Quantizations of Poisson Lie groups as noncommutative manifolds

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On any q-deformation of a simply connected simple compact Poisson Lie group we construct an equivariant spectral triple which is an isospectral deformation of that defined by the Dirac operator on the original group. Our quantum Dirac operator is defined using a Drinfeld twist which relates the q-deformed compact quantum group to the original group, and thus a priori depends on the choice of the twist, but it turns out that the spectral triple is nevertheless unique up to unitary equivalence.

Introduction

By work of Drinfeld and Belavin all simply connected simple compact Poisson Lie groups have been classified. Drinfeld further showed that all of these do admit quantization by q-deformation. The quantization of the standard Poisson bracket on G is the compact quantum group G_q one usually encounters. The quantizations corresponding to the other brackets are obtained by twisting the coproduct of G_q or G with a 2-cocycle q^{iu}, where u is a self-adjoint element of $\mathfrak{h} \wedge \mathfrak{h}$.

In [1] we constructed a quantum Dirac operator D_q on G_q that defines a biequivariant spectral triple which is an isospectral deformation of that defined by the Dirac operator D on G. To do this we relied on the existence of a special element $\mathcal{F}$ in the group von Neumann algebra $W^*(G\times G)$, and an isomorphism $\varphi\colon W^*(G_q) \to W^*(G)$ satisfying certain properties. The existence of the pair $(\varphi, \mathcal{F})$ follows from work by Kazhdan and Lusztig [2, 3] and is an analytic version of a result by Drinfeld [4, 5]. From the outset D_q and the associated spectral triple depend on the choice of $(\varphi, \mathcal{F})$, but by a uniqueness result for Drinfeld twists, which we established in [6], we showed that the spectral triple is unique up to unitary equivalence.

*The note is different in two lines from the published version, correcting an annoying mistake in the definition of the r-matrix; January 6, 2012

In this note we briefly describe the above results and show that one can construct Dirac operators providing equivariant spectral triples for all the twisted versions as well.

1. Quantum groups

Let G be a compact connected simply connected simple Lie group, $\mathfrak{g}$ its complexified Lie algebra. Let $\mathfrak{h}\subset\mathfrak{g}$ be the Cartan subalgebra defined by a maximal torus T in G. Fix a system $\{\alpha_1,\dots,\alpha_r\}$ of simple roots. Let $(a_{ij})_{1\le i,j\le r}$ be the Cartan matrix and $d_1,\dots,d_r$ be the coprime positive integers such that $(d_ia_{ij})_{i,j}$ is symmetric. Let $h_i\in\mathfrak{h}$ be such that $\alpha_j(h_i)=a_{ij}$. Denote by $\mathfrak{h}_\mathbb{R}$ the $\mathbb{R}$-linear span of h_i, $i=1,\dots,r$. Define a bilinear form on $\mathfrak{h}^*$ by $(\alpha_i,\alpha_j)=d_ia_{ij}$. The dual form on $\mathfrak{h}$ extends to a symmetric invariant form on $\mathfrak{g}$. Denote by $t\in\mathfrak{g}\otimes\mathfrak{g}$ the corresponding $\mathfrak{g}$-invariant element. Consider the decomposition $\mathfrak{g}=\mathfrak{n}_+\oplus\mathfrak{h}\oplus\mathfrak{n}_-$ and write $t=t_{+-}+t_0+t_{-+}$ with $t_{+-}\in\mathfrak{n}_+\otimes\mathfrak{n}_-$, $t_{-+}\in\mathfrak{n}_-\otimes\mathfrak{n}_+$ and $t_0\in\mathfrak{h}\otimes\mathfrak{h}$. Define $r_0\in\mathfrak{g}\otimes\mathfrak{g}$ by

