KMS STATES OF QUASI-FREE DYNAMICS ON PIMSNER ALGEBRAS

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Abstract. A continuous one-parameter group of unitary isometries of a right-Hilbert C*-bimodule induces a quasi-free dynamics on the Cuntz-Pimsner C*-algebra of the bimodule and on its Toeplitz extension. The restriction of such a dynamics to the algebra of coefficients of the bimodule is trivial, and the corresponding KMS states of the Toeplitz-Cuntz-Pimsner and Cuntz-Pimsner C*-algebras are characterized in terms of traces on the algebra of coefficients. This generalizes and sheds light onto various earlier results about KMS states of the gauge actions on Cuntz algebras, Cuntz-Krieger algebras, and crossed products by endomorphisms. We also obtain a more general characterization, in terms of KMS weights, for the case in which the inducing isometries are not unitary, and accordingly, the restriction of the quasi-free dynamics to the algebra of coefficients is nontrivial.

Introduction

Soon after the introduction of the Cuntz algebras in [C] it was noticed that the gauge action on $\mathcal{O}_n$ had the unique equilibrium inverse temperature $\beta = \log n$, [OP, Ev, BEK]. Along the same lines, the gauge action on the Cuntz-Krieger algebra $\mathcal{O}_A$ was also shown to have a unique KMS state, at inverse temperature equal to the logarithm of the spectral radius of the irreducible matrix A, [EFW]. Here we are interested in the KMS states of the C*-algebras $\mathcal{T}_X$ and $\mathcal{O}_X$ associated by Pimsner to a right Hilbert bimodule X, [Pim]. Having a rich, yet tractable structure, they provide a convenient framework in which to study the interesting phenomena that characterize the examples mentioned above and many others; see e.g. [PWY] and [EL2]. Specifically, we start with a C*-algebra A and a right Hilbert A bimodule X in which the left action is non-degenerate. Given a continuous one-parameter group of isometries on X, we induce quasi-free dynamics on $\mathcal{T}_X$ and on $\mathcal{O}_X$ via their universal properties, cf [Z]. We then proceed to study the equilibrium states of these quasi-free dynamics associated to groups of isometries in terms of their restrictions to the coefficient algebra A. Our approach underlines the role of the Toeplitz algebra $\mathcal{T}_X$ in its own right and not as a mere preliminary step from which to obtain $\mathcal{O}_X$ as a quotient. The key point, inspired in Evans construction [Ev], is that $\mathcal{T}_X$ acts naturally on the full Fock module over X, and the quasi-free action is implemented there by the Fock quantization of the given group of isometries on X. We use this as a guidance in writing KMS states as quasi–free states, but do not rely on it directly in our arguments. Since $\mathcal{O}_X$ does not act in general on the Fock module, this type of spatial (modular) implementation is lost when one looks at quasi-free dynamics on $\mathcal{O}_X$ alone. However, it is easy to characterize the KMS states on $\mathcal{O}_X$ as those on $\mathcal{T}_X$ that factor through the quotient.

A brief summary of the contents follows. In Section 1 we collect some necessary results about inducing traces from the coefficient algebra A to the algebra of adjointable operators on X. In Section 2, after introducing the Pimsner algebra of a bimodule and the quasi-free dynamics associated to a one-parameter group of isometries of the bimodule, we study the special case of dynamics that fix the elements of A. Under a positivity assumption on the infinitesimal

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generator D of the given group of isometries, we show that the KMS_β states of the quasi-free dynamics are induced from the traces on A that satisfy a certain inequality. This inequality is formulated in terms of a transfer operator between traces (or KMS weights) on A and, essentially, ensures that the Fock quantization of the contraction $e^{-\beta D}$ is an appropriate density operator. In the special case of the gauge action on the Toeplitz Cuntz algebras, the coefficient algebra is $\mathbb{C}$, and its unique trace satisfies the inequality if and only if $\beta \geq \log n$; the resulting KMS_β state of $\mathcal{T}_n$ factors through $\mathcal{O}_n$ only for $\beta = \log n$. At the end of the section we show how to derive, in a unified way, several other examples of KMS states of C*-dynamical systems previously studied under different guises. In Section 3 we extend our results to the more general situation in which the dynamics on the coefficient algebra A is allowed to be nontrivial. Most of the section is devoted to inducing KMS states on A. There is no great simplification at this point in restricting ourselves to states, so in fact we consider KMS weights on A. For von Neumann algebras this problem was studied by Combes and Zettl in [CZ, Section 3], who deduced the existence of induced weights from the well-known cocycle theorem of Connes. Besides giving a slightly different contruction which works equally well for C*-algebras, we provide also a direct proof and indicate how Connes' result can be derived from our results on induced weights. Once the induction procedure for weights has been settled, the characterization of KMS states in terms of their restrictions to A is entirely analogous to that of KMS states in Section 2.

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1. Preliminary results on induced traces.

If A is an algebra and X is a right projective A-module of finite type, then the algebra $\mathrm{End}_A(X)$ is isomorphic to $X \otimes_A \mathrm{Hom}_A(X, A)$. Hence there exists a unique linear map $\mathrm{Tr}\colon \mathrm{End}_A(X) \to A/[A,A]$ such that $\mathrm{Tr}(x \otimes f) = f(x) \bmod [A,A]$. The composition of any tracial linear functional τ on A (one for which $\tau(ab) = \tau(ba)$) with Tr yields an (induced) tracial linear functional Tr_τ on $\mathrm{End}_A(X)$. Clearly, this construction can be applied to any unital C*-algebra A and finite Hilbert A-module X. Our first aim in this section is to define Tr_τ for arbitrary C*-algebras and Hilbert modules, and to derive some of its basic properties.

Suppose X is a right Hilbert module over a C*-algebra A and let $K(X)$ be the C*-algebra of generalized compact operators on X, generated by the operators $\theta_{\xi,\zeta}$, given by $\theta_{\xi,\zeta}\eta = \xi\langle\zeta, \eta\rangle$, with $\xi, \zeta, \eta \in X$. Let $B(X)$ be its multiplier algebra, that is, the C*-algebra of adjointable operators on X. Recall that a bounded net $\{S_k\}_k$ in $B(X)$ converges to $S \in B(X)$ strictly if and only if $S_k\xi \to S\xi$ and $S_k^*\xi \to S^*\xi$ for every $\xi \in X$. We shall need the following result about inducing traces from A through X. The existence and some of the properties of the induced trace Tr_τ can be found in [CZ, Section 2], and [CP1, Lemma 4.6], where they are derived from previous results of Pedersen [P] about extending traces from hereditary subalgebras. We state the relevant properties in a way that is convenient for our purposes, and we include a self-contained, direct proof for completeness.

Theorem 1.1. *Let τ be a finite trace on A. For $T \in B(X)$, $T \geq 0$, set*

$$\mathrm{Tr}_\tau(T) = \sup_I \sum_{\xi \in I} \tau(\langle \xi, T\xi\rangle),$$

where the supremum is taken over all finite subsets I of X such that $\sum_{\xi\in I} \theta_{\xi,\xi} \leq 1$. Define

$$\mathcal{M}_\tau^+ = \{T \geq 0 \mid \mathrm{Tr}_\tau(T) < \infty\}, \quad \mathcal{N}_\tau = \{T \mid T^*T \in \mathcal{M}_\tau^+\}, \quad \mathcal{M}_\tau = \mathrm{span}\, \mathcal{M}_\tau^+ = \mathcal{N}_\tau^* \mathcal{N}_\tau.$$

Then

(i) Tr_τ *is strictly lower semicontinuous, moreover, if* $\liminf_k \tau(\langle \xi, T_k\xi\rangle) \ge \tau(\langle \xi, T\xi\rangle)$ *for every* $\xi \in X$*, then* $\liminf_k \mathrm{Tr}_\tau(T_k) \ge \mathrm{Tr}_\tau(T)$*;*

(ii) *if* $\{e_k = \sum_{\xi\in I_k} \theta_{\xi,\xi}\}_k$ *is a net such that* $e_k \le 1$ *and* $\tau(\langle \eta, e_k\eta\rangle) \to \tau(\langle \eta,\eta\rangle)$ *for every* $\eta \in X$ *(e.g. if* $\{e_k\}_k$ *is an approximate unit in* $K(X)$*), then* $\mathrm{Tr}_\tau(T) = \lim_k \sum_{\xi\in I_k} \tau(\langle \xi, T\xi\rangle)$ *for* $T \in B(X)_+$*; in particular,* Tr_τ *can be extended to a positive linear functional on* $\mathcal{M}_\tau$*;*

(iii) *for every pair* $\xi, \eta \in X$*, we have that* $\theta_{\xi,\eta} \in \mathcal{M}_\tau$ *and* $\mathrm{Tr}_\tau(\theta_{\xi,\eta}) = \tau(\langle \eta, \xi\rangle)$*;*

(iv) Tr_τ *is a semifinite trace; thus* $\mathcal{N}_\tau$ *and* $\mathcal{M}_\tau$ *are two-sided ideals in* $B(X)$*,* $\mathcal{M}_\tau$ *is essential, and if* $S, T \in \mathcal{N}_\tau$*, or if* $S \in B(X)$ *and* $T \in \mathcal{M}_\tau$*, then* $\mathrm{Tr}_\tau(ST) = \mathrm{Tr}_\tau(TS)$*.*

Proof. The proof of (i) is trivial. To prove (ii) we shall first prove that $\mathrm{Tr}_\tau(\theta_{\xi,\xi}) = \tau(\langle \xi,\xi\rangle)$. Suppose $S = \sum_{\eta\in I} \theta_{\eta,\eta} \le 1$. Then

$$\sum_{\eta\in I} \tau(\langle \eta, \theta_{\xi,\xi}\eta\rangle) = \sum_{\eta\in I} \tau(\langle \eta,\xi\rangle\langle \xi,\eta\rangle) = \sum_{\eta\in I} \tau(\langle \xi,\eta\rangle\langle \eta,\xi\rangle) = \tau(\langle \xi, S\xi\rangle),$$

and the equality $\mathrm{Tr}_\tau(\theta_{\xi,\xi}) = \tau(\langle \xi,\xi\rangle)$ follows. The same proof shows that $\mathrm{Tr}_\tau(\sum_{\xi\in I} \theta_{\xi,\xi}) = \sum_{\xi\in I} \tau(\langle \xi,\xi\rangle)$ for any finite set I. Hence if $\{e_k\}_k$ is as in the formulation of (ii) and $T \in B(X)_+$, then

$$\mathrm{Tr}_\tau(T^{\frac12} e_k T^{\frac12}) = \mathrm{Tr}_\tau(\sum_{\xi\in I_k} \theta_{T^{\frac12}\xi, T^{\frac12}\xi}) = \sum_{\xi\in I_k} \tau(\langle \xi, T\xi\rangle).$$

Since $\lim_k \mathrm{Tr}_\tau(T^{\frac12} e_k T^{\frac12}) = \mathrm{Tr}_\tau(T)$ by property (i), (ii) follows. Part (iii) has already been proved for $\xi = \eta$, the general case follows by polarization, and implies that $\mathcal{M}_\tau$ is essential. Since $\theta_{u\xi,u\xi} = u\theta_{\xi,\xi}u^*$, it is obvious that $\mathrm{Tr}_\tau(uTu^*) = \mathrm{Tr}_\tau(T)$ for any unitary $u \in B(X)$. Thus Tr_τ is a trace. □

Suppose now Y is a right Hilbert A-bimodule, that is, Y is a right Hilbert A-module together with a left action of A given by a $*$-homomorphism of A into $B(Y)$. Denote by $B_A(Y)$ the subalgebra of $B(Y)$ consisting of A-bimodule maps. Let X be another right Hilbert A-module. Then the tensor product $X \otimes_A Y$ is a right Hilbert A-module and for any $S \in B(X)$ and $T \in B_A(Y)$ there is an operator $S \otimes T \in B(X \otimes_A Y)$. We shall need the following property about the induction in stages of a trace τ on A through a tensor product of modules. We indicate explicitly in the proposition the module used to induce the trace in each case, but we drop this notation later for simplicity, and rely on the context to indicate the relevant module.

