# ON BOST-CONNES TYPE SYSTEMS FOR NUMBER FIELDS

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Abstract. We give a complete description of the phase transition of the Bost-Connes type systems for number fields recently introduced by Connes-Marcolli-Ramachandran and Ha-Paugam. We also introduce a notion of $K$-lattices and discuss an interpretation of these systems in terms of 1-dimensional $K$-lattices.

## Introduction

The generalization of the results of Bost and Connes [1] to general number fields has received significant attention for more than ten years, but was only recently formulated in detail as an explicit problem, see [4, Problem 1.1]. We refer to [1] for a very nice motivation and explanation of the quantum statistical mechanical systems arising from number theory and to [2] for the operator algebra approach to quantum statistical mechanics.

The basic components are the following. A quantum statistical mechanical dynamical system, or C$^*$-dynamical system, $(\mathcal{A}, \sigma)$ consists of a C$^*$-algebra $\mathcal{A}$ whose self-adjoint elements represent the physical observables and which is endowed with a continuous one-parameter group of automorphisms $\sigma$ representing the time evolution of the system. The states of $\mathcal{A}$, that is to say, the positive linear functionals of norm one on $\mathcal{A}$, represent the physical states of the system, and one is particularly interested in the equilibrium states, which are defined in terms of the Kubo-Martin-Schwinger (KMS) condition and depend on an inverse temperature parameter $\beta$, see Section 1 below for a precise formulation. In general, for a given value of $\beta$, the system $(\mathcal{A}, \sigma)$ may not have any KMS$_\beta$-states at all, but when it is nonempty, the set of KMS$_\beta$-states is a compact convex subset of the state space having the Choquet simplex property: each KMS$_\beta$-state is generated uniquely as a generalized convex linear combination of the extremal points of the simplex. A group of automorphisms of $\mathcal{A}$ that commutes with the dynamics $\sigma$ can be seen as a group of symmetries of the system, and the appearance of KMS$_\beta$-states that are not invariant under such a group is interpreted as a phase transition with spontaneous symmetry breaking.

Next we paraphrase the explicit formulation of [4, Problem 1.1] for easy reference:

*Given an algebraic number field $K$, construct a C$^*$-dynamical system $(\mathcal{A}, \sigma)$ such that*

(i) *the partition function of the system is the Dedekind zeta function of $K$;*
(ii) *the quotient of the idele class group $C_K$ by the connected component $D_K$ of the identity acts as symmetries of the system;*
(iii) *for each inverse temperature $0 < \beta \le 1$ there is a unique KMS$_\beta$-state;*
(iv) *for each $\beta > 1$ the action of the symmetry group $C_K/D_K$ on the extremal KMS$_\beta$-states is free and transitive;*
(v) *there is a $K$-subalgebra $\mathcal{A}_0$ of $\mathcal{A}$ such that the values of the extremal KMS$_\infty$-states on elements of $\mathcal{A}_0$ are algebraic numbers that generate the maximal abelian extension $K^{ab}$ of $K$; and*
(vi) *the Galois action of $\mathcal{G}(K^{ab}/K)$ on these values is realized by the action of $C_K/D_K$ on the extremal KMS$_\infty$-states via the class field theory isomorphism $s\colon C_K/D_K \to \mathcal{G}(K^{ab}/K)$.*

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Systems with properties (i)-(iv) have been constructed by several authors for various classes of number fields, see [3, Section 1.4] for a discussion of these constructions and an extensive list of references. However, the last two properties have proven quite elusive. This should not come as a surprise, since a system satisfying (v) and (vi) has the potential to shed light onto Hilbert's 12th problem about the explicit class field theory of $K$, although this will ultimately depend on the specific expressions obtained for the extremal $\mathrm{KMS}_\infty$-states and the generators of the subalgebra $\mathcal{A}_0$. Since imaginary quadratic fields are the only fields beyond $\mathbb{Q}$ for which explicit class field theory is completely understood, it is natural that they should be the first case to be solved, and indeed, Connes, Marcolli, and Ramachandran have produced a complete solution of the problem for these fields, see [4, Theorem 3.1]. It should be noticed also that properties (v) and (vi) are intrinsically related so that the 'right' Galois symmetries and the 'right' arithmetic subalgebra must match each other for the system to have genuine class field theory content. This failed for instance in the system constructed in [10], where it was natural to include certain cyclotomic elements in the arithmetic subalgebra $\mathcal{A}_0$, but the Galois action on the corresponding values of extremal $\mathrm{KMS}_\infty$ states did not match the symmetry action of the idele class group on these elements, see [10, Theorem 4.4].

In this paper we study a generalization of the system from [4] to all algebraic number fields; this construction is also isomorphic to a particular case, for Shimura data arising from a number field, of a general construction due to Ha and Paugam [6].

In Section 1 we show how to reduce the study of KMS states of certain restricted groupoid C*-algebras to measures of the (unrestricted) transformation group $(G, X)$ satisfying a scaling condition. Similar results are well known when $G$ acts freely, see [15]; the key result in this section allows a certain degree of non-freeness and is motivated by our considerations in [12]. We remark that for our applications in Section 2 we could use instead the earlier results from [8] on crossed products by lattice semigroups. However, our present results can be applied to a wider class of systems, e.g. those studied in [10] for fields of class number bigger than one.

Under further assumptions, in Proposition 1.2, we give a natural parametrization of the extremal $\mathrm{KMS}_\beta$-states in terms of a specific subset of the space $X$. In Proposition 1.3 we prove a similar result for ground states.

In Section 2 we discuss the dynamical system $(\mathcal{A}, \sigma)$. We initially construct the C*-algebra $\mathcal{A}$ along the lines of [3, 6], using the restricted groupoid obtained from an action of the group of fractional ideals on the cartesian product of $\mathcal{G}(K^{ab}/K)$ by the finite adeles, balanced over the integral ideles. Since we choose to incorporate the class field theory isomorphism in the construction, the symmetry group of the system is $\mathcal{G}(K^{ab}/K)$ itself. We also indicate that $\mathcal{A}$ is a semigroup crossed product of the type discussed in [8], and that, for totally imaginary fields of class number one, the resulting system is isomorphic to the one constructed in [10] using Hecke algebras.

Using the results of Section 1 we show in Theorem 2.1 that for an arbitrary number field $K$ the system $(\mathcal{A}, \sigma)$ satisfies properties (i) through (iv) above. The description of the symmetry action and classification of the $\mathrm{KMS}_\beta$-states generalize the corresponding results of [4] and complete the initial results of [6, Section 6]. The argument goes along familiar lines [8, 13], but we include a complete proof to make the paper self-contained.

Finally, we introduce in Section 3 a notion of $n$-dimensional $K$-lattices, which generalizes the $n$-dimensional $\mathbb{Q}$-lattices from [3] and the 1-dimensional $K$-lattices for imaginary quadratic fields $K$ from [4]. After discussing some of their basic properties we show in Corollary 3.7 how 1-dimensional $K$-lattices are related to the systems of Section 2. Therefore these systems can be introduced without using any class field theory data. This may turn out to be useful in verifying properties (v) and (vi), as was the case for $\mathbb{Q}$ and imaginary quadratic fields [3, 4].

## 1. KMS states and measures

Throughout this section we suppose that $G$ is a countable discrete group acting on a second countable, locally compact, Hausdorff topological space $X$ and that $Y$ is a clopen subset of $X$

satisfying $GY = X$. The C*-algebra $C_0(X) \rtimes_r G$ is the reduced C*-algebra of the transformation groupoid $G \times X$. Consider the subgroupoid

$$G \boxtimes Y = \{(g,x) \mid x \in Y,\ gx \in Y\}$$

and denote by $C^*_r(G\boxtimes Y)$ its reduced C*-algebra. In other words, $C^*_r(G\boxtimes Y) = \mathbb{1}_Y(C_0(X) \rtimes_r G)\mathbb{1}_Y$, where $\mathbb{1}_Y$ is the characteristic function of $Y$.

We endow $C^*_r(G\boxtimes Y)$ with the dynamics $\sigma$ associated to a given homomorphism $N\colon G \to (0,+\infty)$, so

$$\sigma_t(f)(g,x) = N(g)^{it} f(g,x)$$

for $t \in \mathbb{R}$ and $f \in C_c(G\boxtimes Y) \subset C^*_r(G\boxtimes Y)$. Recall that a KMS state for $\sigma$ at inverse temperature $\beta \in \mathbb{R}$, or $\sigma$-KMS$_\beta$-state, is a $\sigma$-invariant state $\varphi$ such that $\varphi(ab) = \varphi(b\sigma_{i\beta}(a))$ for $a$ and $b$ in a set of $\sigma$-analytic elements with dense linear span.

