

The Martin Boundary of a Discrete Quantum Group

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Abstract

We consider the Markov operator P_ϕ on a discrete quantum group given by convolution with a q -tracial state ϕ . In the study of harmonic elements x , $P_\phi(x) = x$, we define the Martin boundary A_ϕ . It is a separable C^* -algebra carrying canonical actions of the quantum group and its dual. We establish a representation theorem to the effect that positive harmonic elements correspond to positive linear functionals on A_ϕ . The C^* -algebra A_ϕ has a natural time evolution, and the unit can always be represented by a KMS state. Any such state gives rise to a u.c.p. map from the von Neumann closure of A_ϕ in its GNS representation to the von Neumann algebra of bounded harmonic elements, which is an analogue of the Poisson integral. Under additional assumptions this map is an isomorphism which respects the actions of the quantum group and its dual. Next we apply these results to identify the Martin boundary of the dual of $SU_q(2)$ with the quantum homogeneous sphere of Podleś. This result extends and unifies previous results by Ph. Biane and M. Izumi.

Introduction

Hopf algebras play an important role in quantum probability as they provide a natural setting for generalizations of Lévy processes [27]. The convolution operators on Hopf algebras also lead to one of the simplest non-trivial examples of non-commutative Markov processes. In a series of papers [4, 5, 6, 7] Biane studied such operators on the dual \hat{G} of a compact group G . He considered the group von Neumann algebra $L(G)$ of G with comultiplication $\Delta(\lambda_g) = \lambda_g \otimes \lambda_g$ and looked at the convolution operator $P_\phi = (\phi \otimes \iota)\Delta$, where ϕ is a normal tracial state on $L(G)$. One is particularly interested in P_ϕ -harmonic elements, that is, the elements x affiliated with $L(G)$ such that $P_\phi(x) = x$. A significant part of the theory of random walks on abelian groups [26, 28] can be generalized to this context. For example, one can prove that the constants are the only bounded harmonic elements [4]. This result also follows from a generalization of the Choquet-Deny theorem, which states that extremal normalized harmonic elements are exponentials, that is, positive group-like elements [5]. In the classical theory the set of extremal elements constitutes a part of the boundary of an appropriate compactification of \hat{G} . This boundary, coined the Martin boundary, is constructed by completing \hat{G} with respect to a metric depending on the asymptotic behavior of a function K called the Martin kernel. Having described the set of extremal harmonic elements, one could ask for a geometric realization of this set as a boundary of \hat{G} . The boundary should then be understood in the sense of non-commutative geometry, so it should be a unital C^* -subalgebra of $L(G)/C^*(G)$. This problem was solved for $SU(2)$ in [7], where Biane introduced a non-commutative analogue of the Martin kernel, proved that the corresponding boundary of $\widehat{SU(2)}$ is the 2-dimensional sphere, and showed that this sphere could be naturally identified with the set of harmonic exponentials. Thus he obtained an analogue of the Ney-Spitzer theorem which describes the Martin compactification of \mathbb{Z}^n , see e.g. [32]. Note, however, that for this result Biane assumes $\|\phi\| < 1$, so he considers sub-Markov operators for which all harmonic elements are unbounded. If ϕ is a state, then there are, in fact, no non-trivial harmonic elements on $\widehat{SU(2)}$. Another problem is that in general one expects a boundary of a non-commutative space to be non-commutative. So if

one wants to generalize Biane's result, one needs not only to describe the pure states of the algebra corresponding to extremal harmonic elements, but also to justify its multiplicative structure.

One of the main properties of the Martin boundary is that any harmonic element can be represented as an integral of the Martin kernel by some measure. The unit can be represented by a canonical measure, and the Martin boundary, considered as a measure space, is called the Poisson boundary, see e.g. [15, 14]. It turns out that the algebra of bounded measurable functions on the Poisson boundary is canonically isomorphic to the space of bounded harmonic elements, which a priori is just an operator space, but actually has a unique structure of a von Neumann algebra. This allows to define the Poisson boundary in the non-commutative setting, namely by postulating that the algebra of bounded measurable functions on it is the algebra of bounded harmonic elements. This approach was suggested by Izumi [12], who was motivated by subfactor theory and ITP-actions of compact quantum groups. As we already mentioned, there are no non-constant harmonic elements in $L(G)$ [4, 5, 12], so the Poisson boundary of the dual of a compact group is trivial. The situation changes drastically if we remove the assumption of cocommutativity and instead of $L(G)$ consider the algebra of bounded functions on a discrete quantum group. Izumi proved that the Poisson boundary is non-trivial for any non-Kac algebra. One of the principal results in [12] is the computation of the Poisson boundary of $\widehat{SU_q(2)}$. For this purpose Izumi carried out a detailed study of the Markov operator P_ϕ for the q -trace ϕ associated with the fundamental corepresentation of $SU_q(2)$. This allowed him to prove that the Poisson boundary is (the weak operator closure of) the quantum homogeneous sphere of Podleś. Then he extended this result to q -traces with finite support.

The appearance of 2-spheres in the works of Biane and Izumi is of course no coincidence. However, their interpretations of the spheres as boundaries are quite different, and in the case covered by both authors (ϕ is a state and $q = 1$) the boundaries are trivial. The missing link is the theory of the Martin boundary for discrete quantum groups. The main objective in this paper is to develop such a theory.

The paper is organized as follows.

In Section 1 we gather some facts about quantum groups, with proofs included in cases we could not find a good reference. Here we also discuss Radon-Nikodym cocycles for actions of discrete quantum groups. They will play an important role in our considerations for the same reason as in the classical theory, where the Poisson kernel can be described as the Radon-Nikodym cocycle of a measure representing the unit.

In Section 2 we study the Markov operator P_ϕ given by convolution with a q -tracial state ϕ . We develop the part of the potential theory needed to construct the boundary, featuring the balayage theorem, which gives a canonical approximation of harmonic elements by potentials $\sum_{n=0}^{\infty} P_\phi^n(x)$. Although the adaptation of this theorem to our setting is fairly straightforward, it is nevertheless striking that the lattice property for selfadjoint elements is not needed. Note that to talk about potentials we need to assume transience, i.e., that the expected number of returns of the random walk to the origin is finite. This condition is also necessary for the existence of non-constant harmonic elements. It turns out that it is fulfilled automatically in the generic quantum group case. Namely, the Markov operators we consider always have a positive eigenvector given by quotients of the classical and the quantum dimensions with eigenvalue strictly less than one in the generic case. This does not only imply transience, but also shows that the probability of return to the origin at the n th step decreases exponentially. In the last subsection of Section 2 we briefly review the results of Izumi for the Poisson boundary with emphasis on the quantum path spaces of our random walks and on the non-commutative 0-2 law rather than on fusion algebras and ITP-actions.

In Section 3 we define the Martin kernel K . It is a completely positive map from the algebra of finitely supported functions to the algebra of bounded functions on the discrete quantum group. Then we define the Martin compactification \hat{A}_ϕ as the C*-algebra generated by the algebra \hat{A}

of functions vanishing at infinity and the image of K . The Martin boundary A_ϕ is then the quotient C*-algebra \tilde{A}_ϕ/\hat{A} . We prove that any harmonic element gives rise to a positive linear functional on A_ϕ . The algebra A_ϕ has a canonical time evolution, and there always exists a state which represents the unit and has the KMS property with respect to this evolution. Any state ν representing the unit gives rise to a normal u.c.p. map K^* from the von Neumann algebra $\pi_\nu(A_\phi)''$ to the Poisson boundary. We give sufficient conditions for K^* to be an isomorphism which respects the canonical actions of the quantum group and its dual.

In Section 4 we consider the case of $SU_q(2)$. Under a certain summability assumption on ϕ , we prove that the Martin boundary A_ϕ together with the two actions of the quantum groups $SU_q(2)$ and $\widehat{SU}_q(2)$ on it can be identified with the quantum homogeneous sphere of Podleś. The proof is inspired by Biane's argument for ordinary $SU(2)$ [7]. The first step is to realize the sphere as the quotient by the compacts of the algebra of invariant formal pseudodifferential operators of order zero associated to the $4D_+$ -calculus of Woronowicz. Then using the classical theory for \mathbb{Z} we show that the Martin boundary of the center consists of one point. Combining this with detailed knowledge of the representation theory of $SU_q(2)$, including the Clebsch-Gordan coefficients, we compute explicitly the Martin kernel on certain generating elements. The main new problem in our context is to identify the actions of $\widehat{SU}_q(2)$. This is solved by studying the Radon-Nikodym cocycles for the actions. As a corollary we recover (and, in fact, extend to states with infinite support) the results of Izumi.

1 Quantum Groups

In Subsections 1.1–1.4 we collect a number of definitions and results on compact and discrete quantum groups stated in a unified notation. For more details see [36, 33, 25, 31, 30, 18, 24, 9].

In the following $M(A)$ will denote the multiplier algebra of a C*-algebra A . We usually use the same symbol for a map and its extension to the multiplier algebra. The C*-algebra of compact operators on a Hilbert space H is denoted by $B_0(H)$. We shall use \odot to distinguish the algebraic tensor product from the spatial tensor product \otimes , but we always use \otimes to denote tensor products of maps, then understood according to context.

1.1 Compact Quantum Groups

A compact quantum group (A, Δ) is a unital C*-algebra A and a unital *-homomorphism $\Delta: A \rightarrow A \otimes A$ such that $(\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta$ and that both $\Delta(A)(A \otimes 1)$ and $\Delta(A)(1 \otimes A)$ are dense in $A \otimes A$. To any compact quantum group there exists a unique state φ of A which is left- and right-invariant, i.e., $(\iota \otimes \varphi)\Delta = \varphi(\cdot)1$ and $(\varphi \otimes \iota)\Delta = \varphi(\cdot)1$, respectively. This functional is called the Haar functional and is always assumed to be faithful, so we are considering a reduced compact quantum group in the sense of [16].

A unitary corepresentation U of (A, Δ) on a Hilbert space H_U is a unitary element of $M(A \otimes B_0(H_U))$ such that $(\Delta \otimes \iota)(U) = U_{13}U_{23}$. Here we have used the leg-numbering convention. Let \mathcal{A} consist of the elements $(\iota \otimes \omega)(U)$, where U is a finite dimensional unitary corepresentation and ω is a linear functional of $B(H_U)$. Then \mathcal{A} is a dense unital *-subalgebra of A such that $\Delta(\mathcal{A}) \subset \mathcal{A} \odot \mathcal{A}$. Let \mathcal{A}' denote the space of all linear functionals on \mathcal{A} . There exist $\varepsilon \in \mathcal{A}'$ and a linear map $S: \mathcal{A} \rightarrow \mathcal{A}$ such that (\mathcal{A}, Δ) is a Hopf *-algebra with counit ε and coinverse S , so $S(a^*) = S^{-1}(a)^*$ for $a \in \mathcal{A}$.

Let $\omega \in \mathcal{A}'$ and $a \in \mathcal{A}$. Define $\omega*a, a*\omega \in \mathcal{A}$ and $\bar{\omega} \in \mathcal{A}'$ by $\omega*a = (\iota \otimes \omega)\Delta(a)$, $a*\omega = (\omega \otimes \iota)\Delta(a)$ and $\bar{\omega}(a) = \omega(a^*)$. There exists a family $\{f_z\}_{z \in \mathbb{C}}$ of unital, multiplicative functionals on \mathcal{A} uniquely

determined by the following properties:

(F1) $z \mapsto f_z(a)$ is an entire function of exponential growth on the right-half plane for $a \in \mathcal{A}$.

(F2) $(f_z \otimes f_w)\Delta = f_{z+w}$ and $f_0 = \varepsilon$.

(F3) $f_z S = f_{-z}$ and $\bar{f}_z = f_{-\bar{z}}$.

(F4) $S^2(a) = f_{-1} * a * f_1$ for $a \in \mathcal{A}$.

(F5) $\varphi(ab) = \varphi(b(f_1 * a * f_1))$ for $a, b \in \mathcal{A}$.

The modular group $\{\sigma_t^\varphi\}_t$ of φ is thus determined by $\sigma_t^\varphi(a) = f_{it} * a * f_{it}$ for $a \in \mathcal{A}$ and $t \in \mathbb{R}$. The one-parameter automorphism group $\{\tau_t\}_t$ of A determined by $\tau_t(a) = f_{-it} * a * f_{it}$, for $a \in \mathcal{A}$ and $t \in \mathbb{R}$, is called the scaling group of (A, Δ) . The involutive $*$ -antiautomorphism R of A given by $R(a) = f_{\frac{1}{2}} * S(a) * f_{-\frac{1}{2}}$ for $a \in \mathcal{A}$ is called the unitary antipode. By definition $S = R\tau_{-\frac{i}{2}} = \tau_{-\frac{i}{2}}R$.

Let U be a unitary corepresentation on H_U . Denote by \bar{H}_U the conjugate Hilbert space. Let J be the canonical antilinear isometry $H_U \rightarrow \bar{H}_U$ and $j: B(H_U) \rightarrow B(\bar{H}_U)$, $j(x) = Jx^*J^{-1}$, the corresponding $*$ -antiisomorphism. Then the conjugate unitary corepresentation \bar{U} on $H_{\bar{U}} = \bar{H}_U$ is defined by $\bar{U} = (R \otimes j)(U)$. The tensor product of two unitary corepresentations U and V is the unitary corepresentation $U \times V$ on $H_U \otimes H_V$ defined by $U \times V = U_{12}V_{13}$.

Let $(H_\varphi, \xi_\varphi, \pi_r)$ be the GNS-triple associated with the Haar functional φ . Since φ is assumed to be faithful we can identify A with its image $\pi_r(A)$ in $B(H_\varphi)$. The mapping $\xi \otimes a\xi_\varphi \mapsto \Delta(a)(\xi \otimes \xi_\varphi)$, $\xi \in H_\varphi$, $a \in A$, extends to a unitary operator on $H_\varphi \otimes H_\varphi$. Its adjoint W is called the multiplicative unitary associated with (A, Δ) . It is a unitary corepresentation of (A, Δ) on H_φ such that $\Delta(a) = W^*(1 \otimes a)W$ for $a \in A$. These two facts imply that W satisfies the pentagon equation $W_{12}W_{13}W_{23} = W_{23}W_{12}$.

If G is a compact group, then the C^* -algebra $C(G)$ of continuous functions on G with comultiplication $\Delta: C(G) \rightarrow C(G) \otimes C(G) \cong C(G \times G)$ given by $\Delta(f)(g, h) = f(gh)$ is a compact quantum group. If Γ is a discrete group, then the reduced group C^* -algebra $C_r^*(\Gamma)$ of the group Γ with comultiplication $\Delta(\lambda_g) = \lambda_g \otimes \lambda_g$ is a compact quantum group.