Proposition 1.2. *Let* $S \in B(X)_+$*, and* $T \in B_A(Y)_+$*. Suppose* τ *is a finite trace on* A *such that* $\mathrm{Tr}^Y_\tau(T) < \infty$*, and define a new finite trace* τ_T *on* A *by letting* $\tau_T(a) = \mathrm{Tr}^Y_\tau(aT)$*. Then* $\mathrm{Tr}^{X\otimes Y}_\tau(S \otimes T) = \mathrm{Tr}^X_{\tau_T}(S)$*.*

Proof. We begin by constructing an approximate unit for $K(X \otimes_A Y)$ from approximate units for $K(X)$ and $K(Y)$. For finite subsets $I \subset X$ and $J \subset Y$, define the operators $e_I = \sum_{\xi\in I} \theta_{\xi,\xi}$, $e_J = \sum_{\zeta\in J} \theta_{\zeta,\zeta}$ and $e_{I,J} = \sum_{\xi\in I,\zeta\in J} \theta_{\xi\otimes\zeta,\xi\otimes\zeta}$. We claim that if $e_I \le 1$ and $e_J \le 1$, then $e_{I,J} \le 1$. To verify this we consider a vector $\eta = \sum_k \mu_k \otimes \nu_k$ in $X \otimes_A Y$; and observe that

$$\langle e_{I,J}\eta, \eta\rangle = \sum_{\xi\in I} \langle e_J\delta_\xi, \delta_\xi\rangle,$$

where $\delta_\xi = \sum_k \langle \xi, \mu_k\rangle \nu_k$. If $e_I \le 1$, then $(\langle e_I\mu_k, \mu_l\rangle)_{k,l} \le (\langle \mu_k, \mu_l\rangle)_{k,l}$ in the algebra $\mathrm{Mat}_{|I|}(A)$ of $|I|$ by $|I|$ matrices over A, so if, in addition, $e_J \le 1$ we get

$$\sum_{\xi\in I} \langle e_J\delta_\xi, \delta_\xi\rangle \le \sum_{\xi\in I} \langle \delta_\xi, \delta_\xi\rangle = \sum_{k,l} \langle \nu_k, \langle e_I\mu_k, \mu_l\rangle \nu_l\rangle \le \sum_{k,l} \langle \nu_k, \langle \mu_k, \mu_l\rangle \nu_l\rangle = \langle \eta, \eta\rangle,$$

which proves the claim. Note also that

$$e_{I,J}(\mu \otimes \nu) = e_I\mu \otimes \nu + \sum_{\xi\in I} \xi \otimes (e_J - 1)\langle \xi, \mu\rangle \nu.$$

It follows that there exists an approximate unit in $K(X\otimes_A Y)$ consisting of elements of the form $e_{I,J}$, with $e_I\le 1$ and $e_J\le 1$, so, by Theorem 1.1(ii),

$$\mathrm{Tr}_\tau^{X\otimes Y}(S\otimes T)=\sup_{I,J}\sum_{\xi\in I,\zeta\in J}\tau(\langle\zeta,\langle\xi,S\xi\rangle T\zeta\rangle).$$

Since for fixed I,

$$\sup_J\sum_{\xi\in I,\zeta\in J}\tau(\langle\zeta,\langle\xi,S\xi\rangle T\zeta\rangle)=\sum_{\xi\in I}\mathrm{Tr}_\tau^Y(\langle\xi,S\xi\rangle T)=\sum_{\xi\in I}\tau_T(\langle\xi,S\xi\rangle),$$

recalling that $\sup_I\sum_{\xi\in I}\tau_T(\langle\xi,S\xi\rangle)=\mathrm{Tr}_{\tau_T}^X(S)$, we get $\mathrm{Tr}_\tau^{X\otimes Y}(S\otimes T)=\mathrm{Tr}_{\tau_T}^X(S)$. □

We have shown that any finite trace on A can be induced to a unique, strictly densely defined, strictly lower semicontinuous trace Tr_τ on $B(X)$ such that $\mathrm{Tr}_\tau(\theta_{\xi,\eta})=\tau(\langle\eta,\xi\rangle)$. Clearly one should not expect all strictly densely defined, strictly lower semicontinuous traces on $B(X)$ to be induced from finite traces on A. In Section 3 below we generalize this extension procedure so that it applies to KMS weights of quasi-free dynamics. By setting the dynamics to be trivial, we then obtain, as a corollary, a bijective correspondence between densely defined lower semicontinuous traces on $\overline{\langle X,X\rangle}$ and strictly densely defined, strictly lower semicontinuous traces on $B(X)$.

2. KMS states on Pimsner algebras.

Next we consider a Hilbert A-bimodule X, with the purpose of studying KMS-states on the Toeplitz-Pimsner and Cuntz-Krieger-Pimsner algebras associated in [Pim] to such a bimodule. The only extra assumption that we make on the bimodule is the non-degeneracy of the left action, i.e. $AX=X$. In particular, we do not assume that X is full or that A is unital. We denote by $i_X\colon A\to B(X)$ the homomorphism defining the left action of A on X.

Let $\mathcal{F}(X)=A\oplus X\oplus(X\otimes_A X)\oplus\dots$ be the Fock Hilbert bimodule of X. The Toeplitz-Pimsner algebra $\mathcal{T}_X$ of X is, by definition, the C*-algebra of operators on $\mathcal{F}(X)$ generated by the left multiplication operators $i_{\mathcal{F}(X)}(a)$ for $a\in A$ and the left creation operators T_ξ for $\xi\in X$, which are given by $T_\xi(\xi_1\otimes\dots\otimes\xi_n)=\xi\otimes\xi_1\otimes\dots\otimes\xi_n$. It is shown in [Pim, FR] that $\mathcal{T}_X$ is the universal C*-algebra generated by elements $\pi(a)$ with $a\in A$ and T_ξ with $\xi\in X$, such that $\pi\colon A\to\mathcal{T}_X$ is a $*$-homomorphism, $X\ni\xi\mapsto T_\xi$ is an A-bilinear map, (that is, $T_{\xi a}=T_\xi\pi(a)$ and $T_{a\xi}=\pi(a)T_\xi$), and $T_\xi^*T_\zeta=\pi(\langle\xi,\zeta\rangle)$. More precisely, the Fock realization of these relations, given by the left action of A and the left creation operators on $\mathcal{F}(X)$, determines an isomorphism of the universal C*-algebra of the relations onto $\mathcal{T}_X$.

Let $j_X\colon K(X)\to\mathcal{T}_X$ be the injective homomorphism given by $j_X(\theta_{\xi,\eta})=T_\xi T_\eta^*$. Let also I_X be the ideal in A consisting of elements $a\in A$ such that $i_X(a)\in K(X)$. The Cuntz-Krieger-Pimsner algebra $\mathcal{O}_X$ is, by definition, the quotient of $\mathcal{T}_X$ by the ideal generated by elements of the form $\pi(a)-(j_X\circ i_X)(a)$, for $a\in I_X$. We shall usually omit π and i_X in the computations below.

Let $\mathbb{R}\ni t\to\sigma_t$ be a one-parameter automorphism group of A and let $\mathbb{R}\ni t\mapsto U_t$ be a one-parameter group of isometries on X such that $U_ta\xi=\sigma_t(a)U_t\xi$ and $\langle U_t\xi,U_t\zeta\rangle=\sigma_t(\langle\xi,\zeta\rangle)$; as usual, both σ and U are assumed to be strongly continuous. By the universal property of the Toeplitz-Pimsner algebra there exists, for each $t\in\mathbb{R}$, a unique automorphism γ_t of $\mathcal{T}_X$ such that $\gamma_t(a)=\sigma_t(a)$ and $\gamma_t(T_\xi)=T_{U_t\xi}$. The resulting one-parameter group $t\mapsto\gamma_t$ is strongly continuous and is called the *quasi-free dynamics* associated to the module dynamics U. Since A and $j_X(K(X))$ are invariant under γ_t, so is I_X, and thus there is a quasi-free dynamics at the level of $\mathcal{O}_X$, too. When we view $\mathcal{T}_X$ as acting on the Fock bimodule, the automorphisms γ_t are implemented by the 'Fock quantization' of the isometries U_t; specifically $\gamma_t=\mathrm{Ad}\,\Gamma(U_t)$, where $\Gamma(U_t)=1\oplus U_t\oplus(U_t\otimes U_t)\oplus\dots$. The group of gauge transformations is a particular case of this, corresponding to the trivial dynamics on A and the one-parameter scalar unitary group $\{\xi \mapsto e^{it}\xi\}_{t\in\mathbb{R}}$ on X.

Given a quasi-free dynamics on $\mathcal{T}_X$, we are interested in the relation between the (σ,β)-KMS states on A and the (γ,β)-KMS states on $\mathcal{T}_X$. For simplicity we shall first consider this question under the assumption that the dynamics σ on A is trivial. This covers most examples in the literature, and has the advantage of being tractable using the elementary properties of induced traces from the preceding section. The case of nontrivial σ requires a generalization of these properties to induced KMS-weights, a task that we take up in the next section. Accordingly, we now restrict our attention to one-parameter groups of isometries U_t of X such that $U_t a\xi = aU_t\xi$ and $\langle U_t\xi, U_t\zeta\rangle = \langle \xi,\zeta\rangle$, in other words, we assume that U is a one-parameter *unitary* group in $B_A(X)$.

Theorem 2.1. *Let $\mathbb{R} \ni t \to U_t$ be a one-parameter unitary group in $B_A(X)$ satisfying the following 'positive energy' condition: the vectors $\xi \in X$ such that $\mathrm{Sp}_U(\xi) \subset (0,+\infty)$ form a dense subspace of X, where $\mathrm{Sp}_U(\xi)$ is the Arveson spectrum of ξ with respect to U. Let γ be the corresponding dynamics on $\mathcal{T}_X$, given by $\gamma_t(T_\xi) = T_{U_t\xi}$ and $\gamma_t(a) = a$, and suppose $\beta \in (0,\infty)$. If ϕ is a (γ,β)-KMS state on $\mathcal{T}_X$, then $\tau = \phi|_A$ is a tracial state on A and*

$$\mathrm{Tr}_\tau(ae^{-\beta D}) \le \tau(a) \quad \text{for} \quad a \in A_+, \tag{2.1}$$

where D is the generator of U (so that $U_t = e^{itD}$). Conversely, if τ is a tracial state on A such that (2.1) is satisfied, then there exists a unique (γ,β)-KMS state ϕ on $\mathcal{T}_X$ with $\phi|_A = \tau$. The state ϕ is determined by τ through

$$\phi(T_{\xi_1}\cdots T_{\xi_m}T^*_{\eta_n}\cdots T^*_{\eta_1}) = \begin{cases} \tau(\langle \eta_1\otimes\cdots\otimes\eta_n\,,\, e^{-\beta D}\xi_1\otimes\cdots\otimes e^{-\beta D}\xi_n\rangle) & \text{if } m=n \\ 0 & \text{otherwise.}\end{cases} \tag{2.2}$$

Proof. Note that our positive energy condition is equivalent to the existence of an increasing sequence of U-invariant submodules Y_n of X having dense union and such that $D|_{Y_n} \ge c_n 1$ with $c_n > 0$. It follows from this that $e^{-\beta D}$ is a selfadjoint contraction in $B_A(X)$ for each $\beta > 0$.

Assume first that ϕ is a (γ,β)-KMS state. Since the left action of A on X is non-degenerate, the homomorphism $\pi\colon A \to \mathcal{T}_X$ is non-degenerate. Hence $\tau = \phi|_A$ is a state. Since γ is trivial on A, τ is a trace.

For any $\xi \in X$ we have $\gamma_{\frac{i\beta}{2}}(T_\xi) = T_{e^{-\frac{\beta D}{2}}\xi}$, so by the KMS condition we get

$$\phi(T_\xi T^*_\eta) = \phi(\gamma_{\frac{i\beta}{2}}(T_\eta)^*\gamma_{\frac{i\beta}{2}}(T_\xi)) = \tau(\langle e^{-\frac{\beta D}{2}}\eta, e^{-\frac{\beta D}{2}}\xi\rangle) = \mathrm{Tr}_\tau(\theta_{\xi,\eta}e^{-\beta D}).$$

Thus, for $a \in A_+$ we have

$$\mathrm{Tr}_\tau(a^{\frac12}\theta_{\xi,\xi}a^{\frac12}e^{-\beta D}) = \mathrm{Tr}_\tau(\theta_{a^{\frac12}\xi,a^{\frac12}\xi}e^{-\beta D}) = \phi(T_{a^{\frac12}\xi}T^*_{a^{\frac12}\xi}) = \phi(a^{\frac12}j_X(\theta_{\xi,\xi})a^{\frac12}).$$

Since $\phi(a^{\frac12}j_X(\cdot)a^{\frac12})$ is a positive linear functional on $K(X)$ of norm less than or equal to $\tau(a)$, by the strict lower semicontinuity of Tr_τ we conclude that $\mathrm{Tr}_\tau(ae^{-\beta D}) \le \tau(a)$.