Denote by $E$ the usual conditional expectation from $C_0(X) \rtimes_r G$ onto $C_0(X)$: thus with $u_g$ denoting the element in the multiplier algebra $M(C_0(X) \rtimes_r G)$ corresponding to $g \in G$, we have $E(fu_g) = f$ if $g = e$ and 0 otherwise. Observe that the image under $E$ of the corner $C^*_r(G\boxtimes Y)$ is $C_0(Y)$. By restriction to $C_0(Y)$, a state $\varphi$ on $C^*_r(G\boxtimes Y)$ gives rise to a Borel probability measure $\mu$, and conversely, the state $\mu_*$ of $C_0(Y)$ defined by a Borel probability measure $\mu$ on $Y$ can be extended via the conditional expectation to a state on $C^*_r(G\boxtimes Y)$. Clearly $(\mu_*\circ E)|_{C_0(Y)} = \mu_*$, but in general it is not true that $(\varphi|_{C_0(Y)})\circ E$ will be the same as $\varphi$. We will show that, under certain combined assumptions on $N$ and the action of $G$ on $X$, the $\sigma$-KMS$_\beta$-states do arise from their restrictions to $C_0(Y)$, and are thus in one-to-one correspondence with a class of measures on $Y$ characterized by a scaling condition. We point out that since the correspondences in the next three propositions are via composition with a conditional expectation, they are clearly affine.

**Proposition 1.1.** *Under the general assumptions on $G$, $X$, $Y$ and $N$ listed above, suppose there exist a sequence $\{Y_n\}_{n=1}^\infty$ of Borel subsets of $Y$ and a sequence $\{g_n\}_{n=1}^\infty$ of elements of $G$ such that*
(i) *$\cup_{n=1}^\infty Y_n$ contains the set of points in $Y$ with nontrivial isotropy;*
(ii) *$N(g_n) \ne 1$ for all $n \ge 1$;*
(iii) *$g_nY_n = Y_n$ for all $n \ge 1$.*

*Then for each $\beta \ne 0$ the map $\mu \mapsto \varphi = (\mu_*\circ E)|_{C^*_r(G\boxtimes Y)}$ is an affine isomorphism between Radon measures $\mu$ on $X$ satisfying $\mu(Y) = 1$ and the scaling condition*

$$\mu(gZ) = N(g)^{-\beta}\mu(Z) \tag{1.1}$$

*for Borel $Z \subset X$, and $\sigma$-KMS$_\beta$-states $\varphi$ on $C^*_r(G\boxtimes Y)$.*

*Proof.* It is straightforward to check that any measure satisfying the scaling condition extends via $E$ to a KMS$_\beta$-state.

Conversely, let $\varphi$ be a KMS$_\beta$-state. Denote by $\mu$ the probability measure on $Y$ defined by $\varphi|_{C_0(Y)}$. Applying the KMS-condition to elements of the form $u_gfu_g^* = f(g^{-1}\cdot)$, it is easy to see that (1.1) is satisfied for Borel $Z \subset Y$ such that $gZ \subset Y$. In particular, $\mu(Y_n) = 0$ by conditions (ii) and (iii).

To show that $\varphi = \mu_*\circ E$ fix $g \ne e$. Let $f \in C_c(Y)$ be such that $g^{-1}(\operatorname{supp} f) \subset Y$. Then $fu_g \in C^*_r(G\boxtimes Y)$ and we have to prove that $\varphi(fu_g) = 0$.

Denote by $Y_g$ the set of points of $Y$ left invariant by $g$. If $\operatorname{supp} f \cap Y_g = \emptyset$ then we can write $f$ as a finite sum of functions $h_1h_2$ such that $g(\operatorname{supp} h_1) \cap \operatorname{supp} h_2 = \emptyset$. By the KMS-condition we have

$$\varphi(h_1h_2u_g) = \varphi(h_2u_gh_1) = \varphi(h_2h_1(g^{-1}\cdot)u_g) = 0.$$

Therefore $\varphi(fu_g) = 0$.

Assume now that $\operatorname{supp} f \cap Y_g \ne \emptyset$. As $\mu(Y_n) = 0$, by condition (i) we get $\mu(Y_g) = 0$. Hence there exists a norm-bounded sequence $\{f_n\}_n \subset C_c(Y)$ such that $\operatorname{supp} f_n \cap Y_g = \emptyset$, $g^{-1}(\operatorname{supp} f_n) \subset Y$ and $f_n \to f$ in measure $\mu$. Then $\varphi(f_nu_g) = 0$. On the other hand, by the Cauchy-Schwarz inequality,

$$|\varphi(fu_g) - \varphi(f_nu_g)| \le \varphi(|f-f_n|)^{1/2}\varphi(u_g^*|f-f_n|u_g)^{1/2} \le \|f-f_n\|^{1/2}\varphi(|f-f_n|)^{1/2},$$

whence $\varphi(fu_g)=0$. Therefore $\varphi=\mu_*\circ E$.

To finish the proof it remains to show that the measure $\mu$ extends uniquely to a measure on $X$, which we will still denote by $\mu$, such that (1.1) is satisfied for all $g\in G$ and Borel $Z\subset X$. Since $G$ is countable we can choose $h_i\in G$ and $Z_i\subset Y$ for $i\in\mathbb{N}$ such that $X$ is the disjoint union of the sets $h_i^{-1}Z_i$. Since $h_iZ\cap Z_i\subset Y$ we may extend the measure $\mu$ to $X$ by

$$\mu(Z)=\sum_{i=1}^\infty N(h_i)^\beta\mu(h_iZ\cap Z_i),$$

which satisfies the scaling condition by [11, Lemma 2.2]. □

Our next goal is to classify measures satisfying the scaling condition. The classification depends on convergence of certain Dirichlet series. More precisely, when $S$ is a subset of $G$ the zeta function associated to $S$ is defined to be

$$\zeta_S(\beta):=\sum_{s\in S}N(s)^{-\beta}.$$

**Proposition 1.2.** *Assume the hypotheses of Proposition 1.1. Let $\beta\neq0$, $S$ be a subset of $G$, and $Y_0\subset Y$ a nonempty Borel set such that*

(i) *$gY_0\cap Y_0=\emptyset$ for $g\in G\setminus\{e\}$;*

(ii) *$SY_0\subset Y$;*

(iii) *if $gY_0\cap Y\neq\emptyset$ then $g\in S$;*

(iv) *$Y\setminus SU\subset\cup_nY_n$ for every open set $U$ containing $Y_0$;*

(v) *$\zeta_S(\beta)<\infty$.*

*Then*

(1) *the map $\varphi=\mu_*\circ E\mapsto\zeta_S(\beta)\mu|_{Y_0}$ is an affine isomorphism between the $\sigma$-KMS$_\beta$-states on $C_r^*(G\boxtimes Y)$ and the Borel probability measures on $Y_0$; the inverse map is given by $\nu\mapsto\mu_*\circ E$, where the measure $\mu$ on $Y$ is defined by*

$$\mu(Z)=\zeta_S(\beta)^{-1}\sum_{s\in S}N(s)^{-\beta}\nu(s^{-1}Z\cap Y_0);\tag{1.2}$$

(2) *if $\mu$ is the measure on $Y$ defined by a probability measure $\nu$ on $Y_0$ by (1.2), and $H_S$ is the subspace of $L^2(Y,d\mu)$ consisting of functions $f$ such that $f(sy)=f(y)$ for $y\in Y_0$ and $s\in S$, then for $f\in H_S$ we have*

$$\|f\|_2^2=\zeta_S(\beta)\int_{Y_0}|f(y)|^2d\mu(y);\tag{1.3}$$

*furthermore, the orthogonal projection $P\colon L^2(Y,d\mu)\to H_S$ is given by*

$$Pf|_{Sy}=\zeta_S(\beta)^{-1}\sum_{s\in S}N(s)^{-\beta}f(sy)\quad\text{for}\quad y\in Y_0.\tag{1.4}$$

*Proof.* By Proposition 1.1 any KMS$_\beta$-state is determined by a Radon measure $\mu$ such that $\mu(Y)=1$ and $\mu$ satisfies the scaling condition (1.1). By assumptions (i) and (ii), for such a measure $\mu$ we have

$$1\geq\mu(SY_0)=\sum_{s\in S}N(s)^{-\beta}\mu(Y_0)=\zeta_S(\beta)\mu(Y_0).$$

On the other hand, as $\mu(Y_n)=0$, by assumption (iv) we have

$$1=\mu(Y)\leq\mu(SU)\leq\sum_{s\in S}\mu(sU)=\zeta_S(\beta)\mu(U)$$

for any open set $U$ containing $Y_0$. By regularity of the measure we conclude that $\zeta_S(\beta)\mu(Y_0)\geq1$, and hence $\zeta_S(\beta)\mu(Y_0)=1$. It follows that $SY_0$ is a subset of $Y$ of full measure. Since $\mu$ satisfies the scaling condition, we conclude that $\mu$ is completely determined by its restriction to $Y_0$.

