

ON THE DEFINITION OF L^2 -BETTI NUMBERS OF EQUIVALENCE RELATIONS

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ABSTRACT. We show that the L^2 -Betti numbers of equivalence relations defined by R. Sauer coincide with those defined by D. Gaboriau.

INTRODUCTION

The notion of L^2 -Betti numbers of countable standard equivalence relations was introduced in a celebrated paper of Gaboriau [2]. A few years later a different definition was given by Sauer [4]. While Gaboriau's construction was motivated by Cheeger and Gromov's definition of L^2 -Betti numbers of discrete groups [1], Sauer was inspired by the algebraic framework developed by Lück [3]. Each definition has its own advantages. E.g. the proof of the theorem of Gaboriau that orbit equivalent groups have the same L^2 -Betti numbers is quite short and transparent in his setting. On the other hand, the computational power of homological algebra is better accessible through Sauer's definition [5]. The two approaches are equivalent for equivalence relations generated by free actions of discrete groups [4]. The aim of this note is to show that they are equivalent in general.

1. DIMENSION THEORY AND HOMOLOGICAL ALGEBRA

Let M be a finite von Neumann algebra with a fixed faithful normal tracial state τ . For a finitely generated projective M -module $P \cong M^n p$, where $p \in \text{Mat}_n(M) = M \otimes \text{Mat}_n(\mathbb{C})$ is a projection, its dimension is defined by $\dim_M P = (\tau \otimes \text{Tr})(p)$. Lück extended the dimension function to all M -modules by letting

$$\dim_M Q = \sup\{\dim_M P \mid P \subset Q \text{ is finitely generated projective}\} \in [0, +\infty].$$

The most important properties of \dim_M are additivity and cofinality. Together they imply that if Q is the inductive limit of a sequence of modules Q_i with $\dim_M Q_i < \infty$ then

$$\dim_M Q = \lim_i \lim_j \dim_M \text{im}(Q_i \rightarrow Q_j),$$

see [3] for details.

A morphism $h: Q_1 \rightarrow Q_2$ of M -modules is called a \dim_M -isomorphism if both $\ker h$ and $\text{coker } h$ have dimension zero. By localizing the category $M\text{-Mod}$ of M -modules by the subcategory of zero-dimensional modules one can deal with \dim_M -isomorphisms as with usual isomorphisms. What makes life even better is that the localized category can be embedded back into the category of M -modules using the functor of rank completion introduced by Thom [6]. The definition of this functor is motivated by the following criterion [4]: an M -module Q has dimension zero if and only if for any $\xi \in Q$ and $\varepsilon > 0$ there exists a projection $p \in M$ such that $p\xi = \xi$ and $\tau(p) < \varepsilon$. Now for $Q \in M\text{-Mod}$ and $\xi \in Q$ define

$$[\xi]_M = \inf\{\tau(p) \mid p \text{ is a projection in } M, p\xi = \xi\}.$$

Then $d_M(\xi, \zeta) := [\xi - \zeta]_M$ is a pseudometric on Q . Denote by $c_M(Q)$ the completion of Q in this pseudometric, that is, the quotient of the module of Cauchy sequences by the submodule of sequences converging to zero. Any M -module map $h: Q_1 \rightarrow Q_2$ is a contraction in the pseudometric d_M , hence

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it defines a morphism $c_M(h): c_M(Q_1) \rightarrow c_M(Q_2)$. Therefore c_M is a functor $M\text{-Mod} \rightarrow M\text{-Mod}$, called the functor of rank completion [6]. Notice that although d_M depends on the choice of the trace, the corresponding uniform structure does not, so the functor c_M does not depend on the choice of the trace either.

Lemma 1.1 ([6]). *We have:*

- (i) *for any $Q \in M\text{-Mod}$ the completion map $Q \rightarrow c_M(Q)$ is a \dim_M -isomorphism;*
- (ii) *$\dim_M Q = 0$ if and only if $c_M(Q) = 0$; more generally, a morphism $h: Q_1 \rightarrow Q_2$ is a \dim_M -isomorphism if and only if $c_M(h): c_M(Q_1) \rightarrow c_M(Q_2)$ is an isomorphism;*
- (iii) *the functor c_M is exact.*

We remark that our setting is not the same as that studied by Thom [6]. There, he considers M -bimodules and defines

$$[\xi] = \inf\{\tau(p) + \tau(q) \mid p\xi q = \xi\}.$$

However, all the proofs work equally well if we rather than M -bimodules consider M - N -bimodules. Our situation then corresponds to the case when N consists of the scalars. Furthermore, part (i) and the first part of (ii) in the above lemma follow immediately by definition and the criterion of zero dimensionality, while the second part of (ii) then follows from exactness. Thus the only statement in Lemma 1.1 which requires a proof is part (iii). See [6, Lemma 2.6] for details.

If $Q \subset P$ is dense in the pseudometric d_M then we say that Q is M -dense in P . If $Q = c_M(Q)$, we say that Q is M -complete.

For a pair of algebras $N \subset M$ we shall always assume that both the algebras and the embedding are unital. Furthermore, if N and M are finite von Neumann algebras then we shall assume that the trace on N is the restriction of the trace on M .

Lemma 1.2. *Assume $N \subset \mathfrak{M}$ is a pair of algebras such that N is a finite von Neumann algebra. Then the following conditions are equivalent:*

- (i) *for any $Q \in \mathfrak{M}\text{-Mod}$ and $m \in \mathfrak{M}$ the map $Q \rightarrow Q$, $\xi \mapsto m\xi$, is uniformly continuous with respect to the pseudometric d_N ;*
- (ii) *for any $m \in \mathfrak{M}$ and $\varepsilon > 0$ there exists $\delta > 0$ such that if $p \in N$ is a projection with $\tau(p) < \delta$ then $[mp]_N < \varepsilon$;*
- (iii) *if $Q \in N\text{-Mod}$ is such that $\dim_N Q = 0$ then $\dim_N(\mathfrak{M} \otimes_N Q) = 0$.*

Proof. Applying (i) to $Q = \mathfrak{M}$ we immediately get (ii), so (i) \Rightarrow (ii). Conversely, assume (ii) is satisfied. Let $Q \in \mathfrak{M}\text{-Mod}$, $m \in \mathfrak{M}$ and $\varepsilon > 0$. Choose $\delta > 0$ as in (ii). Then if $\xi \in Q$ and $[\xi]_N < \delta$, we can find a projection $p \in N$ with $p\xi = \xi$ and $\tau(p) < \delta$, and get

$$[m\xi]_N = [mp\xi]_N \leq [mp]_N < \varepsilon.$$

Furthermore, a similar computation shows that $[m \otimes \xi]_N < \varepsilon$. Thus (ii) \Rightarrow (i) and (ii) \Rightarrow (iii).