Let us now prove that ϕ is completely determined by τ. First note that there is an A-bilinear isometry of $X^{\otimes n}$ to $\mathcal{T}_X$ mapping $\xi = \xi_1 \otimes_A \ldots \otimes_A \xi_n$ to $T_\xi = T_{\xi_1}\ldots T_{\xi_n}$, so that A and the elements of the form $T_\xi T^*_\zeta$, with $\xi \in X^{\otimes n}$ and $\zeta \in X^{\otimes m}$, span a dense subspace of $\mathcal{T}_X$. The same computation as above shows that for $\xi,\zeta \in X^{\otimes n}$ we have $\phi(T_\xi T^*_\zeta) = \tau(\langle (e^{-\frac{\beta D}{2}})^{\otimes n}\zeta, (e^{-\frac{\beta D}{2}})^{\otimes n}\xi\rangle)$. So, in order to prove that ϕ is uniquely determined by τ and that (2.2) holds, it suffices to show that $\phi(T_\xi T^*_\zeta) = 0$ when $\xi \in X^{\otimes n}$, $\zeta \in X^{\otimes m}$ and $n \ne m$. We may assume that $n > m$ and $\xi = \xi_1 \otimes \xi_2$ with $\xi_1 \in X^{\otimes m}$ and $\xi_2 \in X^{\otimes (n-m)}$. Using the KMS condition we get

$$\phi(T_\xi T^*_\zeta) = \phi(T_{\xi_1}T_{\xi_2}T^*_\zeta) = \phi(T_{\xi_2}T^*_\zeta T_{(e^{-\beta D})^{\otimes m}\xi_1}) = \phi(T_{\xi_2\langle\zeta,(e^{-\beta D})^{\otimes m}\xi_1\rangle}).$$

Thus, it is enough to show that $\phi(T_\xi) = 0$ for every $\xi \in X^{\otimes n}$ with $n \ge 1$. We claim that the elements of the form $\eta - (U_t)^{\otimes n}\eta$ span a dense subspace of $X^{\otimes n}$; this will finish the proof because $\phi(T_{\eta-(U_t)^{\otimes n}\eta}) = 0$ by virtue of the γ_t-invariance of ϕ. To prove the claim, suppose that $\xi \in X$ is such that $\mathrm{Sp}_U(\xi)$ is compact and $0 \notin \mathrm{Sp}_U(\xi)$, and notice that such elements are dense because of our assumption on the spectrum of U. Choose $t_0 \in \mathbb{R}$ such that the function $t \mapsto 1 - e^{itt_0}$ is non-zero on $\mathrm{Sp}_U(\xi)$; then $\xi = (1 - U_{t_0})U_f\xi$ for every function $f \in L^1(\mathbb{R})$ such that $\hat{f}(t) = (1 - e^{itt_0})^{-1}$ for t in a neighbourhood of $\mathrm{Sp}_U(\xi)$. Hence ξ is of the form $\eta - U_t\eta$, proving the claim for $n = 1$. Since $\mathrm{Sp}_{U^{\otimes n}}(\xi_1 \otimes \ldots \otimes \xi_n) \subset \overline{\mathrm{Sp}_U(\xi_1) + \ldots + \mathrm{Sp}_U(\xi_n)}$ by [A], $U^{\otimes n}$ satisfies the same spectral assumption as U, and the above argument also proves the claim for $n > 1$. This finishes the proof that ϕ is determined by τ.

Denote by F the operator, mapping finite traces on A into possibly infinite traces on A, defined by $(F\tau)(a) = \mathrm{Tr}_\tau(ae^{-\beta D})$. The second part of the theorem says that if $F\tau \le \tau$ for a tracial state τ on A, then there exists a (γ, β)-KMS state ϕ on $\mathcal{T}_X$ such that $\phi|_A = \tau$. Suppose for a moment that the tracial state τ is of the form $\tau = \sum_{n=0}^\infty F^n\tau_0$ for some finite trace τ_0 (such a state clearly satisfies $F\tau \le \tau$, in fact $F^n\tau \searrow 0$). We claim that in this case the extension is given by $\Phi = \mathrm{Tr}_{\tau_0}(\cdot\,\Gamma(e^{-\beta D}))$, where $\Gamma(e^{-\beta D}) = \sum_n (e^{-\beta D})^{\otimes n}$ is the operator on $\mathcal{F}(X)$ obtained by 'Fock quantization' of the contraction $e^{-\beta D}$. Indeed, it is easy to see that Φ is a positive linear functional with the KMS property, but one must still verify that Φ is a state extending τ. Using Proposition 1.2 we see by induction that $\mathrm{Tr}_{\tau_0}(\cdot\,(e^{-\beta D})^{\otimes n})|_A = F^n\tau_0$, whence $\Phi|_A = \tau$. Since the left action of A on $\mathcal{F}(X)$ is non-degenerate, and Φ is strictly lower semicontinuous by Theorem 1.1, this implies that Φ is a state. Thus $\phi = \Phi|_{\mathcal{T}_X}$ is the required (γ, β)-KMS state extending $\tau = \sum_{n=0}^\infty F^n\tau_0$.

Suppose now that τ is an arbitrary tracial state such that $F\tau \le \tau$. For each $\varepsilon > 0$ consider a one-parameter unitary group U^ε defined by $U^\varepsilon_t\xi = e^{i\varepsilon t}U_t\xi$. Let γ^ε be the associated quasi-free dynamics on $\mathcal{T}_X$. For the corresponding operator F_ε on traces of A we have $F_\varepsilon = e^{-\beta\varepsilon}F$. In particular, $F_\varepsilon\tau \le e^{-\beta\varepsilon}\tau$. Then we may write $\tau = \sum_{n=0}^\infty F_\varepsilon^n\tau_\varepsilon$, with $\tau_\varepsilon = \tau - F_\varepsilon\tau$; indeed, since $F_\varepsilon^m\tau \le e^{-\beta\varepsilon m}\tau \to 0$ as $m \to \infty$, we have $\sum_{n=0}^m F_\varepsilon^n\tau_\varepsilon = \tau - F_\varepsilon^{m+1}\tau \to \tau$. Hence there exists a $(\gamma^\varepsilon, \beta)$-KMS state ϕ_ε on $\mathcal{T}_X$ such that $\phi_\varepsilon|_A = \tau$. As in [BR, Proposition 5.3.25], any weak* limit point of the states ϕ_ε as $\varepsilon \to 0^+$ is a (γ, β)-KMS state ϕ extending τ. ☐

The situation for ground states is slightly different, since for $\beta = \infty$ there is no tracial condition on A. For each state ω of A we define a *generalized Fock state* ϕ_ω of $\mathcal{T}_X$ by $\phi_\omega(T) = \lim_\lambda \omega(\langle e_\lambda, Te_\lambda\rangle)$ for $T \in \mathcal{T}_X$; where $(e_\lambda)_{\lambda\in\Lambda}$ is an approximate unit in A. Clearly $\omega = \phi_\omega|_A$ and ϕ_ω is characterized by $\phi_\omega(a) = \omega(a)$ for $a \in A$, and $\phi_\omega(T_\xi T_\eta^*) = 0$ for $\xi \in X^{\otimes m}$, and $\eta \in X^{\otimes n}$ with m or n nonzero.

Theorem 2.2. *Under the assumptions of Theorem 2.1, a state of $\mathcal{T}_X$ is a ground state for γ, if and only if it is a generalized Fock state.*

Proof. Suppose first that ϕ is a generalized Fock state of $\mathcal{T}_X$. To see that whenever $b \in \mathcal{T}_X$, $\xi \in X^{\otimes m}$ and $\eta \in X^{\otimes n}$, the analytic function $z \mapsto \phi(b\gamma_z(T_\xi T_\eta^*))$ is bounded on the upper half plane, we write

$$|\phi(b\gamma_z(T_\xi T_\eta^*))| = |\phi(bT_{U_z\xi}T^*_{U_{\bar{z}}\eta})| \le \phi(bb^*)^{\frac{1}{2}}\phi(T_{U_{\bar{z}}\eta}T^*_{U_z\xi}T_{U_z\xi}T^*_{U_{\bar{z}}\eta})^{\frac{1}{2}};$$

the right hand side vanishes for $n > 0$ because $\phi(T_{U_{\bar{z}}\eta}T^*_{U_{\bar{z}}\eta}) = 0$, and it is bounded for $n = 0$ because $\|U_z\xi\| \le \|\xi\|$ for z in the upper half plane. Hence ϕ is a ground state.

Suppose next that ϕ is a ground state. By [BR, Proposition 5.3.19(4)],

$$\phi(\gamma_f(T)^*\gamma_f(T)) = 0, \qquad (T \in \mathcal{T}_X)$$

for every function $f \in L^1(\mathbb{R})$ such that $\operatorname{supp}\hat{f} \subset (-\infty, 0)$. By the positivity condition, the set of all $\xi \in X$ such that $\mathrm{Sp}_U(\xi)$ is a compact subset of $(0, \infty)$ is dense in X. For each such ξ one has that $\mathrm{Sp}_\gamma(T_\xi^*)$ is a compact subset of $(-\infty, 0)$, and there exists a function f as above such that $\gamma_f(T_\xi^*) = T_\xi^*$. Putting $T = T_\xi^*$ one sees that $\phi(T_\xi T_\xi^*) = 0$ for every ξ in a dense set and hence for all $\xi \in X$. Since the one-parameter group $U^{\otimes n}$ on $X^{\otimes n}$ satisfies the same positivity condition, this implies that ϕ is a generalized Fock state. □

Suppose $\gamma = \operatorname{Ad}\Gamma(U)$ is a quasi-free dynamics satisfying the hypothesis of Theorem 2.1, let ϕ be a (γ,β)-KMS state on $\mathcal{T}_X$, and let $\tau = \phi|_A$. In the course of the proof of Theorem 2.1 we let $(F\tau)(a) = \operatorname{Tr}_\tau(ae^{-\beta D})$ and showed that if $\tau = \sum_{n=0}^\infty F^n\tau_0$ for some finite trace τ_0 on A, then ϕ has a canonical extension to a strictly continuous state on $B(\mathcal{F}(X))$. We shall see later in Theorem 3.2 that, in general, the existence of τ_0 is also necessary for ϕ to have a strictly continuous state extension to $B(\mathcal{F}(X))$. If $\tilde{\phi}$ is the strongly continuous extension of ϕ to $B(\mathcal{F}(X))$, then $\tau_0(a) = \tilde{\phi}(P_0aP_0)$, where P_0 is the projection onto the 'vacuum', i.e. the zeroth component $X^0 = A \subset \mathcal{F}(X)$.

Definition 2.3. Let ϕ be a (γ,β)-KMS state and set $\tau = \phi|_A$. Following [EL2] we say that ϕ is of *finite type* if $\tau = \sum_{n=0}^\infty F^n\tau_0$ for some finite trace τ_0 and we say that ϕ is of *infinite type* if $F(\tau) = \tau$. Since $F\tau \le \tau$ and $F^n\tau(1) = \operatorname{Tr}_\tau(1\,(e^{-\beta D})^{\otimes n})$, we see that ϕ is of finite type iff $\operatorname{Tr}_\tau((e^{-\beta D})^{\otimes n}) \to 0$, and of infinite type iff $\operatorname{Tr}_\tau(e^{-\beta D}) = 1$.

Note that in the last part of the proof of Theorem 2.1 we showed that every (γ,β)-KMS state is a weak* limit of $(\gamma^\varepsilon,\beta)$-KMS states of finite type as the perturbation ε tends to zero.

With the appropriate convention, the above definition makes sense also for $\beta = \infty$, and it is clear that KMS_∞-states are necessarily of finite type. As in several other similar contexts, there is a 'Wold decomposition' for (γ,β)-KMS states of $\mathcal{T}_X$.

Proposition 2.4. *Under the assumptions of Theorem 2.1 let ϕ be a (γ,β)-KMS state on $\mathcal{T}_X$. Then there exists a unique convex decomposition $\phi = \lambda\phi_1 + (1-\lambda)\phi_2$ such that ϕ_1 is a (γ,β)-KMS state of finite type and ϕ_2 is a (γ,β)-KMS state of infinite type.*

Proof. By Theorem 2.1 we may carry out the decomposition at the level of traces on A; that is, we prove that for any finite trace τ on A such that $F\tau \le \tau$ there exists a unique decomposition $\tau = \tau_1 + \tau_2$ with $\tau_1 = \sum_{n=0}^\infty F^n\tau_0$ and $F\tau_2 = \tau_2$. The uniqueness is obvious, since τ_0 must equal $\tau - F\tau$. To prove existence, set $\tau_0 = \tau - F\tau$ and $\tau_2 = \lim_n F^n\tau$ (the limit exists because $F^{n+1}\tau \le F^n\tau$). Then $\sum_{n=0}^m F^n\tau_0 = \tau - F^{m+1}\tau \to \tau - \tau_2$ weakly.