To finish the proof of (1) we have to construct the inverse map. Let $\nu$ be a Borel measure on $Y_0$ with $\nu(Y_0)=1$. Similarly to the proof of Proposition 1.1 define a measure $\mu$ on $X$ by

$$\mu(Z)=\zeta_S(\beta)^{-1}\sum_{g\in G}N(g)^\beta\nu(gZ\cap Y_0)\ \text{ for Borel }\ Z\subset X.$$

Then $\zeta_S(\beta)\mu$ extends $\nu$ by assumption (i) and satisfies (1.1). Furthermore, by assumptions (ii) and (iii) we have $gY\cap Y_0\ne\emptyset$ if and only if $g^{-1}\in S$, and in the latter case $Y_0\subset gY$. It follows that for $Z\subset Y$ we have (1.2). In particular, $\mu(Y)=\zeta_S(\beta)^{-1}\sum_{s\in S}N(s)^{-\beta}\nu(Y_0)=1$.

Turning to the proof of (2), suppose $\mu$ is the measure on $Y$ defined by a probability measure $\nu$ on $Y_0$ by (1.2) and recall that we have already shown that $SY_0$ is a subset of $Y$ of full $\mu$-measure. Then (2) is a particular case of [11, Lemma 2.9]. For the reader's convenience we sketch a proof.

Equality (1.3) follows from the identity

$$\int_{sY_0}|f|^2d\mu(y)=N(s)^{-\beta}\int_{Y_0}|f(s\,\cdot)|^2d\mu(y),$$

valid for $f\in L^2(Y,d\mu)$, on summing over $s\in S$. Furthermore, as

$$\sum_{s\in S}N(s)^{-\beta}|f(sy)|^2\ge\zeta_S(\beta)\left|\zeta_S(\beta)^{-1}\sum_{s\in S}N(s)^{-\beta}f(sy)\right|^2$$

the above identity and (1.3) show that the operator $T$ on $L^2(Y,d\mu)$ defined by the right hand side of (1.4) is a contraction. Since $Tf=f$ for $f\in H_S$, and the image of $T$ is $H_S$, we conclude that $T=P$. □

In our applications the set $S$ will be a subsemigroup of $\{g\in G\mid N(g)\ge1\}$ and $Y_0$ the complement of the union of the sets $gY$, $g\in S\setminus\{e\}$.

We next give a similar classification of ground states. Recall that a $\sigma$-invariant state $\varphi$ is called a ground state if the holomorphic function $z\mapsto\varphi(a\sigma_z(b))$ is bounded on the upper half-plane for $a$ and $b$ in a set of $\sigma$-analytic elements spanning a dense subspace. If a state $\varphi$ is a weak* limit point of a sequence of states $\{\varphi_n\}_n$ such that $\varphi_n$ is a $\sigma$-KMS$_{\beta_n}$-state and $\beta_n\to+\infty$ as $n\to\infty$, then $\varphi$ is a ground state. Such ground states are called $\sigma$-KMS$_\infty$-states [3].

**Proposition 1.3.** *Under the general assumptions on $G$, $X$, $Y$ and $N$ listed before Proposition 1.1, define $Y_0=Y\setminus\cup_{\{g\colon N(g)>1\}}gY$. Assume $Y_0$ has the property that if $gY_0\cap Y_0\ne\emptyset$ for some $g\in G$ then $g=e$. Then the map $\mu\mapsto\mu_*\circ E$ is an affine isomorphism between the Borel probability measures on $Y$ supported on $Y_0$ and the ground states on $C^*_r(G\boxtimes Y)$.*

*Proof.* Assume first that $\mu$ is a probability measure on $Y$ supported on $Y_0$, $\varphi=\mu_*\circ E$. If $a=f_1u_g$ and $b=f_2u_h$ with $g^{-1}(\operatorname{supp}f_1), h^{-1}(\operatorname{supp}f_2)\subset Y$, then $E(a\sigma_z(b))$ is nonzero only if $h=g^{-1}$. In the latter case the function $\varphi(a\sigma_z(b))=N(g)^{-iz}\varphi(ab)$ is clearly bounded on the upper half-plane if $N(g)\le1$. So assume $N(g)>1$. As $u_gf_2u_g^{-1}=f_2(g^{-1}\,\cdot)$ is supported on $gY$, we see that the support of $f_1f_2(g^{-1}\,\cdot)$ is contained in $Y\setminus Y_0$, whence $\varphi(a\sigma_z(b))=0$.

Conversely, assume $\varphi$ is a ground state. Let $\mu$ be the probability measure on $Y$ defined by $\varphi|_{C_0(Y)}$. Take an element $g\in G$ with $N(g)>1$. If $f\in C_c(Y\cap g^{-1}Y)$ is positive, $a=u_gf^{1/2}$ and $b=f^{1/2}u_{g^{-1}}$, then the function $z\mapsto\varphi(a\sigma_z(b))$ can be bounded on the upper half-plane only if it is identically zero. Therefore $\varphi(f(g^{-1}\,\cdot))=0$. Hence $\mu(gY\cap Y)=0$. Thus $\mu$ is supported on $Y_0$.

It remains to show that $\varphi(fu_g)=0$ for all $g\ne e$ and $f\in C_c(Y)$ with $g^{-1}(\operatorname{supp}f)\subset Y$. If $x\in\operatorname{supp}f\cap Y_0$ then $g^{-1}x\notin Y_0$ by our assumptions on $Y_0$. Hence there exists $h\in G$ with $N(h)>1$ such that $g^{-1}x\in hY$. This shows that the sets $Y\setminus Y_0$ and $ghY$ with $N(h)>1$ form an open cover of $\operatorname{supp}f$. Using a partition of unit subordinate to this cover we decompose $f$ into a finite sum of functions with supports contained in these sets. Therefore we may assume that either $\operatorname{supp}f\subset Y\setminus Y_0$ or $g^{-1}(\operatorname{supp}f)\subset hY$ for some $h$ with $N(h)>1$. In the first case we have $\varphi(fu_g)=0$

as $\mu$ is supported on $Y_0$. In the second case write $f$ as a product $f_1f_2$ of continuous functions with the same support, letting e.g. $f_1 = |f|^{1/2}$ and $f_2 = f|f|^{-1/2}$. Consider the elements $a = f_1u_{gh}$ and $b = f_2(gh\cdot)u_{h^{-1}}$ of $C^*_r(G \boxtimes Y)$, so that $fu_g = ab$. Since $N(h) > 1$, the function $z \mapsto \varphi(a\sigma_z(b))$ can be bounded on the upper half-plane only if it is identically zero. Therefore $\varphi(fu_g) = 0$. □

## 2. Bost-Connes systems for number fields

Suppose $K$ is an algebraic number field with subring of integers $\mathcal{O}$. Recall some notation. Denote by $V_K$ the set of places of $K$, and by $V_{K,f} \subset V_K$ the subset of finite places. For $v \in V_K$ denote by $K_v$ the corresponding completion of $K$. If $v$ is finite, let $\mathcal{O}_v$ be the closure of $\mathcal{O}$ in $K_v$. The ring of finite integral adeles is $\hat{\mathcal{O}} = \prod_{v\in V_{K,f}} \mathcal{O}_v$, and $\mathbb{A}_{K,f} = K \otimes_{\mathcal{O}} \hat{\mathcal{O}}$ is the ring of finite adeles. Denoting by $K_\infty = \prod_{v|\infty} K_v$ the completion of $K$ at all infinite places, we get the ring $\mathbb{A}_K = K_\infty \times \mathbb{A}_{K,f}$ of adeles. The idele group is $I_K = \mathbb{A}^*_K$.