It remains to show that (iii) implies (ii). Assume (ii) is not true. Then there exist $m \in \mathfrak{M}$, $\varepsilon > 0$ and a sequence of projections $p_n \in N$ such that $\tau(p_n) \rightarrow 0$ but $[mp_n]_N \geq \varepsilon$ for all n . Passing to a subsequence we may assume that $\sum_n \tau(p_n) < \infty$. Consider the N -module $Q = (\prod_n Np_n)/(\oplus_n Np_n)$. Observe that if $\xi = (\xi_n)_{n \geq k} \in \prod_{n \geq k} Np_n$ then

$$[\xi]_N \leq \sum_{n=k}^{\infty} [\xi_n]_N \leq \sum_{n=k}^{\infty} \tau(p_n).$$

This implies that $\dim_N Q = 0$. So assuming (iii) we have $\dim_N(\mathfrak{M} \otimes_N Q) = 0$. In particular, by considering the image of $\xi := m \otimes (p_n)_n \in \mathfrak{M} \otimes_N (\prod_n Np_n)$ in $\mathfrak{M} \otimes_N Q$, we can find a projection $p \in N$ such that $\tau(p) < \varepsilon$ and $\xi - p\xi$ lies in the image of $\mathfrak{M} \otimes_N (\oplus_n Np_n)$. By considering the projection $\prod_n Np_n \rightarrow Np_k$ onto the k -th factor we conclude that $m \otimes p_k = pm \otimes p_k \in \mathfrak{M} \otimes_N Np_k$ for all k sufficiently large. As $\mathfrak{M} \otimes_N Np_k = \mathfrak{M}p_k$, this shows that $[mp_k]_N = [pm p_k]_N \leq \tau(p) < \varepsilon$ for all k big enough. This contradicts our choice of the sequence $\{p_n\}_n$. Therefore (iii) \Rightarrow (ii). \square

Under the equivalent conditions of the above lemma, the multiplication by $m \in \mathfrak{M}$ on $Q \in \mathfrak{M}\text{-Mod}$ extends by continuity to a map on $c_N(Q)$. Therefore the functor c_N of rank completion on $N\text{-Mod}$ defines a functor $\mathfrak{M}\text{-Mod} \rightarrow \mathfrak{M}\text{-Mod}$ which we denote, slightly abusing notation, by the same symbol c_N . It follows from [6] that if P is a projective \mathfrak{M} -module then $c_N(P)$ is projective in the category $\mathfrak{M}\text{-Mod}_c$ of N -complete \mathfrak{M} -modules, so that if $Q \in \mathfrak{M}\text{-Mod}_c$ then any surjective morphism $h: Q \rightarrow c_N(P)$ has a right inverse. Indeed, the completion morphism $P \rightarrow c_N(P)$ lifts to a morphism $s: P \rightarrow Q$ by projectivity of P , and then $c_N(s): c_N(P) \rightarrow c_N(Q) = Q$ is a right inverse of h . It follows that any exact sequence of \mathfrak{M} -modules of the form $0 \leftarrow c_N(P_0) \leftarrow c_N(P_1) \leftarrow \dots$, where the P_n are projective \mathfrak{M} -modules, is split-exact.

Lemma 1.3. *Let $N \subset \mathfrak{M} \subset M$ be a triple of algebras such that N and M are finite von Neumann algebras and the pair $N \subset \mathfrak{M}$ satisfies the equivalent conditions of Lemma 1.2. Then any morphism $Q_1 \rightarrow Q_2$ of \mathfrak{M} -modules which is a \dim_N -isomorphism induces a \dim_M -isomorphism $\text{Tor}_n^{\mathfrak{M}}(M, Q_1) \rightarrow \text{Tor}_n^{\mathfrak{M}}(M, Q_2)$ for all $n \geq 0$.*

Proof. This is proved in [4] and in a different form in [6]. We shall nevertheless sketch a proof for the reader's convenience.

Consider the case $n = 0$. It suffices to show that for any \mathfrak{M} -module Q the completion map $Q \rightarrow c_N(Q)$ induces a \dim_M -isomorphism $M \otimes_{\mathfrak{M}} Q \rightarrow M \otimes_{\mathfrak{M}} c_N(Q)$. Since $[mp]_M \leq \tau(p)$ for any $m \in M$ and any projection $p \in M \supset N$, we have $[m \otimes \xi]_M \leq [\xi]_N$. Hence the image of $M \otimes_{\mathfrak{M}} Q$ is M -dense in $M \otimes_{\mathfrak{M}} c_N(Q)$, and we get a surjective morphism

$$c_M(M \otimes_{\mathfrak{M}} Q) \rightarrow c_M(M \otimes_{\mathfrak{M}} c_N(Q)). \quad (1.1)$$

On the other hand, if $\{\xi_n\}_n$ is a Cauchy sequence in Q , then for any $m \in M$ the sequence $\{m \otimes \xi_n\}_n$ is Cauchy in $M \otimes_{\mathfrak{M}} Q$ (in the pseudometric d_M), so it defines an element of $c_M(M \otimes_{\mathfrak{M}} Q)$. Moreover, if $[\xi_n]_N \rightarrow 0$ as $n \rightarrow \infty$ then $[m \otimes \xi_n]_M \rightarrow 0$. Since $c_N(Q)$ is the quotient of the module of Cauchy sequences by the submodule of sequences converging to zero, we therefore get a well-defined map $M \otimes_{\mathfrak{M}} c_N(Q) \rightarrow c_M(M \otimes_{\mathfrak{M}} Q)$ with M -dense image, and hence a surjective morphism

$$c_M(M \otimes_{\mathfrak{M}} c_N(Q)) \rightarrow c_M(M \otimes_{\mathfrak{M}} Q).$$

Clearly, it is the inverse of (1.1).