1.2 Dual Discrete Quantum Groups

Suppose (A, Δ) is a compact quantum group. Consider \mathcal{A}' as a unital $*$ -algebra with product $\omega\eta = (\omega \otimes \eta)\Delta$, unit ε and $*$ -operation $\omega^* = \bar{\omega}S$. Define the $*$ -subalgebra \hat{A} of \mathcal{A}' by $\hat{A} = \{\hat{a} = a\varphi \mid a \in \mathcal{A}\}$, where $(a\varphi)(b) = \varphi(ba)$ for $b \in \mathcal{A}$. For each $\omega \in \hat{A}$ the linear operator $b\xi_\varphi \mapsto (b*\omega S^{-1})\xi_\varphi$ extends to a bounded operator $\hat{\pi}_r(\omega)$ on H_φ and $\hat{\pi}_r$ is a faithful $*$ -representation of \hat{A} on H_φ . In the sequel we suppress the $*$ -isomorphism $\hat{\pi}_r$ of \hat{A} onto its image $\hat{\pi}_r(\hat{A}) \subset B(H_\varphi)$. Let \hat{A} denote the norm closure of \hat{A} in $B(H_\varphi)$. The formula $\hat{\Delta}(x) = W(x \otimes 1)W^*$ defines a comultiplication $\hat{\Delta}: \hat{A} \rightarrow M(\hat{A} \otimes \hat{A})$, so $(\hat{\Delta} \otimes \iota)\hat{\Delta} = (\iota \otimes \hat{\Delta})\hat{\Delta}$ and both $\hat{\Delta}(\hat{A})(\hat{A} \otimes 1)$ and $\hat{\Delta}(\hat{A})(1 \otimes \hat{A})$ are dense in $\hat{A} \otimes \hat{A}$. Henceforth $(\hat{A}, \hat{\Delta})$ is called the discrete quantum group dual to (A, Δ) . Note that $W \in M(A \otimes \hat{A})$. Any element $a \in \mathcal{A}$ extends to a bounded linear functional on \hat{A} , so the latter can be considered as a subspace of \mathcal{A}' . Moreover, \hat{A} is a $*$ -subalgebra of \mathcal{A}' and $\hat{\Delta}$ extends to a unital $*$ -homomorphism $\hat{\Delta}: \mathcal{A}' \rightarrow (\mathcal{A} \odot \mathcal{A})'$ given by $\hat{\Delta}(\omega) = \omega m$, where $m: \mathcal{A} \odot \mathcal{A} \rightarrow \mathcal{A}$ is the multiplication. The map $\hat{\Delta}$ satisfies the coassociativity identity with counit and coinverse given by $\hat{\varepsilon}(\omega) = \omega(1)$ and $\hat{S}(\omega) = \omega S$, respectively. There exists a unique right-invariant Haar weight $\hat{\psi}$ for $(\hat{A}, \hat{\Delta})$ which is a lower semicontinuous extension of the linear functional on \hat{A} given by $\hat{a} \mapsto \varepsilon(a)$. It satisfies the Plancherel formula $\hat{\psi}(\hat{a}^*\hat{b}) = \varphi(a^*b)$ for $a, b \in \mathcal{A}$. Let $\rho = f_1$. It is a positive self-adjoint operator affiliated with the von Neumann algebra \hat{M} generated by $\hat{A} \subset B(H_\varphi)$. Since f_1 is multiplicative, we have $\hat{\Delta}(\rho) = \rho \otimes \rho$ and $\hat{S}(\rho) = \rho^{-1} = f_{-1}$. Let $\hat{\varphi}(x) = \hat{\psi}(\rho x \rho)$ for $x \in \hat{A}_+$, where \hat{A}_+ is the cone of positive elements of the C^* -algebra \hat{A} . Then $\hat{\varphi}$ is the (unique up to a scalar) left-invariant Haar weight of $(\hat{A}, \hat{\Delta})$. The modular group of $\hat{\varphi}$ is given by $\sigma_t^{\hat{\varphi}}(x) = \rho^{it} x \rho^{-it}$ for $x \in \hat{A}$, whereas that of $\hat{\psi}$ is given by $\sigma_t^{\hat{\psi}}(x) = \rho^{-it} x \rho^{it}$. The scaling group $\hat{\tau}$ coincides with $\sigma^{\hat{\varphi}}$,

so the unitary antipode \hat{R} for $(\hat{A}, \hat{\Delta})$ is given by $\hat{R}(x) = \rho^{-\frac{1}{2}} \hat{S}(x) \rho^{\frac{1}{2}}$ for $x \in \hat{A}$.

Assume U is a unitary corepresentation of (A, Δ) . Then the formula $\pi_U(\omega) = (\omega \otimes \iota)(U)$ for $\omega \in \hat{A}$, defines a non-degenerate $*$ -representation of $\hat{A} \subset \mathcal{A}'$ on H_U . The map $U \mapsto \pi_U$ is an equivalence of the category of unitary corepresentations of (A, Δ) and the category of non-degenerate $*$ -representations of \hat{A} . Note that $\pi_W = \hat{\pi}_r$, and that the formula $\pi \mapsto (\iota \otimes \pi)(W)$ defines an inverse functor. Moreover, these functors preserve tensor products, where the tensor product of two non-degenerate $*$ -representations π_1 and π_2 is given by $\pi_1 \times \pi_2 = (\pi_1 \otimes \pi_2) \hat{\Delta}$, so $\pi_{U \times V} = \pi_U \times \pi_V$.

Suppose G is a compact group and Γ is a discrete group. The discrete quantum group dual to $(C(G), \Delta)$ is the group C^* -algebra $C^*(G)$ of the group G with comultiplication $\hat{\Delta}(\int_G f(g) \lambda_g dg) = \int_G f(g) (\lambda_g \otimes \lambda_g) dg$ under the identification $\hat{f} = \int_G f(g) \lambda_g dg$. The discrete quantum group dual to $(C_r^*(\Gamma), \Delta)$ is the C^* -algebra $c_0(\Gamma)$ of functions on Γ vanishing at infinity with comultiplication $\hat{\Delta}: c_0(\Gamma) \rightarrow M(c_0(\Gamma) \otimes c_0(\Gamma)) = l^\infty(\Gamma \times \Gamma)$ given by $\hat{\Delta}(f)(g, h) = f(gh)$ under the identification $\hat{\lambda}_g = \delta_{g^{-1}}$.

1.3 Matrix Units and the Fourier Transform

Denote by I the set of equivalence classes of irreducible (and thus finite dimensional) unitary corepresentations of a compact quantum group (A, Δ) , and let $U^s \in A \otimes B(H_s)$ denote a fixed representative for the equivalence class $s \in I$. The corresponding irreducible $*$ -representation of \hat{A} is denoted by π_s . Then $\bigoplus_s \pi_s$ is a $*$ -isomorphism of \hat{A} and the C^* -algebraic direct sum $\bigoplus_s B(H_s)$. Under this isomorphism $\hat{\mathcal{A}}$ is the algebraic direct sum of $B(H_s)$, $s \in I$, and \mathcal{A}' is the algebraic direct product of $B(H_s)$, $s \in I$. Denote by I_s the unit of $B(H_s)$ considered as an element of $\hat{\mathcal{A}}$, so $\pi_s(x) = x I_s = I_s x$ for $x \in \mathcal{A}'$. Then $\hat{\psi}(x) = \sum_s d_s \text{Tr} \pi_s(x \rho^{-1})$, where $d_s = \text{Tr} \pi_s(\rho) = \text{Tr} \pi_s(\rho^{-1})$ is the quantum dimension of U^s and Tr is the canonical, non-normalized trace on $B(H_s)$. For each $s \in I$, we pick an orthonormal basis $\{\xi_i^s\}_i$ for H_s . Let m_{ij}^s denote the corresponding matrix units in $B(H_s)$ with respect to $\{\xi_i^s\}_i$, so $m_{ij}^s \xi_k^s = \delta_{jk} \xi_i^s$ and therefore $m_{ij}^s m_{kl}^s = \delta_{jk} m_{il}^s$ and $m_{ij}^{s*} = m_{ji}^s$. Let $u_{ij}^s \in \mathcal{A}$ be the matrix coefficients of U^s , so $U^s = \sum_{i,j} u_{ij}^s \otimes m_{ij}^s$. The fact that U^s is a corepresentation means then that $\Delta(u_{ij}^s) = \sum_k u_{ik}^s \otimes u_{kj}^s$. Note that $U^s = (\iota \otimes \pi_s)W$ implies

$$W = \sum_s \sum_{i,j} u_{ij}^s \otimes m_{ij}^s. \quad (1.1)$$

The following orthogonality relations hold:

$$\varphi((u_{kl}^t)^* u_{ij}^s) = \delta_{st} \delta_{jl} \frac{f_{-1}(u_{ik}^s)}{d_s}. \quad (1.2)$$

To simplify some formulas we pick the basis $\{\xi_i^s\}_i$ such that the matrix $\pi_s(\rho)$ is diagonal. Then $f_z(u_{ij}^s) = 0$ if $i \neq j$, and $\pi_s(\rho^z) = \sum_i f_z(u_{ii}^s) m_{ii}^s$. We also have

$$\tau_t(u_{kl}^s) = f_{it}(u_{kk}^s) f_{-it}(u_{ll}^s) u_{kl}^s \quad \text{and} \quad \sigma_t^\varphi(u_{kl}^s) = f_{it}(u_{kk}^s) f_{it}(u_{ll}^s) u_{kl}^s. \quad (1.3)$$

Let \bar{s} be the equivalence class of $\overline{U^s}$. Since $\overline{U^s}$ is equivalent to $U^{\bar{s}}$, there exists an antilinear isometry $J_s: H_s \rightarrow H_{\bar{s}}$ uniquely defined up to a scalar of modulus one. So $j_s(x) = J_s x^* J_s^{-1}$ is a well-defined $*$ -antiisomorphism from $B(H_s)$ to $B(H_{\bar{s}})$. Let $n_{ij}^{\bar{s}} = j_s(m_{ji}^s)$ be the new system of matrix units in $B(H_{\bar{s}})$ and $v_{ij}^{\bar{s}}$ be the corresponding matrix coefficients of $U^{\bar{s}}$, so $U^{\bar{s}} = \sum_{i,j} v_{ij}^{\bar{s}} \otimes n_{ij}^{\bar{s}}$.

The Fourier transform $\mathcal{F}: \mathcal{A} \rightarrow \hat{\mathcal{A}}$ is the bijection defined by $\mathcal{F}(a) = \hat{a}$.

Lemma 1.1 *The following formulas hold:*

(i) $m_{ij}^s (u_{kl}^t) = \delta_{st} \delta_{ik} \delta_{jl}$;

- (ii) $v_{ij}^{\bar{s}} = f_{\frac{1}{2}}(u_{ii}^s) f_{-\frac{1}{2}}(u_{jj}^s) u_{ij}^{s*}$;
(iii) $\mathcal{F}(u_{ij}^s) = d_s^{-1} f_{-\frac{1}{2}}(u_{ii}^s) f_{-\frac{1}{2}}(u_{jj}^s) n_{ij}^{\bar{s}}$;
(iv) $\hat{R}(x) = j_s(x)$ for $x \in B(H_s)$, so $\hat{S}(m_{ij}^s) = f_{\frac{1}{2}}(u_{ii}^s) f_{-\frac{1}{2}}(u_{jj}^s) n_{ji}^{\bar{s}}$.

Proof. Part (i) is obvious, because by definition we have $\delta_{st} m_{ij}^s = \pi_t(m_{ij}^s) = \sum_{k,l} m_{ij}^s(u_{kl}^t) m_{kl}^t$.

By definition of \bar{U}^s we have $v_{ij}^{\bar{s}} = R(u_{ij}^s)$. Since $S(u_{ij}^s) = u_{ij}^{s*}$, $S = R\tau_{-\frac{i}{2}}$ and $\tau_{\frac{i}{2}}(u_{ji}^s) = f_{-\frac{1}{2}}(u_{jj}^s) f_{\frac{1}{2}}(u_{ii}^s) u_{ji}^s$ by (1.3), we get (ii).

By definition of the Fourier transform the orthogonality relations can be rewritten as

$$\mathcal{F}(u_{ij}^s)(u_{kl}^{t*}) = \delta_{st} \delta_{jl} \delta_{ik} f_{-1}(u_{ii}^s) d_s^{-1},$$

or, in view of (ii), as $\mathcal{F}(u_{ij}^s)(v_{kl}^{\bar{t}}) = \delta_{st} \delta_{jl} \delta_{ik} f_{-\frac{1}{2}}(u_{ii}^s) f_{-\frac{1}{2}}(u_{jj}^s) d_s^{-1}$. Since $n_{ij}^{\bar{s}}(v_{kl}^{\bar{t}}) = \delta_{st} \delta_{ik} \delta_{jl}$ by (i), we get (iii).

By definition of the conjugate unitary corepresentation we have

$$\pi_{\bar{U}}(\omega) \bar{\xi} = \overline{\pi_U(\omega R)^* \xi} = \overline{\pi_U(\hat{R}(\omega^*)) \xi} \quad \text{for } \xi \in H_U \text{ and } \omega \in \hat{A}.$$

This implies the first part of (iv). The second part follows from $\hat{S} = \hat{R}(\rho^{\frac{1}{2}} \cdot \rho^{-\frac{1}{2}})$. ■

We denote by 0 the equivalence class of the trivial one-dimensional corepresentation, so $\pi_0 = \hat{\varepsilon}$. Note that $\varphi = \mathcal{F}(1) = I_0$ because $I_0^2 = I_0$ and $\hat{a}I_0 = \hat{\varepsilon}(\hat{a})I_0$, for $a \in \mathcal{A}$, determine I_0 uniquely.

Lemma 1.2 *We have*

$$\hat{\Delta}(I_0) = \sum_s \sum_{i,j} \mathcal{F}(u_{ij}^s) \otimes m_{ij}^s = \sum_s \sum_{i,j} d_s^{-1} f_{-\frac{1}{2}}(u_{ii}^s) f_{-\frac{1}{2}}(u_{jj}^s) n_{ij}^{\bar{s}} \otimes m_{ij}^s.$$

Proof. This is easily verified by applying the functionals on both sides of the identity to the linear basis $\{v_{ij}^s \otimes u_{kl}^t\}_{s,t,i,j,k,l}$ of $\mathcal{A} \odot \mathcal{A}$ and using the orthogonality relations (1.2) together with Lemma 1.1. Alternatively, as $(\iota \otimes b)(W) = b$ for any $b \in \mathcal{A}$, we have $(a \otimes b)(\mathcal{F} \otimes \iota)(W) = a\mathcal{F}(b) = \varphi(ab)$ and $(a \otimes b)\hat{\Delta}(I_0) = I_0(ab) = \varphi(ab)$ for all $a, b \in \mathcal{A}$. Hence $\hat{\Delta}(I_0) = (\mathcal{F} \otimes \iota)(W)$. ■

1.4 Coactions and Invariant States

Suppose (A, Δ) is a compact or discrete quantum group. A left (resp. right) coaction α of (A, Δ) on a C^* -algebra B is a $*$ -homomorphism $\alpha: B \rightarrow M(A \otimes B)$ such that $\alpha(B)(A \otimes 1)$ (resp. $\alpha(B)(1 \otimes A)$) is dense in $A \otimes B$ (resp. $B \otimes A$) and that $(\iota \otimes \alpha)\alpha = (\Delta \otimes \iota)\alpha$ (resp. $(\alpha \otimes \iota)\alpha = (\iota \otimes \Delta)\alpha$).

The fixed point algebra B^α for a left coaction α is the C^* -subalgebra of B consisting of elements $x \in B$ such that $\alpha(x) = 1 \otimes x$.

Proposition 1.3 *Let $\alpha: B \rightarrow M(B \otimes A)$ be a right coaction of (A, Δ) on B . Define $\mathcal{B} = B$ in the discrete case and $\mathcal{B} = \text{span}\{(\iota \otimes \varphi)(\alpha(b)(1 \otimes a)) \mid a \in \mathcal{A}, b \in B\}$ in the compact case. Then \mathcal{B} is a dense $*$ -subalgebra of B and $\mathcal{B} \odot \mathcal{A} = \alpha(\mathcal{B})(1 \otimes \mathcal{A}) = (1 \otimes \mathcal{A})\alpha(\mathcal{B})$. Moreover, the operator $r: \mathcal{B} \odot \mathcal{A} \rightarrow \mathcal{B} \odot \mathcal{A}$, $r(b \otimes a) = \alpha(b)(1 \otimes a)$, is invertible with inverse given by $s(b \otimes a) = (\iota \otimes S)((1 \otimes S^{-1}(a))\alpha(b))$.*

Proof. In the compact case the orthogonality relations (1.2) imply (see e.g. [24, Theorem 1.5]) that \mathcal{B} is dense in B , $\alpha(\mathcal{B}) \subset \mathcal{B} \odot \mathcal{A}$ and

$$\mathcal{B} = \text{span}\{(\iota \otimes \varphi)(\alpha(b)(1 \otimes a)) \mid a \in \mathcal{A}, b \in \mathcal{B}\}.$$

In the discrete case $(B \otimes A)(1 \otimes I_s) = B \odot B(H_s)$, so $\alpha(\mathcal{B})(1 \otimes \mathcal{A}) \subset \mathcal{B} \odot \mathcal{A}$ and $(1 \otimes \mathcal{A})\alpha(\mathcal{B}) \subset \mathcal{B} \odot \mathcal{A}$.

Let us prove that $(\iota \otimes \varepsilon)\alpha(b) = b$ for $b \in \mathcal{B}$. Note that in the discrete case ε is defined on A , whereas in the compact case the formula makes sense due to the property $\alpha(\mathcal{B}) \subset \mathcal{B} \odot \mathcal{A}$. Apply $\iota \otimes \varepsilon \otimes \iota$ to the identity

$$(\iota \otimes \Delta)(\alpha(b)(1 \otimes a)) = (\alpha \otimes \iota)\alpha(b)(1 \otimes \Delta(a)),$$

which yields

$$\alpha(b)(1 \otimes a) = (\iota \otimes \varepsilon \otimes \iota)((\alpha \otimes \iota)\alpha(b))(1 \otimes a) = ((\iota \otimes \varepsilon)\alpha \otimes \iota)(\alpha(b)(1 \otimes a)).$$

In the discrete case the density of $\alpha(\mathcal{B})(1 \otimes \mathcal{A})$ in $B \otimes A$ gives the result, whereas in the compact case we apply $\iota \otimes \varphi$ and use the description of \mathcal{B} given above.

