb{Z}}^2$. Since $\overline{L+L'}=\bar L+\bar L'$, the identity $\pi_L\pi_{L'}=\pi_{L+L'}$ follows.

Denote by B the universal C*-algebra generated by projections χ_L, $L\supset\mathbb{Z}^2$, satisfying the relations $\chi_L\chi_{L'}=\chi_{L+L'}$. Then we have a homomorphism $B\to C(\mathrm{GL}_2(\hat{\mathbb{Z}})\backslash\mathrm{Mat}_2(\hat{\mathbb{Z}}))$ which maps χ_L onto π_L. Since any finite set of the χ_L's generate a finite dimensional algebra, to prove injectivity it is enough to check that the map is injective on the linear span of the χ_L's. Assume $\sum_i\lambda_i\pi_{L_i}=0$. Choose i_0 such that L_i is not contained in L_{i_0} for $i\ne i_0$, and write $L_{i_0}=s^{-1}\mathbb{Z}^2$ for $s\in S$. Then $\pi_{L_{i_0}}(s)=1$

and $\pi_{L_i}(s) = 0$ for $i \neq i_0$, since $s\bar{L}_i$ is not contained in $\hat{\mathbb{Z}}^2$. Hence $\lambda_{i_0} = 0$. Thus we inductively get $\lambda_i = 0$ for all i.

To prove surjectivity we have to check that the functions π_L separate points of $\mathrm{GL}_2(\hat{\mathbb{Z}})\backslash \mathrm{Mat}_2(\hat{\mathbb{Z}})$. For this we claim that $m, n \in \mathrm{Mat}_2(\hat{\mathbb{Z}})$ belong to the same $\mathrm{GL}_2(\hat{\mathbb{Z}})$-orbit if and only if $m^t\hat{\mathbb{Z}}^2 = n^t\hat{\mathbb{Z}}^2$, where m^t and n^t denote the transposed matrices. Indeed, assume $m^t\hat{\mathbb{Z}}^2 = n^t\hat{\mathbb{Z}}^2$. Then for each p the $\mathbb{Z}_p$-module $m_p^t\mathbb{Z}_p^2$ is finitely generated and torsion-free, hence it is free. Therefore $\mathbb{Z}_p^2$ decomposes into the direct sum of $\mathbb{Z}_p^2 \cap \operatorname{Ker} m_p^t$ and a module V_p' such that $V_p' \cap \operatorname{Ker} m_p^t = 0$. Similarly find a module V_p'' for n. Since $m_p^t\mathbb{Z}_p^2 = n^t\mathbb{Z}_p^2$, the ranks of V_p' and V_p'' coincide. Define $r_p \in \mathrm{GL}_2(\mathbb{Z}_p)$ such that $r_p(\operatorname{Ker} n_p^t \cap \mathbb{Z}_p^2) = \operatorname{Ker} m_p^t \cap \mathbb{Z}_p^2$, $r_pV_p'' = V_p'$ and $m_p^t r_p = n_p^t$. Then $m^t r = n^t$, so that $n = r^t m$. The converse statement is clear.

Now if m and n belong to different $\mathrm{GL}_2(\hat{\mathbb{Z}})$-orbits, we may assume that $n^t\hat{\mathbb{Z}}^2$ is not contained in $m^t\hat{\mathbb{Z}}^2$. Then there exists a lattice in $\hat{\mathbb{Z}}^2$ which contains $m^t\hat{\mathbb{Z}}^2$ but does not contain $n^t\hat{\mathbb{Z}}^2$, and so the dual lattice $\bar{L}$ has the properties that $\hat{\mathbb{Z}}^2 \subset \bar{L}$, $m\bar{L} \subset \hat{\mathbb{Z}}^2$ but $n\bar{L}$ is not contained in $\hat{\mathbb{Z}}^2$. In other words, $\pi_L(m) = 1$ and $\pi_L(n) = 0$, and surjectivity is proved. □

We note that in the representation π considered in (3.10), the projections π_L act as follows:

$$\pi(\pi_L)\delta_{L'} = \begin{cases} \delta_{L'}, & \text{if } L \subset L', \\ 0, & \text{otherwise.} \end{cases}$$

To introduce the second important subalgebra of $C_r^*(P, P_0)^{\hat{\mathbb{Z}}^*}$, recall that by Proposition 1.6 and the remark following it, the map $[s]_\Gamma \mapsto \det(s)^{-1}[s]_{P_0}$ gives an embedding of the classical Hecke algebra $\mathcal{H}(S,\Gamma) = \mathcal{H}(\mathrm{Mat}_2^+(\mathbb{Z}), \mathrm{SL}_2(\mathbb{Z}))$ into $\mathcal{H}(P, P_0)$. Note that the right hand side of (3.6) with $f = \det(s)^{-1}[s]_{P_0}$ is the characteristic function of the set $\mathrm{SL}_2(\hat{\mathbb{Z}})s\mathrm{SL}_2(\hat{\mathbb{Z}}) \times \mathrm{Mat}_2(\hat{\mathbb{Z}})$. In terms of pairs (L, m), the element $[s]_\Gamma$ corresponds to the characteristic function of the set $\Gamma L_s \times \mathrm{Mat}_2(\hat{\mathbb{Z}})$, where $L_s = s^{-1}\mathbb{Z}^2$. Since s can be assumed to be diagonal, we see that $\mathcal{H}(S,\Gamma)$ is contained in $C_r^*(P, P_0)^{\hat{\mathbb{Z}}^*}$.

The algebra $\mathcal{H}(S,\Gamma)$ is the tensor product of its subalgebras $\mathcal{H}(S_p,\Gamma)$, see [7, Chapter IV]. Our Theorem 3.2 can be regarded as a generalization of this fact. Moreover, by e.g. [7], $\mathcal{H}(S_p,\Gamma)$ is generated by two elements

$$u_p = \left[\begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}\right] \quad \text{and} \quad v_p = \left[\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}\right].$$

Since $\Gamma \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \Gamma$ is the set of all matrices in $\mathrm{Mat}_2^+(\mathbb{Z})$ with determinant p, the element v_p is the characteristic function of the set of pairs (L, m) such that $|L/\mathbb{Z}^2| = p$. Thus for $s \in S$ we have $v_p(sL', sm) = 1$ if and only if $|L'/s^{-1}\mathbb{Z}^2| = p$. Therefore from (3.10) we get

$$\pi(v_p)\delta_L = \sum_{L' \supset L:\, |L'/L| = p} \delta_{L'}. \tag{3.15}$$

We also have

$$\pi(u_p)\delta_L = \delta_{p^{-1}L}. \tag{3.16}$$

These are classical formulas for Hecke operators.

Note that the algebras $\mathcal{H}(S_p,\Gamma)$ and $C(\mathrm{GL}_2(\mathbb{Z}_p)\backslash \mathrm{Mat}_2(\mathbb{Z}_p)) \subset C(\mathrm{GL}_2(\hat{\mathbb{Z}})\backslash \mathrm{Mat}_2(\hat{\mathbb{Z}}))$ are subalgebras of $\mathfrak{A}_p^{\mathbb{Z}_p^*} = C_r^*(\mathrm{Mat}_2(\mathbb{Z}[p^{-1}]) \rtimes G_p, \mathrm{Mat}_2(\mathbb{Z}) \rtimes \Gamma)^{\mathbb{Z}_p^*}$. Recall that equation (3.11) defines a representation π_p of $\mathfrak{A}_p$ on $\ell^2(\Gamma\backslash S_p)$. We can now establish the following.

