GIBBS STATES FOR AF-ALGEBRAS

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Abstract

We consider a special class of C^*-systems containing asymptotically abelian binary shifts and shifts of Temperley-Lieb algebras. We study Gibbs states for these systems corresponding to potentials with finite range interaction, and obtain the same results as well-known Araki's results for a one-dimensional quantum lattice. In particular, it is proved that a Gibbs state in the infinite volume is a translation invariant KMS-state having the exponential uniform clustering property. Entropic properties of the Gibbs states are also discussed. This allows, in particular, to construct new examples of quantum K-systems.

I Introduction

In [1] Araki studied a one-dimensional infinite quantum spin lattice system with a finite range interaction. He showed that the Gibbs state ϕ of the system in the infinite volume is a limit of local Gibbs states, is a factor-state of the UHF-algebra A corresponding to the quantum spin lattice system, is invariant under time and lattice translation, satisfies the KMS boundary condition and has the exponential uniform clustering property.

Developing the transfer matrix technique in a fashion analogous to [2] Araki introduced an auxiliary operator $\mathcal{L}$ acting on A, and found an eigenstate ν for $\mathcal{L}$. It turned out that the Gibbs state ϕ is a perturbation of the state ν. To realize this approach Araki developed the Tomonaga-Shwinger-Dyson perturbation theory to the case of UHF-algebras (see [3],[4] for a more detailed exposition and applications).

It is a natural problem to extend Araki's theory to a more general class of AF-algebras. We would like to include in this class such important algebras as Temperley-Lieb algebras and asymptotically abelian binary shifts. So we will consider quadruples $(A, \{A_{[n,m]}\}, \tau, \gamma)$, where A is a unital C^*-algebra, $A_{[n,m]}$, $n, m \in \mathbb{Z}$, $n \le m$, is a finite-dimensional subalgebra of A such that $A_{[n,m]} \subset A_{[n',m']}$ for $n' \le n \le m \le m'$, $\cup_n A_{[-n,n]}$ is dense in A, τ is a faithful trace on A, and γ is a τ-preserving automorphism of A such that $\gamma(A_{[n,m]}) = A_{[n+1,m+1]}$. Moreover, we suppose that the trace τ has some multiplicativity property, and the local algebras satisfy Popa's commuting square condition (see Section II below). Nevertheless there are difficulties on this way. For example, we cannot define an analogue of Araki's operator $\mathcal{L}$ on the whole algebra A, since in general the algebra cannot be "cut" into two commuting pieces as in [1]. Fortunately, it is possible to define such an operator on the subalgebra $A_{[0,\infty)}$ as in Ruelle's paper [2]. Then basing on Araki's perturbation technique we can prove the existence of an eigenstate ν on $A_{[0,\infty)}$. The Gibbs state ϕ on A can be restored from this state ν. Again, as in [1], the Gibbs state is the limit of local Gibbs states, invariant under γ, satisfies the KMS boundary condition and has the exponential uniform clustering property. These results are proved in Section III.

The constructed Gibbs state is a σ_t-KMS state ($\beta = 1$), where σ_t is the time evolution. For a one-dimensional quantum lattice it is a unique σ_t-KMS state (see [5], [4]). The same result holds for binary shifts, or for the C^*-tensor product of a binary shift and a quantum spin lattice system. To prove this one can apply the same arguments as in [5], or [4] (see Remark III.13), but in general this problem apparently requires new arguments.

The present research has been undertaken to construct new examples of quantum Kolmogorov systems in the sense of Narnhofer and Thirring [6]. Indeed, let ϕ be a Gibbs state, π_ϕ the GNS-representation of A corresponding to ϕ. Then (M, ϕ, γ), where $M = \pi_\phi(A)''$, is a quantum K-system. We deduce this result in Section IV from the proved clustering property and from the sufficient condition for the K-property in our paper [7].

In Section IV we also compute the mean entropy of a Gibbs state for certain C^*-dynamical systems. In particular, for a binary shift and for a Temperley-Lieb algebra of index≤ 4. It is a very interesting problem to investigate when this mean entropy is equal to the dynamical entropy $h_\phi(\gamma)$. But this problem has not been solved even for a quantum spin lattice system [8] (see also Remark IV.7 below).

It is important to note that Hiai and Petz [9] considered quantum statistical thermodynamics in AF C^*-systems. They studied, in particular, such problems as the Gibbs conditions, the variational principle for states, their relations with the KMS-conditions, and presented a lot of interesting examples. But Araki's approach to Gibbs states was not considered.

II Notations and Examples

We consider quadruples $(A, \{A_{[n,m]}\}_{n\leq m}, \tau, \gamma)$, $n, m \in \mathbb{Z}$, where A is a unital C^*-algebra, $A_{[n,m]}$ is a finite-dimensional C^*-subalgebra of A such that $A_{[n',m']} \subset A_{[n'',m'']}$ for $n'' \leq n' \leq m' \leq m''$ and $\cup_n A_{[-n,n]}$ is dense in A, τ a faithful trace of A, γ a τ-preserving automorphism of A such that $\gamma(A_{[n,m]}) = A_{[n+1,m+1]}$. For any subset Λ of $\mathbb{Z}$, we denote by A_Λ the C^*-subalgebra of A generated by $A_{[k,n]}$, $[k,n] \subset \Lambda$, and write A_n instead of $A_{[n,n]}$.

Throughout the paper we suppose that the following conditions are satisfied.

Assumptions II.1

(i) There exists $n_0 \geq 0$ *such that* $A_{(-\infty,0]}$ *and* $A_{[n_0+1,\infty)}$ *commute.*

(ii) $\tau(xy) = \tau(x)\tau(y)$ *for* $x \in A_{(-\infty,0]}$, $y \in A_{[n_0+1,\infty)}$.

(iii) There exists a τ-preserving conditional expectation $\mathcal{E}\colon A_{[0,\infty)} \to A_{[1,\infty)}$, and $\mathcal{E}(A_{[0,n]}) = A_{[1,n]}$ for any $n \geq 1$. Equivalently,

$$\begin{array}{ccc} A_{[0,n]} & \subset & A_{[0,n+1]} \\ \cup & & \cup \\ A_{[1,n]} & \subset & A_{[1,n+1]} \end{array}$$

is a commuting square [10], [11] for any n.

Besides a one-dimensional quantum spin lattice system we have the following standard examples.

Example II.2 *Automorphism θ_λ [11].*
Let $\lambda \in \{(4\cos^2\frac{\pi}{n})^{-1}\}_{n>3} \cup (0, \frac{1}{4}]$, A the C^*-algebra generated by projections e_n, $n \in \mathbb{Z}$, satisfying $e_i e_j = e_j e_i$ for $|i-j| \geq 2$, and $e_i e_{i\pm1} e_i = \lambda e_i$, $A_{[n,m]} = C^*(e_n, e_{n+1}, \ldots, e_m)$, τ the λ-Markov trace on A, i. e. $\tau(we_n) = \lambda\tau(w)$ for any $w \in A_{(-\infty,n-1]}$, $\gamma(e_n) = e_{n+1}$.

Example II.3 *Canonical shift on the tower of relative commutants [12].*
Let M be a II_1-factor, $N \subset M$ its subfactor of finite index, $N \subset M \subset^{e_1} M_1 \subset^{e_2} M_2 \subset \ldots$ the Jones tower, $\ldots \subset^{e_{-2}} M_{-2} \subset^{e_{-1}} M_{-1} = N \subset^{e_0} M_0 = M$ a tunnel, $A_{[n,m]} = M'_{2n} \cap M_{2m}$, $A = \overline{\cup_n A_{[-n,n]}}$, τ the restriction of the trace on $\cup_n M_n$. For any $n \in \mathbb{N}$, the triple $M_{-n} \subset M_n \subset M_{3n}$ is a basic construction in a canonical way. In particular, a (not necessarily τ-preserving) antiautomorphism $\gamma_n(x) = J_n x^* J_n$ of $M'_{-n} \cap M_{3n}$ is defined, where J_n is the canonical conjugation on $L^2(M_n)$. Then $\gamma(x) = \gamma_{n+1}(\gamma_n(x))$ for $x \in M'_{-n} \cap M_{3n}$.

Example II.4 *Asymptotically abelian binary shift [13], [14].*
Let X be a non-empty finite subset of $\mathbb{N}$, $a(x)$ the characteristic function of X, A the C^*-algebra with unit generated by symmetries s_n, $n \in \mathbb{Z}$, satisfying $s_i s_j = (-1)^{a(|i-j|)} s_j s_i$, $A_{[n,m]} = C^*(s_n, s_{n+1}, \ldots, s_m)$, $\tau(w) = 0$ for any non-empty word w in s_n's, $\gamma(s_n) = s_{n+1}$.

It is worth to note that the C^*-tensor product of systems satisfying Assumptions II.1 also satisfies these assumptions.