Next consider $a \in \mathcal{A}$ and $b \in \mathcal{B}$. Observe that $r = (\iota \otimes m)(\alpha \otimes \iota)$. Thus since $(\iota \otimes \varepsilon)\alpha(b) = b$ and $m(\iota \otimes S)\Delta = \varepsilon(\cdot)1$ we get

$$\begin{aligned} rs(b \otimes a) &= (\iota \otimes m)(\alpha \otimes \iota)(\iota \otimes S)((1 \otimes S^{-1}(a))\alpha(b)) \\ &= (\iota \otimes m)(\iota \otimes \iota \otimes S)((\iota \otimes \iota \otimes S^{-1}(a))(\alpha \otimes \iota)\alpha(b)) \\ &= (\iota \otimes m)(\iota \otimes \iota \otimes S)((\iota \otimes \iota \otimes S^{-1}(a))(\iota \otimes \Delta)\alpha(b)) \\ &= (\iota \otimes \varepsilon(\cdot)1)\alpha(b)(1 \otimes a) = b \otimes a, \end{aligned}$$

so $rs = \iota$. The equality $sr = \iota$ is proved analogously. ■

The proof of the previous proposition shows the following.

Corollary 1.4

(i) *Let (A, Δ) be a compact quantum group and $\alpha: B \rightarrow B \otimes A$ a $*$ -homomorphism such that $(\alpha \otimes \iota)\alpha = (\iota \otimes \Delta)\alpha$. Then α is a coaction if and only if there exists a dense $*$ -subalgebra \mathcal{B} of B such that $\alpha(\mathcal{B}) \subset \mathcal{B} \odot \mathcal{A}$ and $(\iota \otimes \varepsilon)\alpha = \iota$ on \mathcal{B} .*

(ii) *Let (A, Δ) be a discrete quantum group and $\alpha: B \rightarrow M(B \otimes A)$ a non-degenerate $*$ -homomorphism such that $(\alpha \otimes \iota)\alpha = (\iota \otimes \Delta)\alpha$. Then α is a coaction if and only if $(\iota \otimes \varepsilon)\alpha = \iota$. ■*

An invariant state for a right coaction α of a compact or discrete quantum group (A, Δ) on B is a state η on B such that $(\eta \otimes \iota)\alpha = \eta(\cdot)1$.

Proposition 1.5 *Let η be an invariant state for a right coaction α of a compact or discrete quantum group. Then*

(i) *$(\eta \otimes \omega)(\alpha(b_1)(b_2 \otimes 1)) = (\eta \otimes \omega S)((b_1 \otimes 1)\alpha(b_2))$ for any bounded functional ω of A such that the functional ωS on \mathcal{A} extends to a bounded functional on A ;*

(ii) *$\alpha\sigma_t^\eta = (\sigma_t^\eta \otimes \tau_{-t})\alpha$ whenever η is a faithful KMS-state (at inverse temperature $\beta = -1$).*

Proof. Let $(H_\eta, \xi_\eta, \pi_\eta)$ be a GNS-triple for η , and as usual we suppress π_η . It is easy to check that the formula

$$U(b\xi_\eta \otimes \xi) = \alpha(b)(\xi_\eta \otimes \xi),$$

for $b \in B$, $\xi \in H_\varphi$, defines a unitary corepresentation U of (A, Δ) on H_η such that $\alpha(b) = U(b \otimes 1)U^*$. Then $(\iota \otimes \omega S)(U) = (\iota \otimes \omega)(U^*)$ for example by [16, Proposition 5.2]. Since any bounded linear functional on A can be weakly approximated by vector functionals, by definition of U we conclude that

$$(\iota \otimes \omega)(U)b\xi_\eta = (\iota \otimes \omega)\alpha(b)\xi_\eta.$$

Hence

$$\begin{aligned}
\eta((\iota \otimes \omega)\alpha(b_1)b_2) &= (b_2\xi_\eta, (\iota \otimes \bar{\omega})\alpha(b_1^*)\xi_\eta) = (b_2\xi_\eta, (\iota \otimes \bar{\omega})(U)b_1^*\xi_\eta) \\
&= ((\iota \otimes \omega)(U^*)b_2\xi_\eta, b_1^*\xi_\eta) = ((\iota \otimes \omega S)(U)b_2\xi_\eta, b_1^*\xi_\eta) \\
&= ((\iota \otimes \omega S)\alpha(b_2)\xi_\eta, b_1^*\xi_\eta) = (\eta \otimes \omega S)((b_1 \otimes 1)\alpha(b_2)).
\end{aligned}$$

This proves (i). Then (ii) follows for instance from the proof of [9, Theorem 2.9]. ■

If η is an invariant state for a left coaction α , we have the following analogous formulas:

- (i) $(\omega \otimes \eta)(\alpha(b_1)(1 \otimes b_2)) = (\omega S^{-1} \otimes \eta)((1 \otimes b_1)\alpha(b_2))$,
- (ii) $\alpha\sigma_t^\eta = (\tau_t \otimes \sigma_t^\eta)\alpha$ whenever η is a faithful KMS-state.

Let η be a KMS-state on a C^* -algebra B . The formula

$$(x, y)_\eta = \eta(x\sigma_{-\frac{i}{2}}^\eta(y^*)) = (xJ_\eta y\xi_\eta, \xi_\eta),$$

where J_η is the modular involution, defines an inner product $(\cdot, \cdot)_\eta$ on the von Neumann algebra $N = \pi_\eta(B)''$. If we are given two C^* -algebras B_1 and B_2 with KMS-states η_1 and η_2 , and a completely positive contraction $T: B_1 \rightarrow B_2$ such that $\eta_2 T = \eta_1$, then there exists a unique normal unital completely positive map $T^*: N_2 \rightarrow N_1$ such that $\eta_1 T^* = \eta_2$ and $(x, T^*y)_{\eta_1} = (Tx, y)_{\eta_2}$ for $x \in B_1$ and $y \in B_2$ [1].

Proposition 1.6 *Suppose η is a faithful invariant KMS-state for a right coaction α of a discrete or compact quantum group (A, Δ) on a C^* -algebra B . Define $T_\omega: B \rightarrow B$ by $T_\omega = (\iota \otimes \omega)\alpha$, where ω is a state on A . Then T_ω is a completely positive contraction, $\eta T_\omega = \eta$ and $T_\omega^* = T_{\omega R}$.*

Proof. By density of analytic elements it is sufficient to consider x and y to be σ^η -analytic and ω to be τ -analytic. Then Proposition 1.5(i) says that

$$\eta(T_\omega(x)\sigma_t^\eta(y^*)) = \eta(xT_\omega S(\sigma_t^\eta(y^*))),$$

while part (ii) of that proposition implies $T_\omega S\sigma_t^\eta = \sigma_t^\eta T_\omega S\tau_{-t}$. Thus

$$\eta(T_\omega(x)\sigma_t^\eta(y^*)) = \eta(x\sigma_t^\eta T_\omega S\tau_{-t}(y^*)) = \eta(\sigma_{-t}^\eta(x)T_\omega S\tau_{-t}(y^*)).$$

Taking analytic continuation to $t = -\frac{i}{2}$ and using $\omega S\tau_{-t} = \omega\tau_{-t-\frac{i}{2}}R$ completes the proof. ■

From this point onwards (A, Δ) will always denote a compact quantum group.

We may clearly regard Δ as a left coaction of (A, Δ) on A and $\hat{\Delta}$ as a right coaction of $(\hat{A}, \hat{\Delta})$ on \hat{A} . We shall consider two more coactions.

Proposition 1.7 *The formulas*

$$\Phi(x) = W^*(1 \otimes x)W \quad \text{and} \quad \hat{\Phi}(a) = W(a \otimes 1)W^*,$$

for $x \in \hat{A}$ and $a \in A$, define a left coaction Φ of (A, Δ) on \hat{A} and a right coaction $\hat{\Phi}$ of $(\hat{A}, \hat{\Delta})$ on A .

Proof. Coassociativity follows from the pentagon equation for W . Note that $\Phi(x) = U^{s*}(1 \otimes x)U^s$ for $x \in B(H_s)$, so the result follows from Corollary 1.4 and the properties $(\varepsilon \otimes \iota)U^s = I_s$ and $(\iota \otimes \varepsilon)W = 1$. ■

The coactions Φ and $\hat{\Phi}$ are analogues of the adjoint action of a group on its group C^* -algebra. Suppose again that G is a compact group and Γ is a discrete group. If $(A, \Delta) = (C(G), \Delta)$, then $\hat{\Phi}$ is trivial, $\hat{\Phi}(a) = a \otimes 1$, while $\Phi: C^*(G) \rightarrow M(C(G) \otimes C^*(G)) \subset L^\infty(G, W^*(G))$ is given by $\Phi(\int_G f(g) \lambda_g dg)(h) = \int_G f(g) \lambda_{h^{-1}gh} dg$. On the other hand, if $(A, \Delta) = (C_r^*(\Gamma), \Delta)$, then Φ is trivial, $\Phi(x) = 1 \otimes x$, while $\hat{\Phi}: C_r^*(\Gamma) \rightarrow M(C_r^*(\Gamma) \otimes c_0(\Gamma)) = l^\infty(\Gamma, C_r^*(\Gamma))$ is given by $\hat{\Phi}(\lambda_g)(h) = \lambda_{hgh^{-1}}$.

Consider the left coaction $\alpha_s = \Phi|_{B(H_s)}$ of (A, Δ) on $B(H_s)$. The restriction of the q -trace $\phi_s = d_s^{-1} \text{Tr} \pi_s(\cdot \rho^{-1})$ to $B(H_s)$ is an invariant state for this coaction. To see this, first note that

$$(S^2 \otimes \iota)U^s = (1 \otimes \pi_s(\rho))U^s(1 \otimes \pi_s(\rho^{-1}))$$

by property (F4) for the family $\{f_z\}$, so since $(S \otimes \iota)U^s = U^{s*}$, we get

$$(S^{-1} \otimes \iota)(U^s) = (S^{-2} \otimes \iota)(U^{s*}) = (S^2 \otimes \iota)(U^s)^* = (1 \otimes \pi_s(\rho^{-1}))U^{s*}(1 \otimes \pi_s(\rho)).$$

Thus

$$\begin{aligned} d_s(\iota \otimes \phi_s)\alpha_s(x) &= (\iota \otimes \text{Tr})(U^{s*}(1 \otimes x)U^s(1 \otimes \pi_s(\rho^{-1}))) \\ &= (\iota \otimes \text{Tr})((1 \otimes x)(S \otimes \iota)((S^{-1} \otimes \iota)(U^s)(1 \otimes \pi_s(\rho^{-1}))(S^{-1} \otimes \iota)(U^{s*}))) \\ &= (\iota \otimes \text{Tr})((1 \otimes x)(S \otimes \iota)((1 \otimes \pi_s(\rho^{-1}))U^{s*}U^s)) = d_s\phi_s(x)1. \end{aligned}$$

Note that since U^s is irreducible, we have $B(H_s)^{\alpha_s} = \mathbb{C}I_s$. The formula $E_s = (\varphi \otimes \iota)\alpha_s$ defines a conditional expectation of $B(H_s)$ onto $B(H_s)^{\alpha_s}$, and any invariant state η must satisfy the property $\eta E_s = \eta$. As a consequence $E_s = \phi_s(\cdot)I_s$ and $\phi_s|_{B(H_s)}$ is the unique invariant state for α_s .

Let $\mathcal{C} \subset \hat{A}^*$ denote the norm closure of the linear span of $\{\phi_s\}_{s \in I}$, so \mathcal{C} consists of the functionals $\sum_s \lambda_s \phi_s$, $\{\lambda_s\}_s \in l^1(I)$. Any state in \mathcal{C} is an invariant state for Φ . Conversely, since $\phi_s|_{B(H_s)}$ is the only invariant state for α_s , any invariant state for Φ belongs to \mathcal{C} .

For any unitary corepresentation U of (A, Δ) , the formula $\alpha_U(x) = U^*(1 \otimes x)U$, for $x \in B_0(H_U)$, defines a left coaction α_U of (A, Δ) on $B_0(H_U)$, so $\alpha_s = \alpha_{U^s}$. Again by uniqueness of invariant states for α_s , for any invariant state η on $B_0(H_U)$ we have $\tilde{\eta}\pi_U \in \mathcal{C}$, where $\tilde{\eta}$ is the unique normal extension of η to $B(H_U)$.

Lemma 1.8

- (i) The state ϕ_s considered as a linear functional on \hat{A} lies in \mathcal{A} and $\phi_s = \sum_i d_s^{-1} f_{-1}(u_{ii}^s) u_{ii}^s$.
- (ii) The linear space \mathcal{C} is a subalgebra of \hat{A}^* . Namely, if $U^s \times U^t \cong \sum_w N_{s,t}^w U^w$ is the decomposition of the corepresentation $U_s \times U_t$ into irreducible ones, then $\phi_s \phi_t = \sum_w \frac{d_w}{d_s d_t} N_{s,t}^w \phi_w$.

Proof. Statement (i) follows from Lemma 1.1(i). Since $\hat{\Delta}(\rho) = \rho \otimes \rho$, property (ii) follows from $\sum_w N_{s,t}^w \text{Tr} \pi_w = \text{Tr} \pi_{U^s \times U^t} = (\text{Tr} \otimes \text{Tr})(\pi_s \otimes \pi_t) \hat{\Delta}$. ■

Note that in general \mathcal{C} is not a $*$ -subalgebra because $\phi_s^* = d_s^{-1} \text{Tr} \pi_s(\cdot \rho)$.

1.5 Radon-Nikodym Cocycle

Let $\alpha: B \rightarrow M(B \otimes \hat{A})$ be a right coaction of a discrete quantum group $(\hat{A}, \hat{\Delta})$ on a unital C^* -algebra B . Denote by $M(B \odot \hat{A})$ the algebraic multiplier algebra of $B \odot \hat{A}$ [30], so

$$M(B \odot \hat{A}) = \prod_{s \in I} B \otimes B(H_s),$$

where we use the algebraic direct product. Thus $M(B \odot \hat{A})$ is a $*$ -algebra, but not a normed algebra.

Definition 1.9 A state η on B is called *quasi-invariant* (with respect to a right coaction $\alpha: B \rightarrow M(B \otimes \hat{A})$) with Radon-Nikodym cocycle $y \in M(B \odot \hat{A})$ if

- (i) $(\eta \otimes \iota)\alpha(b) = (\eta \otimes \iota)((b \otimes 1)(\iota \otimes \hat{S})(y))$ for $b \in B$;
- (ii) $(\iota \otimes \hat{\Delta})(y) = (\alpha \otimes \iota)(y)(y \otimes 1)$.

Note that these formulas make sense if we extend all appropriate homomorphisms to algebraic multiplier algebras. A more concrete way to proceed is as follows. First observe that $M(\hat{A}) = \mathcal{A}' = \prod_{s \in I} B(H_s)$, so both $\hat{S}: \mathcal{A}' \rightarrow \mathcal{A}'$ and the homomorphism $\hat{\Delta}: \mathcal{A}' \rightarrow (\mathcal{A} \odot \mathcal{A})' = \prod_{s,t \in I} B(H_s) \otimes B(H_t)$ are well-defined, see Subsection 1.2. Since $\hat{S}(B(H_s)) = B(H_{\bar{s}})$, clearly $\iota \otimes \hat{S}$ is well-defined. Thus identity (i) makes sense. Concerning (ii), notice that for fixed $s, t \in I$, there exists only finitely many $w \in I$ such that $\hat{\Delta}(x)$, $x \in B(H_w)$, has a non-zero component in $B(H_s) \otimes B(H_t)$, namely, those $w \in I$ for which U^w is a subcorepresentation of $U^s \times U^t$. Thus $(\iota \otimes \hat{\Delta})(y)$ is a well-defined element of $M(B \odot \hat{A} \odot \hat{A}) = \prod_{s,t \in I} B \otimes B(H_s) \otimes B(H_t)$.

Observe that whenever η is faithful, condition (i) of this definition determines y uniquely.

Proposition 1.10 Suppose η is a state on B and $y \in M(B \odot \hat{A})$ satisfies identity (i) in Definition 1.9. Then:

- (i) if η is faithful, the element y satisfies identity (ii) in Definition 1.9, so η is quasi-invariant with Radon-Nikodym cocycle y ;
- (ii) $(\eta \otimes \iota)(\alpha(b_1)(b_2 \otimes 1)) = (\eta \otimes \hat{S})((b_1 \otimes 1)\alpha(b_2)y)$ for $b_1, b_2 \in B$.