Proposition 3.6. *For each prime p, the C^*-algebra generated by $\mathcal{H}(S_p,\Gamma)$ and $C(\mathrm{GL}_2(\mathbb{Z}_p)\backslash \mathrm{Mat}_2(\mathbb{Z}_p))$ in the representation π_p contains the algebra of compact operators on $\ell^2(\Gamma\backslash S_p)$. In particular, the restriction of π_p to $\mathfrak{A}_p^{\mathbb{Z}_p^*}$ is irreducible.*

Proof. First of all note that for the representation π_p both formulas (3.15)-(3.16) are valid. Given $L=s^{-1}\mathbb{Z}^2$ with $s\in S_p$, we denote by $e_L\in C(\mathrm{GL}_2(\mathbb{Z}_p)\backslash\operatorname{Mat}_2(\mathbb{Z}_p))$ the characteristic function of the set $\mathrm{GL}_2(\mathbb{Z}_p)s$, and view it as an element of $C(\mathrm{GL}_2(\hat{\mathbb{Z}})\backslash\operatorname{Mat}_2(\hat{\mathbb{Z}}))$. Then

$$e_L=\pi_L-\bigvee_{L'\supset L\colon |L'/L|=p}\pi_{L'}=\prod_{L'\supset L\colon |L'/L|=p}(\pi_L-\pi_{L'}).$$

Since $\pi_p(e_L)(\delta_{L''})$ can be nonzero only if $L\subset L''\subsetneqq L'$, and $|L'/L|=p$, it follows that $\pi_p(e_L)$ is the projection onto $\mathbb{C}\delta_L\subset\ell^2(\Gamma\backslash S_p)$.

Now suppose that $\det(s)=p^n$. Then by (3.15) the vector $\pi_p(v_p^n)\delta_{\mathbb{Z}^2}$ is a linear combination with nonzero coefficients of the vectors $\delta_{L'}$, where L' runs over all lattices such that $[L'\colon\mathbb{Z}^2]=p^n$. It follows that $\pi_p(e_Lv_p^n)\delta_{\mathbb{Z}^2}=\lambda\delta_L$ for a nonzero scalar λ. Then $\lambda^{-1}\pi_p(e_Lv_p^ne_{\mathbb{Z}^2})$ is a partial isometry with initial space $\mathbb{C}\delta_{\mathbb{Z}^2}$ and range $\mathbb{C}\delta_L$, and the proposition follows. □

It seems natural to refer to the C*-subalgebra of $\mathfrak{A}_p^{\mathbb{Z}_p^*}$ generated by $\mathcal{H}(S_p,\Gamma)$ as *the Toeplitz-Hecke algebra at prime p*. The representation π_p restricted to this algebra is no longer irreducible for instance because the operators in the image commute with the action of Γ on $\ell^2(\Gamma\backslash S_p)$ defined by $\gamma\delta_L=\delta_{\gamma L}$. In the rest of the section we shall show that nevertheless the image under π_p of the Toeplitz-Hecke algebra at prime p contains the projection onto $\mathbb{C}\delta_{\mathbb{Z}^2}$. We begin with a lemma.

Lemma 3.7. *Assume $L=s^{-1}\mathbb{Z}^2$ for $s\in S_p$ is such that $L\ne\mathbb{Z}^2$ and L does not contain $p^{-1}\mathbb{Z}^2$. Then*

(i) *there is a unique lattice $L'\supset L$ such that $[L'\colon L]=p$ and $p^{-1}\mathbb{Z}^2\subset L'$;*

(ii) *there is a unique lattice $L''\subset L$ such that $[L\colon L'']=p$ and $\mathbb{Z}^2\subset L''$.*

Proof. Since $L/\mathbb{Z}^2$ is a nontrivial p-group, it has elements of order p. In other words, $L\cap p^{-1}\mathbb{Z}^2$ strictly contains $\mathbb{Z}^2$ and is strictly contained in $p^{-1}\mathbb{Z}^2$. Hence $(L\cap p^{-1}\mathbb{Z}^2)/\mathbb{Z}^2$ is a group of order p. Then $L'=L+p^{-1}\mathbb{Z}^2$ has the properties $[L'\colon L]=[p^{-1}\mathbb{Z}^2\colon L\cap p^{-1}\mathbb{Z}^2]=p$ and $L'\supset p^{-1}\mathbb{Z}^2$. If there exists another lattice L'' with such properties then $L=L'\cap L''$ and hence $p^{-1}\mathbb{Z}^2\subset L$, which is a contradiction. Thus (i) is proved.

Since $L/\mathbb{Z}^2$ is a nontrivial p-group, it has a subgroup of index p. The preimage of such a subgroup in L is a lattice with the properties required in (ii). Moreover, we have a one-to-one correspondence between such lattices and subgroups of $L/\mathbb{Z}^2$ of index p. Since any double coset in S_p has a diagonal representative, $L/\mathbb{Z}^2$ has the form $\mathbb{Z}/p^a\mathbb{Z}\oplus\mathbb{Z}/p^b\mathbb{Z}$, $a\le b$. Then the condition that L does not contain $p^{-1}\mathbb{Z}^2$ means exactly that $a=0$, or equivalently, $L/\mathbb{Z}^2$ is a cyclic p-group (indeed, to say that L does not contain $p^{-1}\mathbb{Z}^2$ is the same as saying that the group of elements $x\in L/\mathbb{Z}^2$ such that $px=0$ contains strictly fewer than p^2 elements). But if $L/\mathbb{Z}^2$ is cyclic then it contains only one subgroup of index p. Thus (ii) is also proved. □

We can now prove our claim about the projection.

Proposition 3.8. *The image of $v_p^*v_p-v_pv_p^*-p(1-u_pu_p^*)$ under π_p is the orthogonal projection onto $\mathbb{C}\delta_{\mathbb{Z}^2}$ in $B(\ell^2(\Gamma\backslash S_p))$.*

Proof. Let $T:=v_p^*v_p-v_pv_p^*-p(1-u_pu_p^*)$. By virtue of (3.15) we have

$$\pi_p(v_p^*)\delta_L=\sum_{L''\subset L\colon\,\mathbb{Z}^2\subset L'',\,|L/L''|=p}\delta_{L''}.$$