The only thing left to check is that $F\tau_2 = \tau_2$. Since $\tau_2 \le F^n\tau$, we have $F\tau_2 \le F^{n+1}\tau$, and so $F\tau_2 \le \tau_2$. For $a \in A_+$ and $\varepsilon > 0$ we can find a finite subset I of X such that $S_I := \sum_{\xi\in I}\theta_{\xi,\xi} \le 1$ and $\operatorname{Tr}_\tau((1-S_I)ae^{-\beta D}) < \varepsilon$. Since $F^n\tau \le \tau$, we have $\operatorname{Tr}_{F^n\tau}((1-S)ae^{-\beta D}) < \varepsilon$ for every n. Since $F^n\tau$ converges to τ_2 weakly, there exists n such that

$$\operatorname{Tr}_{F^n\tau}(Sae^{-\beta D}) < \operatorname{Tr}_{\tau_2}(Sae^{-\beta D}) + \varepsilon,$$

so

$$\begin{aligned}(F\tau_2)(a) &\ge \operatorname{Tr}_{\tau_2}(Sae^{-\beta D}) > \operatorname{Tr}_{F^n\tau}(Sae^{-\beta D}) - \varepsilon > \operatorname{Tr}_{F^n\tau}(ae^{-\beta D}) - 2\varepsilon \\ &= (F^{n+1}\tau)(a) - 2\varepsilon \ge \tau_2(a) - 2\varepsilon,\end{aligned}$$

since ε was arbitrary, this yields $F\tau_2 \ge \tau_2$, and hence $F\tau_2 = \tau_2$. □

We now turn our attention to the KMS states of $\mathcal{O}_X$. Notice that if ϕ is a (γ,β)-KMS state on $\mathcal{O}_X$ and we compose it with the quotient map $\mathcal{T}_X \to \mathcal{O}_X$, we obtain a (γ,β)-KMS state on $\mathcal{T}_X$. Thus in order to describe the KMS-states on $\mathcal{O}_X$ we only need to describe the KMS-states on $\mathcal{T}_X$ that vanish on the kernel of the quotient map.

Theorem 2.5. *Under the assumptions of Theorem 2.1 suppose ϕ is a (γ,β)-KMS state on $\mathcal{T}_X$ and let $\tau = \phi|_A$. Then ϕ defines a state on $\mathcal{O}_X$ if and only if $\operatorname{Tr}_\tau(ae^{-\beta D}) = \tau(a)$ for every $a \in I_X$ (where $\operatorname{Tr}_\tau(ae^{-\infty D}) = 0$, by convention).*

Proof. Suppose first $\beta < \infty$. From the proof of Theorem 2.1 we know that $\phi \circ j_X = \mathrm{Tr}_\tau(\cdot\, e^{-\beta D})$ on $K(X)$. Thus, $\mathrm{Tr}_\tau(ae^{-\beta D}) = \tau(a)$ is equivalent to $\phi(j_X(a)) = \phi(a)$ for $a \in I_X$. This is clearly a necessary condition for ϕ to define a state of $\mathcal{O}_X$. To see that it is also sufficient, let P be the projection onto the zeroth component of the Fock module $\mathcal{F}(X)$; then $b - j_X(b) = PbP = bP$ for $b \in I_X$ so that $(a - j_X(a))^*(a - j_X(a)) = a^*a - j_X(a^*a)$ and hence $\phi((a - j_X(a))^*(a - j_X(a))) = 0$ for every $a \in I_X$. Since ϕ is a KMS state, the set $N = \{x \in \mathcal{T}_X \mid \phi(x^*x) = 0\}$ is a two-sided ideal in $\mathcal{T}_X$. Hence it contains the ideal generated by the elements $a - j_X(a)$ with $a \in I_X$, which is, by definition, the kernel of the map $\mathcal{T}_X \to \mathcal{O}_X$.

For $\beta = \infty$, ϕ is a generalized Fock state of $\mathcal{T}_X$, so $\phi \circ j_X = 0$. If ϕ defines a state on $\mathcal{O}_X$, then $\phi(a) = \phi(a - j_X(a)) = 0$ for $a \in I_X$. Conversely, suppose ϕ vanishes on I_X. The generalized Fock state ϕ extends to a state on $B(\mathcal{F}(X))$ with the property $\phi(x) = \phi(PxP)$. Then for any $x, y \in B(\mathcal{F}(X))$ and $a \in I_X$ we get $\phi(x(a - j_X(a))y) = \phi(PxPaPyP) = 0$, since $PxPIPyP \subset PIP$ for any ideal I in A and since ϕ vanishes on I_X. □

Remark 2.6. If X is finite over A, then $I_X = A$, so a (γ, β)-KMS state on $\mathcal{T}_X$ gives one on $\mathcal{O}_X$ if and only if it is of infinite type. In this case, the Wold decomposition of a KMS state corresponds to the usual essential-singular decomposition of a state relative to the kernel of the quotient map $\mathcal{T}_X \to \mathcal{O}_X$. However, we point out that in the case of $\mathcal{O}_\infty$ and many other interesting situations, cf [FR, EL2], the ideal I_X is trivial so $\mathcal{O}_X$ is actually equal to $\mathcal{T}_X$.

Considering the way in which the quasi-free states of the CAR algebra are determined by their two-point functions, it is natural to refer to states given by (2.2) as *quasi–free states* of $\mathcal{T}_X$. Following Evans [Ev], we wish to consider next a slightly more general notion of quasi-free states; specifically, we wish to allow for a different positive trace-class operator on each tensor factor. Not surprisingly, the appropriate formula is easy to guess, and the crux of the matter is to determine sufficient compatibility conditions on the various ingredients for it to actually define a state.

Proposition 2.7. *Let $\{\tau_n\}_{n=0}^\infty$ be a sequence of traces on A such that τ_0 is a tracial state and let $\{S_n\}_{n=1}^\infty$ be a sequence of positive operators in $B_A(X)$ such that $\mathrm{Tr}_{\tau_n}(aS_n) \le \tau_{n-1}(a)$ for every $a \in A_+$ and $n \ge 1$. Then there exists a unique gauge-invariant state ϕ on $\mathcal{T}_X$ such that $\phi|_A = \tau_0$ and*

$$\phi(T_{\xi_1} \dots T_{\xi_n} T^*_{\zeta_n} \dots T^*_{\zeta_1}) = \tau_n(\langle \zeta_1 \otimes \dots \otimes \zeta_n, S_1\xi_1 \otimes \dots \otimes S_n\xi_n \rangle)$$

for every $\xi_1, \dots, \xi_n, \zeta_1, \dots, \zeta_n \in X$. If, in addition, $\mathrm{Tr}_{\tau_n}(aS_n) = \tau_{n-1}(a)$ for every $a \in I_X$ and $n \ge 1$, then ϕ defines a state on $\mathcal{O}_X$.

Proof. We shall use an argument borrowed from the proof of [EL2, Proposition 12.6]. Let $\mathcal{T}_0$ be the subalgebra of gauge-invariant elements of $\mathcal{T}_X$; since we are looking for a gauge-invariant state, that is, one that factors through the conditional expectation $E = \frac{1}{2\pi}\int_0^{2\pi} \gamma_t(\cdot)dt$ of $\mathcal{T}_X$ onto $\mathcal{T}_0$, it is enough to define ϕ on $\mathcal{T}_0$. Let I_n be the C*-algebra generated by elements $T_\xi T^*_\eta$ with $\xi, \eta \in X^{\otimes n}$, and let $A_n = A + I_1 + \dots + I_n$ (where $A_0 = I_0 = A$). Then I_n is an ideal in A_n and $\mathcal{T}_0 = \overline{\cup_n A_n}$. The submodules $X^{\otimes n}$ are $\mathcal{T}_0$-invariant, and we denote by π_n the natural representation of $\mathcal{T}_0$ on $X^{\otimes n}$. Then $\pi_0 \oplus \dots \oplus \pi_{n-1}$ is faithful on A_{n-1} and zero on I_n, so $A_{n-1} \cap I_n = 0$. Consider the positive linear functional ψ_n on A_n defined by

$$\psi_n(x) = \mathrm{Tr}_{\tau_n}(\pi_n(x)(S_1 \otimes \dots \otimes S_n))$$

for $n \ge 1$, with $\psi_0 = \tau_0$. Since $\pi_n(x) = \pi_{n-1}(x) \otimes 1$ for every $x \in A_{n-1}$, we have $\psi_n|_{A_{n-1}} \le \psi_{n-1}$ for $n \ge 1$ by Proposition 1.2, and we may define selfadjoint linear functionals ϕ_n on $A_n = A_{n-1} \oplus I_n$ by induction: $\phi_n = \phi_{n-1} \oplus \psi_n$, with $\phi_0 = \tau_0$. Since $\phi_n|_A = \tau_0$ is a state and the left action of A is non-degenerate, to prove that ϕ_n is a state it is enough to check that it is positive. We shall prove by induction that $\phi_n \ge \psi_n$. This is true for $n = 0$, since $\psi_0 = \phi_0$. If this is true for $n-1$, then $\psi_n|_{A_{n-1}} \leq \psi_{n-1} \leq \phi_{n-1}$. For $x \in A_{n-1}$ and $y \in I_n$ we have

$$\phi_n((x+y)^*(x+y)) = \phi_{n-1}(x^*x) + \psi_n(x^*y + y^*x + y^*y) \geq \psi_n(x^*x + x^*y + y^*x + y^*y),$$

so $\phi_n \geq \psi_n$. Thus each ϕ_n is a state, and since the sequence is coherent in the sense that $\phi_n|_{A_{n-1}} = \phi_{n-1}$, there exists a unique state ϕ on $\mathcal{T}_0$ such that $\phi|_{I_n} = \phi_n|_{I_n} = \psi_n|_{I_n}$.

Suppose now that $\mathrm{Tr}_{\tau_n}(aS_n) = \tau_{n-1}(a)$ for every $a \in I_X$; we shall prove that ϕ is zero on the two-sided ideal generated by the elements $a - j_X(a)$, for $a \in I_X$. Let $\mathcal{T}_{n,m}$ be the linear span of elements of the form $T_\xi T_\zeta^*$, with $\xi \in X^{\otimes n}$ and $\zeta \in X^{\otimes m}$ (where $\mathcal{T}_{0,0} = A$). We must prove that $\phi(xay) = \phi(xj_X(a)y)$ for $x \in \mathcal{T}_{n,m}$, $y \in \mathcal{T}_{n',m'}$. Since $aT_\xi = j_X(a)T_\xi$ for every $\xi \in X$, we have $xay = xj_X(a)y$ if either $n' > 0$, or $m > 0$. Thus we may restrict our attention to the case $n' = m = 0$. Because of gauge-invariance we also have $\phi(xay) = \phi(xj_X(a)y) = 0$ if $n \neq m'$. So it remains to consider only the case when $n = m'$, $m = n' = 0$. Let $x = T_{\xi_0}$ and $y = T_{\zeta_0}^*$, with $\xi_0, \zeta_0 \in X^{\otimes n}$. Denoting $S_1 \otimes \ldots \otimes S_n$ by $\tilde{S}_n$, for any $\xi, \zeta \in X$ we have

$$\phi(T_{\xi_0} j_X(\theta_{\xi,\zeta}) T_{\zeta_0}^*) = \phi(T_{\xi_0} T_\xi T_\zeta^* T_{\zeta_0}^*) = \tau_{n+1}(\langle \zeta_0 \otimes \zeta, \tilde{S}_n \xi_0 \otimes S_{n+1} \xi \rangle) = \mathrm{Tr}_{\tau_{n+1}}(\langle \zeta_0, \tilde{S}_n \xi_0 \rangle S_{n+1} \theta_{\xi,\zeta}),$$

so for any $a \in I_X$ we get

$$\begin{aligned} \phi(T_{\xi_0} j_X(a) T_{\zeta_0}^*) &= \mathrm{Tr}_{\tau_{n+1}}(\langle \zeta_0, \tilde{S}_n \xi_0 \rangle S_{n+1} a) = \tau_n(a \langle \zeta_0, \tilde{S}_n \xi_0 \rangle) = \tau_n(\langle \zeta_0 a^*, \tilde{S}_n \xi_0 \rangle) \\ &= \phi(T_{\xi_0} T_{\zeta_0 a^*}^*) = \phi(T_{\xi_0} a T_{\zeta_0}^*). \end{aligned}$$

□

In order to illustrate the range of application of the general theory we discuss several situations that have appeared in the literature and which are unified by the present approach. Other systems such as those studied in [E1, E2, MWY] can be analyzed in a similar way.