Proof. We begin by proving (ii). Let $z = b \otimes a$ for $b \in B$ and $a \in \hat{A}$, and consider the linear maps r, s on $B \odot \hat{A}$ introduced in Proposition 1.3. Then

$$\begin{aligned} (\eta \otimes \iota)(\alpha(b_1)r(z)) &= (\eta \otimes \iota)(\alpha(b_1b)(1 \otimes a)) \\ &= (\eta \otimes \iota)((b_1b \otimes 1)(\iota \otimes \hat{S})(y)(1 \otimes a)) \\ &= (\eta \otimes \hat{S})((b_1 \otimes 1)(\iota \otimes \hat{S}^{-1})(z)y). \end{aligned}$$

If we apply this identity to $z = s(b_2 \otimes I_t)$, and use $rs = \iota$, we get

$$\begin{aligned} I_t(\eta \otimes \iota)(\alpha(b_1)(b_2 \otimes 1)) &= (\eta \otimes \iota)(\alpha(b_1)(b_2 \otimes I_t)) \\ &= (\eta \otimes \hat{S})((b_1 \otimes 1)(\iota \otimes \hat{S}^{-1})s(b_2 \otimes I_t)y) \\ &= (\eta \otimes \hat{S})((b_1 \otimes 1)(1 \otimes I_t)\alpha(b_2)y) \\ &= I_t(\eta \otimes \hat{S})((b_1 \otimes 1)\alpha(b_2)y). \end{aligned}$$

Since this holds for any $t \in I$, assertion (ii) is proved.

To prove (i), apply $\hat{\Delta}$ to the identity

$$(\eta \otimes \iota)\alpha(b) = (\eta \otimes \iota)((b \otimes 1)(\iota \otimes \hat{S})(y)).$$

The right hand side yields $(\eta \otimes \chi)(\iota \otimes \hat{S} \otimes \hat{S})((b \otimes 1 \otimes 1)(\iota \otimes \hat{\Delta})(y))$, where χ denotes the flip on $(\mathcal{A} \odot \mathcal{A})'$. Whereas the left hand side gives

$$\begin{aligned} (\eta \otimes \iota \otimes \iota)(\iota \otimes \hat{\Delta})\alpha(b) &= (\eta \otimes \iota \otimes \iota)(\alpha \otimes \iota)\alpha(b) \\ &= (\eta \otimes \iota \otimes \iota)(\alpha(b)_{13}(\iota \otimes \hat{S} \otimes \iota)(y \otimes 1)) \\ &= (\eta \otimes \chi)((\alpha(b) \otimes 1)(\iota \otimes \iota \otimes \hat{S})(y_{13})) \\ &\stackrel{(ii)}{=} (\eta \otimes \chi)(\iota \otimes \hat{S} \otimes \iota)((b \otimes 1 \otimes 1)(\alpha \otimes \iota)(\iota \otimes \hat{S})(y)(y \otimes 1)) \\ &= (\eta \otimes \chi)(\iota \otimes \hat{S} \otimes \hat{S})((b \otimes 1 \otimes 1)(\alpha \otimes \iota)(y)(y \otimes 1)). \end{aligned}$$

Thus

$$(\eta \otimes \chi)(\iota \otimes \hat{S} \otimes \hat{S})((b \otimes 1 \otimes 1)(\iota \otimes \hat{\Delta})(y)) = (\eta \otimes \chi)(\iota \otimes \hat{S} \otimes \hat{S})((b \otimes 1 \otimes 1)(\alpha \otimes \iota)(y)(y \otimes 1)).$$

Since b is arbitrary and η is faithful, assertion (i) now follows. ■

We call property (ii) strong quasi-invariance, so (i) in Proposition 1.5 follows from Proposition 1.10 (at least in the discrete case).

In general y is not self-adjoint. But it will be self-adjoint and have other nice properties under additional assumptions which will always be satisfied in our examples.

Proposition 1.11 *Let η be a quasi-invariant state with Radon-Nikodym cocycle y . Suppose in addition that η is a faithful KMS-state and that $\alpha\sigma_t^\eta = (\sigma_t^\eta \otimes \hat{\tau}_{-t})\alpha$. Then*

- (i) $(\sigma_t^\eta \otimes \hat{\tau}_{-t})(y) = y$;
- (ii) y is positive and invertible with $y^{-1} = (\iota \otimes m)(\alpha \otimes \iota)(\iota \otimes \hat{S})(y)$.

Proof. Apply $\hat{\tau}_{-t}$ to the identity $(\eta \otimes \iota)\alpha(b) = (\eta \otimes \iota)((b \otimes 1)(\iota \otimes \hat{S})(y))$, and use σ_t^η -invariance of η . Thus

$$(\eta \otimes \iota)\alpha\sigma_t^\eta(b) = (\eta \otimes \iota)((\sigma_t^\eta(b) \otimes 1)(\sigma_t^\eta \otimes \hat{\tau}_{-t})(\iota \otimes \hat{S})(y)) = (\eta \otimes \iota)((\sigma_t^\eta(b) \otimes 1)(\iota \otimes \hat{S})(\sigma_t^\eta \otimes \hat{\tau}_{-t})(y)),$$

for $b \in B$, and (i) follows by uniqueness.

To see that $y \geq 0$, it is clearly sufficient to show that $\sum_{k,j}(\eta \otimes \iota)((b_k^* \otimes a_k^*)y(b_j \otimes a_j)) \geq 0$ for $a_k \in \hat{A}$ and $b_k \in B$. Moreover, we can assume that b_k is σ^η -analytic. Write $x_k = \sigma_{\frac{i}{2}}^\eta(b_k)$. Using (i) and the KMS-condition, we thus get

$$\begin{aligned} & \sum_{k,j}(\eta \otimes \iota)((b_k^* \otimes a_k^*)y(b_j \otimes a_j)) \\ &= \sum_{k,j} a_k^*(\eta \otimes \iota)((\sigma_i^\eta(b_j)b_k^* \otimes 1)y)a_j = \sum_{k,j} a_k^*(\eta \otimes \iota)((x_j x_k^* \otimes 1)(\sigma_{-\frac{i}{2}}^\eta \otimes \iota)(y))a_j \\ &= \sum_{k,j} a_k^*(\eta \otimes \iota)((x_j x_k^* \otimes 1)(\iota \otimes \hat{\tau}_{-\frac{i}{2}})(y))a_j = \sum_{k,j} a_k^*(\eta \otimes \hat{R})((x_j x_k^* \otimes 1)(\iota \otimes \hat{S})(y))a_j \\ &= \sum_{k,j} a_k^*(\eta \otimes \hat{R})\alpha(x_j x_k^*)a_j = \sum_{k,j}(\eta \otimes \hat{R})((1 \otimes \hat{R}(a_j))\alpha(x_j x_k^*)(1 \otimes \hat{R}(a_k^*))) \\ &= \hat{R}(\eta \otimes \iota)(zz^*) \geq 0, \end{aligned}$$

where $z = \sum_j(1 \otimes \hat{R}(a_j))\alpha(x_j)$.

To see that y is invertible, note that $(\iota \otimes \hat{\varepsilon})(y) = 1$. For this apply $\iota \otimes \hat{\varepsilon}$ to the identity $(\eta \otimes \iota)\alpha(b) = (\eta \otimes \iota)((b \otimes 1)(\iota \otimes \hat{S})(y))$, use the property $(\iota \otimes \hat{\varepsilon})\alpha = \iota$ from Proposition 1.3 and faithfulness of η . Now as y is self-adjoint, we may write the cocycle identity as $(\iota \otimes \hat{\Delta})(y) = (y \otimes 1)(\alpha \otimes \iota)(y)$. Next apply the map $(\iota \otimes m)(\iota \otimes \iota \otimes \hat{S})$ on both sides. This gives

$$\begin{aligned} 1 \otimes 1 &= (1 \otimes \hat{\varepsilon}(\cdot)1)(y) = (\iota \otimes m)(\iota \otimes \iota \otimes \hat{S})(\iota \otimes \hat{\Delta})(y) \\ &= (\iota \otimes m)(\iota \otimes \iota \otimes \hat{S})((y \otimes 1)(\alpha \otimes \iota)(y)) = y(\iota \otimes m)(\alpha \otimes \iota)(\iota \otimes \hat{S})(y). \end{aligned}$$

Since $y^* = y$, we conclude that y is invertible with two-sided inverse $(\iota \otimes m)(\alpha \otimes \iota)(\iota \otimes \hat{S})(y)$. ■

Consider now the right coaction $\hat{\Phi}: A \rightarrow M(A \otimes \hat{A})$ introduced above.

Proposition 1.12 *For the Haar state φ on A , the following properties hold:*

- (i) $(\sigma_t^\varphi \otimes \hat{\tau}_{-t})\hat{\Phi} = \hat{\Phi}\sigma_t^\varphi$;
- (ii) the state φ is quasi-invariant with Radon-Nikodym cocycle $y = W(1 \otimes \rho^{-2})W^*$.

Proof. Recall that $\hat{\tau}_t(x) = \rho^{it}x\rho^{-it}$ and $\hat{\Phi}(a) = W(a \otimes 1)W^*$. By $\sigma_t^\varphi(a) = f_{it} * a * f_{it}$ and (1.1), we get $(\sigma_t^\varphi \otimes \iota)(W) = (1 \otimes \rho^{it})W(1 \otimes \rho^{it})$. This implies (i).

For $a \in A$, we have

$$(\varphi \otimes \iota)\hat{\Phi}(a) = (\varphi \otimes \iota)(W(a \otimes 1)W^*) = (\varphi \otimes \iota)((a \otimes 1)(\iota \otimes \hat{S})(\iota \otimes \hat{S}^{-1})(W^*)(\sigma_{-i}^\varphi \otimes \hat{S}^{-1})(W)),$$

so φ is quasi-invariant with Radon-Nikodym cocycle $(\iota \otimes \hat{S}^{-1})(W^*)(\sigma_{-i}^\varphi \otimes \hat{S}^{-1})(W)$. Because $(\iota \otimes \hat{S}^{-1})(W^*) = W$, we thus must show that $(\sigma_{-i}^\varphi \otimes \hat{S}^{-1})(W) = (1 \otimes \rho^{-2})W^*$. We have

$$\begin{aligned} (\sigma_{-i}^\varphi \otimes \hat{S}^{-1})(W) &= (\iota \otimes \hat{S}^{-1})((1 \otimes \rho)W(1 \otimes \rho)) = (1 \otimes \rho^{-1})(\iota \otimes \hat{S}^{-1})(W)(1 \otimes \rho^{-1}) \\ &= (1 \otimes \rho^{-2})(1 \otimes \rho)(\iota \otimes \hat{S}^{-1})(W)(1 \otimes \rho^{-1}) = (1 \otimes \rho^{-2})(1 \otimes \hat{S}^2)(\iota \otimes \hat{S}^{-1})(W) \\ &= (1 \otimes \rho^{-2})W^* \end{aligned}$$

as desired. Note that the cocycle identity, positivity and invertibility for y follow from Propositions 1.10 and 1.11, but can easily be checked directly. ■

2 Random Walks on Discrete Quantum Groups

2.1 Quantum Markov Chains

Let $(\hat{A}, \hat{\Delta})$ be a discrete quantum group. For any bounded linear functional ϕ on \hat{A} , we define the convolution operator $P_\phi: M(\hat{A}) \rightarrow M(\hat{A})$ by $P_\phi = (\phi \otimes \iota)\hat{\Delta}$. Here $M(\hat{A})$ is the multiplier algebra of \hat{A} , so $M(\hat{A})$ is the C*-algebraic direct product of $B(H_s)$, $s \in I$. Note that $P_\phi(\hat{A}) \subset \hat{A}$ and if η is another bounded linear functional on \hat{A} , then $P_\phi P_\eta = P_{\eta\phi}$. If ϕ is a state, then P_ϕ is a unital completely positive map, and we call it the Markov operator associated with ϕ .

We shall usually consider ϕ in the space \mathcal{C} of Φ -invariant linear functionals on \hat{A} introduced in Subsection 1.4. The following proposition partially justifies this. Let $Z(\hat{A})$ denote the center of \hat{A} , so $Z(\hat{A})$ is the C*-algebra generated by I_s , $s \in I$, so $Z(\hat{A}) \cong c_0(I)$.

Proposition 2.1 *For a bounded linear functional ϕ on \hat{A} , we have $P_\phi(Z(\hat{A})) \subset Z(\hat{A})$ if and only if $\phi \in \mathcal{C}$.*

Proof. By Lemma 1.2, we see that $P_\phi(I_0) \in Z(\hat{A})$ if and only if $\phi(n_{ij}^{\bar{s}}) = 0$ for $i \neq j$ and the number $\lambda_{\bar{s}} = f_{-1}(u_{ii}^s)\phi(n_{ii}^{\bar{s}})$ is independent of i , for any $s \in I$. Since by Lemma 1.1(ii), we have $f_{-1}(u_{ii}^s) = f_1(v_{ii}^{\bar{s}})$, these conditions mean that $\phi|_{B(H_{\bar{s}})} = d_{\bar{s}}\lambda_{\bar{s}}\phi_{\bar{s}}$, so $\phi \in \mathcal{C}$. We have thus proved the forward implication. To prove that $P_\phi(Z(\hat{A})) \subset Z(\hat{A})$ whenever $\phi \in \mathcal{C}$, note that by the above argument, we see that $P_{\phi_s}(I_0) = \frac{1}{d_s^2}I_{\bar{s}}$ and therefore $P_\phi(I_0) \in Z(\hat{A})$ for all $\phi \in \mathcal{C}$. Also $P_\phi(I_s) = d_s^2 P_\phi P_{\phi_{\bar{s}}}(I_0) = d_s^2 P_{\phi_{\bar{s}}\phi}(I_0)$, and as \mathcal{C} is an algebra by Lemma 1.8, we see that $P_\phi(I_s) \in Z(\hat{A})$. ■

Suppose $\phi \in \mathcal{C}$ is a state. As in [4] we can construct the associated quantum Markov chain (with initial distribution $\hat{\varepsilon}$ and generator P_ϕ) in the sense of [2]. It consists of:

- (i) a von Neumann algebra $M(\hat{A})^\infty$ and a normal state ϕ^∞ given by $\otimes_{-\infty}^{-1}(M(\hat{A}), \phi)$;
- (ii) unital *-homomorphisms $j_k: M(\hat{A}) \rightarrow M(\hat{A})^\infty$ given by $j_k(a) = \dots \otimes 1 \otimes \hat{\Delta}^{k-1}(a)$ for $k \geq 1$ and $a \in M(\hat{A})$, and $j_0 = \hat{\varepsilon}$.

Here $\hat{\Delta}^k$ is defined inductively by $\hat{\Delta}^0 = \iota$, $\hat{\Delta}^1 = \hat{\Delta}$ and $\hat{\Delta}^{k+1} = (\hat{\Delta} \otimes \iota)\hat{\Delta}^k$. Using $\phi = \hat{\varepsilon}P_\phi$, the crucial property

$$\phi^\infty(j_0(a_0) \dots j_k(a_k)) = \hat{\varepsilon}(a_0 P_\phi(\dots P_\phi(a_{k-1}) P_\phi(a_k) \dots)) \quad (2.1)$$

is then easily checked.

The algebra $M(\hat{A})^\infty$ should be thought of as the algebra of measurable functions on the path space of our quantum Markov chain. Since $P_\phi(Z(\hat{A})) \subset Z(\hat{A}) = c_0(I)$, the operator P_ϕ determines a classical Markov chain on I with kernel $\{p_\phi(s, t)\}_{s, t \in I}$, so $P_\phi(I_t)I_s = p_\phi(s, t)I_s$. Let (Ω, \mathbb{P}_0) be the corresponding path space. Thus $\Omega = \prod_{-\infty}^{-1} I$ and the measure \mathbb{P}_0 is defined on cylindrical sets as follows:

$$\mathbb{P}_0(\{\omega \in \Omega \mid \omega_{-n} = s_{-n}, \dots, \omega_{-1} = s_{-1}\}) = p_\phi(0, s_{-1})p_\phi(s_{-1}, s_{-2}) \dots p_\phi(s_{-n+1}, s_{-n}). \quad (2.2)$$

The following proposition is essentially from [4].