Since any lattice L'' containing a lattice L as a subgroup of index p is contained in $p^{-1}L$, and any sublattice of $p^{-1}L$ of index p contains L, we get by faithfulness of π_p that $v_p^*u_p=v_p$. We can then write

$$v_p^*v_p-v_pv_p^*=v_p^*(1-u_pu_p^*)v_p.$$

To compute the action of $\pi_p(T)$ on $\delta_L \in \ell^2(\Gamma\backslash S_p)$, assume first that L contains $p^{-1}\mathbb{Z}^2$. Since the operator $1-\pi_p(u_pu_p^*)$ is the projection onto the space spanned by the $\delta_{L'}$'s such that L' does not contain $p^{-1}\mathbb{Z}^2$,

$$\pi_p((1-u_pu_p^*)v_p)\delta_L = 0 = \pi_p(1-u_pu_p^*)\delta_L.$$

Thus $\pi_p(T)\delta_L = 0$. Assume now that L does not contain $p^{-1}\mathbb{Z}^2$. Since $R_\Gamma\begin{pmatrix}1 & 0\\ 0 & p\end{pmatrix} = p+1$, see e.g. [7, Chapter IV], there exist exactly $p+1$ lattices containing L as a subgroup of index p, and by Lemma 3.7(i) only one of them contains $p^{-1}\mathbb{Z}^2$. Let $L_1,\dots,L_p$ denote the remaining p lattices. Then

$$\pi_p((1-u_pu_p^*)v_p)\delta_L = \sum_{i=1}^{p}\delta_{L_i}.$$

By Lemma 3.7(ii) applied to L_i for each i, the lattice L is the unique lattice with the properties that $[L_i\colon L]=p$ and $\mathbb{Z}^2\subset L$. Thus $\pi_p(v_p^*)\delta_{L_i} = \delta_L$, and hence

$$\pi_p(v_p^*(1-u_pu_p^*)v_p)\delta_L = p\delta_L = p\pi_p(1-u_pu_p^*)\delta_L,$$

giving $\pi_p(T)\delta_L = 0$. For $L=\mathbb{Z}^2$ the computation is similar, but now all $p+1$ lattices containing $\mathbb{Z}^2$ as a sublattice of index p do not contain $p^{-1}\mathbb{Z}^2$. Hence

$$\pi_p(v_p^*(1-u_pu_p^*)v_p)\delta_{\mathbb{Z}^2} = (p+1)\delta_{\mathbb{Z}^2},$$

and the result follows. □

4. KMS-states

In view of Proposition 1.4, the canonical dynamics, as defined in [1, Proposition 4] for a general Hecke algebra, is determined by $\sigma_t(f)(g,m) = \det(g)^{2it}f(g,m)$ on the finite part of the Connes-Marcolli system, but for simplicity we shall omit the factor 2 and consider instead the dynamics

$$\sigma_t(f)(g,m) = \det(g)^{it}f(g,m)$$

for every f in the dense subalgebra $C_c(\mathrm{SL}_2(\hat{\mathbb{Z}})\backslash \mathrm{GL}_2^+(\mathbb{A}_f)\boxtimes_{\mathrm{SL}_2(\hat{\mathbb{Z}})}\mathrm{Mat}_2(\hat{\mathbb{Z}}))$ of $C_r^*(P,P_0)$.

4.1. **KMS-states on corners of crossed products.** Before we turn our attention to the classification of KMS-states on $C_r^*(P,P_0)$, we first prove some general results on KMS-states on crossed products, which will be needed later, and may prove useful elsewhere.

Suppose that X is a locally compact space, X_0 a clopen subset of X, G a locally compact group acting on X, and G_0 a compact open subgroup of G. Assume $G_0X_0 = X_0$ and $\bigcup_{g\in G} gX_0 = X$. Let $p_{G_0} = \int_{G_0}\lambda_g\,dg$, and denote by p the projection $\mathbb{1}_{X_0}p_{G_0}$ in the multiplier algebra of $C_0(X)\rtimes_r G$.

For the purpose of this subsection we have to assume that the action of G or G_0 is free. Although this assumption is not satisfied in our main example, it will be satisfied for certain subsystems, see the proof of Lemma 4.5.

Lemma 4.1. *Suppose that the action of G_0 on X_0 is free. Then the projection p is full.*

Proof. It is well-known that the projection p_{G_0} is full in the multiplier algebra of $C_0(X_0)\rtimes_r G_0$ (this is essentially equivalent to the fact that $C_0(X_0)\rtimes_r G_0$ and $C_0(G_0\backslash X_0)$ are Morita equivalent), but for completeness we provide a short argument. The ideal generated by p_{G_0} is the closed linear span of functions of the form $(g,x)\mapsto f_1(x)f_2(gx)$, $f_i\in C_0(X_0)$. Since by the assumption on freeness such functions separate points, and thus span a dense subspace of $C_0(G_0\times X_0)$, it follows that p_{G_0} is full in $C_0(X_0)\rtimes_r G_0$.

Hence the ideal generated by p in $C_0(X)\rtimes_r G$ contains $C_0(X_0)\rtimes_r G_0$. Since $\bigcup_{g\in G} gX_0 = X$, the elements $\lambda_{g_1}a\lambda_{g_2}$ for $a\in C_0(X_0)\rtimes_r G_0$ and $g_i\in G$ span a dense subspace of $C_0(X)\rtimes_r G$. Thus p is full. □

Assume now that $N\colon G\to\mathbb{R}^*_+$ is a homomorphism whose kernel contains G_0. Define a dynamics σ on $C_0(X)\rtimes_r G$ by $\sigma_t(f)(g,x)=N(g)^{it}f(g,x)$. We want to classify σ-KMS$_\beta$-states on

$$A:=p(C_0(X)\rtimes_r G)p.$$

Recall, see e.g. [8], that a semifinite σ-invariant weight φ is called a σ-KMS$_\beta$-weight, where $\beta\in\mathbb{R}$, if

$$\varphi(aa^*)=\varphi(\sigma_{i\beta/2}(a)^*\sigma_{i\beta/2}(a))$$

for all σ-analytic elements $a\in A$. Since the projection p is full, any KMS$_\beta$-state on A extends uniquely to a KMS$_\beta$-weight on $C_0(X)\rtimes_r G$, see e.g. [13, Remark 3.3(i)].

On the other hand, if we let E be the C*-valued weight

$$E\colon C_0(X)\rtimes_r G\to C_0(X)$$

such that $E(f)=f(e,\cdot)$, then any Radon measure μ on X defines a semifinite weight $\mu_*\circ E$ on $C_0(X)\rtimes_r G$. This is a standard way of getting dual weights on von Neumann algebras [21], and can be justified in the C*-algebra setting using [9]. We are not going into details because there are at least two other ways of constructing this weight: either by restricting the dual weight on the von Neumann algebra crossed product to the C*-algebra crossed product, or by using [17].

Note that the equation $E_0(a)=E(a)p$ defines a conditional expectation $A\to C_0(G_0\backslash X_0)p\subset A$, and by identifying $C_0(G_0\backslash X_0)p$ with $C(G_0\backslash X_0)$ we get

$$(\mu_*\circ E)|_A=(\mu|_{X_0})_*\circ E_0. \tag{4.1}$$

Theorem 4.2. *Assume that the action of G on X is free, $G_0X_0=X_0$ and $\bigcup_{g\in G}gX_0=X$. Then for each $\beta\in\mathbb{R}$ there is a one-to-one correspondence between σ-KMS$_\beta$-states on $A=p(C_0(X)\rtimes_r G)p$ and Radon measures μ on X such that*

$$\mu(X_0)=1\ \text{ and }\ \mu(gY)=N(g)^{-\beta}\mu(Y)\ \text{for compact } Y\subset X \text{ and } g\in G. \tag{4.2}$$

Proof. A standard computation using the covariance relation in A shows that any measure satisfying the conditions in (4.2) defines a KMS$_\beta$-state. This is in fact valid without the assumption that G acts freely on X. We omit the details.