By an interaction potential we mean a mapping $X \mapsto \Phi(X) \in A_X$ defined on finite subsets X of $\mathbb{Z}$ such that $\Phi(X) = \Phi(X)^*$, $\Phi(X+n) = \gamma^n(\Phi(X))$. We suppose that $\Phi(X) = 0$ if X is not within an interval of length $r > 0$. Following Araki, we denote by $U(\Lambda)$ the Hamiltonian for a finite interval Λ,

$$U(\Lambda) = \sum_{X \subset \Lambda} \Phi(X).$$

We write $U(a,b)$ instead of $U([a,b])$.

For a finite interval Λ, the local Gibbs state is

$$\phi_\Lambda(Q) = \frac{\tau(Qe^{-U(\Lambda)})}{\tau(e^{-U(\Lambda)})}.$$

III Existence and clustering properties of Gibbs states

Our main results in this section are as follows.

Theorem III.1
(i) The limit $P(\Phi) = \lim_{b-a\to\infty} \frac{1}{b-a} \log \tau(e^{-U(a,b)})$ *exists and is finite.*
(ii) For any $Q \in A$*, there exists the limit* $\phi(Q) = \lim_{\substack{a\to-\infty \\ b\to+\infty}} \phi_{[a,b]}(Q)$*.*
Moreover, there exist $C, q > 0$ *such that*

$$|\phi_{[-n+a,b+n]}(Q) - \phi(Q)| \leq Ce^{-qn}||Q|| \quad \textit{for } Q \in A_{[a,b]}, \; n \in \mathbb{N}.$$

Theorem III.2 *The state* ϕ *is uniformly exponentially clustering, i. e. there exist* $C, q > 0$ *such that*

$$|\phi(Q_1Q_2) - \phi(Q_1)\phi(Q_2)| \leq Ce^{-qn}||Q_1||\,||Q_2||$$

for $Q_1 \in A_{(-\infty,-n]}$*,* $Q_2 \in A_{[n,\infty)}$*,* $n \in \mathbb{N}$*.*

We shall call the state ϕ the Gibbs state corresponding to the interaction potential $\Phi(X)$.

The proofs of Theorems follow closely the work by Araki [1]. We introduce an auxiliary operator $\mathcal{L}$ and found an eigenstate ν for this operator. Then the Gibbs state can be restored from ν. The main difference of our work from Araki's one is that the operator $\mathcal{L}$ can not be defined on the whole algebra A, but only on $A_{[0,\infty)}$. So ν is a state on $A_{[0,\infty)}$, and the perturbation argument by Araki can't be applied. Our arguments are closer to Ruelle's work [2] (see also [15]).

Following Araki, for $Q \in A_{[0,\infty)}$, we define

$$\begin{aligned} ||Q||_l &= \inf_{Q_l \in A_{[0,l]}} ||Q - Q_l||, \\ |||Q|||_{N,x} &= ||Q|| + \sum_{n=N}^{\infty} ||Q||_n x^n \quad (x > 1, \; N \in \mathbb{N}), \\ A(x) &= \{Q \in A_{[0,\infty)} \mid |||Q|||_{1,x} < \infty\}. \end{aligned}$$

Then $A(x)$ is a Banach space with respect to the norm $|||\;|||_{1,x}$. Every $|||\;|||_{N,x}$ is an equivalent norm on $A(x)$. For $Q > 0$, we also define

$$\alpha_l(Q) = \inf_{Q_l \in A_{[0,l]},\, Q_l > 0} ||Q - Q_l||\,||Q_l^{-1}||,$$

$$\alpha(Q) = ||Q||\,||Q^{-1}|| = \frac{\text{l.u.b. } Spec\,Q}{\text{g.l.b } Spec\,Q}.$$

Lemma III.3 *For $Q > 0$, we have*

(i) $\alpha_l(Q) \leq \frac{\alpha(Q)-1}{\alpha(Q)+1} < 1$;

(ii) there exists $Q_l \in A_{[0,l]}$, $Q_l > 0$, *such that* $\alpha_l(Q) = ||Q - Q_l||\, ||Q_l^{-1}||$, *then*

$$(1-\alpha_l(Q))\alpha(Q) \leq ||Q||\, ||Q_l^{-1}|| \leq (1+\alpha_l(Q))\alpha(Q) \ ;$$

(iii) if $||Q||_l||Q^{-1}|| < 1$, *then* $\alpha_l(Q) \leq \frac{||Q||_l||Q^{-1}||}{1-||Q||_l||Q^{-1}||}$;

(iv) $||Q||_l\, ||Q^{-1}|| \leq \frac{\alpha_l(Q)}{1-\alpha_l(Q)}$.

Proof. Cf. [1, Lemma 3.9]. It is worth only to note that if $||Q||_l||Q^{-1}|| < 1$, then, for $Q_l = Q_l^* \in A_{[0,l]}$, $||Q - Q_l|| = ||Q||_l$, we have $Q_l > 0$. Indeed,

$$||\, ||Q||1 - Q_l|| \leq ||\, ||Q||1 - Q|| + ||Q||_l = ||Q|| - ||Q^{-1}||^{-1} + ||Q||_l < ||Q||.$$

□

Let us also define

$$\Phi = \sum_{I\subset[0,r]} \frac{\Phi(I)}{n(I)} \quad \text{and} \quad H(I) = \sum_{n\,:\,[n,n+r]\subset I} \gamma^n(\Phi),$$

where $n(I)$ is the number of translates $I+a$ of I that are still in $[0,r]$. $H(a,b) = H([a,b])$ and $U(a,b)$ differ only near the two ends. More precisely, there exist $\Delta^- \in A_{[0,r]}$ and $\Delta^+ \in A_{[-r,0]}$ such that, for $b-a > 2r$, we have

$$U(a,b) = H(a,b) + \gamma^a(\Delta^-) + \gamma^b(\Delta^+).$$

For any subset $I \subset \mathbb{Z}$, the sequence $\mathrm{Ad}\exp(itH(I\cap[-n,n]))$ pointwise converges to a strongly continuous one-parameter automorphism group σ_t^I on A, and every $Q \in \cup_n A_{[-n,n]}$ is a σ_t^I-analytic element (see [1], Theorem 4.2, or [4], Theorem 6.2.4). We write σ_t instead of $\sigma_t^{\mathbb{Z}}$. For $Q \in \cup_n A_{[-n,n]}$, $\lambda \in \mathbb{R}$, let us also consider the perturbed dynamics $(\sigma_t^I)^Q$ (see [4], Section 5.4.1). These dynamics are related via a σ_t^I-cocycle Γ_t^Q. Following Araki, an analytic continuation of this cocycle at the point $t = -i\lambda$ is denoted by $E(\lambda Q; \lambda H(I))$,

$$E(\lambda Q; \lambda H(I)) = 1 + \sum_{n=1}^{\infty} \int_0^1 ds_1 \int_0^{s_1} ds_2 \ldots \int_0^{s_{n-1}} ds_n\, \sigma^I_{-i\lambda s_n}(\lambda Q) \ldots \sigma^I_{-i\lambda s_1}(\lambda Q)$$

(see [1], Definition 5.1, or [4], p.149). If I is finite, then

$$E(\lambda Q; \lambda H(I)) = e^{\lambda Q + \lambda H(I)} e^{-\lambda H(I)}.$$

Lemma III.4 *There exist constants $q > 0$ and $C_n > 0$, $n \in \mathbb{N}$, such that*

$$||E(Q; \lambda H(I))|| \leq e^{C_n ||Q||} \qquad \text{and}$$

$$||E(Q; \lambda H(I)) - E(Q; \lambda H(I \cap [-N, n+N]))|| \leq C_n \frac{q^{\left[\frac{N}{r+n_0}\right]}}{\left(1 + \left[\frac{N}{r+n_0}\right]\right)!} e^{C_n ||Q||}$$

for $Q \in A_{[0,n]}$, $n \geq r$, $N > 0$, $|\lambda| \leq 1$, $I \subset \mathbb{Z}$, where n_0 is defined in Assumptions II.1.

Proof. Cf. [1], Theorem 4.2 and Lemma 5.2. We would only mention that if $[\gamma^{j_m}(\Phi),[\ldots,[\gamma^{j_1}(\Phi),Q]\ldots]]\neq\emptyset$ then $[j_k,j_k+r]$ has non-empty intersection with

$$[-n_0,n+n_0]\cup\bigcup_{1\le l<k}[j_l-n_0,j_l+r+n_0]$$

for $k=1,\ldots,m$. But $[j_k,j_k+r]\cap[a-n_0,b+n_0]\neq\emptyset$ iff $[j_k,j_k+r+n_0]\cap[a,b+n_0]\neq\emptyset$. So the estimates by Araki hold with the replacement of r by $r+n_0$. □

We define the transfer operator $\mathcal{L}\colon A_{[0,\infty)}\to A_{[0,\infty)}$ as follows: $\mathcal{L}(Q)=\gamma^{-1}\mathcal{E}(K^*QK)$, where $K=E(-\frac{1}{2}\Phi;-\frac{1}{2}H(1,\infty))$ and $\mathcal{E}\colon A_{[0,\infty)}\to A_{[1,\infty)}$ is the τ-preserving conditional expectation.