$$r_0=i(t_{+-}-t_{-+}).$$

It is known that any self-adjoint skew-symmetric solution of the modified classical Yang-Baxter equation can be written, up to inner automorphisms of G, as ar_0+u for some $a\in\mathbb{R}$ and $u\in\wedge^2\mathfrak{h}_\mathbb{R}$. This means that any Poisson Lie group structure on G is given by the bracket $\{\cdot,\cdot\}\colon\mathbb{C}[G]\times\mathbb{C}[G]\to\mathbb{C}[G]$ defined by

$$\{f,g\}=(f\otimes g)([ar_0+u,\hat\Delta(\cdot)]),$$

up to inner automorphisms of G and a complex rescaling of the bracket, where $\hat\Delta$ is the comultiplication on $U\mathfrak{g}$. They all admit quantization. For $a=1$ and $u=0$ the quantization is the standard q-deformation, and that for $a\ne1$ (and $u=0$) just means a reparametrization, i.e. change of q, of the standard one. We shall focus on the q-deformations for $0<q<1$ associated to $a=1$ and $u\in\wedge^2\mathfrak{h}_\mathbb{R}$; the case $a=0$ is analogous but easier, one essentially has to replace $U_q\mathfrak{g}$ everywhere by $U\mathfrak{g}$.

The quantized universal enveloping algebra $U_q\mathfrak{g}$ is generated by elements E_i, F_i, K_i, $1\le i\le r$, satisfying the relations

$$K_iK_j=K_jK_i,\quad K_iE_jK_i^{-1}=q_i^{a_{ij}}E_j,\quad K_iF_jK_i^{-1}=q_i^{-a_{ij}}F_j,$$

$$E_iF_j-F_jE_i=\delta_{ij}\frac{K_i-K_i^{-1}}{q_i-q_i^{-1}},$$

$$\sum_{k=0}^{1-a_{ij}}(-1)^k\begin{bmatrix}1-a_{ij}\\ k\end{bmatrix}_{q_i}E_i^kE_jE_i^{1-a_{ij}-k}=0,$$

$$\sum_{k=0}^{1-a_{ij}}(-1)^k\begin{bmatrix}1-a_{ij}\\ k\end{bmatrix}_{q_i}F_i^kF_jF_i^{1-a_{ij}-k}=0,$$

where $\begin{bmatrix} m \\ k \end{bmatrix}_{q_i} = \frac{[m]_{q_i}!}{[k]_{q_i}![m-k]_{q_i}!}$, $[m]_{q_i}! = [m]_{q_i}[m-1]_{q_i}\dots[1]_{q_i}$, $[n]_{q_i} = \frac{q_i^n - q_i^{-n}}{q_i - q_i^{-1}}$ and $q_i = q^{d_i}$.

This is a Hopf $*$-algebra with coproduct $\hat\Delta_q$ defined by

$$\hat\Delta_q(K_i) = K_i \otimes K_i, \quad \hat\Delta_q(E_i) = E_i \otimes 1 + K_i \otimes E_i, \quad \hat\Delta_q(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i$$

and involution

$$K_i^* = K_i, E_i^* = F_i K_i, F_i^* = K_i^{-1} E_i.$$

Let P be the lattice of integral weights. If V is a finite dimensional $U_q\mathfrak{g}$-module and $\lambda \in P$, denote by $V(\lambda)$ the space of vectors $v \in V$ of weight λ, that is, $K_i v = q_i^{\lambda(h_i)} v$. The module V is called admissible if $V = \oplus_{\lambda\in P} V(\lambda)$.

The quantized algebra of regular functions $\mathbb{C}[G_q] \subset (U_q\mathfrak{g})^*$ is the Hopf $*$-algebra of matrix coefficients of finite dimensional admissible $U_q\mathfrak{g}$-modules.