The nontrivial part is to show that any KMS-state is determined by a measure. Assume φ is a σ-KMS$_\beta$-state on A. Using [13, Remark 3.3(i)] we extend it to a σ-KMS$_\beta$-weight on $C_0(X)\rtimes_r G$, which we continue to denote by φ. Consider the subsets

$$\mathfrak{N}_\varphi=\{a\,|\,\varphi(a^*a)<\infty\},\quad \mathfrak{M}_\varphi=\operatorname{span}\mathfrak{N}_\varphi^*\mathfrak{N}_\varphi=\operatorname{span}\{a\ge0\,|\,\varphi(a)<\infty\}$$

of $C_0(X)\rtimes_r G$. To continue, recall from e.g. [8], [21] that φ extends uniquely to a linear functional on $\mathfrak{M}_\varphi$, and $\mathfrak{M}_\varphi$ is a bimodule over the algebra of σ-analytic elements. Since $A\subset\mathfrak{M}_\varphi$, and $f\in C_0(X)$ and λ_g are analytic, we conclude as in the proof of Lemma 4.1 that $\mathfrak{M}_\varphi\cap(C_0(X)\rtimes G_0)$ is dense in $C_0(X)\rtimes G_0$. Hence the restriction of φ to $C_0(X)\rtimes G_0$ is a semifinite weight. Because $G_0\subset\ker N$, this weight is in fact a trace on $C_0(X)\rtimes G_0$. Since p_{G_0} is full and $p_{G_0}(C_0(X)\rtimes G_0)p_{G_0}=C_0(G_0\backslash X)p_{G_0}$, this trace is uniquely determined by a weight on $C_0(G_0\backslash X)$, that is, by a G_0-invariant measure μ on X. But $(\mu_*\circ E)|_{C_0(X)\rtimes G_0}$ is also a semifinite trace on $C_0(X)\rtimes G_0$ which extends the weight $fp_{G_0}\mapsto\mu_*(f)$ on $C_0(G_0\backslash X)p_{G_0}$, and so we conclude that

$$\varphi=\mu_*\circ E\ \text{ on }\ C_0(X)\rtimes G_0.$$

In particular, $f\in\mathfrak{M}_\varphi$ for $f\in C_c(G_0\times X)$, and $\varphi(f)=\int_X f(e,\cdot)d\mu$. Using again that λ_g is σ-analytic for every $g\in G$, we conclude that $f\in\mathfrak{M}_\varphi$ for $f\in C_c(G\times X)$. If the support of f does not intersect $\{e\}\times X$ then, since the action is free, we can find functions $f_1,\dots,f_n\in C_c(X)$ such that if (g,x) is in the support of f for some $g\in G\setminus\{e\}$ and $x\in X$, then $\sum_{i=1}^n f_i(x)=1$ and $f_i(gx)=0$ for all i. It follows that $f_i*f=0$, so by using the KMS-condition we get

$$\varphi(f*f_i)=\varphi(f_i*f)=0 \text{ for } i=1,\dots,n.$$

Hence $\varphi(f) = 0$, and because G_0 is open in G, we conclude that

$$\varphi(f) = \int_X f(e, \cdot) d\mu \tag{4.3}$$

for $f \in C_c(G \times X)$. Now it is easy to check that μ satisfies the conditions in (4.2), for example $\mu(X_0) = 1$ because $\varphi(p) = 1$.

The equality (4.3) completely determines the state φ on A. Since μ satisfies the scaling condition in (4.2), we can also conclude that $\mu_* \circ E$ is a σ-KMS$_\beta$-weight on $C_0(X) \rtimes_r G$ which extends φ on A. Since the extension is unique, we must have $\varphi = \mu_* \circ E$ on $C_0(X) \rtimes_r G$. □

4.2. **Classification of KMS-states on $C_r^*(P, P_0)$.** We can not apply Theorem 4.2 directly to $X = \mathrm{Mat}_2(\mathbb{A}_f)$, $X_0 = \mathrm{Mat}_2(\hat{\mathbb{Z}})$, $G = \mathrm{GL}_2^+(\mathbb{A}_f)$, $G_0 = \mathrm{SL}_2(\hat{\mathbb{Z}})$ and $N(g) = \det(g)$, as we would like to, because the action is very far from being free. So we shall impose an additional assumption on KMS-states, and apply the theorem to systems corresponding to finite sets of primes.

Let us first recall that for every $\beta > 1$ there exists a canonical measure $\mu_{\beta,f}$ on $\mathrm{Mat}_2(\mathbb{A}_f)$ satisfying (4.2). As we already remarked in the proof of Theorem 4.2, any such measure defines a KMS$_\beta$-state, but in view of lack of freeness we can not be sure that in this way we get all KMS-states.

The construction of $\mu_{\beta,f}$ is as follows, see [12, Section 4] for details. For each prime number p consider the Haar measure on $\mathrm{GL}_2(\mathbb{Z}_p)$ normalized so that the total mass is $(1 - p^{-\beta})(1 - p^{-\beta+1})$. This measure extends to a unique measure $\mu_{\beta,p}$ on $\mathrm{GL}_2(\mathbb{Q}_p)$ such that

$$\mu_{\beta,p}(gZ) = |\det(g)|_p^\beta \mu_{\beta,p}(Z) \ \text{ for compact } \ Z \subset \mathrm{GL}_2(\mathbb{Q}_p) \ \text{ and } \ g \in \mathrm{GL}_2(\mathbb{Q}_p),$$

where $|a|_p$ denotes the p-adic valuation of a. The total mass of $\mathrm{Mat}_2^i(\mathbb{Z}_p) := \mathrm{Mat}_2(\mathbb{Z}_p) \cap \mathrm{GL}_2(\mathbb{Q}_p)$, which is the set of regular matrices, is one, and we can define a measure on $\mathrm{Mat}_2(\mathbb{A}_f)$ by setting $\mu_{\beta,f} = \prod_p \mu_{\beta,p}$. By construction this measure satisfies $\mu_{\beta,f}(\mathrm{Mat}_2(\hat{\mathbb{Z}})) = 1$ and

$$\mu_{\beta,f}(gZr) = \left(\prod_p |\det(g_p)|_p \right)^\beta \mu_{\beta,f}(Z)$$

for $Z \subset \mathrm{Mat}_2(\mathbb{A}_f)$, $g \in \mathrm{GL}_2(\mathbb{A}_f)$ and $r \in \mathrm{GL}_2(\hat{\mathbb{Z}})$. Note that it is not difficult to show that $\mu_{\beta,f}(Zg) = (\prod_p |\det(g_p)|_p)^\beta \mu_{\beta,f}(Z)$ for $g \in \mathrm{GL}_2(\mathbb{A}_f)$, but we will not need this. Note also that $\mu_{2,f}$ is a Haar measure on $\mathrm{Mat}_2(\mathbb{A}_f)$. Furthermore, by construction of $\mu_{\beta,f}$ the set

$$\mathrm{Mat}_2^i(\hat{\mathbb{Z}}) := \prod_{p \in \mathcal{P}} \mathrm{Mat}_2^i(\mathbb{Z}_p) \tag{4.4}$$

is a subset of $\mathrm{Mat}_2(\hat{\mathbb{Z}})$ of full measure. We denote by φ_β the KMS$_\beta$-state corresponding to $\mu_{\beta,f}$ for $\beta > 1$.