Turning to the general case, consider first an \mathfrak{M} -module Q such that $\dim_N Q = 0$. We have to show that $\dim_M \text{Tor}_n^{\mathfrak{M}}(M, Q) = 0$ for all $n \geq 0$. Consider a projective resolution $0 \leftarrow Q \leftarrow P_{\bullet}$. Since $c_N(Q) = 0$, the complex $0 \leftarrow c_N(P_{\bullet})$ is exact. By the remark before the lemma, it is therefore split-exact. It follows that $0 \leftarrow M \otimes_{\mathfrak{M}} c_N(P_{\bullet})$ is exact. On the other hand, by the first part of the proof this complex is \dim_M -isomorphic to the complex $0 \leftarrow M \otimes_{\mathfrak{M}} P_{\bullet}$. Since $\text{Tor}_n^{\mathfrak{M}}(M, Q) \cong H_n(M \otimes_{\mathfrak{M}} P_{\bullet})$, we conclude that $\dim_M \text{Tor}_n^{\mathfrak{M}}(M, Q) = 0$ for all $n \geq 0$.

Finally, for an arbitrary morphism $h: Q_1 \rightarrow Q_2$ of \mathfrak{M} -modules which is a \dim_N -isomorphism, consider the short exact sequences

$$0 \rightarrow \ker h \rightarrow Q_1 \rightarrow \text{im } h \rightarrow 0 \quad \text{and} \quad 0 \rightarrow \text{im } h \rightarrow Q_2 \rightarrow \text{coker } h \rightarrow 0$$

and the corresponding long exact sequences of Tor-groups. Since $\dim_M \text{Tor}_n^{\mathfrak{M}}(M, \ker h) = 0$ and $\dim_M \text{Tor}_n^{\mathfrak{M}}(M, \text{coker } h) = 0$ for all n , we then see that $\text{Tor}_n^{\mathfrak{M}}(M, Q_1)$ and $\text{Tor}_n^{\mathfrak{M}}(M, Q_2)$ are \dim_M -isomorphic. \square

Lemma 1.4. *Let $N \subset \mathfrak{M} \subset M$ be a triple of algebras such that N and M are finite von Neumann algebras. Assume $0 \leftarrow Q \leftarrow P_{\bullet}$ is a resolution of an \mathfrak{M} -module Q such that $\dim_M \text{Tor}_n^{\mathfrak{M}}(M, P_k) = 0$ for all $n \geq 1$ and $k \geq 0$. Then*

$$\dim_M \text{Tor}_n^{\mathfrak{M}}(M, Q) = \dim_M H_n(M \otimes_{\mathfrak{M}} P_{\bullet}) \quad \text{for all } n \geq 0.$$

In particular, if the pair $N \subset \mathfrak{M}$ satisfies the equivalent conditions of Lemma 1.2, then to compute $\dim_M \text{Tor}_{\bullet}^{\mathfrak{M}}(M, Q)$ one can use any resolution of Q by \mathfrak{M} -modules that contain N -dense projective \mathfrak{M} -submodules.

Proof. Consider the functor $F = c_M(M \otimes_{\mathfrak{M}} \cdot): \mathfrak{M}\text{-Mod} \rightarrow M\text{-Mod}$. Since c_M is exact, for the derived functors of F we have $L_n F = c_M \circ L_n(M \otimes_{\mathfrak{M}} \cdot) = c_M(\text{Tor}_n^{\mathfrak{M}}(M, \cdot))$. Therefore the assumption of the lemma says that $0 \leftarrow Q \leftarrow P_{\bullet}$ is an F -acyclic resolution of Q . Hence

$$c_M(\text{Tor}_n^{\mathfrak{M}}(M, Q)) = L_n F(Q) \cong H_n(F(P_{\bullet})) = H_n(c_M(M \otimes_{\mathfrak{M}} P_{\bullet})) \cong c_M(H_n(M \otimes_{\mathfrak{M}} P_{\bullet})),$$

which proves the first part of the lemma.

The second part follows from Lemma 1.3, since if an \mathfrak{M} -module P contains an N -dense projective submodule \tilde{P} , then by that lemma the modules $\text{Tor}_n^{\mathfrak{M}}(M, P)$ and $\text{Tor}_n^{\mathfrak{M}}(M, \tilde{P}) = 0$ (for $n \geq 1$) are \dim_M -isomorphic. \square

The following remarks will not be used later, but may be of independent interest.

Remark 1.5.

(i) Lemma 1.3 can be strengthened as follows. Assume M is a finite von Neumann algebra, $N \subset \mathfrak{M}$ is a pair satisfying the equivalent conditions of Lemma 1.2, and R is an M - \mathfrak{M} -bimodule satisfying the following equivalent (by an analogue of Lemma 1.2) conditions:

- for any $r \in R$ and $\varepsilon > 0$ there exists $\delta > 0$ such that if $p \in N$ is a projection with $\tau(p) < \delta$ then $[rp]_M < \varepsilon$;
- if $Q \in N\text{-Mod}$ is such that $\dim_N Q = 0$ then $\dim_M(R \otimes_N Q) = 0$.

Then any morphism $Q_1 \rightarrow Q_2$ of \mathfrak{M} -modules which is a \dim_N -isomorphism induces a \dim_M -isomorphism $\text{Tor}_n^{\mathfrak{M}}(R, Q_1) \rightarrow \text{Tor}_n^{\mathfrak{M}}(R, Q_2)$ for all $n \geq 0$. The proof is essentially the same as above. This is [4, Lemma 4.10], but we see that the flatness assumption there is not needed.