Example 2.8. [OP, BEH, Ev] Let $X = H$ be a Hilbert space considered as a Hilbert bimodule over $A = \mathbb{C}$. Then $\mathcal{T}_X$ is the Toeplitz-Cuntz algebra $\mathcal{T}_n$ and $\mathcal{O}_X$ is the Cuntz algebra $\mathcal{O}_n$ corresponding to $n = \dim H$. If τ is the unique tracial state on A, then Tr_τ is the usual trace on $B(H)$. Let $\{U_t = e^{itD}\}_{t\in\mathbb{R}}$ be a one-parameter unitary group on H, and let γ be the corresponding quasi-free dynamics. Then our results say that a (γ, β)-KMS state on $\mathcal{T}_H$ exists if and only if $\mathrm{Tr}(e^{-\beta D}) \leq 1$, and such a state extends to a normal state on $B(\mathcal{F}(H))$ if and only if $\mathrm{Tr}(e^{-\beta D}) < 1$. If $\dim H \geq 2$ and $\beta \geq 0$ (or $\beta \leq 0$), then the condition $\mathrm{Tr}(e^{-\beta D}) \leq 1$ implies that D is positive (resp. negative) and non-singular, so the (γ, β)-KMS state is unique. If H is infinite-dimensional, then $I_H = 0$ and $\mathcal{O}_H = \mathcal{T}_H$. If $\dim H < \infty$, then $I_H = \mathbb{C}$, so a (γ, β)-KMS state on $\mathcal{O}_H$ exists if and only if $\mathrm{Tr}(e^{-\beta D}) = 1$.

Example 2.9. [EL2] Let $T = (T(j,k))_{j,k\in I}$ be a (possibly infinite) $0-1$ matrix with no identically zero columns and rows. Consider the rows $q_j = (T(j,k))_{k\in I}$ as elements of $l^\infty(I)$, and let A be the C*-algebra they generate. Let X be the Hilbert A-bimodule generated as a right Hilbert A-module by vectors ξ_j, $j \in I$, such that $\langle \xi_j, \xi_k \rangle = \delta_{jk} q_j$, $q_j \xi_k = T(j,k)\xi_k$. By [Sz] the algebra $\mathcal{O}_X$ is the Cuntz-Krieger algebra corresponding to the matrix T as in [EL1]. If F is a finite subset of I, then $\sum_{j\in F} \theta_{\xi_j,\xi_j}$ is the projection onto the right submodule generated by ξ_j, $j \in F$. It follows that if τ is a trace on A, then $\mathrm{Tr}_\tau(S) = \sum_j \tau(\langle \xi_j, S\xi_j \rangle)$ for any $S \in B(X)_+$.

Let the dynamics on X be given by $U_t \xi_j = N_j^{it} \xi_j$ for some choice $N_j > 1$ for $j = 1, 2, \ldots$, which ensures that U satisfies the positivity condition. Then (γ, β)-KMS states on $\mathcal{T}_X$ are in one-to-one correspondence with states on A such that $\sum_j N_j^{-\beta} \tau(\langle \xi_j, a\xi_j \rangle) \leq \tau(a)$ for any $a \in A_+$. Any positive element in A can be approximated by a linear combination with positive coefficients of projections $q(Y,Z) = \prod_{l\in Y} q_l \prod_{k\in Z}(1 - q_k)$, where Y and Z are finite subsets of I. Note that $q(Y,Z)\xi_j = T(Y,Z,j)\xi_j$, where

$$T(Y,Z,j) = \prod_{l\in Y} T(l,j) \prod_{k\in Z} (1 - T(k,j)).$$

Thus (γ,β)-KMS states on $\mathcal{T}_X$ correspond to states τ on A such that

$$\sum_j N_j^{-\beta} T(Y,Z,j)\tau(q_j) \le \tau(q(Y,Z)) \quad \text{for all finite } Y,Z \subset I, \tag{2.3}$$

namely, to β-subinvariant states of A as defined in [EL2, 12.3]. Notice that our result bypasses the intermediate step of having to consider measures on path space that are rescaled by the partial action and goes straight to the coefficient algebra. The linear span of projections $q(Y,Z)$ such that the function $I \ni j \mapsto T(Y,Z,j)$ has finite support is dense in I_X. So to obtain a state on $\mathcal{O}_X$ the inequality (2.3) must be equality for all Y and Z such that the function $T(Y,Z,\cdot)$ has finite support. We point out that the dynamics arising from such infinite matrices do not satisfy in general the fullness assumption of [PWY].

Example 2.10. [BEH, BEK] Let A be a unital C*-algebra, p a full projection in A, α an injective $*$-endomorphism on A such that $\alpha(A) = pAp$. Consider the semigroup crossed product C*-algebra $A \rtimes_\alpha \mathbb{N}$. Let γ be the periodic dynamics on $A \rtimes_\alpha \mathbb{N}$ defined by the dual action of $\mathbb{T}$. It is known that $A \rtimes_\alpha \mathbb{N}$ can be considered as a Cuntz-Krieger-Pimsner algebra: the module X is the space Ap with left and right actions of A given by $a\cdot\xi\cdot b = a\xi\alpha(b)$, and with inner product $\langle\xi,\zeta\rangle = \alpha^{-1}(\xi^*\zeta)$. Then γ is the gauge action. Since p is full, the module is finite, so $I_X = A$. Thus (γ,β)-KMS states on $\mathcal{T}_X$ (resp. $\mathcal{O}_X$) correspond to traces τ such that $e^{-\beta}\mathrm{Tr}_\tau(a) \le \tau(a)$ (resp. $e^{-\beta}\mathrm{Tr}_\tau(a) = \tau(a)$) for $a \in A_+$. Since p is full, to prove an equality/inequality for traces on A it is enough to check it on $pAp = \alpha(A)$. Noting that $p = \theta_{p,p}$ in $B(X)$, for any $a \in A$ we get $\mathrm{Tr}_\tau(\alpha(a)) = \mathrm{Tr}_\tau(\alpha(a)\theta_{p,p}) = \tau(\langle p, \alpha(a)p\rangle) = \tau(a)$. Thus (γ,β)-KMS states on $\mathcal{T}_X$ (resp. $\mathcal{O}_X$) correspond to traces τ on A such that $e^{-\beta}\tau \le \tau\circ\alpha$ (resp. $e^{-\beta}\tau = \tau\circ\alpha$). It is proved in [BEH] that any closed subset of $(0,+\infty)$ can be realized as the set of possible temperatures of the system $(\mathcal{O}_X,\gamma)$ for a convenient choice of AF-algebra A and endomorphism α with $\mathcal{O}_X$ is simple. In [BEK] a similar construction yields quite arbitrary choices of the simplex of β-KMS states for each β.

Remark 2.11. A question raised in the introduction of [EL2] is whether the Toeplitz Cuntz-Krieger algebra of an infinite matrix can have KMS states of finite and of infinite type coexisting at a given finite inverse temperature. We would like to shed some light on the analogous question for the quasi–free dynamics on the Toeplitz–Pimsner algebras. Under the same assumptions of Theorem 2.1, let $\beta_0 < \infty$ and suppose that ϕ is a (γ,β_0)-KMS state with restriction τ to A. For each $\beta > \beta_0$, we then have that $F_\beta\tau \le F_{\beta_0}\tau \le \tau$ so τ determines a (γ,β)-KMS state ϕ_β of $\mathcal{T}_X$, which cannot be of infinite type because of our positivity assumption. The parameters of Example 2.10 can be adjusted to get also a (γ,β)-KMS state of $\mathcal{O}_X$, and the resulting infinite type (γ,β)-KMS state of $\mathcal{T}_X$ will thus coexist, at inverse temperature β, with the finite part of the decomposition of ϕ_β. A further strengthening of the positivity assumption, namely, the assumption that $D \ge c1$ for some $c > 0$, yields $F_\beta^n\tau \le e^{-n(\beta-\beta_0)c}F_{\beta_0}^n\tau$, from which one readily sees that the state ϕ_β above is of finite type for $\beta > \beta_0$. This gives a Pimsner algebra version of the 'cooling lemma' [EL2, Lemma 15.1], which implies that KMS states of finite type are weak* dense in all KMS states.

3. KMS weights on module algebras

The theory of KMS weights on C*-algebras is similar to (and is based on) the theory of normal weights on von Neumann algebras. We refer the reader to [St] and [K] for the basic definitions. Suppose $\mathbb{R} \ni t \mapsto \sigma_t$ is a one-parameter automorphism group of a C*-algebra A. We assume that σ is continuous in the sense that the function $\mathbb{R} \ni t \mapsto \sigma_t(a)$ is norm-continuous for all $a \in A$. The same assumption is made for more general one-parameter group of isometries on Banach spaces. In several places where we consider von Neumann algebras the continuity assumption is weaker: the function $\mathbb{R} \ni t \mapsto \sigma_t(a)$ is weakly (operator) continuous.

Let ϕ be a weight on A. As usual, we set

$$\mathcal{M}_\phi^+ = \{a \in A_+ \,|\, \phi(a) < \infty\}, \quad \mathcal{N}_\phi = \{a \in A \,|\, a^*a \in \mathcal{M}_\phi^+\}, \text{ and } \quad \mathcal{M}_\phi = \operatorname{span} \mathcal{M}_\phi^+ = \mathcal{N}_\phi^* \mathcal{N}_\phi,$$

and extend ϕ to a linear functional on $\mathcal{M}_\phi$. We say that ϕ is a (σ, β)-KMS weight if ϕ is lower semicontinuous on A_+, densely defined (i.e. $\mathcal{M}_\phi^+$ is dense in A_+), σ-invariant, and satisfies the KMS-condition in the form

$$\phi(x^*x) = \phi(\sigma_{-\frac{i\beta}{2}}(x)\sigma_{-\frac{i\beta}{2}}(x)^*) \quad \text{for every } x \in D(\sigma_{-\frac{i\beta}{2}}),$$

where $D(\sigma_{-\frac{i\beta}{2}})$ is the domain of definition of $\sigma_{-\frac{i\beta}{2}}$.

Given such a weight ϕ, the well known GNS construction produces a Hilbert space H_ϕ and a linear map $\Lambda_\phi \colon \mathcal{N}_\phi \to H_\phi$ such that $\Lambda_\phi(\mathcal{N}_\phi)$ is dense in H_ϕ and $(\Lambda_\phi(x), \Lambda_\phi(y)) = \phi(y^*x)$. There is a representation π_ϕ of A on H_ϕ, defined by letting $\pi_\phi(x)\Lambda_\phi(y) = \Lambda_\phi(xy)$, and the set $\mathcal{U}_\phi = \Lambda_\phi(\mathcal{N}_\phi \cap \mathcal{N}_\phi^*)$ is a left Hilbert algebra with operations

$$\Lambda_\phi(x)\Lambda_\phi(y) = \Lambda_\phi(xy), \quad \Lambda_\phi(x)^\# = \Lambda_\phi(x^*).$$

The associated von Neumann algebra $\mathcal{L}(\mathcal{U}) = \pi_\phi(A)''$ is equipped with a canonical normal semifinite faithful (n.s.f.) weight Φ. Then $\phi = \Phi \circ \pi_\phi$ and $\sigma_t^\Phi \circ \pi_\phi = \pi_\phi \circ \sigma_{-\beta t}$, where σ_t^Φ is the modular group of Φ.

Any lower semicontinuous densely defined weight on A extends uniquely to a strictly lower semicontinuous weight $\bar\phi$ on the multiplier algebra $M(A)$. If ϕ is a KMS-weight, then $\bar\phi = \Phi \circ \bar\pi_\phi$, where $\bar\pi_\phi$ is the canonical extension of π_ϕ to a representation of $M(A)$. The weight $\bar\phi$ is only strictly densely defined, and the one-parameter automorphism group σ is only strictly continuous on $M(A)$, so $\bar\phi$ is not a KMS-weight in the sense of the definition above. However, since $\bar\phi = \Phi \circ \bar\pi_\phi$, $\bar\phi$ satisfies a form of the KMS-condition, so that its restriction to a σ-invariant subalgebra B of $M(A)$ is a KMS-weight if $\sigma|_B$ is continuous and $\bar\phi|_B$ is densely defined.

We now consider a right Hilbert A-module X and a continuous one-parameter group of isometries $\mathbb{R} \ni t \mapsto U_t$ of X such that $\langle U_t\xi, U_t\zeta\rangle = \sigma_t(\langle \xi, \zeta\rangle)$. Then $U_t(\xi a) = (U_t\xi)\sigma_t(a)$, and we define a dynamics $\gamma = \operatorname{Ad} U$ on $K(X)$ by $\gamma_t(T) = U_t T U_{-t}$. We shall extend Theorem 1.1 to the present situation, for which we need to associate an induced weight κ_ϕ on $B(X)$ to each (σ, β)-KMS weight ϕ on A. We give two equivalent constructions of this induced weight.