Proposition 2.2 *There exists an embedding $j^\infty: L^\infty(\Omega, \mathbb{P}_0) \hookrightarrow (M(\hat{A})^\infty, \phi^\infty)$ uniquely determined by $j^\infty(a_{-n} \otimes \dots \otimes a_{-1}) = j_n(a_{-n}) \dots j_1(a_{-1})$ for $a_{-n}, \dots, a_{-1} \in Z(\hat{A})$.*

Proof. We first prove that $j_{k+1}(a)$ commutes with $j_{l+1}(b)$ for arbitrary $k, l \geq 0$ and $a, b \in Z(\hat{A})$. We can assume that $l = k+n$ for some $n \in \mathbb{N}$. Thus we need to show that $1 \otimes \dots \otimes 1 \otimes \hat{\Delta}^k(a)$ and $\hat{\Delta}^{k+n}(b)$ commute in $M(\otimes_{-k-n}^{-1} \hat{A})$. This is true because $1 \otimes \dots \otimes 1 \otimes \hat{\Delta}^k(a) = (\iota \otimes \dots \otimes \iota \otimes \hat{\Delta}^k)(1 \otimes \dots \otimes 1 \otimes a)$, $\hat{\Delta}^{k+n}(b) = (\iota \otimes \dots \otimes \iota \otimes \hat{\Delta}^k) \hat{\Delta}^n(b)$ by coassociativity of $\hat{\Delta}$, and because $1 \otimes \dots \otimes 1 \otimes a$ commutes with $\hat{\Delta}^n(b)$. So there exists a unital $*$ -homomorphism $j^\infty: \otimes_{-\infty}^{-1} l^\infty(I) \rightarrow M(\hat{A})^\infty$ determined by $j^\infty(a_{-n} \otimes \dots \otimes a_{-1}) = j_n(a_{-n}) \dots j_1(a_{-1})$ for $a_{-n}, \dots, a_{-1} \in Z(\hat{A})$. Using equalities (2.1) and (2.2) it is easy to check that $\phi^\infty j^\infty = \mathbb{P}_0$. Thus j^∞ extends to the required normal embedding. ■

2.2 Transience

Definition 2.3 *Suppose $\phi \in \mathcal{C}$ is a state.*

- (i) *We say that ϕ is transient if the corresponding classical random walk on I is transient, that is, if the sum $\sum_{n=0}^\infty p_{\phi^n}(s, t)$ is finite for all $s, t \in I$.*
- (ii) *By $\text{supp } \phi$ we mean the set $\{s \in I \mid \phi(I_s) \neq 0\}$.*
- (iii) *We say that ϕ is generating if for any $s \in I$, there exists $n \in \mathbb{N}$ such that $\phi^n(I_s) > 0$, that is $\cup_n \text{supp } \phi^n = I$.*

Define an anti-linear isometric operator $\phi \mapsto \check{\phi}$ on \mathcal{C} by $\check{\phi}_s = \phi_{\bar{s}}$ for $s \in I$.

Let U be a finite dimensional unitary corepresentation of (A, Δ) . Consider the state ϕ_U given by $\phi_U = d_U^{-1} \text{Tr } \pi_U(\cdot \rho^{-1})$, where $d_U = \text{Tr } \pi_U(\rho^{-1}) = \text{Tr } \pi_U(\rho)$ is the quantum dimension. Then $\phi_U \in \mathcal{C}$. Define $N_{U,s}^t$ to be the multiplicity of U^t in $U \times U^s$.

Lemma 2.4 *With the above notation the following properties hold:*

- (i) $\phi_U \phi_V = \phi_{U \times V}$ for any finite dimensional unitary corepresentations U and V ;
- (ii) $\phi_U = \phi_{\bar{U}}$;
- (iii) *the mapping $\phi \mapsto \check{\phi}$ on \mathcal{C} is anti-multiplicative;*
- (iv) $p_{\phi_U}(s, t) = \frac{d_t}{d_U d_s} N_{U,s}^t$;
- (v) $p_{\check{\phi}}(s, t) = \left(\frac{d_t}{d_s}\right)^2 p_\phi(t, s)$ for any state $\phi \in \mathcal{C}$.

Proof. Parts (i) and (ii) follow immediately from definitions. Part (iii) follows from (i) and (ii) and the property $\overline{U \times V} \cong \overline{V} \times \overline{U}$. For part (iv) apply ϕ_s to the identity $P_{\phi_U}(I_t)I_s = p_{\phi_U}(s, t)I_s$. This yields (see the proof of Lemma 1.8)

$$p_{\phi_U}(s, t) = \phi_s P_{\phi_U}(I_t) = (\phi_U \phi_s)(I_t) = \phi_{U \times U^s}(I_t) = \sum_w \frac{d_w}{d_U d_s} N_{U, s}^w \phi_w(I_t) = \frac{d_t}{d_U d_s} N_{U, s}^t.$$

Part (v) now follows for $\phi = \phi_U$ by the Frobenius reciprocity $N_{U, s}^t = N_{\overline{U}, t}^s$, and for general ϕ by linearity. ■

Corollary 2.5 *A state $\phi \in \mathcal{C}$ is generating if and only if the classical random walk is irreducible in the sense that for all $s, t \in I$, there exists $n \in \mathbb{N}$ such that $p_{\phi^n}(s, t) > 0$.*

Proof. Since $p_{\phi}(0, s) = \phi(I_s)$, we see that irreducibility of the classical random walk implies the generating property for ϕ . Conversely, suppose ϕ is generating. Since $(\check{\phi})^n = (\phi^n)^\sim$ by Lemma 2.4(iii), we have $\text{supp } (\check{\phi})^n = \overline{\text{supp } \phi^n}$, so $\check{\phi}$ is generating. Now given $s, t \in I$ we can find $k, l \in \mathbb{N}$ such that $(\check{\phi})^k(I_s) > 0$ and $\phi^l(I_t) > 0$. By Lemma 2.4(v) we thus get $p_{\phi^k}(s, 0) = d_s^{-2} p_{(\check{\phi})^k}(0, s) > 0$ and $p_{\phi^l}(0, t) > 0$. Since

$$p_{\phi^{k+l}}(s, t) = \sum_{w \in I} p_{\phi^k}(s, w) p_{\phi^l}(w, t),$$

we conclude that $p_{\phi^{k+l}}(s, t) > 0$. ■

Note also that Lemma 2.4 implies that ϕ is generating if and only if for any $s \in I$ there exist $s_1, \dots, s_n \in \text{supp } \phi$ such that U^s is a subcorepresentation of $U^{s_1} \times \dots \times U^{s_n}$. In particular, if (A, Δ) is a compact matrix pseudogroup with fundamental corepresentation U [33], the state ϕ_U is generating.

In the classical theory it is usually difficult to check the transience condition. The following result shows, however, that in the generic quantum group case transience is automatic.

Theorem 2.6 *Suppose $\phi \in \mathcal{C}$ is a state for which there exists $w \in \text{supp } \phi$ with $\pi_w(\rho) \neq I_w$ (or equivalently, $\dim H_w < d_w$). Then there exists $\lambda < 1$ such that $p_{\phi^n}(s, t) \leq \frac{d_t \dim H_s}{d_s \dim H_t} \lambda^n$ for any $s, t \in I$ and $n \in \mathbb{N}$. In particular, ϕ is transient.*

Proof. Recall that since $\text{Tr } \pi_s(\rho) = \text{Tr } \pi_s(\rho^{-1})$, we have by Schwarz inequality

$$\dim H_s = \text{Tr } \pi_s(\rho^{\frac{1}{2}} \rho^{-\frac{1}{2}}) \leq d_s,$$

with equality if and only if $\pi_s(\rho^{\frac{1}{2}}) = \pi_s(\rho^{-\frac{1}{2}})$, that is $\pi_s(\rho) = I_s$.

Let $\phi = \sum_r \lambda_r \phi_r$, where $\lambda_r \geq 0$ and $\sum_r \lambda_r = 1$. By Lemma 2.4(i) we have

$$\phi^n = \sum_{r_1, \dots, r_n} \lambda_{r_1} \dots \lambda_{r_n} \phi_{U^{r_1} \times \dots \times U^{r_n}}.$$

Thus Lemma 2.4(iv) and the inequality $N_{U, s}^t \leq \frac{\dim H_U \dim H_s}{\dim H_t}$ entails

$$\begin{aligned} p_{\phi^n}(s, t) &= \sum_{r_1, \dots, r_n} \lambda_{r_1} \dots \lambda_{r_n} \frac{d_t}{d_{r_1} \dots d_{r_n} d_s} N_{U^{r_1} \times \dots \times U^{r_n}, s}^t \\ &\leq \sum_{r_1, \dots, r_n} \lambda_{r_1} \dots \lambda_{r_n} \frac{\dim H_{r_1}}{d_{r_1}} \dots \frac{\dim H_{r_n}}{d_{r_n}} \frac{d_t \dim H_s}{d_s \dim H_t} = \lambda^n \frac{d_t \dim H_s}{d_s \dim H_t}, \end{aligned}$$

where $\lambda = \sum_r \lambda_r \frac{\dim H_r}{d_r} < 1$. ■

In particular, if (A, Δ) is a q -deformation of (the algebra of continuous functions on) a semisimple compact Lie group (see e.g. [19]), then any state $\phi \neq \hat{\varepsilon}$ in \mathcal{C} is transient. The assumptions of Theorem 2.6 are also satisfied for any non-Kac algebra $(\hat{A}, \hat{\Delta})$ with a generating state $\phi \in \mathcal{C}$.

Our next goal is to extend P_ϕ to a larger subspace of $\mathcal{A}' = M(\hat{\mathcal{A}})$. We consider \mathcal{A}' with weak* topology, which coincides with the Tikhonov topology on the algebraic direct product $M(\hat{\mathcal{A}}) = \prod_s B(H_s)$. So $M(\hat{\mathcal{A}})$ is a complete locally convex space. Let \mathcal{I} be the collection of all finite subsets of I . For $X \subset I$ denote by F_X the projection from $M(\hat{\mathcal{A}})$ onto $\prod_{s \in X} B(H_s)$. So a net $\{x_i\}_i$ converges to x in $M(\hat{\mathcal{A}})$ if and only if $F_X(x_i) \rightarrow F_X(x)$ for all $X \in \mathcal{I}$.

Let ϕ be a positive linear functional in \mathcal{C} . Denote by $D(P_\phi)_+$ the set of all $x \in M(\hat{\mathcal{A}})_+$ such that the net $\{P_\phi F_X(x)\}_{X \in \mathcal{I}}$ in $M(\hat{\mathcal{A}})$ is Cauchy, or equivalently, such that the set $\{\pi_s P_\phi F_X(x)\}_{X \in \mathcal{I}}$ is bounded for any $s \in I$. For $x \in D(P_\phi)_+$ we set $P_\phi(x) = \lim_{X \in \mathcal{I}} P_\phi F_X(x)$, and then form the linear extension $P_\phi: D(P_\phi) \rightarrow M(\hat{\mathcal{A}})$, where $D(P_\phi)$ is the linear span of $D(P_\phi)_+$. Note that $D(P_\phi) \cap M(\hat{\mathcal{A}})_+ = D(P_\phi)_+$. If the support of ϕ is finite, then $D(P_\phi) = M(\hat{\mathcal{A}})$. Moreover, as $\phi \in \mathcal{A}$ in this case, the formula $P_\phi = (\phi \otimes \iota)\hat{\Delta}$ is meaningful.

Lemma 2.7

- (i) If $0 \leq x \leq y$ for $x \in M(\hat{\mathcal{A}})$ and $y \in D(P_\phi)$, then $x \in D(P_\phi)$ and $0 \leq P_\phi(x) \leq P_\phi(y)$.
- (ii) If $x_i \rightarrow x \in M(\hat{\mathcal{A}})$ and $0 \leq x_i \leq y$ for some $y \in D(P_\phi)$, then $x \in D(P_\phi)$ and $P_\phi(x_i) \rightarrow P_\phi(x)$.

Proof. Assertion (i) is obvious from definitions. As for (ii), note that $x \in D(P_\phi)$ by (i). Fix $X \in \mathcal{I}$ and $\varepsilon > 0$. Then there exists $Y \in \mathcal{I}$ such that $\|F_X P_\phi(y - F_Y(y))\| < \varepsilon$. There also exists i_0 such that $\|F_X P_\phi F_Y(x - x_i)\| < \varepsilon$ for all $i \geq i_0$. Thus

$$\|F_X(P_\phi(x) - P_\phi(x_i))\| \leq \|F_X P_\phi(x - F_Y(x))\| + \|F_X P_\phi(x_i - F_Y(x_i))\| + \|F_X P_\phi F_Y(x - x_i)\| < 3\varepsilon,$$

since $0 \leq F_X P_\phi(x - F_Y(x)) \leq F_X P_\phi(y - F_Y(y))$ and $0 \leq F_X P_\phi(x_i - F_Y(x_i)) \leq F_X P_\phi(y - F_Y(y))$. ■

We proceed to define the potential operator $G_\phi = \sum_{n=0}^{\infty} P_\phi^n: D(G_\phi) \rightarrow M(\hat{\mathcal{A}})$. By definition its domain $D(G_\phi) \subset M(\hat{\mathcal{A}})$ is the linear span of $D(G_\phi)_+$, where $x \in D(G_\phi)_+$ whenever $x \geq 0$, $x \in \cap_n D(P_\phi^n)$ and the series $\sum_{n=0}^{\infty} P_\phi^n(x)$ converges.

By definition ϕ is transient if and only if $I_s \in D(G_\phi)$ for all $s \in I$, if and only if $\hat{\mathcal{A}} \subset D(G_\phi)$. Thus in this case the domain of G_ϕ is dense, but G_ϕ is by no means continuous.

Note also that if ϕ is transient, then $G_\phi(\hat{\mathcal{A}}) \subset M(\hat{\mathcal{A}})$ by complete maximum principle (see e.g. [26, Theorem 2.1.12]).

Definition 2.8 We say that an element $x \in M(\hat{\mathcal{A}})$ is

- (i) harmonic if $P_\phi(x) = x$;
- (ii) superharmonic if $x \geq 0$ and $P_\phi(x) \leq x$;
- (iii) a potential of an element $y \in D(G_\phi)_+$ if $x = G_\phi(y)$.

Recall that there exists a strong connection between transience and superharmonic elements: ϕ is transient if and only if there exist non-constant central (that is, lying in $Z(\mathcal{A}')$) superharmonic elements (see e.g. [32, Theorem 1.16]).

Lemma 2.9

- (i) If $x \in D(G_\phi)$, then $G_\phi(x) \in D(P_\phi)$ and $P_\phi G_\phi(x) = G_\phi(x) - x$. In particular, any potential is superharmonic.
- (ii) A superharmonic element x is a potential if and only if $P_\phi^n(x) \rightarrow 0$ in $M(\hat{\mathcal{A}})$. In particular, any superharmonic element which is majorized by a potential, is a potential itself.

Proof. In proving (i) we can suppose that $x \geq 0$. For any $Y \in \mathcal{I}$ the set $\{F_Y P_\phi F_X G_\phi(x)\}_X$ is bounded. Indeed, if $x_n = \sum_{k=0}^n P_\phi^k(x)$, then $F_Y P_\phi F_X x_n \nearrow F_Y P_\phi F_X G_\phi(x)$ and $F_Y P_\phi F_X x_n \leq F_Y P_\phi(x_n) = F_Y(x_{n+1} - x) \leq F_Y G_\phi(x)$. Hence $G_\phi(x) \in D(P_\phi)$. Then by Lemma 2.7(ii) we deduce $P_\phi G_\phi(x) = \lim_{n \rightarrow \infty} P_\phi(x_n) = \lim_{n \rightarrow \infty} (x_{n+1} - x) = G_\phi(x) - x$.

To prove (ii) consider a potential $x = G_\phi(y)$. Then by (i) $P_\phi^n(x) = G_\phi(y) - y_{n-1}$, where $y_{n-1} = \sum_{k=0}^{n-1} P_\phi^k(y)$. Thus $P_\phi^n(x) \rightarrow 0$. Conversely, if $P_\phi(x) \leq x$ and $P_\phi^n(x) \rightarrow 0$, then $x = G_\phi(y)$ with $y = x - P_\phi(x)$, since $\sum_{k=0}^n P_\phi^k(y) = x - P_\phi^{n+1}(x) \rightarrow x$. ■

2.3 Balayage Theorem

The aim of this subsection is to show that any superharmonic element can be canonically approximated by potentials. The proof is essentially the same as in the classical theory. However, since most classical proofs use some simplifications arising from probabilistic interpretations and the fact the Markov operator acts on functions (so the space of self-adjoint elements is a lattice), we present a detailed argument.