Consider the topological space $\mathcal{G}(K^{ab}/K) \times \mathbb{A}_{K,f}$, where $\mathcal{G}(K^{ab}/K)$ is the Galois group of the maximal abelian extension of $K$. On this space there is an action of the group $\mathbb{A}^*_{K,f}$ of finite ideles, via the Artin map $s\colon I_K \to \mathcal{G}(K^{ab}/K)$ on the first component and via multiplication on the second component:

$$j(\gamma, m) = (\gamma s(j)^{-1}, jm) \quad \text{for} \quad j \in \mathbb{A}^*_{K,f}, \quad \gamma \in \mathcal{G}(K^{ab}/K), \quad m \in \mathbb{A}_{K,f}.$$

Following [4] we consider the quotient space

$$X := \mathcal{G}(K^{ab}/K) \times_{\hat{\mathcal{O}}^*} \mathbb{A}_{K,f}$$

in which the direct product is balanced over the compact open subgroup of integral ideles $\hat{\mathcal{O}}^* \subset \mathbb{A}^*_{K,f}$, in the sense that one takes the quotient by the action given by $u(\gamma, m) = (\gamma s(u)^{-1}, um)$ for $u \in \hat{\mathcal{O}}^*$. This enables a quotient action of the quotient group $\mathbb{A}^*_{K,f}/\hat{\mathcal{O}}^*$, which is isomorphic to the (discrete) group $J_K$ of fractional ideals in $K$.

Finally we restrict to the clopen subset $Y := \mathcal{G}(K^{ab}/K) \times_{\hat{\mathcal{O}}^*} \hat{\mathcal{O}}$ of $X$, and we consider the dynamical system $(C^*_r(J_K \boxtimes Y), \sigma)$, in which the dynamics $\sigma$ is defined in terms of the absolute norm $N\colon J_K \to (0, +\infty)$. Denote by $J^+_K \subset J_K$ the subsemigroup of integral ideals, and recall that the norm of such an ideal $\mathfrak{a}$ is given by $|\mathcal{O}/\mathfrak{a}|$. Remark that by Theorem 2.1 and Theorem 2.4 of [9] the corner $C^*_r(J_K \boxtimes Y) = \mathbb{1}_Y(C_0(X) \rtimes J_K)\mathbb{1}_Y$ is the semigroup crossed product $C(Y) \rtimes J^+_K$. We also point out that this system is isomorphic to the one that arises from the construction of Ha and Paugam when applied to the Shimura data associated to the number field $K$, see [6, Definition 5.5].

In this situation the zeta function of the semigroup $J^+_K$ is precisely the Dedekind zeta function $\zeta_K(\beta) = \sum_{\mathfrak{a}\in J^+_K} N(\mathfrak{a})^{-\beta}$; the series converges for $\beta > 1$ and diverges for $\beta \in (0, 1]$.

**Theorem 2.1.** *For the system $(C(\mathcal{G}(K^{ab}/K) \times_{\hat{\mathcal{O}}^*} \hat{\mathcal{O}}) \rtimes J^+_K, \sigma)$ we have:*

(i) *for $\beta < 0$ there are no KMS$_\beta$-states;*

(ii) *for each $0 < \beta \le 1$ there is a unique KMS$_\beta$-state;*

(iii) *for each $1 < \beta < \infty$ the extremal KMS$_\beta$-states are indexed by $Y_0 := \mathcal{G}(K^{ab}/K) \times_{\hat{\mathcal{O}}^*} \hat{\mathcal{O}}^* \cong \mathcal{G}(K^{ab}/K)$, with the state corresponding to $w \in Y_0$ given by*

$$\varphi_{\beta,w}(f) = \frac{1}{\zeta_K(\beta)} \sum_{\mathfrak{a}\in J^+_K} N(\mathfrak{a})^{-\beta} f(\mathfrak{a}w) \quad \text{for} \quad f \in C(\mathcal{G}(K^{ab}/K) \times_{\hat{\mathcal{O}}^*} \hat{\mathcal{O}}); \qquad (2.1)$$

(iv) *the extremal ground states are indexed by $Y_0$, with the state corresponding to $w \in Y_0$ given by $\varphi_{\infty,w}(f) = f(w)$, and all ground states are KMS$_\infty$-states.*

*Proof.* We apply Proposition 1.1 to $G = J_K$, $X = \mathcal{G}(K^{ab}/K)\times_{\hat{\mathcal{O}}^*}\mathbb{A}_{K,f}$ and $Y = \mathcal{G}(K^{ab}/K)\times_{\hat{\mathcal{O}}^*}\hat{\mathcal{O}}$. If the image of a point $(\alpha, a) \in \mathcal{G}(K^{ab}/K)\times\mathbb{A}_{K,f}$ in $X$ has nontrivial isotropy then $a_v = 0$ for some $v$, since this is true already for the action of $J_K = \mathbb{A}^*_{K,f}/\hat{\mathcal{O}}^*$ on $\mathbb{A}_{K,f}/\hat{\mathcal{O}}^*$. Therefore for the sequence $\{(g_n, Y_n)\}_n$ we can take the pairs $(\mathfrak{p}_v, Y_v)$ indexed by the finite places $v$, where $\mathfrak{p}_v$ is the prime ideal of $\mathcal{O}$ corresponding to $v$ and $Y_v \subset Y$ consists of the images in $X$ of all pairs $(\alpha, a) \in \mathcal{G}(K^{ab}/K)\times\hat{\mathcal{O}}$ with $a_v = 0$. By Proposition 1.1 we conclude that the KMS$_\beta$-states for $\beta \ne 0$ correspond to the measures $\mu$ on $X$ such that $\mu(Y) = 1$ and $\mu$ satisfies the scaling condition (1.1).

Clearly there are no such measures for $\beta < 0$, since otherwise the inclusion $\mathfrak{a}Y \subset Y$ would imply $N(\mathfrak{a})^{-\beta} \le 1$. This proves (i).

To prove part (iii) notice that $S = J_K^+$ and $Y_0 = \mathcal{G}(K^{ab}/K)\times_{\hat{\mathcal{O}}^*}\hat{\mathcal{O}}^* \subset Y$ satisfy conditions (i), (ii) and (iii) of Proposition 1.2. In order to verify condition (iv) let $A \subset V_{K,f}$ be a finite set and denote by $\mathcal{O}_A$ the product of $\mathcal{O}_v$ over $v \in A$, and by $\hat{\mathcal{O}}_A$ the product of $\mathcal{O}_v$ over $v \notin A$, so that $\hat{\mathcal{O}} = \mathcal{O}_A \times \hat{\mathcal{O}}_A$. Consider the open subset

$$W_A = \mathcal{G}(K^{ab}/K)\times_{\hat{\mathcal{O}}^*}\left(\mathcal{O}_A^* \times \hat{\mathcal{O}}_A\right)$$

of $Y$. The intersection of these sets over all finite $A$ coincides with $Y_0$. Since $Y$ is compact and the sets $W_A$ are closed, it follows that any neighborhood of $Y_0$ contains $W_A$ for some $A$. The complement of $J_K^+W_A$ in $Y$ consists of the images of points $(\alpha, a) \in \mathcal{G}(K^{ab}/K)\times\hat{\mathcal{O}}$ such that $a_v = 0$ for some $v \in A$, so it is covered by the sets $Y_v$, $v \in A$, introduced above. Thus by Proposition 1.2 for each $\beta > 1$ there is a one-to-one affine correspondence between the KMS$_\beta$-states and the probability measures on $Y_0$. In particular, the extremal KMS$_\beta$-states correspond to points of $Y_0$ via (2.1), which is a particular case of (1.2). This finishes the proof of part (iii).

To prove part (iv) we first apply Proposition 1.3 to conclude that there is a one-to-one correspondence between ground states and Borel probability measures on $Y_0$. To show that every ground state is KMS$_\infty$ let $\mu$ be a Borel probability measure on $Y$ supported on $Y_0$ and for $\beta > 1$ consider the measure

$$\mu_\beta(Z) = \zeta_K(\beta)^{-1}\sum_{\mathfrak{a}\in J_K^+} N(\mathfrak{a})^{-\beta}\mu(\mathfrak{a}^{-1}Z \cap Y_0),$$

which defines a KMS$_\beta$-state $\varphi_{\beta,\mu}$. It is clear that if $\beta \to \infty$, then $\mu_\beta \to \mu$ in norm, and therefore $\varphi_{\beta,\mu}$ converges to the ground state defined by $\mu$.