(ii) Lemma 1.4 provides a mildly alternative route to [4, Theorem 4.11], which is a key point in Sauer's approach to Gaboriau's theorem on the L^2 -Betti numbers of orbit equivalent groups. Namely, assume $N \subset \mathfrak{N} \subset \mathfrak{M} \subset M$ is a quadruple of algebras such that N and M are finite von Neumann algebras, \mathfrak{N} is N -dense in \mathfrak{M} and the pair $N \subset \mathfrak{M}$ satisfies the equivalent conditions of Lemma 1.2. Then

$$\dim_M \text{Tor}_n^{\mathfrak{M}}(M, Q) = \dim_M \text{Tor}_n^{\mathfrak{N}}(M, Q)$$

for any \mathfrak{M} -module Q and all $n \geq 0$. Indeed, to compute $\text{Tor}_n^{\mathfrak{M}}(M, Q)$ we use a resolution $0 \leftarrow Q \leftarrow P_{\bullet}$ of Q by free \mathfrak{M} -modules. Since \mathfrak{N} is N -dense in \mathfrak{M} , by Lemma 1.4 this resolution can also be used to compute $\dim_M \text{Tor}_n^{\mathfrak{N}}(M, Q)$. Thus we only need to check that the canonical map $M \otimes_{\mathfrak{N}} P_{\bullet} \rightarrow M \otimes_{\mathfrak{M}} P_{\bullet}$ is a \dim_M -isomorphism. Since $M \otimes_{\mathfrak{N}} P = M \otimes_{\mathfrak{M}} (\mathfrak{M} \otimes_{\mathfrak{N}} P)$, by Lemma 1.3 it is enough to check that the map $h: \mathfrak{M} \otimes_{\mathfrak{N}} P \rightarrow P$, $m \otimes \xi \mapsto m\xi$, is a \dim_N -isomorphism for any \mathfrak{M} -module P . But this is clear, since the N -module map $P \rightarrow \mathfrak{M} \otimes_{\mathfrak{N}} P$, $\xi \mapsto 1 \otimes \xi$, is a right inverse to h and has N -dense image by virtue of density of \mathfrak{N} in \mathfrak{M} .

2. L^2 -BETTI NUMBERS

Let X be a standard Borel space, $R \subset X \times X$ a countable Borel equivalence relation on X preserving a probability measure μ . The measure μ will usually be omitted in our notation, e.g. we write $L^\infty(X)$ instead of $L^\infty(X, \mu)$. As usual denote by $[R]$ the group of invertible Borel transformations of X with graphs in R .

A standard fiber space over X is a standard Borel space U together with a Borel map $\pi: U \rightarrow X$ with at most countable fibers. There is then a natural measure ν_U on U given by

$$\nu_U(C) = \int_X \#(\pi^{-1}(x) \cap C) d\mu(x).$$

The example that we will be the most concerned with in the following is that where $U = R$ and π is either π_l or π_r , the projections onto the first and second coordinates respectively. Since μ is invariant, the induced measures on R are the same, denoted simply by ν .

Given two standard fiber spaces over X , (U, π) and (V, π') , their fiber product is

$$U * V = \{(u, v) \in U \times V \mid \pi(u) = \pi'(v)\},$$

which is again a standard fiber space.

A left R -action on a standard fiber space U over X is a Borel map $(R, \pi_r) * U \rightarrow U$ denoted $((x, y), u) \mapsto (x, y)u$, where $y = \pi(u)$, satisfying

$$(x, y)((y, z)u) = (x, z)u, \quad (z, z)u = u$$

whenever this makes sense. This implies that $\pi((y, z)u) = y$, and that (x, y) is a bijection between $\pi^{-1}(y)$ and $\pi^{-1}(x)$.

Consider the subspace $\mathbb{C}[R]$ of $L^\infty(R, \nu)$ consisting of functions that are supported on finitely many graphs of $\phi \in [R]$. Equivalently, a function $f \in L^\infty(R, \nu)$ belongs to $\mathbb{C}[R]$ if

$$x \mapsto \#\{y \mid f(x, y) \neq 0\} + \#\{y \mid f(y, x) \neq 0\}$$

is in $L^\infty(X)$. Then $\mathbb{C}[R]$ is an involutive algebra with product

$$(fg)(x, z) = \sum_{y \sim x} f(x, y)g(y, z)$$

and involution $f^*(x, y) = \overline{f(y, x)}$.

If (U, π) is a standard fiber space over X , denote by $\Gamma(U)$ the space of Borel functions f on U , considered modulo sets of ν_U -measure zero, such that the support of $f|_{\pi^{-1}(x)}$ is finite for a.e. $x \in X$. Furthermore, denote by $\Gamma^b(U)$ the space of functions $f \in \Gamma(U) \cap L^\infty(U, \nu_U)$ such that

$$x \mapsto \#\{u \in \pi^{-1}(x) \mid f(u) \neq 0\}$$

is in $L^\infty(X)$. We shall also denote the space $L^2(U, \nu_U)$ by $\Gamma^{(2)}(U)$. If R acts on U then all three spaces $\Gamma(U)$, $\Gamma^b(U)$ and $\Gamma^{(2)}(U)$ are $\mathbb{C}[R]$ -modules in a natural way. In particular, if $(U, \pi) = (R, \pi_l)$, we get a representation of $\mathbb{C}[R]$ on $L^2(R, \nu)$, and we let $L(R)$ be the von Neumann algebra generated by $\mathbb{C}[R]$ in this representation. The characteristic function χ_Δ of the diagonal $\Delta \subset R$ is a cyclic and separating vector for $L(R)$ defining a normal tracial state τ on $L(R)$, so that $L^2(R, \nu) = L^2(L(R), \tau)$. Note also that for $U = X$ we have $\Gamma^b(U) = L^\infty(X)$, $\Gamma^{(2)}(U) = L^2(X, \mu)$ and $\Gamma(U) = M(X)$, the space of measurable functions on X . In particular, $L^\infty(X)$, $L^2(X, \mu)$ and $M(X)$ are left $\mathbb{C}[R]$ -modules.

The results of the previous section will be applied to the triple $L^\infty(X) \subset \mathbb{C}[R] \subset L(R)$, where $L^\infty(X)$ is identified with $L^\infty(\Delta, \nu)$. The equivalent conditions of Lemma 1.2 are satisfied. Indeed, if $f \in \mathbb{C}[R]$ is supported on the graph of $\phi \in [R]$ then $f\chi_Z = \chi_{\phi^{-1}(Z)}f$ for any Borel $Z \subset X$, so that

$$[f\chi_Z]_{L(R)} = [\chi_{\phi^{-1}(Z)}f]_{L(R)} \leq \mu(\phi^{-1}(Z)) = \mu(Z) = \tau(\chi_Z),$$

and thus condition (ii) in Lemma 1.2 is satisfied for $m = f$ with $\delta = \varepsilon$.