On the other hand, for $\beta > 2$ and every $w \in \mathrm{GL}_2(\hat{\mathbb{Z}})$ we can construct a σ-KMS$_\beta$-state as follows. Consider the representation π_w of $C_r^*(P, P_0)$ on $\ell^2(\Gamma \backslash S)$ introduced in Theorem 2.14. Define an unbounded positive selfadjoint operator H on $\ell^2(\Gamma \backslash S)$ by

$$H\delta_L = \log[L \colon \mathbb{Z}^2]\delta_L. \tag{4.5}$$

Then, see e.g. [3, Lemma 1.18], we have

$$\mathrm{Tr}(e^{-\beta H}) = \zeta(\beta)\zeta(\beta - 1), \tag{4.6}$$

where ζ is the Riemann ζ-function. The dynamics σ is implemented in the representation π_w by the one-parameter unitary group with generator H, that is,

$$\pi_w(\sigma_t(a)) = e^{itH} \pi_w(a) e^{-itH}.$$

Lemma 4.3. (*cf.* [3, Section 1.7]) *The formula*

$$\varphi_{\beta,w}(a)=\zeta(\beta)^{-1}\zeta(\beta-1)^{-1}\operatorname{Tr}(\pi_w(a)e^{-\beta H})$$

defines a σ-KMS$_\beta$-state that depends only on $\det(w)\in\hat{\mathbb{Z}}^*$.

Proof. Given $w,w'\in \mathrm{GL}_2(\hat{\mathbb{Z}})$ such that $\det(w)=\det(w')$, similarly to Remark 3.4 we conclude that the unitary operator U which maps δ_L onto $\delta_{w'w^{-1}L}$ has the property $U\pi_w(\cdot)U^*=\pi_{w'}$. Since U commutes with H, we conclude that $\varphi_{\beta,w}=\varphi_{\beta,w'}$. □

We shall classify the following class of KMS-states: given a KMS-state φ on $C^*_r(P,P_0)$, we restrict it to the subalgebra $C(\mathrm{SL}_2(\hat{\mathbb{Z}})\backslash\mathrm{Mat}_2(\hat{\mathbb{Z}}))$, obtaining an $\mathrm{SL}_2(\hat{\mathbb{Z}})$-invariant probability measure μ_φ on $\mathrm{Mat}_2(\hat{\mathbb{Z}})$. We say that φ is *regular at every prime* if the set $\mathrm{Mat}_2^i(\hat{\mathbb{Z}})$ from (4.4) is a subset of full measure for μ_φ. Equivalently, for each p in $\mathcal{P}$, the push-forward of μ_φ under the projection $\mathrm{Mat}_2(\hat{\mathbb{Z}})\to\mathrm{Mat}_2(\mathbb{Z}_p)$ gives a measure on $\mathrm{Mat}_2(\mathbb{Z}_p)$ for which the set of singular matrices $\mathrm{Mat}_2(\mathbb{Z}_p)\setminus\mathrm{Mat}_2^i(\mathbb{Z}_p)$ has measure zero. Apart from the fact that then φ is in some sense supported on regular matrices, another reason to call φ regular is the following result.

Lemma 4.4. *A σ-KMS$_\beta$-state φ on $C^*_r(P,P_0)$ is regular at every prime if and only if its restriction to $\mathfrak{A}_p^{\mathbb{Z}_p^*}=C^*_r(\mathrm{Mat}_2(\mathbb{Z}[p^{-1}])\rtimes G_p,\mathrm{Mat}_2(\mathbb{Z})\rtimes\Gamma)^{\mathbb{Z}_p^*}$ is normal with respect to the representation π_p defined by* (3.11) *for every p (in other words, since the representation is irreducible by Proposition 3.6, φ extends to a normal state on $B(\ell^2(\Gamma\backslash S_p))$).*

Proof. We have $\mathrm{Mat}_2^i(\mathbb{Z}_p)=\bigcup_{s\in S_p}\mathrm{GL}_2(\mathbb{Z}_p)s$, which can be established similarly to (2.6). Recall from the proof of Proposition 3.6 that for a prime p and a lattice $L=s^{-1}\mathbb{Z}^2$ with $s\in S_p$, we denoted by e_L the characteristic function of $\mathrm{GL}_2(\mathbb{Z}_p)s$. Then to say that φ is regular is the same as requiring that

$$\sum_{L\supset\mathbb{Z}^2\colon L/\mathbb{Z}^2 \text{ is a } p\text{-group}}\varphi(e_L)=1.$$

But since $\pi_p(e_L)$ is the projection onto $\mathbb{C}\delta_L$, this condition just means that the restriction of φ to the algebra of compact operators on $\ell^2(\Gamma\backslash S_p)$ is a state. It remains to recall that if we have a state on a C*-algebra $A\subset B(H)$ containing the algebra $K(H)$ of compact operators then this state extends to a normal state on $B(H)$ if and only if its restriction to $K(H)$ is a state. □

We next prove that a regular KMS-state is completely determined by μ_φ on $\mathrm{Mat}_2(\hat{\mathbb{Z}})$.

Lemma 4.5. *For all $\beta\in\mathbb{R}$ there is a one-to-one correspondence between σ-KMS$_\beta$-states on $C^*_r(P,P_0)$ which are regular at every prime, and measures μ on $\mathrm{Mat}_2(\mathbb{A}_f)$ such that $\mu(\mathrm{Mat}_2(\hat{\mathbb{Z}}))=\mu(\mathrm{Mat}_2^i(\hat{\mathbb{Z}}))=1$ and $\mu(gY)=\det(g)^{-\beta}\mu(Y)$ for compact $Y\subset\mathrm{Mat}_2(\mathbb{A}_f)$ and $g\in\mathrm{GL}_2^+(\mathbb{A}_f)$.*

Proof. Certainly a measure μ with the properties described in the lemma gives rise to a KMS-state which is regular.

Conversely, suppose that φ is a regular KMS-state. We then have a measure μ_φ on $\mathrm{Mat}_2(\hat{\mathbb{Z}})$ such that $\mu_\varphi(\mathrm{Mat}_2^i(\hat{\mathbb{Z}}))=1$, and to get the extra properties of μ_φ we shall use the push-forwards of μ_φ under the projection maps from $\mathrm{Mat}_2(\hat{\mathbb{Z}})$ onto the coordinates corresponding to finite subsets of primes, whereby we will be in a position to apply Theorem 4.2.

For each finite subset F of primes denote by $\mathbb{Q}_F$ the ring $\prod_{p\in F}\mathbb{Q}_p$, and by $\mathbb{Z}_F$ its subring $\prod_{p\in F}\mathbb{Z}_p$. Consider the subgroup G_F of $\mathrm{GL}_2^+(\mathbb{Q})$ generated by the subgroups G_p for $p\in F$. Denote by $\mathrm{GL}_2^+(\mathbb{Q}_F)$ the closure of G_F in $\mathrm{GL}_2(\mathbb{Q}_F)$. Similarly to Lemma 2.4 one can show that $\mathrm{GL}_2^+(\mathbb{Q}_F)$ equals $G_F\mathrm{SL}_2(\mathbb{Z}_F)$, and is the group of elements $g\in\mathrm{GL}_2(\mathbb{Q}_F)$ such that $\det(g)$ is an element of the multiplicative group generated by elements $p\in F$ considered as a subgroup of $\mathbb{Q}_F$. Then $\mathrm{GL}_2^+(\mathbb{Q}_F)$ acts on $\mathrm{Mat}_2(\mathbb{Q}_F)$ by multiplication on the left, and we define

$$\mathfrak{A}_F=p_F(C_0(\mathrm{Mat}_2(\mathbb{Q}_F))\rtimes_r\mathrm{GL}_2^+(\mathbb{Q}_F))p_F,$$