Turning to (ii), we shall first explicitly construct for each $\beta \in (0, 1]$ a measure $\mu_\beta$ on $X$ such that $\mu_\beta(Y) = 1$ and $\mu_\beta$ satisfies the scaling condition (1.1). Define $\mu_\beta$ as the push-forward of the product measure $\mu_{\mathcal{G}} \times \prod_{v\in V_{K,f}} \mu_{\beta,v}$ on $\mathcal{G}(K^{ab}/K)\times\mathbb{A}_{K,f}$, where $\mu_{\mathcal{G}}$ is the normalized Haar measure on $\mathcal{G}(K^{ab}/K)$ and the measures $\mu_{\beta,v}$ on $K_v$ are defined as follows. The measure $\mu_{1,v}$ is the additive Haar measure on $K_v$ normalized by $\mu_{1,v}(\mathcal{O}_v) = 1$. The measure $\mu_{\beta,v}$ is defined so that it is equivalent to $\mu_{1,v}$ and

$$\frac{d\mu_{\beta,v}}{d\mu_{1,v}}(a) = \frac{1 - N(\mathfrak{p}_v)^{-\beta}}{1 - N(\mathfrak{p}_v)^{-1}}\|a\|_v^{\beta-1},$$

where $\|\cdot\|_v$ is the normalized valuation in the class $v$, so $\|\pi\|_v = N(\mathfrak{p}_v)^{-1}$ for any uniformizing parameter $\pi \in \mathfrak{p}_v$. Equivalently, $\mu_{\beta,v}$ is the unique measure on $K_v$ such that the restriction of $\mu_{\beta,v}$ to $\mathcal{O}_v^*$ is the (multiplicative) Haar measure normalized by $\mu_{\beta,v}(\mathcal{O}_v^*) = 1 - N(\mathfrak{p}_v)^{-\beta}$, and $\mu_{\beta,v}(\pi Z) = N(\mathfrak{p}_v)^{-\beta}\mu_{\beta,v}(Z)$.

To show that the measure $\mu_\beta$ is unique it suffices to show that the action of $J_K$ on $(X, \mu)$ is ergodic for every measure $\mu$ on $X$ such that $\mu(Y) = 1$ and $\mu$ satisfies the scaling condition (1.1). Indeed, since a nontrivial convex combination of measures is never ergodic, if all measures are ergodic the set must consist of one point.

Equivalently, we have to show that the subspace $H$ of $L^2(Y, d\mu)$ of $J_K^+$-invariant functions consists of scalars. Denote by $P$ the projection onto this space. It is enough to compute how $P$ acts on the pull-backs of functions on $\mathcal{G}(K^{ab}/K)\times_{\hat{\mathcal{O}}^*}\mathcal{O}_A$ for finite $A\subset V_{K,f}$. Denote by $J^+_{K,A}$ the unital subsemigroup of $J^+_K$ generated by $\mathfrak{p}_v$, $v\in A$. Modulo a set of measure zero $\mathcal{G}(K^{ab}/K)\times_{\hat{\mathcal{O}}^*}\mathcal{O}_A$ is the union of the sets $\mathfrak{a}(\mathcal{G}(K^{ab}/K)\times_{\hat{\mathcal{O}}^*}\mathcal{O}^*_A)$, $\mathfrak{a}\in J^+_{K,A}$. The compact set $\mathcal{G}(K^{ab}/K)\times_{\hat{\mathcal{O}}^*}\mathcal{O}^*_A$ is a group isomorphic to $\mathcal{G}(K^{ab}/K)/s(\hat{\mathcal{O}}^*_A)$. Therefore it suffices to compute $Pf$ for the pull-back $f$ of the function

$$\mathcal{G}(K^{ab}/K)\times_{\hat{\mathcal{O}}^*}\mathcal{O}_A\ni a\mapsto\begin{cases}\tilde\chi(\mathfrak{a}^{-1}a), & \text{if } a\in\mathfrak{a}(\mathcal{G}(K^{ab}/K)\times_{\hat{\mathcal{O}}^*}\mathcal{O}^*_A),\\ 0, & \text{otherwise,}\end{cases}$$

where $\tilde\chi$ is a character of $\mathcal{G}(K^{ab}/K)\times_{\hat{\mathcal{O}}^*}\mathcal{O}^*_A$. The character $\tilde\chi$ is defined by a Dirichlet character $\chi$ mod $\mathfrak{m}$ with $\mathfrak{m}\in J^+_{K,A}$, see e.g. [14], Ch. VII, §6.

For a finite set $B\subset V_{K,f}$ denote by $P_B$ the projection onto the subspace $H_B\subset L^2(Y,d\mu)$ of $J^+_{K,B}$-invariant functions. Apply Proposition 1.2(2) with $G=J_{K,B}:=(J^+_{K,B})^{-1}J^+_{K,B}$, $S=J^+_{K,B}$ and $Y_0=W_B=\mathcal{G}(K^{ab}/K)\times_{\hat{\mathcal{O}}^*}(\mathcal{O}^*_B\times\hat{\mathcal{O}}_B)$. Note that $\zeta_{J^+_{K,B}}(\beta)=\prod_{v\in B}(1-N(\mathfrak{p}_v)^{-\beta})^{-1}$. Furthermore, for $\mathfrak{b}\in J^+_{K,B}$ the set $\mathfrak{b}W_B$ intersects the support of $f$ only if $\mathfrak{a}|\mathfrak{b}$ and the ideals $\mathfrak{a}$ and $\mathfrak{b}\mathfrak{a}^{-1}$ are relatively prime, or equivalently, $\mathfrak{a}\in J^+_{K,B}$ and $\mathfrak{b}\in\mathfrak{a}J^+_{K,B\setminus A}$. Therefore, assuming $A\subset B$, by (1.4) we get

$$\begin{aligned}P_Bf|_{J^+_{K,B}a}&=\prod_{v\in B}(1-N(\mathfrak{p}_v)^{-\beta})\sum_{\mathfrak{c}\in J^+_{K,B\setminus A}}N(\mathfrak{a}\mathfrak{c})^{-\beta}\tilde\chi(\mathfrak{c}a)\\&=N(\mathfrak{a})^{-\beta}\tilde\chi(a)\prod_{v\in B}(1-N(\mathfrak{p}_v)^{-\beta})\sum_{\mathfrak{c}\in J^+_{K,B\setminus A}}N(\mathfrak{c})^{-\beta}\chi(\mathfrak{c})\\&=N(\mathfrak{a})^{-\beta}\tilde\chi(a)\frac{\prod_{v\in B}(1-N(\mathfrak{p}_v)^{-\beta})}{\prod_{v\in B\setminus A}(1-\chi(\mathfrak{p}_v)N(\mathfrak{p}_v)^{-\beta})}\end{aligned}$$

for $a\in W_B$. If $\chi$ is trivial we see that $P_Bf$ is constant, and hence so is $Pf$. On the other hand, for nontrivial $\chi$ we get

$$\|Pf\|_2=\lim_B\|P_Bf\|_2=N(\mathfrak{a})^{-\beta}\lim_B\frac{\prod_{v\in B}|1-N(\mathfrak{p}_v)^{-\beta}|}{\prod_{v\in B\setminus A}|1-\chi(\mathfrak{p}_v)N(\mathfrak{p}_v)^{-\beta}|}.$$

The right hand side divided by $N(\mathfrak{a})^{-\beta}$ is an increasing function in $\beta$ on $(0,+\infty)$. For $\beta>1$ it equals $|L(\chi,\beta)|/\zeta_K(\beta)$. As $L(\chi,\cdot)$ does not have a pole at 1, see e.g. [14, Lemma VII.13.3], we conclude that the right hand side is zero for $\beta\in(0,1]$. Therefore in either case we see that $Pf$ is constant. ☐

*Remark* 2.2.

(i) There is an obvious action of the Galois group $\mathcal{G}(K^{ab}/K)$ of the maximal abelian extension of $K$ on $Y$, given by $\alpha(\gamma,m)=(\alpha\gamma,m)$, and this gives rise to an action of $\mathcal{G}(K^{ab}/K)$ as symmetries of $(C^*_r(J_K\boxtimes Y),\sigma)$. This action is clearly free and transitive on the set $Y_0$ parametrizing the extreme KMS$_\beta$-states.

(ii) Recall that if $\varphi$ is an extremal KMS$_\beta$-state on a C$^*$-algebra $A$ such that the von Neumann algebra $M$ generated by $A$ in the GNS-representation defined by $\varphi$ has type I, then $\varphi(a)=\operatorname{Tr}(ae^{-\beta H})/\operatorname{Tr}(e^{-\beta H})$ for a unique positive operator $H$ affiliated with $M$ with zero in the spectrum, where Tr is the unique trace on $M$ satisfying $\operatorname{Tr}(p)=1$ for minimal projections $p\in M$. In this case $\operatorname{Tr}(e^{-\beta H})$ is called the partition function. In practice it is more convenient to reformulate this spatially as follows. The assumption on $\varphi$ is equivalent to existence of an irreducible representation $\pi\colon A\to B(\mathcal{K})$ and a positive operator $H$ on $\mathcal{K}$ with zero in the spectrum such that $\varphi(a)=\operatorname{Tr}(\pi(a)e^{-\beta H})/\operatorname{Tr}(e^{-\beta H})$, where Tr is the usual operator trace on $B(\mathcal{K})$. The partition function is then $\operatorname{Tr}(e^{-\beta H})$.