For a general standard fiber space U with an R -action the $\mathbb{C}[R]$ -module structure on $\Gamma^{(2)}(U)$ does not extend to an action of $L(R)$. But it extends for the following class of spaces. An R -action on U is called discrete if there is a Borel fundamental domain, that is, if there is a Borel set $F \subset U$ intersecting each R -orbit once and only once. For the case $(U, \pi) = (R, \pi_l)$, the diagonal Δ is a fundamental domain for the standard R -action. For general discrete R -spaces U , by choosing sections of $F \rightarrow X$ we can embed U into $\bigsqcup_{n=1}^\infty R$, see [2, Lemma 2.3], that is, any discrete R -space is R -equivariantly isomorphic to $\bigsqcup_{n=1}^N R\Delta(X_n)$, where $N \in \mathbb{N} \cup \{+\infty\}$, the X_n are Borel subsets of X , $\Delta(X_n) = \{(x, x) \mid x \in X_n\}$ and therefore $R\Delta(X_n) = \{(y, x) \mid x \in X_n, y \sim x\}$. In particular, the $\mathbb{C}[R]$ -module $\Gamma^{(2)}(U)$ is isomorphic to $\oplus_n L^2(L(R))\chi_{X_n}$, and hence the action of $\mathbb{C}[R]$ on it extends to an action of $L(R)$.

If U is a discrete R -space and $F \subset U$ is a Borel fundamental domain, then we write $\nu_U(R \setminus U)$ for $\nu_U(F)$. If $U \cong \bigsqcup_n R\Delta(X_n)$ then $\nu_U(R \setminus U) = \sum_n \mu(X_n)$. Since $\dim_{L(R)} L^2(L(R))\chi_Z = \tau(\chi_Z) = \mu(Z)$, we also get

$$\nu_U(R \setminus U) = \dim_{L(R)} \Gamma^{(2)}(U).$$

If U and V are standard fiber spaces with R -action, then $U * V$ is again a standard fiber space with diagonal action of R . Furthermore, if F is a Borel fundamental domain for U , then $F * V$ is a Borel fundamental domain for $U * V$.

A simplicial R -complex Σ consists of a discrete R -space Σ^0 and Borel sets $\Sigma^1, \Sigma^2, \dots$ with

$$\Sigma^n \subset \overbrace{\Sigma^0 * \dots * \Sigma^0}^{n+1}$$

satisfying, for $n > 0$,

- (i) $R\Sigma^n = \Sigma^n$;
- (ii) if $(v_0, \dots, v_n) \in \Sigma^n$ then $(v_{\sigma(0)}, \dots, v_{\sigma(n)}) \notin \Sigma^n$ for any nontrivial permutation σ ;
- (iii) if $(v_0, \dots, v_n) \in \Sigma^n$ then for all $i = 0, \dots, n$ a permutation of $(v_0, \dots, \hat{v}_i, \dots, v_n)$ is in Σ^{n-1} .

Note that this definition is slightly different from that in [2], as we prefer to fix an order on the vertices of every simplex. Therefore if Σ is a simplicial R -complex in the sense of [2], to get a simplicial complex in the above sense one has to choose, for every $n \geq 1$, an R -invariant fundamental domain for the action of the symmetric group S_{n+1} on Σ^n , see [2, Section 3.1].

Given a simplicial R -complex Σ , we may associate to it a field of simplicial complexes Σ_x by letting Σ_x^n be the fiber of $\Sigma^n \rightarrow X$ over x . One says that Σ is n -dimensional, contractible, and so on, if these properties hold for Σ_x for μ -a.e. $x \in X$.

For a simplicial R -complex Σ we put

$$C_n^b(\Sigma) = \Gamma^b(\Sigma^n), \quad C_n(\Sigma) = \Gamma(\Sigma^n), \quad C_n^{(2)}(\Sigma) = \Gamma^{(2)}(\Sigma^n).$$

Since the simplices of Σ_x are oriented, we have well-defined boundary operators $\partial_{n,x}: C_n(\Sigma_x) \rightarrow C_{n-1}(\Sigma_x)$ on the simplicial chains with complex coefficients. These operators define $\mathbb{C}[R]$ -module maps $\partial_n: C_n(\Sigma) \rightarrow C_{n-1}(\Sigma)$, which map $C_n^b(\Sigma)$ into $C_{n-1}^b(\Sigma)$.

A simplicial R -complex Σ is called uniformly locally bounded (ULB) if there is an integer m such that every vertex of Σ_x is contained in no more than m simplices for almost every $x \in X$, and if furthermore Σ^0 has a fundamental domain of finite measure. The first condition guarantees that the boundary operators $\partial_{n,x}$ define a bounded $L(R)$ -module map $\partial_n: C_n^{(2)}(\Sigma) \rightarrow C_{n-1}^{(2)}(\Sigma)$. The second condition is equivalent to $\dim_{L(R)} C_0^{(2)}(\Sigma) < \infty$. The two conditions together imply that Σ^n has a fundamental domain of finite measure for every n , that is, $\dim_{L(R)} C_n^{(2)}(\Sigma) < \infty$.

For a ULB simplicial R -complex Σ , its n -th reduced L^2 -homology is defined by

$$\bar{H}_n^{(2)}(\Sigma, R) = \ker(\partial_n: C_n^{(2)}(\Sigma) \rightarrow C_{n-1}^{(2)}(\Sigma)) / \overline{\text{im}(\partial_{n+1}: C_{n+1}^{(2)}(\Sigma) \rightarrow C_n^{(2)}(\Sigma))},$$

and then its n -th L^2 -Betti number is defined by

$$\beta_n^{(2)}(\Sigma, R) = \dim_{L(R)} \bar{H}_n^{(2)}(\Sigma, R).$$

For a general simplicial R -complex Σ consider an exhaustion $\{\Sigma_i\}_{i=1}^\infty$ of Σ by ULB complexes, that is, $\Sigma_i^n \subset \Sigma_{i+1}^n \subset \Sigma^n$ and $\cup_i \Sigma_{i,x}^n = \Sigma_x^n$ for a.e. $x \in X$ and all $n \geq 0$. Then define

$$\beta_n^{(2)}(\Sigma, R) = \lim_i \lim_j \dim_{L(R)} \overline{\text{im}(\bar{H}_n^{(2)}(\Sigma_i, R) \rightarrow \bar{H}_n^{(2)}(\Sigma_j, R))}.$$

It is shown in [2] that $\beta_n^{(2)}(\Sigma, R)$ does not depend on the choice of exhaustion. This will also follow from the proof of the next result.