For a positive linear functional $\phi \in \mathcal{C}$ and $Y \in \mathcal{I}$, consider the linear operator

$$P_{\phi, Y} = \sum_{n=0}^{\infty} [(\iota - F_Y) P_\phi]^n F_Y.$$

Note that if $F_Y(x) \in D(G_\phi)_+$, then $x \in D(P_{\phi, Y})$ and $P_{\phi, Y}(x) \leq G_\phi F_Y(x)$.

Define also

$$G_\phi^Y = \sum_{n=0}^{\infty} [(\iota - F_Y) P_\phi]^n (\iota - F_Y).$$

Then $D(G_\phi) \subset D(G_\phi^Y)$ and $G_\phi^Y(x) \leq G_\phi(x)$ for $x \in D(G_\phi)_+$.

Lemma 2.10 *For any $x \in D(G_\phi)$ and $Y \in \mathcal{I}$, we have $G_\phi(x) \in D(P_{\phi, Y})$ and*

$$G_\phi(x) = G_\phi^Y(x) + P_{\phi, Y} G_\phi(x).$$

Proof. Let $x \geq 0$. Set $x_n = \sum_{k=0}^n P_\phi^k(x)$. Note that

$$P_\phi^k = \sum_{m=0}^k [(\iota - F_Y) P_\phi]^{k-m} F_Y P_\phi^m + [(\iota - F_Y) P_\phi]^k (\iota - F_Y).$$

This can be verified by induction as follows:

$$\begin{aligned} (\iota - F_Y) P_\phi^k &= (\iota - F_Y) P_\phi \left(\sum_{m=0}^{k-1} [(\iota - F_Y) P_\phi]^{k-m-1} F_Y P_\phi^m + [(\iota - F_Y) P_\phi]^{k-1} (\iota - F_Y) \right) \\ &= \sum_{m=0}^{k-1} [(\iota - F_Y) P_\phi]^{k-m} F_Y P_\phi^m + [(\iota - F_Y) P_\phi]^k (\iota - F_Y). \end{aligned}$$

Consequently

$$x_n = \sum_{k=0}^n \sum_{m=0}^k [(\iota - F_Y) P_\phi]^{k-m} F_Y P_\phi^m(x) + \sum_{k=0}^n [(\iota - F_Y) P_\phi]^k (\iota - F_Y)(x)$$

$$\begin{aligned}
&= \sum_{k=0}^n [(\iota - F_Y)P_\phi]^k F_Y \sum_{m=0}^{n-k} P_\phi^m(x) + \sum_{k=0}^n [(\iota - F_Y)P_\phi]^k (\iota - F_Y)(x) \\
&= \sum_{k=0}^n [(\iota - F_Y)P_\phi]^k F_Y(x_{n-k}) + \sum_{k=0}^n [(\iota - F_Y)P_\phi]^k (\iota - F_Y)(x).
\end{aligned}$$

This identity already shows that $G_\phi(x) \in D(P_{\phi,Y})$. Indeed, since $G_\phi(x) \in D(P_\phi^k)$ for any $k \in \mathbb{N}$ by Lemma 2.9, it follows by Lemma 2.7(ii) that $[(\iota - F_Y)P_\phi]^k F_Y(x_{n-k}) \nearrow [(\iota - F_Y)P_\phi]^k F_Y G_\phi(x)$ when $n \rightarrow \infty$. Thus the identity above implies $\sum_{k=0}^m [(\iota - F_Y)P_\phi]^k F_Y G_\phi(x) \leq G_\phi(x)$ for any $m \in \mathbb{N}$, whence $G_\phi(x) \in D(P_{\phi,Y})$.

To complete the proof of Lemma it is enough to show that $\sum_{k=0}^n [(\iota - F_Y)P_\phi]^k F_Y(x_{n-k} - G_\phi(x)) \rightarrow 0$ as $n \rightarrow \infty$. Since $G_\phi(x) - x_{n-k} \leq G_\phi(x)$ and $G_\phi(x) \in D(P_{\phi,Y})$, there exists $n_0 \in \mathbb{N}$ such that $\sum_{k=n_0}^n [(\iota - F_Y)P_\phi]^k F_Y(x_{n-k} - G_\phi(x))$ is close to zero for all $n \geq n_0$. On the other hand, as we have already remarked, we have $[(\iota - F_Y)P_\phi]^k F_Y(x_{n-k} - G_\phi(x)) \rightarrow 0$ for fixed k as $n \rightarrow \infty$. This ends the proof. ■

Theorem 2.11 *Suppose $\phi \in \mathcal{C}$ is transient and consider a superharmonic element $x \in M(\hat{\mathcal{A}})$. Then for any $Y \in \mathcal{I}$, the following properties hold:*

- (i) $P_{\phi,Y}(x) \leq x$ and $F_Y P_{\phi,Y}(x) = F_Y(x)$;
- (ii) if y is a superharmonic element satisfying $F_Y(x) \leq F_Y(y)$, then $P_{\phi,Y}(x) \leq y$;
- (iii) $P_{\phi,Y}(x)$ is a potential.

Thus any superharmonic element x is approximated by potentials $P_{\phi,Y}(x)$ in the topology of $M(\hat{\mathcal{A}})$. Moreover, $P_{\phi,Y}(x)$ is the smallest element in the set of superharmonic elements majorizing x on Y .

Proof of Theorem 2.11. If x is a potential, say $x = G_\phi(y)$, then

$$x = G_\phi(y) = G_\phi^Y(y) + P_{\phi,Y}G_\phi(y) \geq P_{\phi,Y}G_\phi(y) = P_{\phi,Y}(x).$$

For general superharmonic x , consider the operator $P_{\lambda\phi} = \lambda P_\phi$ for $0 < \lambda < 1$. Clearly x is also superharmonic with respect to $P_{\lambda\phi}$. Since $P_{\lambda\phi}^n(x) \leq \lambda^n x \rightarrow 0$, by Lemma 2.9(ii) we deduce that x is a potential with respect to $P_{\lambda\phi}$. Hence $P_{\lambda\phi,Y}(x) \leq x$. This is equivalent to the fact that $\sum_{k=0}^n \lambda^k [(\iota - F_Y)P_\phi]^k F_Y(x) \leq x$ for all $n \in \mathbb{N}$ and $\lambda < 1$. Letting $\lambda \rightarrow 1 - 0$ and $n \rightarrow \infty$, we conclude that $P_{\phi,Y}(x) \leq x$. The equality $F_Y P_{\phi,Y}(x) = F_Y(x)$ follows by definition. Thus (i) is proved.

If $y \geq 0$ is such that $P_\phi(y) \leq y$ and $F_Y(x) \leq F_Y(y)$, we get

$$P_{\phi,Y}(x) = P_{\phi,Y}F_Y(x) \leq P_{\phi,Y}F_Y(y) = P_{\phi,Y}(y) \leq y,$$

where the last inequality follows from (i) applied to y . This proves (ii).

To prove (iii), we shall first check that $P_{\phi,Y}(x)$ is superharmonic. Since $P_{\phi,Y}(x) \leq x$, we get

$$F_Y P_\phi P_{\phi,Y}(x) \leq F_Y P_\phi(x) \leq F_Y(x) = F_Y P_{\phi,Y}(x).$$

The same proof as in Lemma 2.9(i) (applied to $(\iota - F_Y)P_\phi$ and element $F_Y(x)$ instead of P_ϕ and x) shows that $(\iota - F_Y)P_\phi P_{\phi,Y}(x) = P_{\phi,Y}(x) - F_Y(x)$, so $(\iota - F_Y)P_\phi P_{\phi,Y}(x) = (\iota - F_Y)P_{\phi,Y}(x)$. Thus

$$P_\phi P_{\phi,Y}(x) = F_Y P_\phi P_{\phi,Y}(x) + (\iota - F_Y)P_\phi P_{\phi,Y}(x) \leq F_Y P_{\phi,Y}(x) + (\iota - F_Y)P_{\phi,Y}(x) = P_{\phi,Y}(x),$$

so $P_{\phi, Y}(x)$ is superharmonic. Hence to prove that it is a potential, by Lemma 2.9(ii) it is enough to show that it is majorized by a potential. But we obviously have $P_{\phi, Y}(x) \leq G_{\phi} F_Y(x)$. Thus (iii) is also proved. ■

The previous proof is quite formal. It is applicable to any positive operator on a complete ordered vector space with a given increasing net \mathcal{I} of ordered subspaces. We leave it to the interested reader to formulate the precise setting for such a generalization. It is worth noting that the assumption on the subspaces in \mathcal{I} to be monotonically complete, simplifies the proof but is not at all necessary.

2.4 Poisson Boundary and 0-2 Law

Let $\phi \in \mathcal{C}$ be a generating state. Following Izumi [12] we denote by $H^{\infty}(M(\hat{A}), P_{\phi})$ the set of bounded (that is, the elements belonging to $M(\hat{A})$) harmonic elements with respect to P_{ϕ} and call it the Poisson boundary of $(M(\hat{A}), P_{\phi})$. The operator system $H^{\infty}(M(\hat{A}), P_{\phi}) \subset M(\hat{A})$ has a unique structure of a C^* -algebra. To distinguish the product on $H^{\infty}(M(\hat{A}), P_{\phi})$ from the one on $M(\hat{A})$, we shall write $x \cdot y$ for the product of two harmonic elements x and y . It is proved in [12] that

$$x \cdot y = \lim_{n \rightarrow \infty} P_{\phi}^n(xy),$$

where the limit is in the topology of $M(\hat{A})$. In fact, $M(\hat{A})$ is a von Neumann algebra, and $H^{\infty}(M(\hat{A}), P_{\phi})$ is a weakly operator closed subspace. Thus $H^{\infty}(M(\hat{A}), P_{\phi})$ is a von Neumann algebra.

Following [12] and [13] we shall presently give several other descriptions of $H^{\infty}(M(\hat{A}), P_{\phi})$.

We say that a sequence $\{x_n\}_{n=1}^{\infty} \subset M(\hat{A})$ is harmonic if $F_n P_{\phi}(x_{n+1}) = x_n$, where $F_n = F_{\text{supp } \phi^n}$ is the projection onto the C^* -algebra product of $B(H_s)$, $s \in \text{supp } \phi^n$. Note that the homomorphisms j_n introduced in Subsection 2.1 fulfill $\phi^{\infty} j_n = \phi^n$ and

$$E_n j_{n+1} = j_n P_{\phi}, \tag{2.3}$$

where $E_n: (M(\hat{A})^{\infty}, \phi^{\infty}) \rightarrow \otimes_{-n}^{-1}(M(\hat{A}), \phi)$ is the ϕ^{∞} -preserving conditional expectation. Note also that since $\phi^{\infty} j_n = \phi^n$, we have $j_n = j_n F_n$ and $j_n|_{F_n(M(\hat{A}))}$ is injective. Thus if $\{x_n\}_{n=1}^{\infty}$ is a bounded harmonic sequence, the sequence $\{j_n(x_n)\}_{n=1}^{\infty}$ is a martingale, so it converges in strong* operator topology to an element $x \in M(\hat{A})^{\infty}$ such that $E_n(x) = j_n(x_n)$ for all $n \in \mathbb{N}$. Conversely, if $x \in M(\hat{A})^{\infty}$ satisfies $E_n(x) \in \text{Im } j_n$, and $x_n \in F_n(M(\hat{A}))$ is the element uniquely determined by $E_n(x) = j_n(x_n)$, then $\{x_n\}_{n=1}^{\infty}$ is a bounded harmonic sequence.

Any bounded harmonic element $x \in M(\hat{A})$ defines a bounded harmonic sequence $x_n = F_n(x)$, as $F_n P_{\phi} = F_n P_{\phi} F_{n+1}$, which follows from the equality $\phi^n P_{\phi} = \phi^{n+1}$. Conversely, if $\{x_n\}_{n=1}^{\infty}$ is a bounded harmonic sequence with $F_n(x_m) = F_m(x_n)$ for all $n, m \in \mathbb{N}$, then the unique element $x \in M(\hat{A})$ such that $F_n(x) = x_n$ for all $n \in \mathbb{N}$ is harmonic, because

$$F_n P_{\phi}(x) = F_n P_{\phi} F_{n+1}(x) = F_n P_{\phi}(x_{n+1}) = x_n = F_n(x) \quad \forall n \in \mathbb{N},$$

and $\cup_n \text{supp } \phi^n = I$. It turns out that the assumption $F_n(x_m) = F_m(x_n)$ is automatically fulfilled by the 0-2 law. This law was first proved by Ornstein and Sucheston [22]. The proof was later clarified by Foguel [10]. The same line of arguments works also in the context of non-commutative probability.

Proposition 2.12 *Consider a unital positive map $P: A \rightarrow A$ on a C^* -algebra A . Suppose there exist $m, k \in \mathbb{N}$ and a positive map $S: A \rightarrow A$ such that $S(1)$ is invertible, $P^{m+k} \geq S$ and $P^m \geq S$. Then $\lim_{n \rightarrow \infty} \|P^{n+k} - P^n\| = 0$.*

The name '0-2 law' is due to the fact that if A is an abelian von Neumann algebra, then the existence of S means precisely that $\|P^{m+k} - P^m\| < 2$. Hence, if A is an abelian C^* -algebra, then either $\|P^{n+k} - P^n\| = 2$ for all $n \in \mathbb{N}$, or $\lim_{n \rightarrow \infty} \|P^{n+k} - P^n\| = 0$.

Proof of Proposition 2.12. Set $h = m + k$. We claim that there exist positive maps S_{ij} and T_j on A such that $S_{ij}(1)$ is invertible and

$$P^{ijh} = S_{ij}(\iota + P^k)^j + T_j^i \quad \text{for all } i, j \in \mathbb{N}. \quad (2.4)$$

To this end, take $S_{11} = \frac{1}{2}S$ and $T_1 = P^h - \frac{1}{2}S(\iota + P^k) = \frac{1}{2}(P^h - S) + \frac{1}{2}(P^m - S)P^k$. We define S_{1j} and T_j by induction on j using the equality

$$P^{(j+1)h} = P^{jh}P^h = S_{1j}P^h(\iota + P^k)^j + T_jP^h = S_{1j}S_{11}(\iota + P^k)^{j+1} + (S_{1j}T_1(\iota + P^k)^j + T_jP^h),$$

so $S_{1,j+1} = S_{1j}S_{11}$ and $T_{j+1} = S_{1j}T_1(\iota + P^k)^j + T_jP^h$. Then we define S_{ij} by induction on i using

$$P^{(i+1)jh} = S_{ij}P^{jh}(\iota + P^k)^j + T_j^iP^{jh} = (S_{ij}P^{jh} + T_j^iS_{1j})(\iota + P^k)^j + T_j^{i+1},$$

which proves our claim.

Applying (2.4) to the unit, we conclude that $\|T_j\| < 1$ and $\|S_{ij}\| \leq 2^{-j}$. Equation (2.4) also yields

$$\|P^{ijh}(\iota - P^k)\| \leq \|S_{ij}(\iota + P^k)^j(\iota - P^k)\| + \|T_j^i(\iota - P^k)\|.$$

The first term on the right hand side converges to zero as $j \rightarrow \infty$ uniformly in $i \in \mathbb{N}$, since

$$\begin{aligned} \|S_{ij}(\iota + P^k)^j(\iota - P^k)\| &= \|S_{ij}(\iota + \sum_{r=1}^j \binom{j}{r} P^{rk} - P^{(j+1)k})\| \\ &\leq 2^{-j}(2 + \sum_{r=1}^j |\binom{j}{r} - \binom{j}{r-1}|), \end{aligned}$$

and the latter expression converges to zero by known properties of binomial coefficients [22]. On the other hand, for fixed j , the second term converges to zero as $i \rightarrow \infty$, since $\|T_j^i(\iota - P^k)\| \leq 2\|T_j\|^i$. We see that under an appropriate choice of i and j the norm $\|P^{ijh}(\iota - P^k)\|$ can be made arbitrarily small. Since the sequence $\{\|P^n(\iota - P^k)\|\}_n$ is decreasing, its limit is therefore zero. \blacksquare

Let us return to the proof of $F_n(x_m) = F_m(x_n)$. If $\text{supp } \phi^n \cap \text{supp } \phi^m = \emptyset$, then $F_n(x_m) = F_m(x_n) = 0$. If $s \in \text{supp } \phi^n \cap \text{supp } \phi^m$, then $P_\phi^m \geq cP_{\phi_s}$ and $P_\phi^n \geq cP_{\phi_s}$ for $c > 0$ such that $c\phi_s \leq \phi^m$ and $c\phi_s \leq \phi^n$. Thus if $l = m - n \geq 0$, then $\|P_\phi^{k+l} - P_\phi^k\| \rightarrow 0$ as $k \rightarrow \infty$. Since $F_n(x_m) = F_nF_mP_\phi^k(x_{m+k})$ and $F_m(x_n) = F_mF_nP_\phi^{k+l}(x_{n+l+k}) = F_nF_mP_\phi^{k+l}(x_{m+k})$, we see that $\|F_n(x_m) - F_m(x_n)\| \leq \|P_\phi^{k+l} - P_\phi^k\| \|x_{m+k}\| \rightarrow 0$ as $k \rightarrow \infty$.