It is known [6] and easy to check that the partition function of our system is well-defined for $\beta > 1$ and coincides with the Dedekind zeta function. Briefly, if $\varphi_{\beta,w}$ is the extremal KMS$_\beta$-state corresponding to $w \in Y_0$ for some $\beta > 1$, then as representation $\pi$ one takes the representation of the semigroup crossed product induced from the one-dimensional representation $f \mapsto f(w)$ of $C(Y)$, so that the representation space is $\ell^2(J_K^+)$, and one defines the operator $H$ by $H\delta_{\mathfrak{a}} = \log N(\mathfrak{a})\,\delta_{\mathfrak{a}}$ for $\mathfrak{a} \in J_K^+$.

(iii) For totally imaginary fields of class number one the C$^*$-algebra $C_r^*(J_K \boxtimes Y)$ described above is isomorphic to the Hecke C$^*$-algebra $C^*(\Gamma_K;\Gamma_{\mathcal{O}})$ studied in [10]. To see this, observe first that $\mathcal{G}(K^{ab}/K) \cong \mathbb{A}_{K,f}^*/\overline{K^*} \cong \hat{\mathcal{O}}^*/\overline{\mathcal{O}^*}$. It follows that $Y = \mathcal{G}(K^{ab}/K) \times_{\hat{\mathcal{O}}^*} \hat{\mathcal{O}}$ can be identified with $\hat{\mathcal{O}}/\overline{\mathcal{O}^*}$. From [10, Definition 2.2] and the ensueing discussion, multiplication by an extreme inverse different transforms this identification into a homeomorphism of the orbit space $\Omega = \mathcal{D}^{-1}/\overline{\mathcal{O}^*}$ and $Y$. It is then easy to check that the multiplicative action of $J_K^+ \cong \mathcal{O}^\times/\mathcal{O}^*$ on $C(\Omega)$ described in [10, Proposition 2.4] corresponds to the action of $J_K^+ \cong \mathbb{A}_{K,f}^*/\hat{\mathcal{O}}^*$ inherited by $C(Y)$ from the original transformation group. By [10, Theorem 2.5] it follows that the Hecke C$^*$-algebra $C^*(\Gamma_K;\Gamma_{\mathcal{O}})$ is isomorphic to $C(Y) \rtimes J_K^+ \cong C_r^*(J_K \boxtimes Y)$. The isomorphism respects the semigroup of isometries and thus the dynamics arising from the norm, but the Galois group action is changed via the balancing over $\hat{\mathcal{O}}^*$, and this resolves the incompatibility pointed out in [10, Theorem 4.4].

For higher class numbers the Hecke C$^*$-algebra constructed in [10] is a semigroup crossed product by the semigroup of principal ideals so it is essentially different from the one studied here.

## 3. $K$-lattices

In this section we define $n$-dimensional $K$-lattices relative to all infinite places and interpret the BC-systems for number fields in terms of these $K$-lattices.

Recall the following definition given by Connes and Marcolli [3]. An $n$-dimensional $\mathbb{Q}$-lattice is a pair $(L,\varphi)$, where $L \subset \mathbb{R}^n$ is a lattice and $\varphi\colon \mathbb{Q}^n/\mathbb{Z}^n \to \mathbb{Q}L/L$ is a homomorphism. The notion of a 1-dimensional $K$-lattice for an imaginary quadratic field $K$ is analyzed in [4]. In what follows we generalize $K$-lattices to arbitrary number fields and dimensions. We refer to [5] for a related discussion of the function fields case, see also [7].

Recall that we denote by $K_\infty$ the completion of $K$ at all infinite places, so $K_\infty \cong \mathbb{R}^{[K:\,\mathbb{Q}]}$ as a topological group under addition.

**Definition 3.1.** An *$n$-dimensional $\mathcal{O}$-lattice* is a lattice $L$ in $K_\infty^n$ such that $\mathcal{O}L = L$. An *$n$-dimensional $K$-lattice* is a pair $(L,\varphi)$, where $L \subset K_\infty^n$ is an $n$-dimensional $\mathcal{O}$-lattice and $\varphi\colon K^n/\mathcal{O}^n \to KL/L$ is an $\mathcal{O}$-module map.

The simplest example of an $n$-dimensional $\mathcal{O}$-lattice is $\mathcal{O}^n$. Since $K^n = \mathbb{Q}\mathcal{O}^n$, any two finitely generated $\mathcal{O}$-submodules of $K^n$ of rank $n$ are commensurable, in particular, any such module is an $\mathcal{O}$-lattice. Furthermore, a submodule of $K^n$ of rank $m < n$ is an abelian group of rank $m[K:\mathbb{Q}]$, so it cannot be a lattice in $K_\infty^n$. Thus for submodules of $K^n$ we get the usual definition of an $\mathcal{O}$-lattice: an $\mathcal{O}$-submodule $M \subset K^n$ is an $n$-dimensional $\mathcal{O}$-lattice if and only if it is finitely generated and has rank $n$.

We now want to give a parametrization of the set of $n$-dimensional $\mathcal{O}$-lattices in $K_\infty^n$ defined above. It is well-known that the set of $n$-dimensional $\mathcal{O}$-lattices in $K^n$ can be identified with $\mathrm{GL}_n(\mathbb{A}_{K,f})/\mathrm{GL}_n(\hat{\mathcal{O}})$ and that, correspondingly, the set of isomorphism classes of such lattices is parametrized by $\mathrm{GL}_n(K)\backslash\mathrm{GL}_n(\mathbb{A}_{K,f})/\mathrm{GL}_n(\hat{\mathcal{O}})$. Recall that this identification is based on the one-to-one correspondence between finitely generated $\mathcal{O}$-submodules of $K^n$ of rank $n$ and $\hat{\mathcal{O}}$-submodules $\mathcal{L} = \prod_{v\in V_{K,f}} L_v \subset \mathbb{A}_{K,f}^n$ such that $L_v$ is a compact open $\mathcal{O}_v$-submodule of $K_v^n$ with $L_v = \mathcal{O}_v^n$ for all but a finite number of places $v$. Namely, starting from an $\mathcal{O}$-lattice define $\mathcal{L}$ as its closure. The inverse map is $\mathcal{L} \mapsto \cap_v(L_v \cap K^n)$. Using this we parametrize the set of $n$-dimensional $\mathcal{O}$-lattices in $K_\infty^n$ as follows. Given an element $s=(s_\infty,s_f)\in \mathrm{GL}_n(\mathbb{A}_K)=\mathrm{GL}_n(K_\infty)\times\mathrm{GL}_n(\mathbb{A}_{K,f})$, we get an $\mathcal{O}$-lattice $s_f\hat{\mathcal{O}}^n\cap K^n$ in $K^n$, and then an $\mathcal{O}$-lattice $s_\infty^{-1}(s_f\hat{\mathcal{O}}^n\cap K^n)$ in $K_\infty^n$.

**Lemma 3.2.** *The map* $\mathrm{GL}_n(\mathbb{A}_K)\ni s\mapsto s_\infty^{-1}(s_f\hat{\mathcal{O}}^n\cap K^n)$ *induces a bijection between*

$$\mathrm{GL}_n(K)\backslash\mathrm{GL}_n(\mathbb{A}_K)/\mathrm{GL}_n(\hat{\mathcal{O}})$$

*and the set of $n$-dimensional $\mathcal{O}$-lattices in $K_\infty^n$.*

*Proof.* It is easy to see that the map from $\mathrm{GL}_n(K)\backslash\mathrm{GL}_n(\mathbb{A}_K)/\mathrm{GL}_n(\hat{\mathcal{O}})$ to $\mathcal{O}$-lattices is well-defined. To see that it is injective, assume $r_\infty^{-1}(r_f\hat{\mathcal{O}}^n\cap K^n)=s_\infty^{-1}(s_f\hat{\mathcal{O}}^n\cap K^n)$ for some $r,s\in\mathrm{GL}_n(\mathbb{A}_K)$. Multiplying by $K$ we get $r_\infty^{-1}K^n=s_\infty^{-1}K^n$, so $g:=s_\infty r_\infty^{-1}\in\mathrm{GL}_n(K)$. Taking the closure we get from $g(r_f\hat{\mathcal{O}}^n\cap K^n)=s_f\hat{\mathcal{O}}^n\cap K^n$ that $gr_f\hat{\mathcal{O}}^n=s_f\hat{\mathcal{O}}^n$. Hence $gr_fu=s_f$ for some $u\in\mathrm{GL}_n(\hat{\mathcal{O}})$. Since also $gr_\infty=s_\infty$, this means that $s$ is in a $\mathrm{GL}_n(K)$-$\mathrm{GL}_n(\hat{\mathcal{O}})$-orbit of $r$, so the map is injective.