Denote by $\mathcal{C}(\mathfrak{g}, q)$ the category of finite dimensional admissible $U_q\mathfrak{g}$-modules. It is a semisimple tensor category, with simple objects V_λ indexed by dominant integral weights $\lambda \in P_+$. Therefore, if $F\colon \mathcal{C}(\mathfrak{g}, q) \to \mathcal{V}ec$ is the forgetful functor, then

$$\mathcal{U}(G_q) := \mathrm{Nat}(F, F) \cong \prod_{\lambda\in P_+} \mathrm{End}(V_\lambda).$$

The algebra $\mathcal{U}(G_q)$ is a completion of $U_q\mathfrak{g}$. It can also be identified with the algebra of closed densely defined operators affiliated with the von Neumann algebra $W^*(G_q)$ of G_q. The category $\mathcal{C}(\mathfrak{g}, q)$ is braided, with braiding $\sigma = \Sigma\mathcal{R}$, where Σ is the flip and $\mathcal{R} \in \mathcal{U}(G_q \times G_q) := \mathrm{Nat}(F^{\otimes 2}, F^{\otimes 2})$ is the universal R-matrix.

The algebra $\mathcal{U}(T)$ embeds into $\mathcal{U}(G_q)$ by identifying h_i with a unique self-adjoint element $H_i \in \mathcal{U}(G_q)$ such that $K_i = q_i^{H_i}$. If $u \in \wedge^2\mathfrak{h}_\mathbb{R}$ we can therefore consider the element q^{iu} as an element $\mathcal{H} \in \mathcal{U}(G_q \times G_q)$. Since $u^* = u$ and u is skew-symmetric, we have $\mathcal{H}^* = \mathcal{H}^{-1} = \mathcal{H}_{21}$. We also have $u \otimes 1 + (\hat\Delta \otimes \iota)u = 1 \otimes u + (\iota \otimes \hat\Delta)u$, so that by commutativity of $\mathcal{U}(T)$ we get

$$(\mathcal{H} \otimes 1)(\hat\Delta_q \otimes \iota)(\mathcal{H}) = (1 \otimes \mathcal{H})(\iota \otimes \hat\Delta_q)(\mathcal{H}).$$

Therefore $\mathcal{H}$ is a unitary 2-cocycle for $(\mathcal{U}(G_q), \hat\Delta_q)$ with $\mathcal{H}^{-1} = \mathcal{H}_{21}$. Thus we can define a new coproduct $\hat\Delta_{u,q}\colon \mathcal{U}(G_q) \to \mathcal{U}(G_q \times G_q)$ by $\hat\Delta_{u,q} = \mathcal{H}\hat\Delta_q(\cdot)\mathcal{H}^{-1}$. It defines a new tensor structure on finite dimensional admissible $U_q\mathfrak{g}$-modules. Denote by $\mathbb{C}[G_q^u]$ the corresponding Hopf $*$-algebra of matrix coefficients. In other words, $\mathbb{C}[G_q^u]$ coincides with $\mathbb{C}[G_q]$ as a coalgebra, but has a new $*$-algebra structure defined by the twist $\mathcal{H}$, see [7]. The compact quantum group G_q^u is the q-deformation of the Poisson Lie group G with the Poisson structure associated to u (and $a = 1$). The category of finite dimensional representations of G_q^u is braided with braiding defined by the R-matrix $\mathcal{R}^u = \mathcal{H}_{21}\mathcal{R}\mathcal{H}^{-1}$.

2. Drinfeld twist

Let $\hbar \in i\mathbb{R}$ be such that $q = e^{\pi i \hbar}$. Assume V_1, V_2, V_3 are finite dimensional $\mathfrak{g}$-modules and put $V = V_1 \otimes V_2 \otimes V_3$. Consider $\mathrm{End}(V)$-valued solutions of the equation

$$w' = \hbar \left(\frac{t_{12}}{x} + \frac{t_{23}}{x-1} \right) w$$

on $(0, 1)$. There exist unique solutions G_0 and G_1 such that the functions $G_0(x)x^{-\hbar t_{12}}$ and $G_1(1-x)x^{-\hbar t_{23}}$ extend to holomorphic functions in the unit disc with value 1 at $x = 0$. Hence there exists $\Phi_{KZ} \in \mathrm{GL}(V)$ such that

$$G_0(x) = G_1(x)\Phi_{KZ} \quad \text{for all} \quad x \in (0, 1).$$

The operators Φ_{KZ} for different V_1, V_2, V_3 define an element of $\mathcal{U}(G \times G \times G)$. It is a unitary 3-cocycle for $(\mathcal{U}(G), \hat{\Delta})$.