where p_F is the projection in the crossed product corresponding to the characteristic function of the compact open subset $\mathrm{SL}_2(\mathbb{Z}_F) \times \mathrm{Mat}_2(\mathbb{Z}_F)$ of $\mathrm{GL}_2^+(\mathbb{Q}_F) \times \mathrm{Mat}_2(\mathbb{Q}_F)$. We view $\mathfrak{A}_F$ as a subalgebra of $p_0(C_0(\mathrm{Mat}_2(\mathbb{A}_f)) \rtimes_r \mathrm{GL}_2^+(\mathbb{A}_f))p_0$ by identifying it with the closure of $(\mathrm{SL}_2(\hat{\mathbb{Z}}) \times \mathrm{SL}_2(\hat{\mathbb{Z}}))$-invariant functions f on

$$\{(g,m) \mid g \in \mathrm{GL}_2^+(\mathbb{A}_f),\ m \in \mathrm{Mat}_2(\hat{\mathbb{Z}}),\ gm \in \mathrm{Mat}_2(\hat{\mathbb{Z}})\}$$

such that $f(g,m) = 0$ if $g \notin G_F\mathrm{SL}_2(\hat{\mathbb{Z}})$, and $f(g,m) = f(g,m')$ if $m_p = m'_p$ for all $p \in F$. Alternatively, with $N_F := \prod_{p\in F} p$, one can show that $\mathfrak{A}_F$ is the reduced Hecke C*-algebra of the pair

$$\left(\begin{pmatrix}1 & \mathrm{Mat}_2(\mathbb{Z}[N_F^{-1}]) \\ 0 & \mathrm{GL}_2^+(\mathbb{Z}[N_F^{-1}])\end{pmatrix}, \begin{pmatrix}1 & \mathrm{Mat}_2(\mathbb{Z}) \\ 0 & \mathrm{SL}_2(\mathbb{Z})\end{pmatrix}\right).$$

The restriction of φ to $\mathfrak{A}_F$ is a KMS-state, which upon further restriction to the subalgebra $C(\mathrm{SL}_2(\mathbb{Z}_F)\backslash \mathrm{Mat}_2(\mathbb{Z}_F))$ defines an $\mathrm{SL}_2(\mathbb{Z}_F)$-invariant measure μ_F on $\mathrm{Mat}_2(\mathbb{Z}_F)$. Note that μ_F is the push-forward of μ_φ to $\mathrm{Mat}_2(\mathbb{Z}_F)$. Thus, since φ is regular, $\mu_F(\mathrm{Mat}_2(\mathbb{Z}_F)\setminus\prod_{p\in F}\mathrm{Mat}_2^i(\mathbb{Z}_p)) = 0$, and therefore the restriction of φ to the ideal

$$I_F = p_F(C_0(\mathrm{GL}_2(\mathbb{Q}_F)) \rtimes_r \mathrm{GL}_2^+(\mathbb{Q}_F))p_F$$

is still a state. In particular, $\varphi|_{\mathfrak{A}_F}$ is completely determined by $\varphi|_{I_F}$. Since $\mathrm{GL}_2^+(\mathbb{Q}_F)$ certainly acts freely on $\mathrm{GL}_2(\mathbb{Q}_F)$, Theorem 4.2 applied to the algebra I_F says that the measure μ_F uniquely extends to a measure on $\mathrm{Mat}_2(\mathbb{Q}_F)$, which we still denote by μ_F, such that $\mathrm{GL}_2(\mathbb{Q}_F)$ is a subset of $\mathrm{Mat}_2(\mathbb{Q}_F)$ of full measure, $\mu_F(\mathrm{Mat}_2(\mathbb{Z}_F)) = 1$, and

$$\mu_F(gY) = \det(g)^{-\beta}\mu_F(Y)$$

for $g \in \mathrm{GL}_2^+(\mathbb{Q}_F)$ and $Y \subset \mathrm{Mat}_2(\mathbb{Q}_F)$ compact.

If F' is another finite set of prime numbers containing F, then the push-forward of $\mu_{F'}$ under the projection map

$$\mathrm{Mat}_2(\mathbb{Q}_{F'}) \supset \mathrm{Mat}_2(\mathbb{Q}_F) \times \mathrm{Mat}_2(\mathbb{Z}_{F'\setminus F}) \to \mathrm{Mat}_2(\mathbb{Q}_F)$$

must coincide with μ_F. It follows that there exists a unique measure μ on $\mathrm{Mat}_2(\mathbb{A}_f)$ such that its push-forward under

$$\mathrm{Mat}_2(\mathbb{A}_f) \supset \mathrm{Mat}_2(\mathbb{Q}_F) \times \prod_{q\notin F}\mathrm{Mat}_2(\mathbb{Z}_q) \to \mathrm{Mat}_2(\mathbb{Q}_F)$$

coincides with μ_F. By the properties of the μ_F's we conclude that μ is $\mathrm{SL}_2(\hat{\mathbb{Z}})$-invariant, the scaling condition $\mu(gY) = \det(g)^{-\beta}\mu(Y)$ holds for compact $Y \subset \mathrm{Mat}_2(\mathbb{A}_f)$ and $g \in \mathrm{GL}_2^+(\mathbb{Q})$, and that $\mu(\mathrm{Mat}_2(\hat{\mathbb{Z}})) = \mu(\mathrm{Mat}_2^i(\hat{\mathbb{Z}})) = 1$.

Since $\bigcup_F \mathfrak{A}_F$ is dense in $C_r^*(P,P_0)$, the state φ is completely determined by μ: indeed, φ is obtained by composing $\mu|_{\mathrm{Mat}_2(\hat{\mathbb{Z}})}$ with the conditional expectation $E_0 : C_r^*(P,P_0) \to C(\mathrm{SL}_2(\hat{\mathbb{Z}})\backslash \mathrm{Mat}_2(\hat{\mathbb{Z}}))$ from (4.1), because this is true on each $\mathfrak{A}_F$, i.e. $\varphi|_{\mathfrak{A}_F}$ is the composition of μ_F with the similar conditional expectation onto $C(\mathrm{SL}_2(\mathbb{Z}_F)\backslash \mathrm{Mat}_2(\mathbb{Z}_F))$. Therefore the proof is complete. □

It is not difficult to show that for $\beta \neq 0,1$ the condition $\mu(\mathrm{Mat}_2^i(\hat{\mathbb{Z}})) = 1$ follows from the remaining ones. In fact, a stronger result is proved in [12, Corollary 3.6], where it is shown that a weaker condition of relative invariance under Hecke operators implies that the measure of singular matrices must be zero. On the other hand, for $\beta = 0,1$ one does have measures satisfying all conditions of Lemma 4.5 except $\mu(\mathrm{Mat}_2^i(\hat{\mathbb{Z}})) = 1$, see [12, Remark 4.7].

We can now formulate our main classification result.