To prove surjectivity, take an $\mathcal{O}$-lattice $L\subset K_\infty^n$. We have $KL=\mathbb{Q}L\cong\mathbb{Q}^{n[K\colon\mathbb{Q}]}$, so $\dim_K KL=n$. In particular, $L$ is a finitely generated $\mathcal{O}$-module of rank $n$. Therefore it suffices to show that there exists $g\in\mathrm{GL}_n(K_\infty)$ such that $gL\subset K^n$. Let $e_1,\dots,e_n$ be a basis of $KL$ over $K$. Since $KL=\mathbb{Q}L$ is dense in $K_\infty^n$, the image of $KL$ under the projection $K_\infty^n\to K_v^n$ is dense in $K_v^n$ for any infinite place $v$. It follows that the images of $e_1,\dots,e_n$ are linearly independent over $K_v$. So there exists $g_v\in\mathrm{GL}_n(K_v)$ which maps these images onto the standard basis of $K_v^n$. Then $g=(g_v)_{v|\infty}$ is an element in $\mathrm{GL}_n(K_\infty)$ mapping $e_1,\dots,e_n$ onto the standard basis of $K_\infty^n$, so that $gKL=K^n$. □

For $s\in\mathrm{GL}_n(\mathbb{A}_K)$ and $t\in\mathrm{Mat}_n(\hat{\mathcal{O}})$ consider the $\mathcal{O}$-lattice $L=s_f\hat{\mathcal{O}}^n\cap K^n$. The map $s_ft\colon\mathbb{A}_{K,f}^n\to\mathbb{A}_{K,f}^n$ maps $\hat{\mathcal{O}}^n$ into $s_f\hat{\mathcal{O}}^n$, hence induces an $\hat{\mathcal{O}}$-module map $\mathbb{A}_{K,f}^n/\hat{\mathcal{O}}^n\to\mathbb{A}_{K,f}^n/s_f\hat{\mathcal{O}}^n$. Then there exists a unique $\mathcal{O}$-module map $\varphi\colon K^n/\mathcal{O}^n\to KL/L$ such that the diagram

$$\begin{array}{ccc}
\mathbb{A}_{K,f}^n/\hat{\mathcal{O}}^n & \xrightarrow{\;s_ft\;} & \mathbb{A}_{K,f}^n/s_f\hat{\mathcal{O}}^n \\
\uparrow & & \uparrow \\
K^n/\mathcal{O}^n & \xrightarrow[\;\varphi\;]{} & KL/L
\end{array}$$

commutes, where the vertical arrows are the canonical isomorphisms defined by the inclusions $K^n\subset\mathbb{A}_{K,f}^n$, $KL\subset\mathbb{A}_{K,f}^n$. We shall also denote $\varphi$ by $[s_ft]$. Thus $(L,\varphi)$ is a $K$-lattice. Therefore $(s_\infty^{-1}L,s_\infty^{-1}\varphi)$ is also a $K$-lattice, which we denote by $[(s,t)]$.

**Lemma 3.3.** *The map* $\mathrm{GL}_n(\mathbb{A}_K)\times\mathrm{Mat}_n(\hat{\mathcal{O}})\ni(s,t)\to[(s,t)]=(s_\infty^{-1}(s_f\hat{\mathcal{O}}^n\cap K^n),s_\infty^{-1}[s_ft])$ *induces a bijection between*

$$\mathrm{GL}_n(K)\backslash\mathrm{GL}_n(\mathbb{A}_K)\times_{\mathrm{GL}_n(\hat{\mathcal{O}})}\mathrm{Mat}_n(\hat{\mathcal{O}})$$

*and the set of $n$-dimensional $K$-lattices.*

*Proof.* By Lemma 3.2 we only need to check that any $\mathcal{O}$-module map $\mathbb{A}_{K,f}^n/\hat{\mathcal{O}}^n\to\mathbb{A}_{K,f}^n/s_f\hat{\mathcal{O}}^n$, where $s_f\in\mathrm{GL}_n(\mathbb{A}_{K,f})$, is defined by the matrix $s_ft$ for a unique $t\in\mathrm{Mat}_n(\hat{\mathcal{O}})$. It suffices to consider $s_f=1$. The problem then reduces to showing that any $\mathcal{O}$-module map $K_v/\mathcal{O}_v\to K_v/\mathcal{O}_v$ is given by multiplication by a unique element of $\mathcal{O}_v$. If $\pi$ is a uniformizing parameter in $\mathcal{O}_v$, then any $\mathcal{O}$-module map $\mathcal{O}_v\pi^{-m}/\mathcal{O}_v\to K_v/\mathcal{O}_v$ is determined by the image of $\pi^{-m}$, so it is given by multiplication by an element in $\mathcal{O}_v$ which is uniquely determined modulo $\mathcal{O}_v\pi^m$. Since $\mathcal{O}_v$ is complete in the $(\pi)$-adic topology, this gives the result. □

Notice that we have shown in particular that for any $K$-lattice $(L,\varphi)$ with $L\subset K^n$ the homomorphism $\varphi$ lifts to a unique $\mathbb{A}_{K,f}$-module map $\tilde{\varphi}\colon\mathbb{A}_{K,f}^n\to\mathbb{A}_{K,f}^n$.

**Definition 3.4.** Two $n$-dimensional $K$-lattices $(L_1,\varphi_1)$ and $(L_2,\varphi_2)$ are called commensurable if the lattices $L_1$ and $L_2$ are commensurable and $\varphi_1=\varphi_2$ modulo $L_1+L_2$.

If $L_1$ and $L_2$ are commensurable then $KL_1 = \mathbb{Q}L_1 = \mathbb{Q}L_2 = KL_2$. In particular, if $L_1 \subset K^n$ then also $L_2 \subset K^n$. It is clear that then the lifting of the composition of the homomorphisms $\varphi_1\colon K^n/\mathcal{O}^n \to KL_1/L_1$ and $KL_1/L_1 \to K(L_1+L_2)/(L_1+L_2)$ coincides with $\tilde\varphi_1$. Therefore two $K$-lattices $(L_1,\varphi_1)$ and $(L_2,\varphi_2)$ with $L_1, L_2 \subset K^n$ are commensurable if and only if $\tilde\varphi_1 = \tilde\varphi_2$. This implies that commensurability is an equivalence relation.

Denote the equivalence relation of commensurability of $n$-dimensional $K$-lattices by $\mathcal{R}_{K,n}$. Consider now the action of $\mathrm{GL}_n(\mathbb{A}_{K,f})$ on $\mathrm{GL}_n(K)\backslash\mathrm{GL}_n(\mathbb{A}_K) \times \mathrm{Mat}_n(\mathbb{A}_{K,f})$ defined by

$$g(s,t) = (sg^{-1}, gt).$$

Define a subgroupoid

$$\mathrm{GL}_n(\mathbb{A}_{K,f}) \boxtimes (\mathrm{GL}_n(K)\backslash\mathrm{GL}_n(\mathbb{A}_K) \times \mathrm{Mat}_n(\hat{\mathcal{O}})) = \{(g,s,t) \mid t \in \mathrm{Mat}_n(\hat{\mathcal{O}}),\ gt \in \mathrm{Mat}_n(\hat{\mathcal{O}})\}$$

of the transformation groupoid $\mathrm{GL}_n(\mathbb{A}_{K,f}) \times (\mathrm{GL}_n(K)\backslash\mathrm{GL}_n(\mathbb{A}_K) \times \mathrm{Mat}_n(\mathbb{A}_{K,f}))$. We have a groupoid homomorphism

$$\mathrm{GL}_n(\mathbb{A}_{K,f}) \boxtimes (\mathrm{GL}_n(K)\backslash\mathrm{GL}_n(\mathbb{A}_K) \times \mathrm{Mat}_n(\hat{\mathcal{O}})) \to \mathcal{R}_{K,n}$$

defined by

$$(g,s,t) \mapsto ([(sg^{-1}, gt)], [(s,t)]). \tag{3.1}$$

To see that $[(s,t)]$ and $[(sg^{-1},gt)]$ are indeed commensurable recall that by definition we have $[(s,t)] = (s_\infty^{-1}(s_f\hat{\mathcal{O}}^n \cap K^n), s_\infty^{-1}[s_f t])$ and $[(sg^{-1},gt)] = (s_\infty^{-1}(s_f g^{-1}\hat{\mathcal{O}}^n \cap K^n), s_\infty^{-1}[s_f t])$.