Lemma III.5

(i) $\mathcal{L}$ is a faithful completely positive mapping. Moreover, if $Q\geq 0$ is invertible, $\mathcal{L}(Q)$ is invertible too.

(ii) $\mathcal{L}^n(Q)=\gamma^{-n}\mathcal{E}_n(K_n^*QK_ne^{-H(0,n-n_0-1)})$, where $\mathcal{E}_n\colon A_{[0,\infty)}\to A_{[n,\infty)}$ is the τ-preserving conditional expectation,

$$K_n=\begin{cases}E\left(-\frac{1}{2}\sum_{j=0}^{n-1}\gamma^i(\Phi);-\frac{1}{2}H(n,\infty)\right), & n\le r+n_0\\ E\left(-\frac{1}{2}\gamma^{n-1}(\Psi);-\frac{1}{2}[H(0,n-n_0-1)+H(n,\infty)]\right), & n>r+n_0\end{cases};$$

$$\Psi=\sum_{j=1-r-n_0}^{0}\gamma^j(\Phi).$$

Proof. We follow the proof of Lemma 6.3 in [1].

(i) is trivial, since $\mathcal{E}$ is faithful and K is invertible ($K^{-1}=E(\frac{1}{2}\Phi;\frac{1}{2}H(1,\infty))^*$).

(ii) Define $K_n'=K\gamma(K)\ldots\gamma^{n-1}(K)$. Then $\mathcal{L}^n(Q)=\gamma^{-n}\mathcal{E}_n(K_n'^*QK_n')$. Indeed, for $n=1$ it is true by definition. By induction,

$$\begin{aligned}\mathcal{L}^n(Q)&=\gamma^{-n+1}\mathcal{E}_{n-1}(K_{n-1}'^*\mathcal{L}(Q)K_{n-1}')\\&=\gamma^{-n+1}\mathcal{E}_{n-1}\gamma^{-1}\mathcal{E}\left((K\gamma(K_{n-1}'))^*QK\gamma(K_{n-1}')\right)\\&=\gamma^{-n}\mathcal{E}_n(K_n'^*QK_n'),\end{aligned}$$

since $\mathcal{E}_n|_{A_{[1,\infty)}}=\gamma\mathcal{E}_{n-1}\gamma^{-1}$ and $\mathcal{E}_n\mathcal{E}=\mathcal{E}_n$.

Using the identity $E(Q_1;Q_2+R)E(Q_2;R)=E(Q_1+Q_2;R)$, we obtain

$$\begin{aligned}K_n'&=E\left(-\frac{1}{2}\sum_{j=0}^{n-1}\gamma^j(\Phi);-\frac{1}{2}H(n,\infty)\right)\\ (\text{for } n>r+n_0)\quad&=E\left(-\frac{1}{2}\sum_{j=n-r-n_0}^{n-1}\gamma^j(\Phi);-\frac{1}{2}\left[\sum_{j=0}^{n-r-n_0-1}\gamma^j(\Phi)+H(n,\infty)\right]\right)\end{aligned}$$

$$
\begin{aligned}
&\times E\left(-\frac{1}{2}\sum_{j=0}^{n-r-n_0-1}\gamma^j(\Phi);-\frac{1}{2}H(n,\infty)\right)\\
=\ & E\left(-\frac{1}{2}\gamma^{n-1}(\Psi);-\frac{1}{2}\left[H(0,n-n_0-1)+H(n,\infty)\right]\right)\\
&\times E\left(-\frac{1}{2}H(0,n-n_0-1);-\frac{1}{2}H(n,\infty)\right)\\
=\ & E\left(-\frac{1}{2}\gamma^{n-1}(\Psi);-\frac{1}{2}\left[H(0,n-n_0-1)+H(n,\infty)\right]\right)\\
&\times e^{-\frac{1}{2}H(0,n-n_0-1)},
\end{aligned}
$$

since $H(0,n-n_0-1)$ commute with $A_{[n,\infty)}$. It remains to note that $\mathcal{E}_n(xy)=\mathcal{E}_n(yx)$ for any $x\in A_{[0,n-n_0-1]}$ and $y\in A_{[0,\infty)}$ by Assumption II.1(i). □

Lemma III.6 *There exist constants $C,q>0$ such that, for $Q>0$,*

(i) $\alpha(\mathcal{L}^n(Q))\le C\alpha(Q)$;

(ii) $\alpha_l(\mathcal{L}^n(Q))\le C\left(\alpha(Q)\delta_{l-r}+\alpha_{n+l}(Q)\right)$ for $l\ge r$, where $\delta_{l-r}=\dfrac{q^{\left[\frac{l-r+1}{n_0+r}\right]}}{\left(1+\left[\frac{l-r+1}{n_0+r}\right]\right)!}$ (see Lemma III.4) ;

(iii) if $|||Q|||_{M,x}||Q^{-1}||\le a$, then $\alpha(\mathcal{L}^n(Q))\le C$ for any $n\ge N(a,M,x)$ for a constant $N(a,M,x)$ depending on $a>0$, $x>1$, $M\in\mathbb{N}$.

Proof. Cf. [1, Lemma 6.4]. We give a proof to demonstrate that all the Assumptions II.1 are applied.

(i) Let K_n be as in Lemma III.5. By Lemma III.4, there exist a constant C such that $||K_n||,||K_n^{-1}||\le C$.

Let $p_n=\tau(e^{-H(0,n-n_0-1)})$. Then by Assumption II.1(ii), $\mathcal{E}_n(e^{-H(0,n-n_0-1)})=p_n 1$, so that

$$
||(K_n^*QK_n)^{-1}||^{-1}\le\frac{1}{p_n}\mathcal{L}^n(Q)\le||K_n^*QK_n||,
$$

hence $\alpha(\mathcal{L}^n(Q))\le C^4\alpha(Q)$.

(ii) Let $Q_{n+l}\in A_{[0,n+l]}$, $Q_{n+l}>0$, $\alpha_{n+l}(Q)=||Q-Q_{n+l}||\,||Q_{n+l}^{-1}||$ (Lemma III.3(ii)),

$$
K_{(n,l)}=E\left(-\frac{1}{2}\gamma^{n-1}(\Psi);-\frac{1}{2}[H(0,n-n_0-1)+H(n,n+l)]\right),
$$

$Q'_l=\gamma^{-n}\mathcal{E}_n(K_{(n,l)}^*Q_{n+l}K_{(n,l)}e^{-H(0,n-n_0-1)})$. By Assumption II.1(iii), $Q'_l\in A_{[0,l]}$. Let us compute $||\mathcal{L}^n(Q)-Q'_l||\,||(Q'_l)^{-1}||$.

By Lemma III.4,

$$
||K_{(n,l)}||,\ ||K_{(n,l)}^{-1}||\le C\quad\text{and}\quad||K_n-K_{(n,l)}||\le C\delta_{l-r}.
$$

Hence

$$||K_n^*QK_n - K_{(n,l)}^*QK_{(n,l)}|| \le 2C^2\delta_{l-r}||Q|| \text{ and } ||K_{(n,l)}^*(Q-Q_{n+l})K_{(n,l)}|| \le C^2||Q-Q_{n+l}||.$$

Being a completely positive unital mapping, $A_{[0,\infty)} \ni x \mapsto \frac{1}{p_n}\mathcal{E}_n(xe^{-H(0,n-n_0-1)})$ has norm one, so that

$$||\mathcal{L}^n(Q) - Q_l'|| \le p_nC^2(2\delta_{l-r}||Q|| + ||Q-Q_{n+l}||). \tag{1}$$

On the other hand, $K_{(n,l)}^*Q_{n+l}K_{(n,l)} \ge ||(K_{(n,l)}^*Q_{n+l}K_{(n,l)})^{-1}||^{-1} \ge C^{-2}||Q_{n+l}^{-1}||^{-1}$. Hence

$$Q_l' \ge \frac{p_n}{C^2}||Q_{n+l}^{-1}||^{-1}. \tag{2}$$

Using (1) and (2) we obtain

$$\alpha_l(\mathcal{L}^n(Q)) \le ||\mathcal{L}^n(Q) - Q_l'||\,||(Q_l')^{-1}|| \le C^4(2\delta_{l-r}||Q||\,||Q_{n+l}^{-1}|| + \alpha_{n+l}(Q)).$$

Since $||Q||\,||Q_{n+l}^{-1}|| \le 2\alpha(Q)$ by Lemma III.3(i),(ii), the proof of (ii) is complete.