Theorem 4.6. *For $\beta \in \mathbb{R}$ denote by K_β the simplex of σ-KMS$_\beta$-states, and by K'_β the subset of states which are regular at every prime. Then K'_β is a subsimplex of K_β, that is, it is closed and if a probability measure on K_β has barycenter in K'_β then the measure must be supported on K'_β. Furthermore, we have:*

(i) *for* $\beta\le 1$ *the set* K'_β *is empty;*

(ii) *for* $\beta\in(1,2]$ *the set* K'_β *consists of one point* φ_β *corresponding to the measure* $\mu_{\beta,f}$*;*

(iii) *for* $\beta>2$ *the simplex* K'_β *is isomorphic to the simplex of probability measures on* $\mathrm{SL}_2(\hat{\mathbb{Z}})\backslash\mathrm{GL}_2(\hat{\mathbb{Z}})$*; in particular, extremal points of* K'_β *correspond to* $\mathrm{SL}_2(\hat{\mathbb{Z}})$*-orbits in* $\mathrm{GL}_2(\hat{\mathbb{Z}})$*, and the state corresponding to* $\mathrm{SL}_2(\hat{\mathbb{Z}})w$ *is* $\varphi_{\beta,w}$*.*

Proof. For every prime p consider the operator H_p on $\ell^2(\Gamma\backslash S_p)$ defined exactly as H in (4.5). Then similarly to (4.6) we have

$$\mathrm{Tr}(e^{-\beta H_p})=\begin{cases}(1-p^{-\beta})^{-1}(1-p^{-\beta+1})^{-1}, & \text{if } \beta>1,\\ +\infty, & \text{otherwise.}\end{cases}\tag{4.7}$$

The operator H_p implements the dynamics σ on $\mathfrak{A}_p$ in the representation π_p, in the sense that $\pi_p(\sigma_t(a))=e^{itH_p}\pi_p(a)e^{-itH_p}$. It follows that if a σ-KMS$_\beta$-state on $\mathfrak{A}_p^{\mathbb{Z}_p^*}$ extends to a normal state on $B(\ell^2(\Gamma\backslash S_p))$, then this extension is a KMS$_\beta$-state for the dynamics $\mathrm{Ad}\,e^{itH_p}$. By virtue of (4.7) the latter dynamics admits no normal KMS-states for $\beta\le1$, while for $\beta>1$ there exists a unique normal KMS-state on $B(\ell^2(\Gamma\backslash S_p))$ given by

$$\psi_{\beta,p}=(1-p^{-\beta})(1-p^{-\beta+1})\,\mathrm{Tr}(\cdot\, e^{-\beta H_p}).$$

Thus Lemma 4.4 implies that for $\beta>1$, a σ-KMS$_\beta$-state φ is regular at every prime if and only if $\varphi=\psi_{\beta,p}\circ\pi_p$ on $\mathfrak{A}_p^{\mathbb{Z}_p^*}$ for every p, while for $\beta\le1$ there are no regular KMS-states. This proves (i) and shows that K'_β is closed. Furthermore, assuming $\beta>1$, every KMS-state on $C_r^*(P,P_0)$ defines a positive KMS-functional on the algebra $K(\ell^2(\Gamma\backslash S_p))$ of compact operators. Since the latter algebra has a unique KMS-state, this functional is $\lambda\psi_{\beta,p}$ for some $\lambda\le1$. Moreover, since $\psi_{\beta,p}$ is faithful, to check whether $\lambda=1$ it suffices to evaluate the functional on one positive nonzero operator. In other words, if we fix a positive nonzero element $a_p\in\mathfrak{A}_p^{\mathbb{Z}_p^*}$ such that $\pi_p(a_p)$ is compact, then $\varphi(a_p)\le\psi_{\beta,p}(\pi_p(a_p))$, and the equality holds for all p if and only if φ is regular at every prime. It is now clear that if the barycenter of a probability measure on the set of KMS-states is a regular state then almost every state is regular.

It remains to prove (ii) and (iii). In other words, we want to classify all measures μ satisfying the conditions in Lemma 4.5. Since similar results are proved in [12], we will here be somewhat brief.

Assume $1<\beta\le2$. Then by the proof of [12, Lemma 4.5], for every measure μ on $\mathrm{Mat}_2(\mathbb{A}_f)$ such that $\mu(\mathrm{Mat}_2(\hat{\mathbb{Z}}))=1$ and $\mu(gY)=\det(g)^{-\beta}\mu(Y)$ for $Y\subset\mathrm{Mat}_2(\mathbb{A}_f)$ and $g\in\mathrm{GL}_2^+(\mathbb{Q})$, the action of $\mathrm{GL}_2^+(\mathbb{Q})$ on $(\mathrm{Mat}_2(\mathbb{A}_f),\mu)$ is ergodic. Since the set of such measures is convex, this means that this set consists of at most one point. Hence $\mu_{\beta,f}$ is the unique such measure.

Assume now that $\beta>2$, and let μ be a measure on $\mathrm{Mat}_2(\mathbb{A}_f)$ satisfying the conditions of Lemma 4.5. For a finite set F of prime numbers denote

$$Y_F:=\mathrm{GL}_2(\mathbb{Z}_F)\times\prod_{q\notin F}\mathrm{Mat}_2(\mathbb{Z}_q),$$

and $S_F:=G_F\cap\mathrm{Mat}_2^+(\mathbb{Z})$. Then by regularity the set S_FY_F is a subset of $\mathrm{Mat}_2(\hat{\mathbb{Z}})$ of full measure. Therefore

$$1=\mu(S_FY_F)=\sum_{s\in S_F/\Gamma}\mu(sY_F)=\mu(Y_F)\sum_{s\in S_F/\Gamma}\det(s)^{-\beta}=\mu(Y_F)\prod_{p\in F}(1-p^{-\beta})^{-1}(1-p^{-\beta+1})^{-1},$$

where the last step follows e.g. from [12, Equation (3.3)]. Thus $\mu(Y_F)=\prod_{p\in F}(1-p^{-\beta})(1-p^{-\beta+1})$. By taking the intersection of all sets Y_F for different F we obtain

$$\mu(\mathrm{GL}_2(\hat{\mathbb{Z}}))=\prod_{p\in\mathcal{P}}(1-p^{-\beta})(1-p^{-\beta+1})=\zeta(\beta)^{-1}\zeta(\beta-1)^{-1}.$$

Then we have

$$\mu(S\mathrm{GL}_2(\hat{\mathbb{Z}})) = \mu(\mathrm{GL}_2(\hat{\mathbb{Z}})) \sum_{s\in S/\Gamma} \det(s)^{-\beta} = \mu(\mathrm{GL}_2(\hat{\mathbb{Z}}))\zeta(\beta)\zeta(\beta-1) = 1.$$