As irreducible representations of G and G_q are both parametrized by dominant integral weights, we have a canonical identification of the centers of $\mathcal{U}(G)$ and $\mathcal{U}(G_q)$. Since the dimensions of irreducible modules with the same highest weight do not depend on q, this identification extends to a $*$-isomorphism $\varphi\colon \mathcal{U}(G_q) \to \mathcal{U}(G)$. Furthermore, the dimensions of the weight spaces do not depend on q either, which implies that we can arrange that $\varphi(K_i) = q_i^{h_i}$. In particular, $(\varphi \otimes \varphi)(\mathcal{H}) = q^{iu}$.

Theorem 2.1. *There exists a unitary $\mathcal{F} \in \mathcal{U}(G \times G)$ such that*
(i) $(\varphi \otimes \varphi)\hat{\Delta}_{u,q} = \mathcal{F}\hat{\Delta}\varphi(\cdot)\mathcal{F}^{-1}$;
(ii) $(\hat{\varepsilon} \otimes \iota)(\mathcal{F}) = (\iota \otimes \hat{\varepsilon})(\mathcal{F}) = 1$, *where $\hat{\varepsilon}$ is the trivial representation of G;*
(iii) $(\varphi \otimes \varphi)(\mathcal{R}^u) = \mathcal{F}_{21} q^t \mathcal{F}^{-1}$;
(iv) $\Phi_{KZ} = (\iota \otimes \hat{\Delta})(\mathcal{F}^{-1})(1 \otimes \mathcal{F}^{-1})(\mathcal{F} \otimes 1)(\hat{\Delta} \otimes \iota)(\mathcal{F})$.

Proof. For $u = 0$ the existence of $\mathcal{F}$ follows by work of Kazhdan and Lusztig [2, 3], see [8]. Denote this element by $\mathcal{F}_0$. For general u the required element is $\mathcal{F} := q^{iu}\mathcal{F}_0$. Indeed, the only nontrivial property is (iv). Since the elements K_i are group-like, we have $\hat{\Delta} = (\varphi \otimes \varphi)\hat{\Delta}_q \varphi^{-1}$ on $\mathcal{U}(T)$. It follows that $\mathcal{F}_0$ commutes with elements of the form $\hat{\Delta}(w)$, $w \in \mathcal{U}(T)$. Then (iv) for $\mathcal{F}$ follows from the corresponding property of $\mathcal{F}_0$ and the fact that q^{iu} is a 2-cocycle for $(\mathcal{U}(T), \hat{\Delta})$. □

We call $\mathcal{F}$ a unitary Drinfeld twist for G_q^u. We have the following uniqueness result.

Theorem 2.2. *Suppose $\mathcal{E}$ and $\mathcal{F}$ are two unitary Drinfeld twists for G_q^u for the same $*$-isomorphism φ. Then there exists a unitary central element c in $\mathcal{U}(G)$ such that $\mathcal{E} = (c \otimes c)\mathcal{F}\hat{\Delta}(c)^{-1}$.*

Proof. The elements $q^{-iu}\mathcal{E}$ and $q^{-iu}\mathcal{F}$ are unitary Drinfeld twists for G_q. Therefore the result follows from [6, Theorem 5.2]. Briefly, the reason is that

$$\mathcal{G} := (\varphi^{-1} \otimes \varphi^{-1})(q^{-iu}\mathcal{E}\mathcal{F}^{-1}q^{iu})$$

is a symmetric invariant unitary 2-cocycle for $(\mathcal{U}(G_q),\hat\Delta_q)$, that is, $\mathcal{R}\mathcal{G}=\mathcal{G}_{21}\mathcal{R}$ and $\mathcal{G}$ commutes with elements of the form $\hat\Delta_q(w)$. Hence $\mathcal{G}$ is the coboundary of a central element in $\mathcal{U}(G_q)$ by [6, Theorem 2.1]. $\square$