In the first construction, we replace A by $\overline{\langle X, X\rangle}$ so we may assume that X is full. Let $\pi \colon A \to B(H)$ be an arbitrary representation of A such that there exists a n.s.f. weight Φ on $L = \pi(A)''$ with $\phi = \Phi \circ \pi$ and $\sigma_t^\Phi \circ \pi = \pi \circ \sigma_{-\beta t}$. Set $M = \pi(A)'$. Consider the Hilbert space $H_X = X \otimes_A H$ and the induced representation ρ of $B(X)$ on H_X. Set $N = \rho(B(X))''$. Note that M acts faithfully on H_X, and $N' = M$ in $B(H_X)$. Let now Φ' be an arbitrary n.s.f. weight on M and Ψ the unique n.s.f. weight on N such that

$$\Delta(\Psi/\Phi')^{it} = U_{-\beta t} \otimes \Delta(\Phi/\Phi')^{it} \text{ in } B(H_X),$$

where $\Delta(\cdot/\cdot)$ denotes the spatial derivative, cf [Co2, St]. We then set $\kappa_\phi = \Psi \circ \rho$.

Let us compute κ_ϕ explicitly on a dense subalgebra of $K(X)$. Let $\zeta \in H$ be a Φ'-bounded vector, that is, the map $R_\zeta \colon H_{\Phi'} \to H$, given by $R_\zeta(\Lambda_{\Phi'}(x)) = x\zeta$ for $x \in \mathcal{N}_{\Phi'}$, is bounded. For any $\xi \in X$ the vector $\xi \otimes \zeta \in H_X$ is Φ'-bounded as well and $R_{\xi\otimes\zeta}\eta = \xi \otimes R_\zeta \eta$ for $\eta \in H_{\Phi'}$. Then if $\xi \in D(U_{\frac{i\beta}{2}})$ and $\zeta \in D(\Delta(\Phi/\Phi')^{\frac12})$ is Φ'-bounded, we get by definition

$$\begin{aligned}\Psi(R_{\xi\otimes\zeta}R_{\xi\otimes\zeta}^*) &= \|\Delta(\Psi/\Phi')^{\frac12}(\xi \otimes \zeta)\|^2 = \|U_{\frac{i\beta}{2}}\xi \otimes \Delta(\Phi/\Phi')^{\frac12}\zeta\|^2 \\ &= (\Delta(\Phi/\Phi')^{\frac12}\zeta, \pi(\langle U_{\frac{i\beta}{2}}\xi, U_{\frac{i\beta}{2}}\xi\rangle)\Delta(\Phi/\Phi')^{\frac12}\zeta).\end{aligned}$$

At this point it is convenient to introduce the form $\Phi(x\sigma_{-\frac{i}{2}}^\Phi(y))$, whose relevant properties are stated in the following essentially known lemma.

Lemma 3.1. *Let Φ be a n.s.f. weight on a von Neumann algebra M, with modular group σ. Then the bilinear form $(\cdot,\cdot)_\Phi\colon \mathcal{M}_\Phi\times D(\sigma_{-\frac{i}{2}})\to\mathbb{C}$, defined by $(x,y)_\Phi=\Phi(x\sigma_{-\frac{i}{2}}(y))$, extends to a bilinear form on $\mathcal{M}_\Phi\times M$ with the following properties:*

(i) *for $x\in\mathcal{M}_\Phi$, $(x,\cdot)_\Phi$ is a normal linear functional on M; if $x\in\mathcal{M}_\Phi^+$, then $(x,\cdot)_\Phi$ is positive and $(x,\cdot)_\Phi\le\|x\|\Phi$;*

(ii) *for $x,y\in\mathcal{M}_\Phi$, $(x,y)_\Phi=(y,x)_\Phi$;*

(iii) *for $x_1,x_2\in\mathcal{N}_\Phi\cap D(\sigma_{\frac{i}{2}})$, $(x_2^*x_1,\cdot)_\Phi=\Phi(\sigma_{\frac{i}{2}}(x_1)\cdot\sigma_{\frac{i}{2}}(x_2)^*)$;*

(iv) *if $\{x_k\}_k$ is a net in $\mathcal{M}_\Phi^+\cap D(\sigma_{-\frac{i}{2}})$ such that $x_k\le 1$, the set $\{\sigma_{-\frac{i}{2}}(x_k)\}_k$ is bounded in norm, and $\sigma_{-\frac{i}{2}}(x_k)\to 1$ strongly, then $\Phi(x)=\lim_k(x_k,x)_\Phi$ for any $x\in M_+$.*

Proof. Consider the GNS-representation $\Lambda\colon\mathcal{N}_\Phi\to H$ as described above. Recall that, in terms of the modular group σ, the modular involution J is given by $J\Lambda(x)=\Lambda(\sigma_{\frac{i}{2}}(x)^*)$ for $x\in\mathcal{N}_\Phi\cap D(\sigma_{\frac{i}{2}})$. It has the properties

$$\begin{aligned} Jy^*J\Lambda(x) &= \Lambda(x\sigma_{-\frac{i}{2}}(y))\quad \forall x\in\mathcal{N}_\Phi\quad\forall y\in D(\sigma_{-\frac{i}{2}}),\ \text{and}\\ JyJ\Lambda(x) &= xJ\Lambda(y)\quad\forall x,y\in\mathcal{N}_\Phi. \end{aligned}$$

Now if $x=x_2^*x_1$, $x_1,x_2\in\mathcal{N}_\Phi$, $y\in D(\sigma_{-\frac{i}{2}})$, then

$$(x,y)_\Phi=(x_2^*x_1,y)_\Phi=(\Lambda(x_1\sigma_{-\frac{i}{2}}(y)),\Lambda(x_2))=(Jy^*J\Lambda(x_1),\Lambda(x_2)).$$

This shows that $(x,\cdot)_\Phi$ extends to a normal linear functional on M. Moreover, if $x\ge0$, then we can take $x_1=x_2=x^{\frac12}$ and conclude that $(x,y)_\Phi\ge0$ for $y\ge0$. If $x,y\in\mathcal{M}_\Phi^+$,

$$\begin{aligned}(x,y)_\Phi &= (JyJ\Lambda(x^{\frac12}),\Lambda(x^{\frac12}))=(Jy^{\frac12}J\Lambda(x^{\frac12}),Jy^{\frac12}J\Lambda(x^{\frac12}))=(x^{\frac12}J\Lambda(y^{\frac12}),x^{\frac12}J\Lambda(y^{\frac12}))\\ &= (JxJ\Lambda(y^{\frac12}),\Lambda(y^{\frac12}))\le\|x\|(\Lambda(y^{\frac12}),\Lambda(y^{\frac12}))=\|x\|\Phi(y).\end{aligned}$$

Thus part (i) is proved. Part (ii) is already proved for positive x and y, the general case follows by linearity. If $x_2,x_1\in\mathcal{N}_\phi\cap D(\sigma_{\frac{i}{2}})$, we have

$$\begin{aligned}(x_2^*x_1,y)_\Phi &= (Jy^*J\Lambda(x_1),\Lambda(x_2))=(yJ\Lambda(x_2),J\Lambda(x_1))\\ &= (y\Lambda(\sigma_{\frac{i}{2}}(x_2)^*),\Lambda(\sigma_{\frac{i}{2}}(x_1)^*))=\Phi(\sigma_{\frac{i}{2}}(x_1)y\sigma_{\frac{i}{2}}(x_2)^*),\end{aligned}$$

which proves (iii). It remains to prove (iv). Let $x\in M_+$. Since $\{\sigma_{-\frac{i}{2}}(x_k)^*x\sigma_{-\frac{i}{2}}(x_k)\}_k$ converges weakly to x, and $x_k^2\le x_k$, we have

$$\Phi(x)\le\liminf_k\Phi(\sigma_{-\frac{i}{2}}(x_k)^*x\sigma_{-\frac{i}{2}}(x_k))=\liminf_k(x_k^2,x)_\Phi\le\liminf_k(x_k,x)_\Phi.$$

On the other hand, $(x_k,x)_\Phi\le\Phi(x)$ by (i), hence $\Phi(x)=\lim_k(x_k,x)_\Phi$. This finishes the proof of Lemma 3.1. ☐

Returning to the computation of Ψ, for any $a\in L$ we get

$$(\Delta(\Phi/\Phi')^{\frac12}\zeta,a\Delta(\Phi/\Phi')^{\frac12}\zeta)=(\Delta(\Phi/\Phi')^{\frac12}\zeta,\Delta(\Phi/\Phi')^{\frac12}\sigma^\Phi_{\frac{i}{2}}(a)\zeta)=\Phi(R_\zeta R^*_{\sigma^\Phi_{\frac{i}{2}}(a)\zeta})=(R_\zeta R_\zeta^*,a^*)_\Phi.$$

Thus

$$\Psi(R_{\xi\otimes\zeta}R^*_{\xi\otimes\zeta})=(R_\zeta R_\zeta^*,\pi(\langle U_{\frac{i\beta}{2}}\xi,U_{\frac{i\beta}{2}}\xi\rangle))_\Phi. \tag{3.1}$$

Note that $R^*_{\xi\otimes\zeta}(\xi_0\otimes\zeta_0)=R^*_\zeta\pi(\langle\xi,\xi_0\rangle)\zeta_0$. So if we introduce an operator $T_{\xi,a}$ on H_X defined by $T_{\xi,a}(\xi_0\otimes\zeta_0)=\xi\otimes a\pi(\langle\xi,\xi_0\rangle)\zeta_0$, then $R_{\xi\otimes\zeta}R^*_{\xi\otimes\zeta}=T_{\xi,R_\zeta R_\zeta^*}$. Then (3.1) implies that

$$\Psi(T_{\xi,a})=(a,\pi(\langle U_{\frac{i\beta}{2}}\xi,U_{\frac{i\beta}{2}}\xi\rangle))_\Phi$$

for any a in the algebra $\mathrm{span}\{R_{\zeta_1}R_{\zeta_2}^* \,|\, \zeta_i \in D(\Delta(\Phi/\Phi')^{\frac{1}{2}})$ is Φ'-bounded, $i=1,2\}$. If we now apply this to an approximate unit $a_i \nearrow 1$ and use Lemma 3.1(i-ii) together with the property $T_{\xi,a_i} \nearrow \rho(\theta_{\xi,\xi})$, we conclude that

$$\kappa_\phi(\theta_{\xi,\xi}) = \phi(\langle U_{\frac{i\beta}{2}}\xi, U_{\frac{i\beta}{2}}\xi\rangle) \tag{3.2}$$

for every $\xi \in D(U_{\frac{i\beta}{2}})$ such that $\langle U_{\frac{i\beta}{2}}\xi, U_{\frac{i\beta}{2}}\xi\rangle \in \mathcal{M}_\phi$.

In our second construction of κ_ϕ, we shall show directly that there is a linear functional satisfying property (3.2) on a dense subalgebra of $K(X)$, and then extend it to a weight on the whole algebra using the GNS-representation. For this we introduce the following sets:

$$\begin{aligned} C_\phi &= \{a \in A \,|\, a \text{ is } \sigma\text{-analytic and } \sigma_z(a) \in \mathcal{N}_\phi \cap \mathcal{N}_\phi^* \ \forall z \in \mathbb{C}\}, \\ X_0 &= \{\xi \in X \,|\, \xi \text{ is } U\text{-analytic}\}, \\ X_\phi &= X_0 C_\phi, \\ \mathcal{U} &= \mathrm{span}\{\theta_{\xi,\zeta} \,|\, \xi,\zeta \in X_\phi\}. \end{aligned}$$

Note that if $\xi \in D(U_z)$ and $\zeta \in D(U_{\bar z})$, then $\theta_{\xi,\zeta} \in D(\gamma_z)$ and $\gamma_z(\theta_{\xi,\zeta}) = \theta_{U_z\xi, U_{\bar z}\zeta}$; moreover, at the level of the coefficient algebra, $\langle \zeta,\xi\rangle \in D(\sigma_z)$ and $\sigma_z(\langle\zeta,\xi\rangle) = \langle U_{\bar z}\zeta, U_z\xi\rangle$. Thus C_ϕ is a dense $*$-subalgebra of A, X_ϕ is a dense subspace of X, and $\mathcal{U}$ a dense $*$-subalgebra of $K(X)$ consisting of γ-analytic elements. Choose an approximate unit $\{e_k = \sum_{\eta\in I_k}\theta_{\eta,\eta}\}_k$ in $\mathcal{U}$ and define

$$\kappa(x) = \lim_k \sum_{\eta\in I_k} \phi(\langle U_{\frac{i\beta}{2}}\eta, xU_{\frac{i\beta}{2}}\eta\rangle) \tag{3.3}$$

for $x \in \mathcal{U}$. If $\eta,\xi \in D(U_{\frac{i\beta}{2}})$ then

(3.4)

$$\phi(\langle U_{\frac{i\beta}{2}}\eta, \theta_{\xi,\xi}U_{\frac{i\beta}{2}}\eta\rangle) = \phi(\langle U_{\frac{i\beta}{2}}\eta, \xi\rangle\langle\xi, U_{\frac{i\beta}{2}}\eta\rangle) = \phi(\langle U_{\frac{i\beta}{2}}\xi, \eta\rangle\langle\eta, U_{\frac{i\beta}{2}}\xi\rangle) = \phi(\langle U_{\frac{i\beta}{2}}\xi, \theta_{\eta,\eta}U_{\frac{i\beta}{2}}\xi\rangle),$$

since $\langle\xi, U_{\frac{i\beta}{2}}\eta\rangle \in D(\sigma_{-\frac{i\beta}{2}})$ and $\sigma_{-\frac{i\beta}{2}}(\langle\xi, U_{\frac{i\beta}{2}}\eta\rangle) = \langle U_{\frac{i\beta}{2}}\xi, \eta\rangle$. Hence for $\xi \in X_\phi$

$$\kappa(\theta_{\xi,\xi}) = \lim_k \sum_{\eta\in I_k} \phi(\langle U_{\frac{i\beta}{2}}\eta, \theta_{\xi,\xi}U_{\frac{i\beta}{2}}\eta\rangle) = \lim_k \phi(\langle U_{\frac{i\beta}{2}}\xi, e_kU_{\frac{i\beta}{2}}\xi\rangle) = \phi(\langle U_{\frac{i\beta}{2}}\xi, U_{\frac{i\beta}{2}}\xi\rangle),$$

so κ satisfies (3.2). It is easy to check that κ has the following properties:

(i) κ is γ_z-invariant for any $z \in \mathbb{C}$;
(ii) $\kappa(xy) = \kappa(\gamma_{-\frac{i\beta}{2}}(y)\gamma_{\frac{i\beta}{2}}(x))$ for any $x,y \in \mathcal{U}$;
(iii) $\kappa(\theta_{\xi a,\xi}) = (\langle U_{\frac{i\beta}{2}}\xi, U_{\frac{i\beta}{2}}\xi\rangle, a)_\phi$ for any $\xi \in X_\phi$ and $a \in C_\phi$.
(iv) the function $z \mapsto \kappa(\gamma_z(x)y)$ is analytic for any $x,y \in \mathcal{U}$.