(iii) Let $Q_k \in A_{[0,k]}$, $Q_k > 0$, $||Q-Q_k||\,||Q_k^{-1}|| = \alpha_k(Q)$,

$$K_{n,l} = E\left(-\frac{1}{2}\gamma^{n-1}(\Psi); -\frac{1}{2}[H(n-l,n-n_0-1) + H(n,\infty)]\right),$$

$Q' = \gamma^{-n}\mathcal{E}_n(K_{n,l}^*Q_kK_{n,l}e^{-H(0,n-n_0-1)})$. Then as in the proof of (ii) we conclude that, for a constant C depending only on the potential, we have $||K_{n,l}||, ||K_{n,l}^{-1}|| \le C$ and

$$||\mathcal{L}^n(Q) - Q'||\,||(Q')^{-1}|| \le C(\alpha(Q)\delta_{l-r-n_0} + \alpha_k(Q)). \tag{3}$$

We denote the right hand part of (3) by Δ. Then

$$\mathcal{L}^n(Q) \le Q' + ||\mathcal{L}^n(Q) - Q'|| \le Q' + ||\mathcal{L}^n(Q) - Q'||\,||(Q')^{-1}||Q' \le (1+\Delta)Q',$$

$$\mathcal{L}^n(Q) \ge Q' - ||\mathcal{L}^n(Q) - Q'|| \ge Q' - ||\mathcal{L}^n(Q) - Q'||\,||(Q')^{-1}||Q' \ge (1-\Delta)Q',$$

so that

$$\alpha(\mathcal{L}^n(Q)) \le \frac{1+\Delta}{1-\Delta}\alpha(Q') \text{ whenever } \Delta < 1. \tag{4}$$

Since $|||Q|||_{M,x}||Q^{-1}|| \le a$, we have $\alpha(Q) \le a$ and $||Q||_k||Q^{-1}|| \le ax^{-k}$ for $k \ge M$, hence (Lemma III.3(iii)) $\alpha_k(Q) \le \frac{ax^{-k}}{1-ax^{-k}}$. We see that we can choose $l \ge r + n_0$ and k such that $\Delta \le \frac{1}{2}$ independently of Q. Let $N(a,M,x) = k + l + n_0 + 1$. Then, for $n \ge N$, $K_{n,l}$ and Q_k commute. If $p = \tau(Q_ke^{-H(0,n-n_0-1)})$, the mapping $y \mapsto \frac{1}{p}\mathcal{E}_n(Q_k^{\frac{1}{2}}yQ_k^{\frac{1}{2}}e^{-H(0,n-n_0-1)})$ is completely positive and unital, so that

$$\frac{1}{C^2} \le ||(K_{n,l}^*K_{n,l})^{-1}||^{-1} \le \frac{1}{p}\mathcal{E}_n(K_{n,l}^*K_{n,l}Q_ke^{-H(0,n-n_0-1)}) \le ||K_{n,l}^*K_{n,l}|| \le C^2,$$

hence $\alpha(Q') \le C^4$. By virtue of this inequality, the inequality (4) and the choice of N, we obtain $\alpha(\mathcal{L}^n(Q)) \le 3C^4$. □

The operator $\psi \mapsto \psi(\mathcal{L}(1))^{-1}\psi\mathcal{L}$ is defined on the state space of $A_{[0,\infty)}$. By the Schauder theorem, there exists a state ν of $A_{[0,\infty)}$ such that $\nu\mathcal{L} = \lambda\nu$ with $\lambda = \nu(\mathcal{L}(1)) > 0$.

Let $L = \lambda^{-1}\mathcal{L}$.

Corollary III.7 *The sequence $\{L^n\}_{n=1}^{\infty}$ is bounded.*

Proof. First let $Q > 0$ and $\alpha(Q) \leq 2$. By Lemma III.6(i), $\alpha(L^n(Q)) = \alpha(\mathcal{L}^n(Q)) \leq 2C$ for a constant C independent of Q. Then

$$||L^n(Q)|| = \alpha(L^n(Q))||L^n(Q)^{-1}||^{-1} \leq 2C\nu(L^n(Q)) = 2C\nu(Q) \leq 2C||Q||.$$

For arbitrary $Q \geq 0$ let $Q' = Q + ||Q||1$. Then $\alpha(Q') \leq 2$. Hence

$$||L^n(Q)|| \leq ||L^n(Q')|| + ||Q||\,||L^n(1)|| \leq 6C||Q||.$$

So for any $Q \in A_{[0,\infty)}$, we have $||L^n(Q)|| \leq 24C||Q||$. □

Proposition III.8 *There exists an element $h \in \bigcap_{x>1} A(x)$ such that $h > 0$, $\nu(h) = 1$, $Lh = h$. Then*

$$|||L^n(Q) - \nu(Q)h|||_{1,x} \leq C_x e^{-q_x n}|||Q|||_{1,x} \quad \textit{for any } Q \in A(x)$$

for constants $C_x, q_x > 0$ depending on x.

Proof. See Lemmas 6.5, 7.5, 7.6 in [1].

STEP 1. Existence of h.

Let C be chosen as in the formulation of Lemma III.6, and δ_l be as defined there. Let X be the norm-closure of the convex hull of $\{L^n(1)\}_{n=1}^{\infty}$. The set X is L-invariant. We prove that X is compact.

Let $Y = \{Q \in A_{[0,\infty)} \mid ||Q|| \leq C, ||Q||_l \leq \frac{C\delta_{l-r}}{1-C\delta_{l-r}}, l \geq r\}$. The set Y is compact, since the compact ball of radius C in $A_{[0,l]}$ is a $\frac{2C\delta_{l-r}}{1-C\delta_{l-r}}$-net for Y. X is a subset of Y. Indeed, it is enough to show that $L^n(1) \in Y$ for any n. We have

$$||L^n(1)|| = \alpha(L^n(1))||L^n(1)^{-1}||^{-1} \leq \alpha(L^n(1))\nu(L^n(1)) = \alpha(L^n(1)) \leq C \quad \text{and}$$

$$||L^n(1)||_l \leq \frac{\alpha_l(L^n(1))}{1-\alpha_l(L^n(1))}||L^n(1)^{-1}||^{-1} \leq \frac{\alpha_l(L^n(1))}{1-\alpha_l(L^n(1))} \leq \frac{C\delta_{l-r}}{1-C\delta_{l-r}}$$

by Lemma III.6(i),(ii) and Lemma III.3(iv).

By the Schauder theorem, there exists $h \in X$ such that $Lh = h$. We have $h \geq 0$, $\nu(h) = 1$, and $h \in \cap_{x>1}A(x)$, since $Y \subset \cap_{x>1}A(x)$. Since

$$L^n(1) \geq \frac{||L^n(1)||}{\alpha(L^n(1))} \geq \frac{\nu(L^n(1))}{C} = \frac{1}{C},$$

we have $h \geq C^{-1}$.

STEP 2. Continuity of $L|_{A(x)}$.

Consider $Q = Q^*$, $|||Q|||_{1,x} \leq 1$. Let $Q' = 2 + Q$. Then $1 \leq Q' \leq 3$ and $||Q'||_l = ||Q||_l$ for any l. By Lemma III.6(i) we obtain $\alpha(L(Q')) \leq 3C$, and as in Step 1 we have

$$||L(Q')||_l \leq \frac{\alpha_l(L(Q'))}{1-\alpha_l(L(Q'))}\nu(Q') \leq \frac{3}{2}(3C+1)\alpha_l(L(Q')), \tag{5}$$

where the latter inequality follows from Lemma III.3(i). By Lemma III.6(ii), for $l \geq r$,

$$\alpha_l(L(Q')) \leq C(3\delta_{l-r} + \alpha_{l+1}(Q')),$$

and by Lemma III.3(iii),

$$\alpha_{l+1}(Q') \leq \frac{||Q'||_{l+1}||(Q')^{-1}||}{1 - ||Q'||_{l+1}||(Q')^{-1}||} \leq \frac{||Q'||_{l+1}}{1 - ||Q'||_{l+1}} \leq \frac{||Q'||_{l+1}}{1 - x^{-1}},$$

so that

$$\alpha_l(L(Q')) \leq C\left(3\delta_{l-r} + \frac{||Q'||_{l+1}}{1 - x^{-1}}\right). \tag{6}$$

Using (5), (6) and $\sum_l ||Q'||_l x^l \leq 1$ we obtain

$$\begin{aligned} |||L(Q')|||_{r,x} &= ||L(Q')|| + \sum_{l=r}^{\infty} ||L(Q')||_l x^l \\ &\leq 3||L(1)|| + \frac{3}{2}(3C+1)\sum_{l=r}^{\infty} C\left(3\delta_{l-r} + \frac{||Q'||_{l+1}}{1-x^{-1}}\right) x^l \\ &\leq 3||L(1)|| + \frac{9C(3C+1)}{2}\sum_{l=r}^{\infty} \delta_{l-r} x^l + \frac{3C(3C+1)}{2(x-1)}. \end{aligned}$$

In particular, $|||L(1)|||_{r,x} < \infty$. Since $|||L(Q)|||_{r,x} \leq 2|||L(1)|||_{r,x} + |||L(Q')|||_{r,x}$, the continuity of $L|_{A(x)}$ is proved.

STEP 3. Convergence proof.

Let C be chosen as in the formulation of Lemma III.6. Then we take

$$a \geq 4C$$

and chose $M \in \mathbb{N}$ such that

$$2Cax^{-M} \leq \frac{1}{4},\ \ Ca\delta_{l-r} \leq \frac{1}{4} \text{ for } l \geq M,\ \ 4C\sum_{l=M}^{\infty} \delta_{l-r} x^l \leq \frac{1}{4}.$$

Finally, we fix $N \in \mathbb{N}$ such that

$$8Cx^{-N} \leq \frac{1}{4} \quad \text{and} \quad N \geq N(a, M, x),$$

where $N(a, M, x)$ is as in the formulation of Lemma III.6(iii).