By Lemma 3.3 to make the above homomorphism injective we have to factor out the action of $\mathrm{GL}_n(\hat{\mathcal{O}})$. Consider the action of $\mathrm{GL}_n(\hat{\mathcal{O}}) \times \mathrm{GL}_n(\hat{\mathcal{O}})$ on $\mathrm{GL}_n(\mathbb{A}_{K,f}) \boxtimes (\mathrm{GL}_n(K)\backslash\mathrm{GL}_n(\mathbb{A}_K) \times \mathrm{Mat}_n(\hat{\mathcal{O}}))$ defined by

$$(u_1,u_2)(g,s,t) = (u_1 g u_2^{-1}, s u_2^{-1}, u_2 t),$$

and denote by

$$\mathrm{GL}_n(\hat{\mathcal{O}})\backslash\mathrm{GL}_n(\mathbb{A}_{K,f}) \boxtimes_{\mathrm{GL}_n(\hat{\mathcal{O}})} (\mathrm{GL}_n(K)\backslash\mathrm{GL}_n(\mathbb{A}_K) \times \mathrm{Mat}_n(\hat{\mathcal{O}}))$$

the quotient space. This is a special case of a groupoid constructed in [6, Section 4.2.2].

**Proposition 3.5.** *The map (3.1) induces a bijection between*

$$\mathrm{GL}_n(\hat{\mathcal{O}})\backslash\mathrm{GL}_n(\mathbb{A}_{K,f}) \boxtimes_{\mathrm{GL}_n(\hat{\mathcal{O}})} (\mathrm{GL}_n(K)\backslash\mathrm{GL}_n(\mathbb{A}_K) \times \mathrm{Mat}_n(\hat{\mathcal{O}}))$$

*and* $\mathcal{R}_{K,n}$.

*Proof.* By Lemma 3.3 the map

$$\mathrm{GL}_n(\hat{\mathcal{O}})\backslash\mathrm{GL}_n(\mathbb{A}_{K,f}) \boxtimes_{\mathrm{GL}_n(\hat{\mathcal{O}})} (\mathrm{GL}_n(K)\backslash\mathrm{GL}_n(\mathbb{A}_K) \times \mathrm{Mat}_n(\hat{\mathcal{O}})) \to \mathcal{R}_{K,n}$$

is well-defined and injective. To prove surjectivity we have to show that if $(L,\varphi) = [(s,t)]$ is a $K$-lattice then any commensurable $K$-lattice is of the form $[(sg^{-1},gt)]$ for some $g \in \mathrm{GL}_n(\mathbb{A}_{K,f})$. We may assume that $L \subset K^n$ and then that $s_\infty = 1$. Then by Lemma 3.3 and the discussion following Definition 3.4 any commensurable $K$-lattice is of the form $[(q,r)]$ with $q_\infty = 1$ and $q_f r = s_f t$. Letting $g = q_f^{-1} s_f$ we get $(q,r) = (sg^{-1}, gt)$. □

*Remark* 3.6. In the case $K = \mathbb{Q}$, or more generally for fields with class number one, there is a better description due to the fact that any $\mathbb{Z}$-lattice is free. Indeed, by freeness we have $\mathrm{GL}_n(\mathbb{A}_{\mathbb{Q},f}) = \mathrm{GL}_n^+(\mathbb{Q})\mathrm{GL}_n(\hat{\mathbb{Z}})$, where $\mathrm{GL}_n^+(\mathbb{Q})$ is the group of rational matrices with positive determinant. It follows that any $\mathrm{GL}_n(\hat{\mathbb{Z}}) \times \mathrm{GL}_n(\hat{\mathbb{Z}})$-orbit in $\mathrm{GL}_n(\mathbb{A}_{\mathbb{Q},f}) \times (\mathrm{GL}_n(\mathbb{A}_\mathbb{Q}) \times \mathrm{Mat}_n(\hat{\mathbb{Z}}))$ has a representative in $\mathrm{GL}_n^+(\mathbb{Q}) \times ((\mathrm{GL}_n(\mathbb{R}) \times \mathrm{GL}_n^+(\mathbb{Q})) \times \mathrm{Mat}_n(\hat{\mathbb{Z}}))$. Furthermore, the map

$$\mathrm{GL}_n^+(\mathbb{R}) \times \mathrm{GL}_n^+(\mathbb{Q}) \to \mathrm{GL}_n^+(\mathbb{R}), \quad (g,h) \mapsto h^{-1}g,$$

induces a bijection between $\mathrm{GL}_n^+(\mathbb{Q})\backslash(\mathrm{GL}_n^+(\mathbb{R}) \times \mathrm{GL}_n^+(\mathbb{Q}))$ onto $\mathrm{GL}_n^+(\mathbb{R})$. One may then conclude that $\mathcal{R}_{\mathbb{Q},n}$ can be identified with

$$\mathrm{SL}_n(\mathbb{Z})\backslash\mathrm{GL}_n^+(\mathbb{Q}) \boxtimes_{\mathrm{SL}_n(\mathbb{Z})} (\mathrm{GL}_n^+(\mathbb{R}) \times \mathrm{Mat}_n(\hat{\mathbb{Z}})),$$

where the action of $\mathrm{SL}_n(\mathbb{Z})\times\mathrm{SL}_n(\mathbb{Z})$ on $\mathrm{GL}_n^+(\mathbb{Q})\times\mathrm{GL}_n^+(\mathbb{R})\times\mathrm{Mat}_n(\hat{\mathbb{Z}})$ is given by

$$(\gamma_1,\gamma_2)(g,h,m)=(\gamma_1 g\gamma_2^{-1},\gamma_2 h,\gamma_2 m).$$

Consider now the case $n=1$ (and $K$ arbitrary). Then we conclude that there is a bijection between $\mathcal{R}_{K,1}$ and the subgroupoid

$$(\mathbb{A}_{K,f}^*/\hat{\mathcal{O}}^*)\boxtimes((\mathbb{A}_K^*/K^*)\times_{\hat{\mathcal{O}}^*}\hat{\mathcal{O}})$$

of the transformation groupoid $(\mathbb{A}_{K,f}^*/\hat{\mathcal{O}}^*)\times((\mathbb{A}_K^*/K^*)\times_{\hat{\mathcal{O}}^*}\mathbb{A}_{K,f})$. We have an action, called the scaling action, of $K_\infty^*$ on $K$-lattices: if $(L,\varphi)$ is a $K$-lattice and $k\in K_\infty^*$ then $k(L,\varphi)=(kL,k\varphi)$. It defines an action of $K_\infty^*$ on $\mathcal{R}_{K,1}$. In our transformation groupoid picture of $\mathcal{R}_{K,1}$ it corresponds to the action of $K_\infty^*$ by multiplication on $\mathbb{A}_K^*/K^*$. Denote by $(K_\infty^*)^\circ$ the connected component of the identity in $K_\infty^*$. Then we get the following result.

**Corollary 3.7.** *The quotient of the equivalence relation $\mathcal{R}_{K,1}$ of commensurability of 1-dimensional $K$-lattices by the scaling action of the connected component of the identity in $K_\infty^*$ is a groupoid that is isomorphic to*

$$(\mathbb{A}_{K,f}^*/\hat{\mathcal{O}}^*)\boxtimes((\mathbb{A}_K^*/K^*(K_\infty^*)^\circ)\times_{\hat{\mathcal{O}}^*}\hat{\mathcal{O}}).$$

Recalling that $\mathbb{A}_{K,f}^*/\hat{\mathcal{O}}^*\cong J_K$ and $\mathbb{A}_K^*/\overline{K^*(K_\infty^*)^\circ}\cong\mathcal{G}(K^{ab}/K)$ by class field theory, we see that the above groupoid is almost the same that we used to define the BC-system. The small nuance is that when we put $\mathcal{G}(K^{ab}/K)$ in our topological groupoid in Section 2 we were effectively taking the quotient of $\mathbb{A}_K^*$ by the *closure* of $K^*(K_\infty^*)^\circ$. In terms of $K$-lattices this means that given a $K$-lattice $(L,\varphi)$ we would have to identify not only all $K$-lattices $(kL,k\varphi)$ with $k\in(K_\infty^*)^\circ$, but also all $K$-lattices of the form $(kL,k\psi)$, where $\psi$ is a limit point of the maps $u\varphi$ with $u\in\mathcal{O}^*\cap(K_\infty^*)^\circ$ in the topology of pointwise convergence.

This nuance does not arise for $\mathbb{Q}$ and for imaginary quadratic number fields because in those cases the group of units is finite, see [6, Section 3.1 and 4.2.2] for more on this.

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