3. Quantum Dirac operator

Denote by $\mathrm{Cl}(\mathfrak{g})$ the complex Clifford algebra of $\mathfrak{g}$ and by $\gamma\colon\mathfrak{g}\to\mathrm{Cl}(\mathfrak{g})$ the canonical embedding, so $\mathrm{Cl}(\mathfrak{g})$ is generated by $\gamma(x)$, $x\in\mathfrak{g}$, and $\gamma(x)^2=(x,x)1$. The adjoint action of G on $\mathfrak{g}$ extends to an action of G on $\mathrm{Cl}(\mathfrak{g})$ which lifts to a homomorphism $G\to\mathrm{Spin}(\mathfrak{g})$. On the Lie algebra level it is given by

$$\mathfrak{g}\ni x\mapsto\widetilde{\mathrm{ad}}(x):=-\frac14\sum_i\gamma(x_i)\gamma([x,x^i]),$$

where $\{x_i\}_i$ is a basis in $\mathfrak{g}$ and $\{x^i\}_i$ is the dual basis. Fix a spin module, an irreducible $*$-representation $s\colon\ \mathrm{Cl}(\mathfrak{g})\to\mathrm{End}(\mathbb{S})$. Identifying the smooth sections of the spin bundle $S=G\times\mathbb{S}$ with $C^\infty(G)\otimes\mathbb{S}$, the Dirac operator $D\colon C^\infty(G)\otimes\mathbb{S}\to C^\infty(G)\otimes\mathbb{S}$ is given by

$$D=\sum_i\left(\partial(x_i)\otimes s\gamma(x^i)-\frac12\otimes s(\gamma(x_i)\widetilde{\mathrm{ad}}(x^i))\right),$$

where ∂ is the representation of $U\mathfrak{g}$ by left-invariant differential operators. This can be written as $D=(\partial\otimes s)(\mathcal{D})$, where

$$\mathcal{D}=-(\iota\otimes\gamma)(t)-\sum_i\frac12\otimes\gamma(x_i)\widetilde{\mathrm{ad}}(x^i)$$

is an element of the non-commutative Weil algebra $U\mathfrak{g}\otimes\mathrm{Cl}(\mathfrak{g})$.

Fix a $*$-isomorphism $\varphi\colon\mathcal{U}(G^u_q)=\mathcal{U}(G_q)\to\mathcal{U}(G)$ as in the previous section and choose a unitary Drinfeld twist $\mathcal{F}\in\mathcal{U}(G\times G)$ for G^u_q. Define

$$\mathcal{D}^u_q=(\varphi^{-1}\otimes\iota)((\iota\otimes\widetilde{\mathrm{ad}})(\mathcal{F})\mathcal{D}(\iota\otimes\widetilde{\mathrm{ad}})(\mathcal{F}^{-1}))\in\mathcal{U}(G^u_q)\otimes\mathrm{Cl}(\mathfrak{g}).$$

The quantum Dirac operator D^u_q is the unbounded operator on $L^2(G^u_q)\otimes\mathbb{S}$ defined by

$$D^u_q=(\partial^u_q\otimes s)(\mathcal{D}^u_q),$$

where ∂^u_q is the right regular representations of $\mathcal{U}(G^u_q)$ on $L^2(G^u_q)$.

Theorem 3.1. *The triple* $(\mathbb{C}[G^u_q],L^2(G^u_q)\otimes\mathbb{S},D^u_q)$ *is a* G^u_q*-biequivariant spectral triple of the same parity as the dimension of* G*. It does not depend on the choice of* φ *and* $\mathcal{F}$ *up to unitary equivalence.*

The proof is identical to that of [1, Theorem 3.7] and [6, Theorem 6.1].

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