We define an auxiliary operator Λ on $A(x)$,

$$\Lambda(Q) = L^N(Q) - \frac{\nu(Q)}{2C}.$$

We state that

$$\text{if } Q > 0,\ |||Q|||_{M,x}||Q^{-1}|| \leq a, \text{ then } \Lambda(Q) > 0,\ |||\Lambda(Q)|||_{M,x}||\Lambda(Q)^{-1}|| \leq a. \tag{7}$$

Suppose the statement is proved. Let $q_x = -\frac{1}{N}\log(1-\frac{1}{2C}) > 0$. Then $\nu(\Lambda(Q)) = (1-\frac{1}{2C})\nu(Q) = e^{-q_x N}\nu(Q)$. In particular, for $\nu(Q)=0$, we have $\Lambda^n(Q) = L^{Nn}(Q)$.

Consider $Q = Q^* \in A(x)$, $|||Q|||_{M,x} \le 1$, $\nu(Q) = 0$. Define $Q_1 = 2+Q$, $Q_2 = 2$. Then $1 \le Q_1 \le 3$, $|||Q_1|||_{M,x}||Q_1^{-1}|| \le |||Q_1|||_{M,x} \le 3 \le a$. By (7),

$$|||\Lambda^n(Q_1)|||_{M,x} \le a||\Lambda^n(Q_1)^{-1}||^{-1} \le a\nu(\Lambda^n(Q_1)) = ae^{-q_x Nn}\nu(Q_1) = 2ae^{-q_x Nn},$$

analogously $|||\Lambda^n(2)|||_{M,x} \le 2ae^{-q_x Nn}$, so that

$$|||L^{Nn}(Q)|||_{M,x} = |||\Lambda^n(Q)|||_{M,x} = |||\Lambda^n(Q_1) - \Lambda^n(Q_2)|||_{M,x} \le 4ae^{-q_x Nn}.$$

For arbitrary $k \in \mathbb{N}$ we obtain

$$|||L^k(Q)|||_{M,x} \le |||L|||_{M,x}^{k-\left[\frac{k}{N}\right]N} 4ae^{-q_x\left[\frac{k}{N}\right]N} \le 4ae^{q_x N}(1+|||L|||_{M,x})^N e^{-q_x k},$$

where $|||L|||_{M,x}$ is the norm of the operator L on the Banach space $(A(x), |||\ |||_{M,x})$ that is finite by Step 2. Let $\tilde{C}_x = 4ae^{q_x N}(1+|||L|||_{M,x})^N$. Then, for $Q = Q^* \in A(x)$, we have

$$\begin{aligned}|||L^k(Q) - \nu(Q)h|||_{M,x} &= |||L^k(Q - \nu(Q)h)|||_{M,x} \le \tilde{C}_x e^{-q_x k}|||Q - \nu(Q)h|||_{M,x} \\ &\le \tilde{C}_x(1+|||h|||_{M,x})e^{-q_x k}|||Q|||_{M,x},\end{aligned}$$

so that, for arbitrary $Q \in A(x)$,

$$|||L^k(Q) - \nu(Q)h|||_{M,x} \le 2\tilde{C}_x(1+|||h|||_{M,x})e^{-q_x k}|||Q|||_{M,x}.$$

It remains to prove (7). We have $\alpha(L^N(Q)) \le C$ by Lemma III.6(iii) and the choice of N. Hence

$$L^N(Q) \ge \frac{||L^N(Q)||}{\alpha(L^N(Q))} \ge \frac{||L^N(Q)||}{C} \ge \frac{\nu(Q)}{C}.$$

In particular, $\Lambda(Q) \ge \frac{\nu(Q)}{2C} > 0$ and $\frac{\nu(Q)}{C}||L^N(Q)^{-1}|| \le 1$. Further,

$$\begin{aligned}\alpha(\Lambda(Q)) &= \frac{||L^N(Q)|| - \frac{\nu(Q)}{2C}}{||L^N(Q)^{-1}||^{-1} - \frac{\nu(Q)}{2C}} = \frac{\alpha(L^N(Q)) - \frac{\nu(Q)}{2C}||L^N(Q)^{-1}||}{1 - \frac{\nu(Q)}{2C}||L^N(Q)^{-1}||} \\ &\le \frac{\alpha(L^N(Q))}{1-\frac{1}{2}} = 2\alpha(L^N(Q)) \le 2C \le \frac{a}{2} \qquad (8)\end{aligned}$$

and

$$\begin{aligned}||\Lambda(Q)||_l||\Lambda(Q)^{-1}|| &= ||L^N(Q)||_l||\Lambda(Q)^{-1}|| = ||L^N(Q)||_l\left(||L^N(Q)^{-1}||^{-1} - \frac{\nu(Q)}{2C}\right)^{-1} \\ &\le 2||L^N(Q)||_l||L^N(Q)^{-1}|| \le \frac{2\alpha_l(L^N(Q))}{1-\alpha_l(L^N(Q))}, \qquad (9)\end{aligned}$$

where we have used Lemma III.3(iv) in the latter inequality. By Lemma III.6(ii),

$$\alpha_l(L^N(Q)) \le C(\alpha(Q)\delta_{l-r} + \alpha_{l+N}(Q)). \tag{10}$$

Since $||Q||_{l+N}||Q^{-1}|| \le ax^{-l-N}$ and $\alpha_{l+N}(Q) \le \frac{||Q||_{l+N}||Q^{-1}||}{1-||Q||_{l+N}||Q^{-1}||}$ by Lemma III.3(iii), $C\alpha_{l+N}(Q) \le C\frac{ax^{-M}}{1-ax^{-M}} \le \frac{1}{4}$. We also have $C\alpha(Q)\delta_{l-r} \le Ca\delta_{l-r} \le \frac{1}{4}$ for $l \ge M$, so that $\alpha_l(L^N(Q)) \le \frac{1}{2}$ for any $l \ge M$. Together with (9) this gives

$$||\Lambda(Q)||_l||\Lambda(Q)^{-1}|| \le 4\alpha_l(L^N(Q)).$$

Now it follows from (10) again that

$$\begin{aligned} ||\Lambda(Q)||_l||\Lambda(Q)^{-1}|| &\le 4C(\alpha(Q)\delta_{l-r} + \alpha_{l+N}(Q)) \le 4Ca\delta_{l-r} + 4C\frac{||Q||_{l+N}||Q^{-1}||}{1-ax^{-M}} \\ &\le 4Ca\delta_{l-r} + 8C||Q||_{l+N}||Q^{-1}||. \end{aligned} \tag{11}$$

Finally, using (8), (11) and the choice of M and N, we obtain

$$\begin{aligned} |||\Lambda(Q)|||_{M,x}||\Lambda(Q)^{-1}|| &= \alpha(\Lambda(Q)) + \sum_{l=M}^{\infty} ||\Lambda(Q)||_l||\Lambda(Q)^{-1}||x^l \\ &\le \frac{a}{2} + 4Ca\sum_{l=M}^{\infty}\delta_{l-r}x^l + 8C\sum_{l=M}^{\infty}||Q||_{l+N}||Q^{-1}||x^l \\ &\le \frac{a}{2} + \frac{a}{4} + 8Cax^{-N} \le a. \end{aligned}$$

□

Corollary III.9 *For any $x > 1$, there exist $C_x, q_x > 0$ such that*

$$|||L^{N+n}(Q) - \nu(Q)h|||_{1,x} \le C_x e^{-q_x n}||Q|| \quad \forall Q \in A_{[0,N]} \quad \forall n, N.$$

Proof. It is enough to prove that there exists a constant C_x independent of N such that $|||L^N(Q)|||_{1,x} \le C_x||Q||\ \forall Q \in A_{[0,N]}$. As in the proof of Proposition III.8, Step 2, it suffices to consider $Q \in A_{[0,N]}$, $1 \le Q \le 3$. By the same arguments as given there, we have $\alpha(L^N(Q)) \le 3C$ and, for $l \ge r$,

$$||L^N(Q)||_l \le \frac{3C(3C+1)}{2}(3\delta_{l-r} + \alpha_{l+N}(Q)).$$

But $\alpha_{l+N}(Q) = 0$ for any $Q \in A_{[0,N]}$, $Q > 0$, and we obtain

$$\begin{aligned} |||L^N(Q)|||_{1,x} &\le rx^{r-1}||L^N(Q)|| + \sum_{l=r}^{\infty}||L^N(Q)||_l x^l \\ &\le 3rx^{r-1}||L^N|| + \frac{9C(3C+1)}{2}\sum_{l=r}^{\infty}\delta_{l-r}x^l. \end{aligned}$$

From Corollary III.7 we conclude that the latter expression is bounded by a constant $\tilde{C}_x$ independent of Q and N. □

Lemma III.10 *The sequence* $\{\nu\gamma^n\}_{n=1}^{\infty}$ *converges to a state* ϕ *on* $A_{[0,\infty)}$ *in norm. Moreover, there exist* $C, q > 0$ *such that* $||\phi - \nu\gamma^n|| \leq Ce^{-qn}$.