Note that by definition $(x,y)_\phi = \phi(x\sigma_{\frac{i\beta}{2}}(y))$ as the modular group of ϕ is $\sigma_t^\phi = \sigma_{-\beta t}$. Since $\kappa(x^*x) \ge 0$ by (3.3), there exist a Hilbert space H and a linear map $\Lambda\colon \mathcal{U} \to H$ such that $\Lambda(\mathcal{U})$ is dense in H, and $(\Lambda(x),\Lambda(y)) = \kappa(y^*x)$. The kernel of this map is the set

$$\operatorname{Ker}\Lambda = \{x \in \mathcal{U} \,|\, \kappa(x^*x) = 0\} = \{x \in \mathcal{U} \,|\, \kappa(yx) = 0 \ \forall y \in \mathcal{U}\},$$

which is obviously a γ_z-invariant left ideal in $\mathcal{U}$. Property (ii) of κ implies that if x is in this ideal, then also $\gamma_{\frac{i\beta}{2}}(x^*)$, and hence also x^*, is in this ideal. Thus the ideal is self-adjoint and two-sided, and it follows that $\Lambda(\mathcal{U})$ has the canonical structure of an algebra with involution, in particular, $\Lambda(x)\Lambda(y) = \Lambda(xy)$, and $\Lambda(x)^\# = \Lambda(x^*)$.

Let us check that $\Lambda(\mathcal{U})$ is a left Hilbert algebra. It is obvious that $(\Lambda(x)\Lambda(y),\Lambda(z)) = (\Lambda(y),\Lambda(x)^\#\Lambda(z))$. The map $\Lambda(y) \mapsto \Lambda(x)\Lambda(y)$ is a bounded map for each $x \in \mathcal{U}$ by (3.3).

In fact, we see that the norm of the mapping is not bigger than $\|x\|$. Since κ is γ_z-invariant, property (ii) of κ can be rewritten as

$$(\Lambda(x^*),\Lambda(y)) = (\Lambda(\gamma_{i\beta}(y^*)),\Lambda(x)). \tag{3.5}$$

It follows $\Lambda(x)\mapsto\Lambda(x^*)$ is a closable operator. It remains to prove that $\Lambda(\mathcal{U})^2$ is dense in $\Lambda(\mathcal{U})$. Let $\{e_k\}_k$ be an approximate unit in $\mathcal{U}$. We claim that $\Lambda(e_k)\Lambda(x)\to\Lambda(x)$ for any $x\in\mathcal{U}$. Let $x=\theta_{\xi,\zeta}$. Then by property (iii) of κ

$$||\Lambda(x)-\Lambda(e_k)\Lambda(x)||^2 = ||\Lambda(\theta_{\xi-e_k\xi,\zeta})||^2 = \kappa(\theta_{\zeta\langle\xi-e_k\xi,\xi-e_k\xi\rangle,\zeta}) = (\langle U_{\frac{i\beta}{2}}\zeta, U_{\frac{i\beta}{2}}\zeta\rangle, \langle\xi-e_k\xi,\xi-e_k\xi\rangle)_\phi.$$

By Lemma 3.1, $(\langle U_{\frac{i\beta}{2}}\zeta, U_{\frac{i\beta}{2}}\zeta\rangle,\cdot)_\phi$ is a bounded linear functional on A, so $\{\Lambda(e_kx)\}_k$ converges to $\Lambda(x)$. This proves the claim and completes the proof that $\Lambda(\mathcal{U})$ is a left Hilbert algebra.

Let Ψ be the canonical n.s.f. weight on the associated von Neumann algebra $\mathcal{L}(\mathcal{U})$. For each $x\in\mathcal{U}$ let $\rho(x)$ be the operator of left multiplication by $\Lambda(x)$. We have already shown that $||\rho(x)||\le||x||$. Hence ρ extends by continuity, first to a representation of $K(X)$, and then to a representation of $B(X)$. Finally, we define $\kappa_\phi=\Psi\circ\rho$.

Let us compute the modular group of Ψ. By property (iv) of κ the vector-valued function $\mathbb{C}\ni z\mapsto\Lambda(\gamma_z(x))$ is analytic. Hence there exists a non-singular positive operator Δ on H such that $\Delta^z\Lambda(x)=\Lambda(\gamma_{i\beta z}(x))$. Let J be the anti-linear involution defined by $J\Lambda(x)=\Lambda(\gamma_{-\frac{i\beta}{2}}(x))^\#$. Then $J\Delta^{\frac12}\Lambda(x)=\Lambda(x)^\#$, from which we conclude that $\Delta_\Psi=\Delta$. Thus $\sigma^\Psi_t\circ\rho=\rho\circ\gamma_{-\beta t}$. Since $\kappa_\phi(y^*x)=(\Lambda(x),\Lambda(y))=\kappa(y^*x)$ by definition of Ψ, we see that $\kappa_\phi|_{K(X)}$ is a (γ,β)-KMS weight such that property (3.2) is satisfied for all $\xi\in X_\phi\langle X_\phi,X_\phi\rangle$.

We can now state and prove the following generalization of Theorem 1.1 for KMS weights. In particular, we will show that both constructions give the same weight.

Theorem 3.2. *Let ϕ be a (σ,β)-KMS weight on A. For $T\in B(X)$, $T\ge0$, set*

$$\kappa_\phi(T)=\sup\sum_{\xi\in I}\phi(\langle U_{\frac{i\beta}{2}}\xi, TU_{\frac{i\beta}{2}}\xi\rangle), \tag{3.6}$$

where the supremum is taken over all finite subsets I of $D(U_{\frac{i\beta}{2}})$ such that $\langle U_{\frac{i\beta}{2}}\xi,U_{\frac{i\beta}{2}}\xi\rangle\in\mathcal{M}^+_\phi$ for every $\xi\in I$ and $\sum_{\xi\in I}\theta_{\xi,\xi}\le1$. Then

(i) *$\kappa_\phi|_{K(X)}$ is a (γ,β)-KMS weight on $K(X)$, and κ_ϕ is its strictly lower semicontinuous extension to $B(X)$;*

(ii) *there exists an approximate unit $\{e_k=\sum_{\xi\in I_k}\theta_{\xi,\xi}\}_k$ in $K(X)$ such that $\xi\in D(U_{\frac{i\beta}{2}})\cap D(U_{-\frac{i\beta}{2}})$ and $\langle U_{\frac{i\beta}{2}}\xi,U_{\frac{i\beta}{2}}\xi\rangle\in\mathcal{M}^+_\phi$ for every $\xi\in I_k$, the net $\{\gamma_{\frac{i\beta}{2}}(e_k)\}_k$ is bounded in norm and converges strictly to 1; for any such approximate unit we have*

$$\kappa_\phi(T)=\lim_k\sum_{\xi\in I_k}\phi(\langle U_{\frac{i\beta}{2}}\xi,TU_{\frac{i\beta}{2}}\xi\rangle)\quad \textit{for each } T\ge0;$$

(iii) *if $\xi\in D(U_{\frac{i\beta}{2}})$, then $\kappa_\phi(\theta_{\xi,\xi})=\phi(\langle U_{\frac{i\beta}{2}}\xi,U_{\frac{i\beta}{2}}\xi\rangle)$, and if this number is finite, then*

$$(\theta_{\xi,\xi},\cdot)_{\kappa_\phi}=\phi(\langle U_{\frac{i\beta}{2}}\xi,\cdot\,U_{\frac{i\beta}{2}}\xi\rangle);$$

moreover, if $\tilde X_\phi\subset X_\phi$ is a dense U_z-invariant C_ϕ-submodule, then $\kappa_\phi|_{K(X)}$ is the unique (γ,β)-KMS weight such that $\kappa_\phi(\theta_{\xi,\xi})=\phi(\langle U_{\frac{i\beta}{2}}\xi,U_{\frac{i\beta}{2}}\xi\rangle)$ for $\xi\in\tilde X_\phi$;

(iv) *if the module is full, then the mapping $\phi\mapsto\kappa_\phi|_{K(X)}$ defines a one-to-one correspondence between (σ,β)-KMS weights on A and (γ,β)-KMS weights on $K(X)$.*

Proof. First note that an approximate unit with the properties stated in part (ii) always exists. Moreover, if $\tilde{X}_\phi \subset X_\phi$ is a dense U_z-invariant C_ϕ-submodule, then such an approximate unit can be found in the algebra $\tilde{\mathcal{U}} = \operatorname{span}\{\theta_{\xi,\zeta} \,|\, \xi,\zeta \in \tilde{X}_\phi\}$. Indeed, let $\{f_k = \sum_{\eta\in I_k} \theta_{\eta,\eta}\}_k$ be an approximate unit in $\tilde{\mathcal{U}}$. It is well-known that if we set $\tilde{e}_k = \frac{1}{\sqrt{\pi}}\int_{\mathbb{R}} e^{-t^2}\gamma_t(f_k)dt$, then $\{\tilde{e}_k\}_k$ is an approximate unit, the net $\{\gamma_z(\tilde{e}_k)\}_k$ is bounded and converges strictly to 1 for any $z \in \mathbb{C}$. But since each f_k is already γ-analytic, we can replace the integral by a finite sum such that the element e_k, which we thus obtain, is arbitrarily close (in norm) to $\tilde{e}_k$, while $\gamma_{\frac{i\beta}{2}}(e_k)$ is close to $\gamma_{\frac{i\beta}{2}}(\tilde{e}_k)$.

We have already shown that there exists a strictly lower semicontinuous weight κ on $B(X)$ such that $\kappa|_{K(X)}$ is (γ,β)-KMS and $\kappa(\theta_{\xi,\xi}) = \phi(\langle U_{\frac{i\beta}{2}}\xi, U_{\frac{i\beta}{2}}\xi\rangle)$ for $\xi \in \tilde{X}_\phi$. To prove the theorem it is enough to show that κ satisfies (iii). Indeed, by Lemma 3.1(i),(iv), κ can then be defined as in part (ii) and (3.6). In particular, $\kappa_\phi = \kappa$, so κ_ϕ has properties (i)-(iii). Part (iv) follows then from the uniqueness result in (iii), since by symmetry for any (γ,β)-KMS weight ψ on $K(X)$ we can define a strictly lower semicontinuous weight κ_ψ on $M(A)$ such that $\kappa_\psi|_A$ is a (σ,β)-KMS weight and

$$\kappa_\psi(\langle\xi,\xi\rangle) = \psi(\theta_{U_{-\frac{i\beta}{2}}\xi, U_{-\frac{i\beta}{2}}\xi}) \ \text{ for any } \xi \in D(U_{-\frac{i\beta}{2}}).$$

To show (iii), note that for any $\xi,\zeta \in \tilde{X}_\phi$

$$(\theta_{\zeta,\zeta}, \theta_{\xi,\xi})_\kappa = \phi(\langle U_{\frac{i\beta}{2}}\zeta, \theta_{\xi,\xi} U_{\frac{i\beta}{2}}\zeta\rangle).$$

Since both sides above are continuous functions of ξ, the equality holds for any $\zeta \in \tilde{X}_\phi$ and $\xi \in X$, and, using (3.4), we obtain that

$$(x, \theta_{\xi,\xi})_\kappa = \phi(\langle U_{\frac{i\beta}{2}}\xi, x U_{\frac{i\beta}{2}}\xi\rangle) \tag{3.7}$$

for every $x \in \tilde{\mathcal{U}}$ and $\xi \in D(U_{\frac{i\beta}{2}})$. Choosing an approximate unit in $\tilde{\mathcal{U}}$ satisfying the conditions of Lemma 3.1(iv) we conclude that

$$\kappa(\theta_{\xi,\xi}) = \phi(\langle U_{\frac{i\beta}{2}}\xi, U_{\frac{i\beta}{2}}\xi\rangle))$$

for any $\xi \in D(U_{\frac{i\beta}{2}})$. If this number is finite, by Lemma 3.1(ii) we can rewrite (3.7) as

$$(\theta_{\xi,\xi}, x)_\kappa = \phi(\langle U_{\frac{i\beta}{2}}\xi, x U_{\frac{i\beta}{2}}\xi\rangle)$$

for any $x \in \tilde{\mathcal{U}}$. Since both sides above are strictly continuous linear functionals on $B(X)$, the equality holds for all $x \in B(X)$. By Lemma 3.1(iv) the weight κ is completely determined by linear functionals $(\theta_{\xi,\xi}, \cdot)_\kappa$ for $\xi \in \tilde{X}_\phi$, from which the uniqueness result follows. □

Remark 3.3.