Proposition 2.1. *For any simplicial R -complex Σ and all $n \geq 0$ we have*

$$\beta_n^{(2)}(\Sigma, R) = \dim_{L(R)} H_n(L(R) \otimes_{\mathbb{C}[R]} C_\bullet^b(\Sigma)) = \dim_{L(R)} H_n(L(R) \otimes_{\mathbb{C}[R]} C_\bullet(\Sigma)).$$

This is analogous to the fact [3] that if a discrete group G acts freely on a simplicial complex Σ then $\beta_n^{(2)}(\Sigma, G) = \dim_{L(G)} H_n(L(G) \otimes_{\mathbb{C}[G]} C_\bullet(\Sigma))$, and the proof is similar, although one needs a bit more care in dealing with different chain spaces. For the proof we will need the following lemma.

Lemma 2.2. *Let U be a discrete R -space. Then*

- (i) $\Gamma(U)$ is the $L^\infty(X)$ -completion of a projective $\mathbb{C}[R]$ -module;
- (ii) if $\nu_U(R \setminus U) < \infty$, then the map $L(R) \otimes_{\mathbb{C}[R]} \Gamma^b(U) \rightarrow \Gamma^{(2)}(U)$, $m \otimes \xi \mapsto m\xi$, is a $\dim_{L(R)}$ -isomorphism.

Proof. We may assume that $U = \bigsqcup_{n=1}^N R\Delta(X_n)$. Consider the projective submodule

$$P = \bigoplus_{n=1}^N \mathbb{C}[R]\chi_{X_n}$$

of $\Gamma^b(U)$. We claim that it is $L^\infty(X)$ -dense in $\Gamma(U)$. Indeed, let $f \in \Gamma(U)$. For $m \in \mathbb{N}$ consider the set

$$Y_m = \{x \in X \mid \text{supp } f|_{\pi^{-1}(x)} \subset \cup_{n=1}^m R\Delta(X_n)\}.$$

Then $\{Y_m\}_m$ is an increasing sequence of Borel sets with union a subset of X of full measure. Thus $\chi_{Y_m} f \rightarrow f$ in the metric $d_{L^\infty(X)}$ as $m \rightarrow \infty$. Furthermore, $\chi_{Y_m} f$ is supported on $\cup_{n=1}^m R\Delta(X_n)$. Therefore we may assume that N is finite. But then it suffices to show that $\mathbb{C}[R]$ is $L^\infty(X)$ -dense in $\Gamma(R)$.

Choose a sequence of transformations $\phi_n \in [R]$ such that R is the union of the graphs of ϕ_n . For $f \in \Gamma(R)$ and $m \in \mathbb{N}$ consider the set

$$Z_m = \{x \in X \mid |f(x, y)| \leq m \text{ for all } y \sim x, \text{supp } f(x, \cdot) \subset \{\phi_1(x), \dots, \phi_m(x)\}\}.$$

Then $\chi_{Z_m} f \rightarrow f$ in the metric $d_{L^\infty(X)}$ as $m \rightarrow \infty$, and $\chi_{Z_m} f \in \mathbb{C}[R]$. This finishes the proof of density of P in $\Gamma(U)$.

To finish the proof of (i) it remains to check that $\Gamma(U)$ is $L^\infty(X)$ -complete. For this one can observe that if Q is an M -module for a finite von Neumann algebra M , then for any Cauchy sequence in Q one can choose a subsequence $\{\xi_n\}_n$ for which there is an increasing sequence of projections $p_n \in M$ converging strongly to the unit such that $p_n \xi_n = p_n \xi_m$ for all $m \geq n$. But if we have a sequence of this form in $\Gamma(U)$, it obviously converges to an element of $\Gamma(U)$.

Turning to (ii), we have

$$\Gamma^{(2)}(U) = \bigoplus_{n=1}^N L^2(L(R))\chi_{X_n} \quad (\text{Hilbert space direct sum}).$$

We claim that $L(R) \otimes_{\mathbb{C}[R]} P \rightarrow \Gamma^{(2)}(U)$, $m \otimes \xi \mapsto m\xi$, is a $\dim_{L(R)}$ -isomorphism. Indeed,

$$L(R) \otimes_{\mathbb{C}[R]} P = \bigoplus_n L(R)\chi_{X_n}.$$

Since $L(R)$ is $L(R)$ -dense in $L^2(L(R))$, we see that $L(R) \otimes_{\mathbb{C}[R]} P$ is $L(R)$ -dense in the algebraic direct sum of $L^2(L(R))\chi_{X_n}$, $n = 1, \dots, N$. On the other hand, since $\sum_n \mu(X_n) < \infty$ by assumption, the algebraic direct sum is $L(R)$ -dense in the Hilbert space direct sum (because if $\xi \in L^2(L(R))p$ for a projection $p \in L(R)$ then $[\xi]_{L(R)} \leq \tau(p)$). This proves our claim. Since P is $L^\infty(X)$ -dense in $\Gamma^b(U)$ by (i), by an application of Lemma 1.3 with $n = 0$ we conclude that $L(R) \otimes_{\mathbb{C}[R]} \Gamma^b(U) \rightarrow \Gamma^{(2)}(U)$ is a $\dim_{L(R)}$ -isomorphism. \square

Proof of Proposition 2.1. We start with the first equality. Assume that Σ is a ULB simplicial R -complex. In this case the dimension (over $L(R)$) of the module $\text{im}(\partial_{n+1}: C_{n+1}^{(2)}(\Sigma) \rightarrow C_n^{(2)}(\Sigma))$

coincides with the dimension of its Hilbert space closure, since $\partial_{n+1}\partial_{n+1}^*$ maps $\overline{\text{im } \partial_{n+1}}$ injectively into $\text{im } \partial_{n+1}$. It follows that the canonical surjection

$$H_n(C_\bullet^{(2)}(\Sigma)) \rightarrow \bar{H}_n^{(2)}(\Sigma, R)$$

is a $\dim_{L(R)}$ -isomorphism. On the other hand, by Lemma 2.2(ii) we have a canonical $\dim_{L(R)}$ -isomorphism $L(R) \otimes_{\mathbb{C}[R]} C_\bullet^b(\Sigma) \rightarrow C_\bullet^{(2)}(\Sigma)$. Therefore we obtain a canonical $\dim_{L(R)}$ -isomorphism

$$H_n(L(R) \otimes_{\mathbb{C}[R]} C_\bullet^b(\Sigma)) \rightarrow \bar{H}_n^{(2)}(\Sigma, R),$$

which gives the desired result for Σ .