Proof. For n and N, $N + n_0 < n$, let

$$K_{n-N,n} = E\left(-\frac{1}{2}\gamma^{n-N-1}(\Psi); -\frac{1}{2}[H(0, n-N-n_0-1) + H(n-N, n-n_0-1)]\right).$$

According to Lemma III.4, Corollary III.7 and Proposition III.8, there exist $C, q > 0$ such that

$$||K_{n-N,n}K_{n-N}^{-1} - 1|| \leq Ce^{-qN},$$

$$||L^n|| \leq C,$$

$$||L^n(1) - h|| \leq Ce^{-qn},$$

where K_{n-N}'s are defined in Lemma III.5.

Since $\gamma^n(Q)$ and $K_{n-N,n}$ commute for any $Q \in A_{[0,\infty)}$, we have

$$L^{n-N}((K_{n-N,n}K_{n-N}^{-1})^*\gamma^n(Q)K_{n-N,n}K_{n-N}^{-1}) = \gamma^N(Q)L^{n-N}((K_{n-N,n}K_{n-N}^{-1})^*K_{n-N,n}K_{n-N}^{-1}).$$

So

$$||L^{n-N}(\gamma^n(Q)) - \gamma^N(Q)L^{n-N}(1)|| \leq C||\gamma^n(Q) - (K_{n-N,n}K_{n-N}^{-1})^*\gamma^n(Q)K_{n-N,n}K_{n-N}^{-1}||$$

$$+C||(K_{n-N,n}K_{n-N}^{-1})^*K_{n-N,n}K_{n-N}^{-1} - 1|| \, ||Q|| \leq C'e^{-qN}||Q||,$$

where $C' = 2C^2(2 + C)$ (note that $||K_{n-N,n}K_{n-N}^{-1}|| \leq 1 + C$). Then

$$||L^{n-N}(\gamma^n(Q)) - \gamma^N(Q)h|| \leq C'e^{-qN}||Q|| + ||L^{n-N}(1) - h|| \, ||Q|| \leq (C'e^{-qN} + Ce^{-q(n-N)})||Q||,$$

hence $|\nu(\gamma^n(Q)) - \nu(\gamma^N(Q)h)| \leq (C'e^{-qN} + Ce^{-q(n-N)})||Q||$, and

$$||\nu\gamma^n - \nu\gamma^m|| \leq 2C'e^{-qN} + Ce^{-q(n-N)} + Ce^{-q(m-N)}$$

for $n, m > N + n_0$. We see that the sequence $\{\nu\gamma^n\}_n$ converges. Denoting its limit by ϕ and letting $m \to \infty$ in the latter inequality we obtain

$$||\nu\gamma^n - \phi|| \leq 2C'e^{-qN} + Ce^{-q(n-N)}.$$

Taking $N = [\frac{1}{2}n]$ we obtain the desired. □

The state ϕ of $A_{[0,\infty)}$ is obviously γ-invariant. Hence there exists a unique γ-invariant state of A that, being restricted to $A_{[0,\infty)}$, coincides with ϕ. We denote this state by the same letter ϕ, and our aim is to prove that ϕ is the required Gibbs state.

First, we shall prove that ϕ satisfies the conclusion of Theorem III.2.

Proposition III.11 *There exist $C, q > 0$ such that*

$$|\nu(Q_1Q_2) - \nu(Q_1)\nu(Q_2)| \leq Ce^{-qn}||Q_1||\,||Q_2||$$

for $Q_1 \in A_{[0,N]}$ and $Q_2 \in A_{[N+n,\infty)}$ for any N, n.

Proof. Replacing Q_1 by $Q_1 - \nu(Q_1)1$ we can suppose that $\nu(Q_1) = 0$.

By the same arguments as in Lemma III.10, we can find C, q such that

$$||L^{N+m}(Q_1Q_2) - \gamma^{-N-m}(Q_2)L^{N+m}(Q_1)|| \leq Ce^{-q(n-m)}||Q_1||\,||Q_2||$$

for $m < n - n_0$. On the other hand, $||L^{N+m}(Q_1)|| \leq Ce^{-qm}||Q_1||$ by Corollary III.9. Hence

$$||L^{N+m}(Q_1Q_2)|| \leq C(e^{-qm} + e^{-q(n-m)})||Q_1||\,||Q_2||,$$

so that

$$|\nu(Q_1Q_2)| \leq C(e^{-qm} + e^{-q(n-m)})||Q_1||\,||Q_2||.$$

Taking $m = [\frac{1}{2}n]$ we obtain the required. □

Now we can prove that ϕ is uniformly exponentially clustering in the sense of Theorem III.2:

If C, q are chosen according to Proposition III.11 then, for $m > N > n$, $Q_1 \in A_{[-N,-n]}$, $Q_2 \in A_{[n,N]}$, we have

$$|\nu(\gamma^m(Q_1Q_2) - \nu(\gamma^m(Q_1))\nu(\gamma^m(Q_2))| \leq Ce^{-2qn}||Q_1||\,||Q_2||.$$

Letting $m \to \infty$ we obtain

$$|\phi(Q_1Q_2) - \phi(Q_1)\phi(Q_2)| \leq Ce^{-2qn}||Q_1||\,||Q_2||.$$

Lemma III.12 *Let $\bar{Z}_n = \tau(e^{-H(0,n)-\gamma^n(\Delta^+)})$ and $\bar{\phi}_n(Q) = \bar{Z}_n^{-1}\tau(Qe^{-H(0,n)-\gamma^n(\Delta^+)})$ for $Q \in A_{[0,\infty)}$. Then*

(i) $\lim_{n\to\infty} \frac{1}{n} \log \bar{Z}_n = \log \lambda$;

(ii) there exist $C, q > 0$ such that

$$|\nu(Q) - \bar{\phi}_{n+N}(Q)| \leq Ce^{-qn}||Q|| \quad \forall Q \in A_{[0,N]}\ \forall N, n.$$

Proof. See Lemma 8.1 in [1].

Let $Q \in A_{[0,N]}$. Proving (ii) we can suppose that $Q > 0$ and $\alpha(Q) \leq 2$ as in Corollary III.7.

For $n > m + r$, let

$$K_{N+m,N+n} =$$
$$= E\left(-\frac{1}{2}\gamma^{N+m-1}(\Psi); -\frac{1}{2}[H(0, N+m-n_0-1) + H(N+m, N+n) + \gamma^{N+n}(\Delta^+)]\right).$$

Then

$$e^{-\frac{1}{2}H(0,N+n)-\frac{1}{2}\gamma^{N+n}(\Delta^+)} = K_{N+m,N+n}e^{-\frac{1}{2}H(0,N+m-n_0-1)-\frac{1}{2}H(N+m,N+n)-\frac{1}{2}\gamma^{N+n}(\Delta^+)},$$

so that

$$\begin{aligned}\bar{\phi}_{N+n}(Q) &= \bar{Z}_{N+n}^{-1}\tau\Big(K_{N+m,N+n}^{*}QK_{N+m,N+n}e^{-H(0,N+m-n_0-1)-H(N+m,N+n)-\gamma^{N+n}(\Delta^+)}\Big)\\ &= \bar{Z}_{N+n}^{-1}\tau\Big(\gamma^{-N-m}\mathcal{E}_{N+m}(K_{N+m,N+n}^{*}QK_{N+m,N+n}e^{-H(0,N+m-n_0-1)}) \times\\ &\qquad \times e^{-H(0,n-m)-\gamma^{n-m}(\Delta^+)}\Big)\\ &= \bar{Z}_{N+n}^{-1}\bar{Z}_{n-m}\bar{\phi}_{n-m}\Big(\gamma^{-N-m}\mathcal{E}_{N+m}(K_{N+m,N+n}^{*}QK_{N+m,N+n}e^{-H(0,N+m-n_0-1)})\Big)\end{aligned}$$

By Corollary III.9 and a version of Lemma III.4, there exist $C, q > 0$ such that

$$||K_{N+m,N+n}||, ||K_{N+m}||, ||K_{N+m}^{-1}|| \le C,$$

$$||L^{N+m}(Q) - \nu(Q)h|| \le Ce^{-qm}||Q||,$$

$$||K_{N+m} - K_{N+m,N+n}|| \le Ce^{-q(n-m)}.$$

Let $p_{N+m} = \tau(e^{-H(0,N+m-n_0-1)})$. Then $x \mapsto p_{N+m}^{-1}\mathcal{E}_{N+m}(xe^{-H(0,N+m-n_0-1)})$, being a unital completely positive mapping, has norm one. From this observation, the choice of C, and Lemma III.5, we infer

$$||\gamma^{-N-m}\mathcal{E}_{N+m}(K_{N+m,N+n}^{*}QK_{N+m,N+n}e^{-H(0,N+m-n_0-1)}) - \mathcal{L}^{N+m}(Q)||$$

$$\le p_{N+m}||K_{N+m,N+n}^{*}QK_{N+m,N+n} - K_{N+m}^{*}QK_{N+m}|| \le p_{N+m}2C^2e^{-q(n-m)}||Q||$$

and $\mathcal{L}^{N+m}(Q) \ge p_{N+m}||K_{N+m}^{-1}||^{-2}\,||Q^{-1}||^{-1}$, so that $||\mathcal{L}^{N+m}(Q)^{-1}|| \le p_{N+m}^{-1}C^2||Q^{-1}||$.