Summarizing the discussion above we obtain (see [12, 13])

Theorem 2.13 *The following linear spaces are canonically isomorphic:*

- (i) *the space $H^\infty(M(\hat{A}), P_\phi)$ of bounded harmonic elements;*
- (ii) *the space of bounded harmonic sequences;*
- (iii) *the space of elements $x \in M(\hat{A})^\infty$ such that $E_n(x) \in \text{Im } j_n$ for all $n \in \mathbb{N}$.*

Explicitly, the correspondence θ between (i) and (iii) associates to each $x \in H^\infty(M(\hat{A}), P_\phi)$ an element $\theta(x) \in M(\hat{A})^\infty$ uniquely determined by $E_n\theta(x) = j_n(x)$. \blacksquare

Since $E_n(\text{Im } j_{n+1}) \subset \text{Im } j_n$, the space described in part (iii) is, in fact, a von Neumann subalgebra of $M(\hat{A})^\infty$. As θ is a unital completely positive and isometric map of $H^\infty(M(\hat{A}), P_\phi)$ onto this subalgebra, it is a $*$ -homomorphism. Thus θ is an embedding of $H^\infty(M(\hat{A}), P_\phi)$ into $M(\hat{A})^\infty$. Since $\phi^n = \hat{\varepsilon}$ on harmonic elements and $\phi^\infty j_n = \phi^n$, we have $\phi^\infty \theta = \hat{\varepsilon}$.

Izumi observed that $H^\infty(M(\hat{A}), P_\phi)$ can be given a nicer description by embedding it into a larger algebra. Namely, let U be a unitary corepresentation such that the set of all its irreducible components (irrespective of multiplicities) coincides with $\text{supp } \phi$. Then there exists a normal state $\tilde{\phi}$ on $B(H_U)$ such that $\tilde{\phi}|_{B_0(H_U)}$ is α_U -invariant and $\tilde{\phi}\pi_U = \phi$. Set $(N, \tilde{\phi}^\infty) = \otimes_{-\infty}^{-1}(B(H_U), \tilde{\phi})$. The coactions $\alpha_{U \times n}$ of (A, Δ) on $\otimes_{-n}^{-1} B_0(H_U)$ define an ITP coaction α of (M, Δ) on the von Neumann algebra N . Here $M = \pi_r(A)''$ is the weak operator closure of A in the regular representation π_r , so we are considering quantum groups and their coactions in the von Neumann setting. The homomorphism $\otimes_{-\infty}^{-1} \pi_U$ defines an embedding of $(M(\hat{A})^\infty, \phi^\infty)$ into $(N, \tilde{\phi}^\infty)$, and if we identify $B(H_U \otimes \dots \otimes H_U)$ with $\otimes_{-n}^{-1} B(H_U) \subset N$, then $j_n = \pi_{U \times n}$. Thus $j_n(M(\hat{A}))$ coincides with the relative commutant

$$(B(H_U \otimes \dots \otimes H_U)^{\alpha_{U \times n}})' \cap B(H_U \otimes \dots \otimes H_U).$$

Hence $\theta(H^\infty(M(\hat{A}), P_\phi)) \subset N$ coincides with the relative commutant $(N^\alpha)' \cap N$.

3 Martin Boundary

Throughout this section ϕ is presumed to be a generating positive linear functional in \mathcal{C} such that $\|\phi\| \leq 1$. If ϕ is a state, we shall furthermore assume that it is transient. Note that if $\|\phi\| < 1$, then $M(\hat{A}) \subset D(G_\phi)$ and $\|G_\phi|_{M(\hat{A})}\| = (1 - \|\phi\|)^{-1}$.

Definition 3.1 *The Martin kernel for P_ϕ is the linear map $K_{\check{\phi}} : \hat{A} \rightarrow M(\hat{A})$ given by*

$$K_{\check{\phi}}(x) = G_{\check{\phi}}(x)G_{\check{\phi}}(I_0)^{-1}.$$

According to Lemma 2.4(v) and the proof of Corollary 2.5, the linear functional $\check{\phi}$ is generating and transient. Thus $G_{\check{\phi}}(x)$ is a well-defined element of $M(\hat{A})$ for any $x \in \hat{A}$. Since $\check{\phi}$ is generating, the element $G_{\check{\phi}}(I_0)I_s$ is a non-zero scalar multiple of I_s , so $G_{\check{\phi}}(I_0)$ is invertible in $M(\hat{A})$ for any $s \in I$. A priori, therefore, we have $K_{\check{\phi}}(x) \in M(\hat{A})$. However, observe that since $\check{\phi}$ is generating, any positive element $x \in \hat{A}$ is majorized by a scalar multiple of $\sum_{k=0}^n P_{\check{\phi}}^n(I_0)$ for some $n \in \mathbb{N}$. As $G_{\check{\phi}}P_{\check{\phi}}^n(I_0) \leq G_{\check{\phi}}(I_0)$, we see that for any $x \in \hat{A}_+$, there exists $c > 0$ (depending on x) such that $G_{\check{\phi}}(x) \leq cG_{\check{\phi}}(I_0)$. Hence $K_{\check{\phi}}(\hat{A}) \subset M(\hat{A})$. As $G_{\check{\phi}}(I_0)$ is central, obviously $K_{\check{\phi}}$ is completely positive.

Definition 3.2 *The Martin compactification of the discrete quantum group $(\hat{A}, \hat{\Delta})$ with respect to P_ϕ is the C^* -subalgebra \tilde{A}_ϕ of $M(\hat{A})$ generated by $K_{\check{\phi}}(\hat{A})$ and \hat{A} . The Martin boundary A_ϕ is the quotient C^* -algebra \tilde{A}_ϕ/\hat{A} .*

We may think of \hat{A} as the algebra of functions on our discrete quantum group tending to zero at infinity, and of $M(\hat{A})$ as the algebra of all bounded functions. With this picture in mind any unital C^* -subalgebra of $M(\hat{A})$ containing \hat{A} plays the role of the algebra of continuous functions on some compactification of the discrete quantum group. Note that \tilde{A}_ϕ is unital since $K_{\check{\phi}}(I_0) = 1$.

Suppose $(A, \Delta) = (C_r^*(\Gamma), \Delta)$, where Γ is a discrete group. A state ϕ on $\hat{A} = c_0(\Gamma)$ is represented by a measure μ . Then $p_\phi(h, g) = \mu(hg^{-1})$. Set $G(h, g) = \sum_n p_{\phi^n}(h, g) = \sum_n (\mu^{*n})(hg^{-1})$ and $K(h, g) = \frac{G(h, g)}{G(e, g)}$, where $e = 0$ is the unit in Γ . We have $K_{\check{\phi}}(I_h) = \sum_g K(h, g)I_g$. Thus $\tilde{A}_\phi = C(\bar{\Gamma})$, where $\bar{\Gamma}$ is the minimal compactification of Γ for which all the functions $g \mapsto K(h, g)$, $h \in \Gamma$, are continuous.

3.1 Integral Representation of Superharmonic Elements

In Subsection 1.4 we introduced a sesquilinear form $(\cdot, \cdot)_\eta$ associated to a KMS-state η . Here we use a similar definition for the Haar weight $\hat{\psi}$, so $(x, y)_{\hat{\psi}} = \hat{\psi}(x\sigma_{-\frac{i}{2}}^{\hat{\psi}}(y^*))$. We need not worry about domain problems, since we shall always assume that either x , or y belongs to $\hat{\mathcal{A}}$.

Theorem 3.3

(i) For any superharmonic element $x \in M(\hat{\mathcal{A}})$, there exists a bounded positive linear functional ω on \tilde{A}_ϕ such that $(y, x)_{\hat{\psi}} = \omega K_{\check{\phi}}(y)$ for $y \in \hat{\mathcal{A}}$.

(ii) Conversely, for any bounded positive linear functional ω on \tilde{A}_ϕ there exists a unique superharmonic element x_ω such that $(y, x_\omega)_{\hat{\psi}} = \omega K_{\check{\phi}}(y)$ for all $y \in \hat{\mathcal{A}}$. If x_ω is harmonic, then $\omega|_{\hat{\mathcal{A}}} = 0$. Moreover, if $\text{supp } \phi$ is finite, then x_ω is harmonic if and only if $\omega|_{\hat{\mathcal{A}}} = 0$.

To prove the theorem, we need the following result.

Lemma 3.4 For $x, y \in \hat{\mathcal{A}}$ we have $(P_\phi(x), y)_{\hat{\psi}} = (x, P_{\check{\phi}}(y))_{\hat{\psi}}$.

Proof. It is enough to consider $\phi \in \mathcal{C} \cap \mathcal{A}$. Strong left invariance of the Haar weight $\hat{\psi}$ reads as

$$\hat{\psi}(yP_\phi(x)) = \hat{\psi}(P_{\phi\hat{S}}(y)x). \quad (3.1)$$

Recall that $\hat{\psi} = \hat{\psi}(\rho^{-2}\cdot)$ on $\hat{\mathcal{A}}$. Since $\hat{\Delta}(\rho^{-2}) = \rho^{-2} \otimes \rho^{-2}$, we have $P_{\phi\hat{S}}(\rho^{-2}y) = \rho^{-2}P_{\phi\hat{S}(\rho^{-2}\cdot)}(y)$. We claim that $\phi\hat{S}(\rho^{-2}\cdot) = \check{\phi}$. To prove this we may suppose $\phi = \phi_s$. Since ϕ_s is $\sigma^{\hat{\psi}}$ -invariant, \hat{R} is a *-antiisomorphism of $B(H_s)$ onto $B(H_{\bar{s}})$, and $\hat{R}(\rho) = \rho^{-1}$, we have

$$\phi_s\hat{S} = \phi_s\hat{\tau}_{-\frac{i}{2}}\hat{R} = \phi_s\sigma_{\frac{i}{2}}^{\hat{\psi}}\hat{R} = \phi_s\hat{R} = \frac{1}{d_s}\text{Tr } \pi_{\bar{s}}(\rho\cdot),$$

so $\phi_s\hat{S}(\rho^{-2}\cdot) = \phi_{\bar{s}} = \check{\phi}_s$. Thus replacing y by $\rho^{-2}y$ in (3.1) we get $\hat{\psi}(yP_\phi(x)) = \hat{\psi}(P_{\check{\phi}}(y)x)$. It remains to replace y by $\sigma_{\frac{i}{2}}^{\hat{\psi}}(y^*)$ and note that $P_{\check{\phi}}\sigma_t^{\hat{\psi}} = \sigma_t^{\hat{\psi}}P_{\check{\phi}}$. The latter equality follows from $\hat{\Delta}\sigma_t^{\hat{\psi}} = (\sigma_t^{\hat{\psi}} \otimes \sigma_t^{\hat{\psi}})\hat{\Delta}$ and the fact that $\check{\phi}$ is $\sigma^{\hat{\psi}}$ -invariant. ■

Proof of Theorem 3.3. To prove (i) first limit to the case when x is a potential of an element $x_0 \in \hat{\mathcal{A}}_+$. Then we can take ω to be the restriction of the positive linear functional $(\cdot G_{\check{\phi}}(I_0), x_0)_{\hat{\psi}}$ to \tilde{A}_ϕ . Indeed, by the previous lemma we have

$$\omega K_{\check{\phi}}(y) = (G_{\check{\phi}}(y), x_0)_{\hat{\psi}} = (y, G_\phi(x_0))_{\hat{\psi}} = (y, x)_{\hat{\psi}}$$

for any $y \in \hat{\mathcal{A}}$. A general superharmonic element can be approximated from below by potentials due to the balayage theorem. Hence it can be approximated by potentials of elements in $\hat{\mathcal{A}}$. Thus there exists a net $\{x_i\}_i$ of positive elements in $\hat{\mathcal{A}}$ such that $G_\phi(x_i) \leq x$ and $G_\phi(x_i) \rightarrow x$ (in the topology

of $M(\hat{\mathcal{A}})$. Let ω_i be any positive linear functional on \tilde{A}_ϕ satisfying $\omega_i K_{\check{\phi}}(y) = (y, G_\phi(x_i))_{\hat{\psi}}$ for all $y \in \hat{\mathcal{A}}$. Note that

$$\omega_i(1) = \omega_i K_{\check{\phi}}(I_0) = (I_0, G_\phi(x_i))_{\hat{\psi}} = \hat{\varepsilon} G_\phi(x_i) \leq \hat{\varepsilon}(x).$$

Thus we can take for ω any weak* limit point of the net $\{\omega_i\}_i$.

To prove (ii) note that the pairing $(\cdot, \cdot)_{\hat{\psi}}$ on $\hat{\mathcal{A}} \times M(\hat{\mathcal{A}})$ defines an antilinear order isomorphism between $(\hat{\mathcal{A}})'$ and $M(\hat{\mathcal{A}})$. Hence, since $\omega K_{\check{\phi}}$ is a positive linear functional on $\hat{\mathcal{A}}$, there exists a unique positive element $x_\omega \in M(\hat{\mathcal{A}})$ such that $(\cdot, x_\omega)_{\hat{\psi}} = \omega K_{\check{\phi}}$. For any $y \in \hat{\mathcal{A}}_+$ and $X \in \mathcal{I}$, we have

$$G_{\check{\phi}}(y) = G_{\check{\phi}} P_{\check{\phi}}(y) + y \geq G_{\check{\phi}} F_X P_{\check{\phi}}(y) + y,$$

so $K_{\check{\phi}}(y) \geq K_{\check{\phi}} F_X P_{\check{\phi}}(y) + y G_{\check{\phi}}(I_0)^{-1}$, whence

$$\begin{aligned} (y, P_\phi F_X(x_\omega))_{\hat{\psi}} &= (F_X P_{\check{\phi}}(y), x_\omega)_{\hat{\psi}} = \omega K_{\check{\phi}} F_X P_{\check{\phi}}(y) \leq \omega K_{\check{\phi}}(y) - \omega(y G_{\check{\phi}}(I_0)^{-1}) \\ &= (y, x_\omega)_{\hat{\psi}} - \omega(y G_{\check{\phi}}(I_0)^{-1}) \leq (y, x_\omega)_{\hat{\psi}}. \end{aligned}$$

Hence $P_\phi F_X(x_\omega) \leq x_\omega$. It follows that $x_\omega \in D(P_\phi)$ and $P_\phi(x_\omega) \leq x_\omega$. If x_ω is harmonic, then $(y, P_\phi F_X(x_\omega))_{\hat{\psi}} \nearrow (y, P_\phi(x_\omega))_{\hat{\psi}} = (y, x_\omega)_{\hat{\psi}}$, so we see that $\omega(y G_{\check{\phi}}(I_0)^{-1}) = 0$, that is $\omega|_{\hat{\mathcal{A}}} = 0$. Conversely, if $\omega|_{\hat{\mathcal{A}}} = 0$ and $\text{supp } \phi$ is finite, then $P_{\check{\phi}}(y) \in \hat{\mathcal{A}}$ and $K_{\check{\phi}}(y) = K_{\check{\phi}} P_{\check{\phi}}(y) + y G_{\check{\phi}}(I_0)^{-1}$, so $(y, P_\phi(x_\omega))_{\hat{\psi}} = (y, x_\omega)_{\hat{\psi}} - \omega(y G_{\check{\phi}}(I_0)^{-1}) = (y, x_\omega)_{\hat{\psi}}$. Thus $P_\phi(x_\omega) = x_\omega$. ■

3.2 Canonical Coactions

In Subsection 1.4 we introduced several coactions. The aim of this subsection is to show that they induce coactions on the Martin compactification and the Martin boundary.

Theorem 3.5

- (i) *The right coaction $\hat{\Delta}$ of $(\hat{A}, \hat{\Delta})$ on \hat{A} has the property $\hat{\Delta}(\tilde{A}_\phi) \subset M(\tilde{A}_\phi \otimes \hat{A})$, so it induces right coactions of $(\hat{A}, \hat{\Delta})$ on \tilde{A}_ϕ and A_ϕ . We denote these coactions by the same letter $\hat{\Delta}$.*
- (ii) *The left coaction Φ of (A, Δ) on \hat{A} has the property $\Phi(\tilde{A}_\phi) \subset A \otimes \tilde{A}_\phi$. It induces left coactions of (A, Δ) on \tilde{A}_ϕ and A_ϕ , which we again denote by Φ .*

The proof is based on the following result.