It follows that $S\mathrm{GL}_2(\hat{\mathbb{Z}})$ is a subset of $\mathrm{Mat}_2(\hat{\mathbb{Z}})$ of full measure, and in view of $\mathrm{Mat}_2(\mathbb{A}_f) = \mathrm{GL}_2^+(\mathbb{Q})\mathrm{Mat}_2(\hat{\mathbb{Z}})$, the subset $\mathrm{GL}_2^+(\mathbb{Q})\mathrm{GL}_2(\hat{\mathbb{Z}}) = \mathrm{GL}_2(\mathbb{A}_f)$ of $\mathrm{Mat}_2(\mathbb{A}_f)$ has full measure. The scaling condition $\mu(gY) = \det(g)^{-\beta}\mu(Y)$ completely determines μ on $\mathrm{GL}_2(\mathbb{A}_f)$ by its restriction to $\mathrm{GL}_2(\hat{\mathbb{Z}})$. Conversely, we know from [12, Lemma 2.4] that any $\mathrm{SL}_2(\hat{\mathbb{Z}})$-invariant measure on $\mathrm{GL}_2(\hat{\mathbb{Z}})$ extends uniquely to a measure on $\mathrm{GL}_2(\mathbb{A}_f)$ satisfying the scaling condition. To get a measure whose value on $\mathrm{Mat}_2(\hat{\mathbb{Z}}) \cap \mathrm{GL}_2(\mathbb{A}_f)$ is 1, as required in Lemma 4.5, we need the total mass of $\mathrm{GL}_2(\hat{\mathbb{Z}})$ to be $\zeta(\beta)^{-1}\zeta(\beta-1)^{-1}$. To summarize, the map

$$\mu \mapsto \zeta(\beta)\zeta(\beta-1)\mu|_{\mathrm{GL}_2(\hat{\mathbb{Z}})}$$

defines a bijection between measures on $\mathrm{Mat}_2(\mathbb{A}_f)$ satisfying the conditions in Lemma 4.5 and $\mathrm{SL}_2(\hat{\mathbb{Z}})$-invariant probability measures on $\mathrm{GL}_2(\hat{\mathbb{Z}})$. In particular, extremal measures μ correspond to measures supported on one $\mathrm{SL}_2(\hat{\mathbb{Z}})$-orbit.

Now fix $w \in \mathrm{GL}_2(\hat{\mathbb{Z}})$ and consider the state $\varphi_{\beta,w}$ defined in Lemma 4.3. Then

$$\varphi_{\beta,w}(f) = \zeta(\beta)^{-1}\zeta(\beta-1)^{-1} \sum_{s\in\Gamma\backslash S} \det(s)^{-\beta} f(sw)$$

for $f \in C(\mathrm{SL}_2(\hat{\mathbb{Z}})\backslash \mathrm{Mat}_2(\hat{\mathbb{Z}}))$. Notice that the sets $\mathrm{SL}_2(\hat{\mathbb{Z}})s$ are closed and disjoint for s lying in different right cosets of Γ in S, and their union is $\mathrm{Mat}_2(\hat{\mathbb{Z}}) \cap \mathrm{GL}_2^+(\mathbb{A}_f)$. We thus see that the $\mathrm{SL}_2(\hat{\mathbb{Z}})$-invariant measure $\mu_{\beta,w}$ on $\mathrm{Mat}_2(\hat{\mathbb{Z}})$ defined by $\varphi_{\beta,w}$ has the property

$$\mu_{\beta,w}(\mathrm{SL}_2(\hat{\mathbb{Z}})sw) = \zeta(\beta)^{-1}\zeta(\beta-1)^{-1}\det(s)^{-\beta} \text{ for } s \in S.$$

Since $\sum_{s\in\Gamma\backslash S} \zeta(\beta)^{-1}\zeta(\beta-1)^{-1}\det(s)^{-\beta} = 1$, we conclude that $(\mathrm{Mat}_2(\hat{\mathbb{Z}}) \cap \mathrm{GL}_2^+(\mathbb{A}_f))w$ is a subset of $\mathrm{Mat}_2(\hat{\mathbb{Z}})$ of full measure, so that $\varphi_{\beta,w}$ is regular at every prime. Moreover, the restriction of $\mu_{\beta,w}$ to $\mathrm{GL}_2(\hat{\mathbb{Z}})$ is supported on one orbit $\mathrm{SL}_2(\hat{\mathbb{Z}})w$. So indeed $\varphi_{\beta,w}$ is a regular extremal KMS-state corresponding to the orbit $\mathrm{SL}_2(\hat{\mathbb{Z}})w$, and the proof of the theorem is complete. □

Remark 4.7.

(i) As we observed in the proof of Theorem 4.6, for $\beta > 1$ and every prime p the state $\varphi_{\beta,p} = \psi_{\beta,p}\circ\pi_p$ is the unique σ-KMS$_\beta$-state on $\mathfrak{A}_p^{\mathbb{Z}_p^*} = C_r^*(\mathrm{Mat}_2(\mathbb{Z}[p^{-1}]) \rtimes G_p, \mathrm{Mat}_2(\mathbb{Z}) \rtimes \Gamma)^{\mathbb{Z}_p^*}$ which is normal with respect to π_p. Since $\varphi_{\beta,p}$ is a factor state, the state $\otimes_{p\in\mathcal{P}}\varphi_{\beta,p}$ on $\bigotimes_{p\in\mathcal{P}} \mathfrak{A}_p^{\mathbb{Z}_p^*}$ is the unique KMS$_\beta$-state whose restriction to the factor corresponding to a prime p coincides with $\varphi_{\beta,p}$. Thus by Theorem 3.2, $C_r^*(P,P_0)^{\hat{\mathbb{Z}}^*}$ has a unique σ-KMS$_\beta$-state regular at every prime. So at least for regular KMS-states the situation is similar to the one dimensional case: the group $\hat{\mathbb{Z}}^*$ acts on the algebra, the fixed point algebra is a tensor product of algebras corresponding to different primes, and for $\beta > 1$ this fixed point algebra has a unique regular σ-KMS$_\beta$-state, which is a product-state.

(ii) We claim that there are no σ-KMS$_\beta$-states on $C_r^*(P,P_0)$ for $\beta < 0$. Indeed, if φ is such a state then for the isometry $u_p = \left[\begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}\right] \in \mathcal{H}(S,\Gamma)$ we have

$$1 = \varphi(u_p^* u_p) = p^{2\beta}\varphi(u_p u_p^*) \le p^{2\beta},$$

which is a contradiction. On the other hand, for $\beta > 0$, $\beta \ne 1$, we conjecture that any σ-KMS$_\beta$-state on $C_r^*(P,P_0)$ is regular at every prime. Indeed, for the full Connes-Marcolli GL_2-system the analogous property for KMS-states holds by [12, Corollary 3.6]. To prove regularity for states on $C_r^*(P,P_0)$ it would be enough to show that $\mathfrak{A}_p^{\mathbb{Z}_p^*}$ has no σ-KMS$_\beta$-states for $\beta \in (0,1)$, and a unique

KMS_β-state for $\beta > 1$, namely, $\varphi_{\beta,p}$. As we remarked in the proof of Theorem 4.6, to check whether a KMS-state coincides with $\varphi_{\beta,p}$, it is enough to evaluate it at one nonzero positive element a_p such that $\pi_p(a_p)$ is compact. E.g. we can take

$$a_p = v_p^* v_p - v_p v_p^* - p(1 - u_p u_p^*),$$

which by Proposition 3.8 is the preimage of the rank one projection onto $\mathbb{C}\delta_{\mathbb{Z}^2} \subset \ell^2(\Gamma\backslash S_p)$. From this, one can conclude that $\varphi_{\beta,p}$ is characterized by the equality

$$\varphi_{\beta,p}(v_p^* v_p) = p + 1.$$

What makes the situation more difficult than the one dimensional case is that the Toeplitz-Hecke algebra at prime p, that is, the C*-algebra generated by $\pi_p(\mathcal{H}(S_p, \Gamma))$, does have other KMS-states, which can be obtained, for example, by representing the Toeplitz-Hecke algebra on $\ell^2(\Gamma\backslash S_p/\Gamma)$. So considerations in the Toeplitz-Hecke algebra alone are not enough to prove the conjecture, and a better understanding of the whole algebra $\mathfrak{A}_p^{\mathbb{Z}_p^*}$ is required.

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