Hence

$$||\lambda^{-N-m}\gamma^{-N-m}\mathcal{E}_{N+m}(K_{N+m,N+n}^{*}QK_{N+m,N+n}e^{-H(0,N+m-n_0-1)}) - L^{N+m}(Q)||\,||L^{N+m}(Q)^{-1}||$$

$$\le 2C^4e^{-q(n-m)}\alpha(Q) \le 4C^4e^{-q(n-m)}.$$

Applying $\bar{\phi}_{n-m}$ we obtain

$$|\lambda^{-N-m}\bar{Z}_{N+n}\bar{Z}_{n-m}^{-1}\bar{\phi}_{N+n}(Q) - \bar{\phi}_{n-m}(L^{N+m}(Q))| \le 4C^4e^{-q(n-m)}||L^{N+m}(Q)^{-1}||^{-1}$$

$$\le 4C^4e^{-q(n-m)}\nu(L^{N+m}(Q)) \le 4C^4e^{-q(n-m)}||Q||.$$

Hence

$$|\lambda^{-N-m}\bar{Z}_{N+n}\bar{Z}_{n-m}^{-1}\bar{\phi}_{N+n}(Q) - \nu(Q)\bar{\phi}_{n-m}(h)| \le (4C^4e^{-q(n-m)} + Ce^{-qm})||Q||. \tag{12}$$

In particular,

$$|\lambda^{-N-m}\bar{Z}_{N+n}\bar{Z}_{n-m}^{-1} - \bar{\phi}_{n-m}(h)| \le 4C^4e^{-q(n-m)} + Ce^{-qm}. \tag{13}$$

Having fixed $n-m$ sufficiently large we see from $||h^{-1}||^{-1} \leq \bar{\phi}_{n-m}(h) \leq ||h||$ and the latter inequality that the sequence $\{-(N+n)\log\lambda + \log \bar{Z}_{N+n}\}_n$ is bounded. In particular, $\lim_{n\to\infty} n^{-1}\log \bar{Z}_n = \log\lambda$. So we have proved (i).

Multiplying (13) by $\bar{\phi}_{N+n}(Q)$ we also deduce from (12) and (13) that

$$|\bar{\phi}_{N+n}(Q) - \nu(Q)| \leq 2(4C^4 e^{-q(n-m)} + Ce^{-qm})||Q||\,||h^{-1}||.$$

Taking $m = [\frac{1}{2}n]$, we obtain (ii). □

Proof of Theorem III.1. Let $Q \in A_{[a,b]}$.

$$\begin{aligned}\phi_{[-n+a,b+n]}(Q) &= \frac{\tau\big(Qe^{-U(-n+a,b+n)}\big)}{\tau\big(e^{-U(-n+a,b+n)}\big)}\\ &= \frac{\tau\big(\gamma^{n-a}(Q)e^{-U(0,b-a+2n)}\big)}{\tau\big(e^{-U(0,b-a+2n)}\big)}\\ &= \frac{\bar{\phi}_{b-a+2n}(\Delta^*_{b-a+2n}\gamma^{n-a}(Q)\Delta_{b-a+2n})}{\bar{\phi}_{b-a+2n}(\Delta^*_{b-a+2n}\Delta_{b-a+2n})},\end{aligned}$$

where $\bar{\phi}_{b-a+2n}$ is introduced in Lemma III.12,

$$\Delta_{b-a+2n} = E\left(-\frac{1}{2}\Delta^-; -\frac{1}{2}[H(0,b-a+2n) + \gamma^{b-a+2n}(\Delta^+)]\right).$$

Let C, q be chosen as in Lemma III.10, Proposition III.11 and Lemma III.12. By a version of Lemma III.4, we can also suppose that

$$||\Delta_m||, ||\Delta_m^{-1}|| \leq C \quad\text{and}\quad ||\Delta_m - \Delta_k|| \leq Ce^{-q\min\{m,k\}}.$$

We have the following inequalities ($m < n - n_0$):

$$|\bar{\phi}_{b-a+2n}(\Delta^*_{b-a+2n}\gamma^{n-a}(Q)\Delta_{b-a+2n}) - \bar{\phi}_{b-a+2n}(\gamma^{n-a}(Q)\Delta^*_m\Delta_m)| \leq 2C^2e^{-qm}||Q||, \tag{14}$$

since $\gamma^{n-a}(Q)$ and Δ_m commute.

$$|\bar{\phi}_{b-a+2n}(\gamma^{n-a}(Q)\Delta^*_m\Delta_m) - \nu(\gamma^{n-a}(Q)\Delta^*_m\Delta_m)| \leq C^3e^{-qn}||Q|| \tag{15}$$

by Lemma III.12.

$$|\nu(\gamma^{n-a}(Q)\Delta^*_m\Delta_m) - \nu(\gamma^{n-a}(Q))\nu(\Delta^*_m\Delta_m)| \leq C^3e^{-q(n-m)}||Q|| \tag{16}$$

by Proposition III.11.

$$|\nu(\gamma^{n-a}(Q))\nu(\Delta^*_m\Delta_m) - \phi(Q)\nu(\Delta^*_m\Delta_m)| \leq C^3e^{-qn}||Q|| \tag{17}$$

by Lemma III.10.

$$|\phi(Q)\nu(\Delta^*_m\Delta_m) - \phi(Q)\bar{\phi}_{b-a+2n}(\Delta^*_m\Delta_m)| \leq C^3e^{-q(b-a+2n-m)}||Q|| \leq C^3e^{-qn}||Q|| \tag{18}$$

by Lemma III.12. Finally,

$$|\phi(Q)\bar{\phi}_{b-a+2n}(\Delta_m^*\Delta_m) - \phi(Q)\bar{\phi}_{b-a+2n}(\Delta_{b-a+2n}^*\Delta_{b-a+2n})| \le 2C^2 e^{-qm}||Q||, \tag{19}$$

by the property of Δ_m's. Summing the inequalities (14)-(19) we obtain

$$|\bar{\phi}_{b-a+2n}(\Delta_{b-a+2n}^*\gamma^{n-a}(Q)\Delta_{b-a+2n}) - \phi(Q)\bar{\phi}_{b-a+2n}(\Delta_{b-a+2n}^*\Delta_{b-a+2n})|$$
$$\le \tilde{C}(e^{-qn} + e^{-qm} + e^{-q(n-m)})$$

for a constant $\tilde{C}$ depending only on the potential.

Since

$$|\bar{\phi}_{b-a+2n}(\Delta_{b-a+2n}^*\Delta_{b-a+2n})| \ge ||(\Delta_{b-a+2n}^*\Delta_{b-a+2n})^{-1}||^{-1} \ge \frac{1}{C^2},$$

we have

$$|\phi_{[-n+a,b+n]}(Q) - \phi(Q)| \le \tilde{C}C^2(e^{-qn} + e^{-qm} + e^{-q(n-m)}).$$

Taking $m = [\frac{1}{2}n]$ we obtain (ii).

$$\tau(e^{-U(a,b)}) = \bar{Z}_{b-a}\bar{\phi}_{b-a}(\Delta_{b-a}^*\Delta_{b-a}).$$

Since $\frac{1}{C^2} \le \bar{\phi}_{b-a}(\Delta_{b-a}^*\Delta_{b-a}) \le C^2$ and $\frac{1}{n}\log\bar{Z}_n \to \log\lambda$, we have (i). □

Remark III.13 The Gibbs state ϕ is a σ_t-KMS-state ($\beta = 1$). It is known that for a one-dimensional quantum lattice this is a unique σ_t-KMS-state (see [5], or [4]). The same result holds for binary shifts (Example II.4). Indeed, by [16] and [17], there exist $p, k \in \mathbb{N}$ such that $A_{[0,k+pn]}$ is a full matrix algebra for any $n \in \mathbb{N}$. Then one can apply [5].

IV Entropic properties of Gibbs states

In the sequel ϕ denotes a Gibbs state on $(A, \{A_{[n,m]}\}_{n\le m}, \tau, \gamma)$ corresponding to an interaction $\Phi(X)$.