For a general simplicial R -complex Σ consider an exhaustion of Σ by ULB R -complexes Σ_i , $i \geq 1$. Then by definition of $\beta_n^{(2)}(\Sigma, R)$, the ULB case and the fact that the image of $\bar{H}_n^{(2)}(\Sigma_i, R)$ in $\bar{H}_n^{(2)}(\Sigma_j, R)$ (for $j > i$) has the same dimension as its Hilbert space closure, we can write

$$\beta_n^{(2)}(\Sigma, R) = \lim_i \lim_j \dim_{L(R)} \text{im}(H_n(L(R) \otimes_{\mathbb{C}[R]} C_\bullet^b(\Sigma_i)) \rightarrow H_n(L(R) \otimes_{\mathbb{C}[R]} C_\bullet^b(\Sigma_j))).$$

Since the inductive limit of $H_n(L(R) \otimes_{\mathbb{C}[R]} C_\bullet^b(\Sigma_i))$ is isomorphic to $H_n(L(R) \otimes_{\mathbb{C}[R]} (\cup_i C_\bullet^b(\Sigma_i)))$, by cofinality and additivity of the dimension function we get

$$\beta_n^{(2)}(\Sigma, R) = \dim_{L(R)} H_n(L(R) \otimes_{\mathbb{C}[R]} (\cup_i C_\bullet^b(\Sigma_i))).$$

Next observe that $\cup_i C_\bullet^b(\Sigma_i)$ is $L^\infty(X)$ -dense in $C_\bullet^b(\Sigma)$ (we had a similar argument in the proof of part (i) of Lemma 2.2). By Lemma 1.3 we conclude that

$$\beta_n^{(2)}(\Sigma, R) = \dim_{L(R)} H_n(L(R) \otimes_{\mathbb{C}[R]} C_\bullet^b(\Sigma)).$$

The second equality in the statement then holds by the $L^\infty(X)$ -density of $C_\bullet^b(\Sigma)$ in $C_\bullet(\Sigma)$, which follows from the proof of Lemma 2.2(i). \square

By a result of Gaboriau [2], the numbers $\beta_n^{(2)}(\Sigma, R)$ are the same for any contractible simplicial R -complex Σ . This will also follow from the next theorem, which is our main result.

Theorem 2.3. *If Σ is a contractible simplicial R -complex then*

$$\beta_n^{(2)}(\Sigma, R) = \dim_{L(R)} \text{Tor}_n^{\mathbb{C}[R]}(L(R), L^\infty(X)) \quad \text{for all } n \geq 0.$$

To prove the theorem we need a particular resolution of $L^\infty(X)$. The obvious candidate is

$$0 \leftarrow L^\infty(X) \xleftarrow{\varepsilon} C_0^b(\Sigma) \xleftarrow{\partial_1} C_1^b(\Sigma) \xleftarrow{\partial_2} \dots, \quad (2.1)$$

where $\varepsilon(f)(x) = \sum_{u \in \pi^{-1}(x)} f(u)$. It is more convenient to work with its $L^\infty(X)$ -completion, the complex

$$0 \leftarrow M(X) \xleftarrow{\varepsilon} C_0(\Sigma) \xleftarrow{\partial_1} C_1(\Sigma) \xleftarrow{\partial_2} \dots$$

Recall that $M(X)$ denotes the space of measurable functions on X . Fiberwise the above complex is contractible, so to check exactness it suffices to find homotopies depending measurably on $x \in X$. This will be done using the following two lemmas.

Lemma 2.4. *Let V be a vector space over \mathbb{Q} of countable dimension. Let $x \mapsto V_x$ be a field of subspaces of V such that for all measurable mappings $s: X \rightarrow V$ the set $\{x \in X \mid s(x) \in V_x\}$ is measurable. Then there is a field of projections $x \mapsto p_x$ onto V_x which is measurable in the sense that for every measurable mapping $s: X \rightarrow V$ the map $x \mapsto p_x s(x) \in V$ is measurable.*

Proof. Let $\{e_1, e_2, \dots\}$ be a basis for V , and set $V_n = \text{Span}_{\mathbb{Q}}\{e_1, \dots, e_n\}$. We claim that there exist unique projections $p_x: V \rightarrow V_x$ such that

- (a) $p_x V_k \subset V_k$ for all $k \geq 1$;
- (b) if $V_k \cap V_x \subset V_{k-1}$ for some k then $p_x e_k = 0$.

To show this we shall prove by induction on n that there exist unique projections $p_x: V_n \rightarrow V_n \cap V_x$ satisfying properties (a) and (b) for $k \leq n$. For $n = 1$ this is trivial, as $p_x e_1 = \chi_{V_x}(e_1)e_1$ is the only possible option for p_x . Assume by induction that p_x is defined on V_{n-1} . We have two possibilities: either $V_n \cap V_x = V_{n-1} \cap V_x$ or $\dim V_n \cap V_x = \dim(V_{n-1} \cap V_x) + 1$. In the first case the condition $p_x e_n = 0$ completely determines an extension of p_x to V_n . In the second case there exists only one extension, since if $v \in (V_n \cap V_x) \setminus V_{n-1}$ then $V_n = V_{n-1} \oplus \mathbb{Q}v$ and $V_n \cap V_x = (V_{n-1} \cap V_x) \oplus \mathbb{Q}v$. Thus our claim is proved.