(i) In the particular case when A is a full corner pBp, $X = Bp$, γ a one-parameter automorphism group of B leaving the projection $p \in M(B)$ invariant, $\sigma_t = \gamma_t|_A$, $U_t x = \gamma_t(x)$, the weight $\kappa_\phi|_{K(X)}$ is an extension of the (σ,β)-KMS weight ϕ to a (γ,β)-KMS weight on B, and the map $\psi \mapsto \kappa_\psi|_A$ going from KMS-weights on B to KMS-weights on A is just the restriction map. Thus Theorem 3.2 says that any (σ,β)-KMS weight on A can be uniquely extended to a (γ,β)-KMS weight on B. Using the linking algebra the general case could be reduced to this situation. Namely, $\kappa_\phi|_{K(X)}$ is a unique weight ψ on $K(X)$ such that $\Phi = \begin{pmatrix}\psi & 0\\ 0 & \phi\end{pmatrix}$ is an (α,β)-KMS weight on the linking algebra $C = \begin{pmatrix}K(X) & X\\ \bar{X} & A\end{pmatrix}$, where $\alpha_t(\begin{pmatrix}x & \xi\\ \bar{\zeta} & a\end{pmatrix}) = \begin{pmatrix}\gamma_t(x) & U_t\xi\\ \overline{U_t\zeta} & \sigma_t(a)\end{pmatrix}$.

(ii) The induction (or extension) results have natural counterparts for von Neumann algebras, which can be proved by the same methods or deduced from our results for C*-algebras (see also [CZ]).

(iii) The fact that the weight given by our first construction of the induced weight κ_ϕ is independent of the choice of representation π is essentially equivalent to the main result of [Sa].

(iv) The main point in our first construction of the induced weight κ_ϕ is an implicit application of Connes' theorem on existence and uniqueness of a weight with given Radon-Nikodym cocycle [Co1, Theorem 1.2.4]. On the other hand, our second construction uses nothing beyond the basic definitions of the modular theory, and the induction results we have obtained can be used in turn to give an alternative proof of Connes' result. Indeed, let M be a von Neumann algebra, ϕ a n.s.f. weight on M with modular group σ, $\mathbb{R}\ni t\mapsto u_t$ a strongly continuous unitary 1-cocycle for σ. Then by our results (and remark (ii) above) there exists a unique n.s.f. weight Φ on $\mathrm{Mat}_2(M)$ with modular group

$$\sigma^\Phi_t(\begin{pmatrix}a & b\\ c & d\end{pmatrix})=\begin{pmatrix}u_t\sigma_t(a)u_t^* & u_t\sigma_t(b)\\ \sigma_t(c)u_t^* & \sigma_t(d)\end{pmatrix}$$

such that $\Phi(\begin{pmatrix}0 & 0\\ 0 & d\end{pmatrix})=\phi(d)$ for any $d\in M_+$. Since $p=e_{22}$ is in the centralizer of Φ, $\Phi(x)=\Phi(pxp)+\Phi((1-p)x(1-p))$ for any $x\in\mathrm{Mat}_2(M)_+$. So if we set $\psi(a)=\Phi(\begin{pmatrix}a & 0\\ 0 & 0\end{pmatrix})$, then $\Phi=\begin{pmatrix}\psi & 0\\ 0 & \phi\end{pmatrix}$. Thus ψ is a n.s.f. weight with Radon-Nikodym cocycle $(D\psi:D\phi)_t=u_t$.

The following result is an analogue of Proposition 1.2 on induction in stages for KMS-weights. We use the notation κ^U_ϕ instead of κ_ϕ to indicate explicitly the dynamics used to induce the weight.

Proposition 3.4. *Let σ (resp. γ) be a one-parameter automorphism group of a C*-algebra A (resp. B), X a right Hilbert A-module, Y a Hilbert A-B-module, U (resp. V) a one-parameter group of isometries of X (resp. Y) such that $\langle U_t\xi,U_t\zeta\rangle=\sigma_t(\langle\xi,\zeta\rangle)$, $\langle V_t\xi,V_t\zeta\rangle=\gamma_t(\langle\xi,\zeta\rangle)$, $V_ta\xi=\sigma_t(a)V_t\xi$. Let ϕ be a (γ,β)-KMS weight on B. Set $\psi=\kappa^V_\phi|_A$. Suppose ψ is densely defined, so it is a (σ,β)-KMS weight on A. Then*

$$\kappa^{U\otimes V}_\phi(S\otimes 1)=\kappa^U_\psi(S)$$

for any $S\in B(X)$, $S\ge 0$.

Proof. We will give a proof based on the properties of spatial derivatives, but we point out that a proof along the lines of that of Proposition 1.2 is also possible.

Replacing A by $\overline{\langle X,X\rangle}$, then Y by $\overline{AY}$ and then B by $\overline{\langle Y,Y\rangle}$ we may assume that the modules are full. Let $\pi\colon B\to B(H)$ be an arbitrary representation of B such that there exists a n.s.f. weight Φ on $L=\pi(B)''$ such that $\phi=\Phi\circ\pi$ and $\sigma^\Phi_t\circ\pi=\pi\circ\gamma_{-\beta t}$. We then consider the induced representations of $B(Y)$ on $H_Y=Y\otimes_B H$ and of $B(X\otimes_A Y)$ on $H_{X\otimes Y}=X\otimes_A Y\otimes_B H$, and we consider the following von Neumann algebras

(1) $M=L'$ in $B(H)$;
(2) $N_Y=B(Y)''$, $N_0=A''$, and $M_0=A'$ in $B(H_Y)$;
(3) $N_{X\otimes Y}=B(X\otimes Y)''$, $N_X=B(X)''$ in $B(H_{X\otimes Y})$.

The algebra M acts faithfully on H_Y and $H_{X\otimes Y}$, M_0 acts faithfully on $H_{X\otimes Y}=(H_Y)_X$, and

$$\begin{aligned}N'_Y &= M \text{ in } B(H_Y);\\ N'_{X\otimes Y} &= M,\ N'_X=M_0 \text{ in } B(H_{X\otimes Y}).\end{aligned}$$

Choose a n.s.f. weight Φ' on M. Let Ψ_Y be the n.s.f. weight on N_Y such that

$$\Delta(\Psi_Y/\Phi')^{it}=V_{-\beta t}\otimes\Delta(\Phi/\Phi')^{it} \text{ on } H_Y.$$

Set $\Psi_0=\Psi_Y|_{N_0}$, so that $\psi=\Psi_0|_A$. Let Φ'_0, $\Psi_{X\otimes Y}$ and Ψ_X be the n.s.f. weights on M_0, $N_{X\otimes Y}$ and N_X, respectively, such that

$$\begin{aligned}\Delta(\Psi_0/\Phi'_0) &= \Delta(\Psi_Y/\Phi') \text{ on } H_Y;\\ \Delta(\Psi_{X\otimes Y}/\Phi')^{it} &= U_{-\beta t}\otimes V_{-\beta t}\otimes\Delta(\Phi/\Phi')^{it} \text{ on } H_{X\otimes Y};\\ \Delta(\Psi_X/\Phi'_0)^{it} &= U_{-\beta t}\otimes\Delta(\Psi_0/\Phi'_0)^{it}=\Delta(\Psi_{X\otimes Y}/\Phi')^{it} \text{ on } H_{X\otimes Y},\end{aligned}$$

so that $\Psi_{X\otimes Y}|_{B(X\otimes Y)}=\kappa^{U\otimes V}_\phi$ and $\Psi_X|_{B(X)}=\kappa^U_\psi$. We have to prove that $\Psi_{X\otimes Y}|_{N_X}=\Psi_X$. Let $E\colon N_Y\to N_0$ be the Ψ_Y-preserving conditional expectation. By considering N_Y and N_0 as subalgebras of $B(H_Y)$ we get an inverse operator-valued weight $E^{-1}\colon M_0\to M$, cf [H] and [St, Corollary 12.11]. Then by considering M_0 and M as subalgebras of $B(H_{X\otimes Y})$ we get an operator-valued weight $F=(E^{-1})^{-1}\colon N_{X\otimes Y}\to N_X$. By definition

$$\Delta(\Psi_0/\Phi'_0)=\Delta(\Psi_Y/\Phi')=\Delta(\Psi_0\circ E/\Phi')=\Delta(\Psi_0/\Phi'\circ E^{-1}) \text{ on } H_Y,$$

whence $\Phi'\circ E^{-1}=\Phi'_0$. Then

$$\Delta(\Psi_X\circ F/\Phi')=\Delta(\Psi_X/\Phi'\circ E^{-1})=\Delta(\Psi_X/\Phi'_0)=\Delta(\Psi_{X\otimes Y}/\Phi') \text{ on } H_{X\otimes Y},$$

so $\Psi_X\circ F=\Psi_{X\otimes Y}$. But the property of an operator-valued weight to have a conditional expectation as the inverse does not depend on the spatial realization, cf [Ko, Theorem 2.2]. Since E is a conditional expectation, we conclude that F is also a conditional expectation. Hence $\Psi_{X\otimes Y}|_{N_X}=\Psi_X$. □

The characterization of KMS states of general quasi-free dynamics is similar to the case of trivial dynamics on the coefficient algebra, but requires the correspondence just established between the KMS weights on A and those on $K(X)$.

Theorem 3.5. *Let σ be a one-parameter automorphism group of a C*-algebra A, U a one-parameter group of isometries of a Hilbert A-bimodule X such that $\langle U_t\xi,U_t\zeta\rangle=\sigma_t(\langle\xi,\zeta\rangle)$ and $U_ta\xi=\sigma_t(a)U_t\xi$, and denote by γ the corresponding quasi-free dynamics on the Toeplitz algebra $\mathcal{T}_X$. For $\beta\in\mathbb{R}$, let F be the operator mapping (σ,β)-KMS states of A into weights on A, defined by*

$$F\phi=\kappa^U_\phi|_A,$$

so that $F\phi$ is a (σ,β)-KMS weight on A when it is densely defined. Then

(i) *if Φ is a (γ,β)-KMS state on $\mathcal{T}_X$, then $\phi=\Phi|_A$ is a (σ,β)-KMS state on A such that $F\phi\le\phi$;*

(ii) *if ϕ is a (σ,β)-KMS state on A such that $F\phi\le\phi$, then there exists a unique gauge-invariant (γ,β)-KMS state Φ on $\mathcal{T}_X$ such that $\Phi|_A=\phi$; if $\phi=\sum_{n=0}^\infty F^n\phi_0$, then $\Phi=\kappa^{\Gamma(U)}_{\phi_0}|_{\mathcal{T}_X}$;*

(iii) *if U satisfies the 'positive energy' condition (i.e. the vectors ξ such that $\mathrm{Sp}_U(\xi)\subset(0,+\infty)$ span a dense subspace of X), then any (γ,β)-KMS state of $\mathcal{T}_X$ is gauge-invariant, so the mapping $\Phi\mapsto\Phi|_A$ defines a one-to-one correspondence between (γ,β)-KMS states on $\mathcal{T}_X$ and (σ,β)-KMS states ϕ on A such that $F\phi\le\phi$;*

(iv) *a (γ,β)-KMS state Φ on $\mathcal{T}_X$ defines a state on $\mathcal{O}_X$ if and only if $F\phi=\phi$ on I_X, where $\phi=\Phi|_A$.*

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