Theorem IV.1 *Let $M = \pi_\phi(A)''$. Then (M, ϕ, γ) is an entropic K-system.*

Proof. Let M_0 be the W^*-subalgebra of M generated by $\pi_\phi(A_{(-\infty,0]})$. Using Theorem III.2 one concludes that $\cap_n\gamma^n(M_0) = \mathbb{C}1$. Hence (M, ϕ, γ) is an entropic K-system by [7, Theorem 3.1]. □

Corollary IV.2 *If $\pi_\phi(A) \ne \mathbb{C}1$, then $h_\phi(\gamma) > 0$.*

□

Corollary IV.3 *For all the examples of Section II we have* $0 < h_\phi(\gamma) < \infty$*.*

Proof. Let d_n be the tracial dimension of $A_{[1,n]}$, i. e. the dimension of a maximal abelian subalgebra in $A_{[1,n]}$. Then

$$h_\phi(\gamma) \leq \liminf_{n\to\infty} \frac{S(\phi|_{A_{[1,n]}})}{n} \leq \sup_n \frac{\log d_n}{n}.$$

So to prove the finiteness it suffices to show that $\sup_n \frac{1}{n} \log d_n < \infty$.

It is obvious for binary shifts (Example II.4), since $\dim A_{[1,n]} = 2^n$. For the automorphism θ_λ (Example II.2) and for the canonical shift on the tower of relative commutants (Example II.3) we note that the tracial dimension of $M' \cap M_n$ does not exceed $[M_n : M] = [M : N]^n$ [18].

To prove that the entropy is non-zero we have to show that each of the algebras under consideration has no ideals of codimension one. The algebra of a binary shift is simple as well as the algebra of relative commutants corresponding to a finite depth inclusion. In general, an algebra of higher relative commutants can be non-simple. Nevertheless it always contains a Temperley-Lieb algebra that has no ideals of codimension one. □

Next we shall discuss mean entropy. First we need a form of the Gibbs condition (called the Gibbs condition in the strong sense in [9]).

Proposition IV.4 *For a finite interval* $\Lambda = [k, l] \subset \mathbb{Z}$*, let* $W(\Lambda)$ *denotes the surface energy, i. e.*

$$W(\Lambda) = \sum_{X:\ \substack{X\cap\Lambda\neq\emptyset \\ X\cap\Lambda^c\neq\emptyset}} \Phi(X) + \sum_{\substack{X\subset\Lambda^c: \\ X\cap([k-n_0,k-1]\cup[l+1,l+n_0])\neq\emptyset}} \Phi(X).$$

Then for the perturbed state $\phi^{W(\Lambda)}$ *(see, for example, [4], Section 5.4.1) we have*

$$\phi^{W(\Lambda)}(ab) = \phi_\Lambda(a)\phi^{W(\Lambda)}(b) \quad \forall a \in A_\Lambda\ \forall b \in A_{(-\infty,k-n_0)\cup(l+n_0,\infty)}.$$

Proof. Let $K = E(\frac{1}{2}W(\Lambda); -\frac{1}{2}U(\mathbb{Z}))$,

$$K_n = E\left(\frac{1}{2}W(\Lambda); -\frac{1}{2}U(-n,n)\right) = e^{\frac{1}{2}W(\Lambda)-\frac{1}{2}U(-n,n)}e^{\frac{1}{2}U(-n,n)}.$$

Then K and K_n are invertible, K_n converges to K in norm, and

$$\phi^{W(\Lambda)} = \frac{\phi(K^* \cdot K)}{\phi(K^*K)},\ \phi^{W(\Lambda)}_{[-n,n]} = \frac{\phi_{[-n,n]}(K_n^* \cdot K_n)}{\phi_{[-n,n]}(K_n^*K_n)}.$$

Since $\phi_{[-n,n]}$ pointwise converges to ϕ, we see that $\phi^{W(\Lambda)}_{[-n,n]}$ pointwise converges to $\phi^{W(\Lambda)}$. For n sufficiently large, we have

$$\phi^{W(\Lambda)}_{[-n,n]}(ab) \quad = \quad \frac{\tau(abe^{-U(-n,n)+W(\Lambda)})}{\tau(e^{-U(-n,n)+W(\Lambda)})}$$

$$
\begin{aligned}
&= \frac{\tau(abe^{-U(\Lambda)}e^{-U([-n,k-n_0-1]\cup[l+n_0+1,n])})}{\tau(e^{-U(-n,n)+W(\Lambda)})} \\
&= \frac{\tau(ae^{-U(\Lambda)})\tau(be^{-U([-n,k-n_0-1]\cup[l+n_0+1,n])})}{\tau(e^{-U(-n,n)+W(\Lambda)})} \\
&= \frac{\tau(ae^{-U(\Lambda)})}{\tau(e^{-U(\Lambda)})}\cdot\frac{\tau(be^{-U(\Lambda)}e^{-U([-n,k-n_0-1]\cup[l+n_0+1,n])})}{\tau(e^{-U(-n,n)+W(\Lambda)})} \\
&= \phi_\Lambda(a)\phi^{W(\Lambda)}_{[-n,n]}(b).
\end{aligned}
$$

□

With the help of the Gibbs condition the mean entropy is computed in a standard way (see [4],[9]). Namely, we have the following.

Corollary IV.5 *Let P_n be the density operator for $\tau|_{A_{[1,n]}}$. Then*

$$\lim_{n\to\infty}\frac{1}{n}(\phi(\log P_n)+S(\phi|_{A_{[1,n]}}))=\phi(\Phi)+P(\Phi)=\phi(E_\Phi)+P(\Phi),$$

where $E_\Phi=\sum_{X\ni 0}\frac{\Phi(X)}{|X|}$. In particular, the mean entropy $s(\phi)=\lim_n\frac{1}{n}S(\phi|_{A_{[1,n]}})$ exists iff the limit $\lim_n\frac{1}{n}\phi(\log P_n)$ exists.

Proof. Let Q_n be the density operator for $\phi|_{A_{[1,n]}}$. The operator $\frac{e^{-U(1,n)}}{\tau(e^{-U(1,n)})}P_n$ is the density operator for $\phi_{[1,n]}|_{A_{[1,n]}}=\phi^{W(1,n)}|_{A_{[1,n]}}$. Then by [4], p. 278,

$$0\le \mathrm{Tr}_{A_{[1,n]}}\left(Q_n\left(\log Q_n-\log\Big(\frac{e^{-U(1,n)}}{\tau(e^{-U(1,n)})}P_n\Big)\right)\right)\le S(\phi^{W(1,n)},\phi)\le 2||W(1,n)||,$$

so that

$$0\le -S(\phi|_{A_{[1,n]}})-\phi(\log P_n)+\phi(U(1,n))+\log\tau(e^{-U(1,n)})\le 2||W(1,n)||.$$

Since $\frac{1}{n}\phi(U(1,n))\to\phi(\Phi)=\phi(E_\Phi)$, $\frac{1}{n}\log\tau(e^{-U(1,n)})\to P(\Phi)$, and the sequence $\{W(1,n)\}_n$ is bounded, we obtain the desired. □

Corollary IV.6 *The mean entropy exists for each of the following cases:*

1) *the automorphism θ_λ, $\lambda\ge\frac{1}{4}$,*

$$s(\phi)=\phi(E_\Phi)+P(\Phi)-\frac{1}{2}\log\lambda\ ;$$

2) *the canonical shift on the tower of relative commutants corresponding to a subfactor of finite depth,*

$$s(\phi)=\phi(E_\Phi)+P(\Phi)+\log[M:N]\ ;$$

3) *a binary shift,*

$$s(\phi)=\phi(E_\Phi)+P(\Phi)+\frac{1}{2}\log 2.$$

Proof. For all these systems the limit $\lim_n \frac{1}{n} \log P_n$ exists in norm. The result is well-known for the systems in 1) and 2) (see e.g. [9], Examples 1.3, 1.4). Concerning binary shifts the sequence $\{A_{[1,n]}\}_n$ is periodic [17]. Hence the limit $\lim_n -\frac{1}{n} \log P_n$ exists, is scalar and equals to the entropy $h_\tau(\gamma)$ of γ with respect to τ [19]. This entropy is equal to $\frac{1}{2}\log 2$ [19].

□

Remark IV.7 It is not known whether the mean entropy equals to the dynamical entropy even for a quantum spin lattice system. But such an equality is possible. This is the case when an interaction potential lies in the diagonal (then, by the general theory [8], the dynamical entropy coincides with entropy of the restriction of the automorphism to the diagonal). A bit less trivial example can be obtained as follows.

Suppose an interaction potential on a one-dimensional quantum lattice has the property $\sigma_t(A_0) \subset A_{[-n,n]}$ for some $n \in \mathbb{N}$. Then it is easy to see that $h_\phi(\gamma) = s(\phi)$ (in fact, the state ϕ is n-Markov, i. e. it would be Markov if we consider A as a quantum lattice system with the one-site algebra $A_{[0,n-1]}$, [20], and we can refer to [21], see also [7]). One can construct potentials with such a property, but that don't lie in the diagonal. For example, we can choose selfadjoint $a_0, a_1, b \in Mat_d(\mathbb{C})$, $d \geq 3$, such that a_0 commutes with a_1 and b, but a_1 and b don't commute. Define $\Phi(\{0\}) = b$, $\Phi(\{0,1\}) = a_0 \otimes a_1$, $\Phi(X) = 0$ for $|X| \geq 3$. Then $\sigma_t(A_0) \subset A_{[-1,1]}$, but the centralizer A^σ does not contain any diagonal of A_0.

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