It remains to show that the field $x \mapsto p_x$ is measurable. For this it suffices to check that the maps $x \mapsto p_x e_n$ are measurable. Let U_1, U_2, \dots be an enumeration of the subspaces of V_n , and let $X_m \subset X$ be the set of x such that $V_n \cap V_x = U_m$. For any m , the set U_m is measurable by assumption and the vector $p_x e_n$ is the same for all $x \in X_m$ by uniqueness of p_x . Hence $x \mapsto p_x e_n$ is measurable. \square

Lemma 2.5. *Let V be a vector space over \mathbb{Q} of countable dimension. Let $T_x, p_x, q_x: V \rightarrow V$ be measurable fields of operators, with p_x and q_x idempotent, such that $\text{im } T_x = \text{im } p_x$ and $\ker T_x = \text{im } q_x$. Denote by S_x the operator which is zero on $\ker p_x$ and is the inverse of $T_x: \ker q_x \rightarrow \text{im } p_x$ on $\text{im } p_x$, so that*

$$T_x S_x = p_x \quad \text{and} \quad S_x T_x = 1 - q_x.$$

Then the field $x \mapsto S_x$ is measurable.

Proof. Let $\{e_1, e_2, \dots\}$ be a basis for V , and enumerate V as $V = \{v_1, v_2, \dots\}$. For $i, j \in \mathbb{N}$, put

$$X_{ij} = \{x \in X : v_i \in \text{im}(1 - q_x), T_x v_i = p_x e_j\}.$$

Then the X_{ij} are measurable with $\bigsqcup_{i=1}^{\infty} X_{ij} = X$ for all $j \in \mathbb{N}$, and so the field of operators given by

$$S_x e_j = v_i \quad \text{for } x \in X_{ij}$$

is measurable. Furthermore, it clearly has the stated properties. \square

We are now ready to prove exactness.

Proposition 2.6. *Let Σ be a contractible simplicial R -complex. Then*

$$0 \leftarrow M(X) \xleftarrow{\varepsilon} C_0(\Sigma) \xleftarrow{\partial_1} C_1(\Sigma) \xleftarrow{\partial_2} \dots$$

is contractible as a complex of $L^\infty(X)$ -modules.

Proof. Consider first the same sequence with rational coefficients. By choosing Borel sections of $\Sigma^n \rightarrow X$ we can embed Σ^n into the trivial fiber space $X \times \mathbb{N}$ over X and then apply Lemma 2.4 to the field of spaces $x \mapsto \ker \partial_{n,x} \subset C_n(\Sigma_x; \mathbb{Q})$. Then we get a field of projections $p_{n,x}: C_n(\Sigma_x; \mathbb{Q}) \rightarrow \ker \partial_{n,x}$ which is measurable in the sense that it determines a well-defined map of $C_n(\Sigma; \mathbb{Q})$ into itself. By contractibility of Σ_x the map $\partial_{n+1,x}$ is an isomorphism of $\ker p_{n+1,x}$ onto $\text{im } p_{n,x}$. By Lemma 2.5 we thus get measurable fields of operators $h_{n,x} = S_x: C_n(\Sigma_x; \mathbb{Q}) \rightarrow C_{n+1}(\Sigma_x; \mathbb{Q})$ such that

$$\text{id} = h_{n-1,x} \partial_{n,x} + \partial_{n+1,x} h_{n,x}. \quad (2.2)$$

on $C_n(\Sigma_x; \mathbb{Q})$ for all $n \geq -1$ (with $C_{-1}(\Sigma_x; \mathbb{Q}) = \mathbb{Q}$, $\partial_{0,x} = \varepsilon_x$ and $h_{-1,x}: \mathbb{Q} \rightarrow C_0(\Sigma_x; \mathbb{Q})$).

Turning to complex coefficients, extend $h_{n,x}$ to operators $C_n(\Sigma_x) \rightarrow C_{n+1}(\Sigma_x)$ by linearity. These operators form a measurable field for every n , since if $f \in C_n(\Sigma)$ is supported on the image of a section of $\Sigma^n \rightarrow X$ then f is an element of $C_n(\Sigma; \mathbb{Q})$ multiplied with a function in $L^\infty(X)$. By (2.2) the maps $h_{n,x}$ define the required homotopy $h_n: C_n(\Sigma) \rightarrow C_{n+1}(\Sigma)$. \square

Notice that the above proposition does not imply that complex (2.1) is exact, only that its homology is zero-dimensional over $L^\infty(X)$.

Proof of Theorem 2.3. Since $L^\infty(X)$ is $L^\infty(X)$ -dense in $M(X)$, by Lemma 1.3 we have

$$\dim_{L(R)} \text{Tor}_n^{\mathbb{C}[R]}(L(R), L^\infty(X)) = \dim_{L(R)} \text{Tor}_n^{\mathbb{C}[R]}(L(R), M(X)).$$

To compute the latter numbers, by Lemma 2.2(i) and Lemma 1.4 we can use the resolution of $M(X)$ given by Proposition 2.6. The result then follows from Proposition 2.1. \square

Gaboriau [2] defined the L^2 -Betti numbers of R by letting $\beta_n^{(2)}(R) = \beta_n^{(2)}(\Sigma, R)$, where Σ is an arbitrary contractible simplicial R -complex. By the above result this definition is equivalent to that of Sauer [4], $\beta_n^{(2)}(R) = \dim_{L(R)} \operatorname{Tor}_n^{\mathbb{C}[R]}(L(R), L^\infty(X))$.

The proof actually shows a bit more. Let Σ be an n -connected simplicial R -complex. Then the sequence

$$0 \leftarrow M(X) \xleftarrow{\varepsilon} C_0(\Sigma) \xleftarrow{\partial_1} C_1(\Sigma) \xleftarrow{\partial_2} \dots \xleftarrow{\partial_{n+1}} C_{n+1}(\Sigma)$$

is exact. Taking a projective $\mathbb{C}[R]$ -resolution of $\ker \partial_{n+1}$ we get a resolution of $M(X)$. Using this resolution to compute $\dim_{L(R)} \operatorname{Tor}_k^{\mathbb{C}[R]}(L(R), L^\infty(X))$ we conclude that

$$\beta_k^{(2)}(\Sigma, R) = \dim_{L(R)} \operatorname{Tor}_k^{\mathbb{C}[R]}(L(R), L^\infty(X)) \quad \text{for } 0 \leq k \leq n$$

and $\beta_{n+1}^{(2)}(\Sigma, R) \geq \dim_{L(R)} \operatorname{Tor}_{n+1}^{\mathbb{C}[R]}(L(R), L^\infty(X))$, which reproves [2, Theorem 3.13].

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