Lemma 3.6 [12, Lemma 2.2] *For any $x \in M(\hat{A})$ we have*

- (i) $\hat{\Delta} P_\phi(x) = (P_\phi \otimes \iota) \hat{\Delta}(x)$;
- (ii) $\Phi P_\phi(x) = (\iota \otimes P_\phi) \Phi(x)$. ■

This lemma immediately implies part (ii) of Theorem 3.5. Indeed, we have $\Phi G_{\check{\phi}}(x) = (\iota \otimes G_{\check{\phi}}) \Phi$ on \hat{A} (this expression makes sense, since $\Phi(\hat{\mathcal{A}}) \subset \mathcal{A} \odot \hat{\mathcal{A}}$). As $G_{\check{\phi}}(I_0) \in Z(M(\hat{A}))$, we have $\Phi G_{\check{\phi}}(I_0) = 1 \otimes G_{\check{\phi}}(I_0)$, so

$$\Phi K_{\check{\phi}}(x) = (\iota \otimes K_{\check{\phi}}) \Phi(x). \tag{3.2}$$

This implies $\Phi(\tilde{A}_\phi) \subset A \otimes \tilde{A}_\phi$. Furthermore, since $(\varepsilon \otimes \iota) \Phi = \iota$ on $\hat{\mathcal{A}}$ (see Proposition 1.7), the assumptions of Corollary 1.4(i) are fulfilled for the *-algebra generated by $K_{\check{\phi}}(\hat{\mathcal{A}})$ and $\hat{\mathcal{A}}$, so $\Phi: \tilde{A}_\phi \rightarrow A \otimes \tilde{A}_\phi$ is a coaction.

Similarly, part (i) of Lemma 3.6 implies $\hat{\Delta}G_{\check{\phi}}(x) = (G_{\check{\phi}} \otimes \iota)\hat{\Delta}(x)$ for $x \in \hat{\mathcal{A}}$. Here we consider $G_{\check{\phi}} \otimes \iota$ as an operator going from the algebraic direct product $\prod_{s \in I} \hat{\mathcal{A}} \otimes B(H_s)$ into $\prod_{s \in I} M(\hat{\mathcal{A}}) \otimes B(H_s) = M(\hat{\mathcal{A}} \odot \hat{\mathcal{A}})$. Thus

$$\hat{\Delta}K_{\check{\phi}}(x) = (K_{\check{\phi}} \otimes \iota)\hat{\Delta}(x)(G_{\check{\phi}}(I_0) \otimes 1)(\hat{\Delta}G_{\check{\phi}}(I_0)^{-1}) = (K_{\check{\phi}} \otimes \iota)\hat{\Delta}(x)((K_{\check{\phi}} \otimes \iota)\hat{\Delta}(I_0))^{-1}. \quad (3.3)$$

Clearly $((K_{\check{\phi}} \otimes \iota)\hat{\Delta}(I_0))^{-1} \in M(\hat{\mathcal{A}} \odot \hat{\mathcal{A}})$. To show that $\hat{\Delta}K_{\check{\phi}}(x)$ belongs to the multiplier algebra of $\tilde{A}_\phi \otimes \hat{\mathcal{A}}$, first notice that $M(\tilde{A}_\phi \otimes \hat{\mathcal{A}})$ is the C^* -algebraic direct product of $\tilde{A}_\phi \otimes B(H_s)$, $s \in I$. Since $\hat{\Delta}K_{\check{\phi}}(x) \in M(\hat{\mathcal{A}} \otimes \hat{\mathcal{A}})$, we just have to show that $(K_{\check{\phi}} \otimes \iota)((1 \otimes I_s)\hat{\Delta}(I_0))$ is invertible in $M(\hat{\mathcal{A}}) \otimes B(H_s)$, which entails $(1 \otimes I_s)((K_{\check{\phi}} \otimes \iota)\hat{\Delta}(I_0))^{-1} \in \tilde{A}_\phi \otimes B(H_s)$. To this end we need the following two lemmas.

Lemma 3.7 *Let $r, t \in I$ and suppose $p_\phi(r, t) \neq 0$. Then $I_r P_\phi: B(H_t) \rightarrow B(H_r)$ is a faithful completely positive map.*

Proof. If $p_\phi(r, t) \neq 0$, then $\phi_r P_\phi = \phi \phi_r$ majorizes a scalar multiple of the state ϕ_t , which obviously is faithful on $B(H_t)$. ■

As a corollary, any non-zero superharmonic element is invertible in $M(\hat{\mathcal{A}})$.

Lemma 3.8 *Consider $t \in I$ and a finite dimensional C^* -algebra B . Suppose $x \in B(H_t) \otimes B$ is a positive element such that $(\omega \otimes \iota)(x)$ is invertible for some state ω on $B(H_t)$ (equivalently, the support of x cannot be majorized by a non-trivial projection $I_t \otimes p$). Then $(K_{\check{\phi}} \otimes \iota)(x)$ is invertible in $\tilde{A}_\phi \otimes B$.*

Proof. Pick $n \in \mathbb{N}$ such that $p_{\check{\phi}^n}(0, t) \neq 0$. By the previous lemma $I_0 P_{\check{\phi}^n}$ is a faithful completely positive map on $B(H_t)$, so $P_{\check{\phi}^n} \geq c\omega(\cdot)I_0$ for some $c > 0$, whence $P_{\check{\phi}^n} \otimes \iota \geq c\omega(\cdot)I_0 \otimes \iota$ on $B(H_t) \otimes B$. Then

$$(G_{\check{\phi}} \otimes \iota)(x) \geq (G_{\check{\phi}} P_{\check{\phi}^n} \otimes \iota)(x) \geq c(G_{\check{\phi}} \otimes \iota)(\omega(\cdot)I_0 \otimes \iota)(x) = cG_{\check{\phi}}(I_0) \otimes ((\omega \otimes \iota)(x)),$$

so $(K_{\check{\phi}} \otimes \iota)(x) \geq c1 \otimes ((\omega \otimes \iota)(x))$ is invertible. ■

In particular, the element $K_{\check{\phi}}(x)$ is invertible in \tilde{A}_ϕ for any non-zero positive $x \in \hat{\mathcal{A}}$.

Recall that by Lemma 1.2, $(1 \otimes I_s)\hat{\Delta}(I_0) \in B(H_{\bar{s}}) \otimes B(H_s)$ and $(\phi_{\bar{s}} \otimes \iota)((1 \otimes I_s)\hat{\Delta}(I_0)) = \frac{1}{\alpha_s^2} I_s$. Therefore $(K_{\check{\phi}} \otimes \iota)((1 \otimes I_s)\hat{\Delta}(I_0))$ is invertible in $\tilde{A}_\phi \otimes B(H_s)$, and thus $\hat{\Delta}(\tilde{A}_\phi) \subset M(\tilde{A}_\phi \otimes \hat{\mathcal{A}})$. Consequently, $\hat{\Delta}: \tilde{A}_\phi \rightarrow M(\tilde{A}_\phi \otimes \hat{\mathcal{A}})$ is a coaction by Corollary 1.4(ii).

3.3 Poisson Boundary and States Representing the Unit

Throughout this subsection we assume in addition that ϕ is a state, so the unit is a harmonic element. Let ν be a state on \tilde{A}_ϕ representing the unit, that is $\nu K_{\check{\phi}} = (\cdot, 1)_{\hat{\psi}} = \hat{\psi}$ on $\hat{\mathcal{A}}$. Since the unit is harmonic, the restriction of ν to $\hat{\mathcal{A}}$ is zero by Theorem 3.3(ii). Thus ν can be considered as a state on the boundary A_ϕ . Suppose in addition that ν is KMS. Then, as in Subsection 1.4, we can define a dual map $K_{\check{\phi}}^*: \pi_\nu(A_\phi)'' \rightarrow M(\hat{\mathcal{A}})$ determined by $(K_{\check{\phi}}^*(x), a)_\nu = (x, K_{\check{\phi}}^*(a))_{\hat{\psi}}$ for all $x \in \hat{\mathcal{A}}$ and $a \in \pi_\nu(A_\phi)''$. As $(x, K_{\check{\phi}}^*(1))_{\hat{\psi}} = \nu K_{\check{\phi}}(x) = \hat{\psi}(x)$, we deduce that $K_{\check{\phi}}^*(1) = 1$. Thus $K_{\check{\phi}}^*$ is, in fact, a normal unital completely positive map from $\pi_\nu(A_\phi)''$ into $M(\hat{\mathcal{A}})$. Equivalently, it can be described as follows: if $a \in \pi_\nu(A_\phi)''$ is positive, then $K_{\check{\phi}}^*(a)$ is a superharmonic element corresponding to the positive linear functional $(\cdot, a)_\nu$.

Lemma 3.9 *We have $\text{Im } K_{\check{\phi}}^* \subset H^\infty(M(\hat{A}), P_\phi)$.*

Proof. Since $(\cdot, a)_\nu \leq \nu(a)\nu$ for $a \geq 0$, it suffices to prove the following: if ω_1 and ω_2 are positive linear functionals on \tilde{A}_ϕ such that $\omega_1 \leq \omega_2$ and x_{ω_2} is harmonic, then x_{ω_1} is also harmonic. Since x_{ω_2} is harmonic and both x_{ω_1} and $x_{\omega_2} - x_{\omega_1} = x_{\omega_2 - \omega_1}$ are superharmonic, the element x_{ω_1} must be harmonic. ■

Notice also that as $K_{\check{\phi}}(I_0) = 1$, we have $\hat{\varepsilon}K_{\check{\phi}}^* = \nu$.

Define a dynamics γ on \tilde{A}_ϕ as the restriction of $\sigma^{\hat{\psi}}$ to \tilde{A}_ϕ . As already remarked in the proof of Lemma 3.4, we have the equality $\sigma_t^{\hat{\psi}}P_{\check{\phi}} = P_{\check{\phi}}\sigma_t^{\hat{\psi}}$, so $\sigma_t^{\hat{\psi}}K_{\check{\phi}} = K_{\check{\phi}}\sigma_t^{\hat{\psi}}$. Hence γ is a strongly continuous one-parameter automorphism group on \tilde{A}_ϕ . Since $\gamma_t(\hat{A}) = \hat{A}$, we also have a dynamics on A_ϕ , which we denote by the same letter γ . For the coactions Φ and $\hat{\Delta}$ on \tilde{A}_ϕ and A_ϕ we have

- (i) $\Phi\gamma_t = (\tau_t \otimes \gamma_t)\Phi$;
- (ii) $\hat{\Delta}\gamma_t = (\gamma_t \otimes \hat{\tau}_{-t})\hat{\Delta}$.

Theorem 3.10 *Let ν be a weak* limit point of the sequence $\{\phi^n|_{\tilde{A}_\phi}\}_{n=1}^\infty$. Then ν is a γ -KMS state representing the unit.*

Proof. Since $\phi^n|_{\tilde{A}_\phi}$ is a γ -KMS state, clearly ν is γ -KMS.

Consider the positive linear functional $\nu K_{\check{\phi}}$ on \hat{A} . We assert that it is $\sigma^{\hat{\psi}}$ -KMS. Since ν is a weak* limit point of ϕ^n , and ϕ^n is a convex combination of the states ϕ_s , it is enough to check that $\phi_s K_{\check{\phi}}$ is $\sigma^{\hat{\psi}}$ -KMS for any $s \in I$. As $G_{\check{\phi}}(I_0)$ is central, the element $I_s G_{\check{\phi}}(I_0)^{-1}$ is a scalar multiple of I_s , so we need only to consider the linear functional $\phi_s G_{\check{\phi}}$. But it is the sum of $\sigma^{\hat{\psi}}$ -KMS functionals $\check{\phi}^n \phi_s$, so it must be $\sigma^{\hat{\psi}}$ -KMS itself. Thus our assertion is proved.

Since ϕ_s is a unique (up to a scalar) $\sigma^{\hat{\psi}}$ -KMS functional on $B(H_s)$, to prove that ν represents the unit, that is $\nu K_{\check{\phi}} = \hat{\psi}$ on \hat{A} , it is enough to verify this equality on $Z(\hat{A})$. In this case we can apply the classical theory [26, Theorem 7.2.7]. Indeed, Lemma 3.4 (or Lemma 2.4(v)) shows that the classical Markov operators $P_\phi|_{Z(\hat{A})}$ and $P_{\check{\phi}}|_{Z(\hat{A})}$ are in duality with respect to $\hat{\psi}|_{Z(\hat{A})}$. Therefore $K_{\check{\phi}}|_{Z(\hat{A})}$ coincides with the operator \hat{K} introduced in [26, Proposition 7.2.3]. Then with the notation of our Subsection 2.1, Theorem 7.2.7 in [26] states that for any $x \in Z(\hat{A})$ the sequence $\{j_n K_{\check{\phi}}(x)\}_{n=1}^\infty \subset L^\infty(\Omega, \mathbb{P}_0) \subset M(\hat{A})^\infty$ converges a.e. to an element $j_\infty K_{\check{\phi}}(x)$ (in particular, it converges in strong* operator topology on $M(\hat{A})^\infty$) and $\phi^\infty j_\infty K_{\check{\phi}} = \hat{\psi}$ on $Z(\hat{A})$. Since $\phi^\infty j_n = \phi^n$, this is the same as saying that $\nu K_{\check{\phi}} = \hat{\psi}$ on $Z(\hat{A})$. ■

It is shown in [12] that if we consider our quantum groups in the von Neumann setting, then $\hat{\Delta}$ and Φ define right and left coactions of $(\hat{M}, \hat{\Delta})$ and (M, Δ) on $H^\infty(M(\hat{A}), P_\phi)$, respectively. Here $\hat{M} = M(\hat{A})$ and M denote the weak operator closures of \hat{A} and A , respectively, in their regular representations.

Proposition 3.11 *Let ν be a γ -KMS state on A_ϕ representing the unit. Regard $K_{\check{\phi}}^*$ as a map from A_ϕ to $H^\infty(M(\hat{A}), P_\phi)$. Then*

- (i) $K_{\check{\phi}}^*$ intertwines the left coactions of (A, Δ) if and only if ν is invariant;
- (ii) $K_{\check{\phi}}^*$ intertwines the right coactions of $(\hat{A}, \hat{\Delta})$ if and only if ν is quasi-invariant with Radon-Nikodym cocycle $(K_{\check{\phi}} \otimes \iota)\hat{\Delta}(I_0)$; moreover, if these two equivalent conditions are satisfied, then $K_{\check{\phi}}^* = (\nu \otimes \iota)\hat{\Delta}$.

Proof. To prove (i) consider for any $\omega \in A^*$ the operator S_ω on $M(\hat{A})$ given by $S_\omega = (\omega \otimes \iota)\Phi$. Because $\hat{\psi}|_{B(H_s)} = d_s^2\phi_s$ is Φ -invariant for any $s \in I$, by Proposition 1.6 we see that $S_\omega^* = S_{\bar{\omega}R}$ with respect to the inner product $(\cdot, \cdot)_{\hat{\psi}}$. Thus $(S_\omega(x), y)_{\hat{\psi}} = (x, S_{\bar{\omega}R}(y))_{\hat{\psi}}$ if x or y is in \hat{A} . Analogously, consider the operator $T_\omega = (\omega \otimes \iota)\Phi$ on A_ϕ . Then K_ϕ^* intertwines the left coactions, that is $\Phi K_\phi^* = (\iota \otimes K_\phi^*)\Phi$ on A_ϕ , if and only if $S_\omega K_\phi^* = K_\phi^* T_\omega$ for all $\omega \in A^*$. On the other hand, the state ν is invariant if and only if $\nu T_\omega = \omega(1)\nu$ for all $\omega \in A^*$. The identity (3.2) can be rewritten as $T_\omega K_\phi^* = K_\phi^* S_\omega$. For $a \in A_\phi$ and $x \in \hat{A}$, we have

$$(S_\omega K_\phi^*(a), x)_{\hat{\psi}} = (K_\phi^*(a), S_{\bar{\omega}R}(x))_{\hat{\psi}} = (a, K_\phi^* S_{\bar{\omega}R}(x))_\nu = (a, T_{\bar{\omega}R} K_\phi^*(x))_\nu$$

and $(K_\phi^* T_\omega(a), x)_{\hat{\psi}} = (T_\omega(a), K_\phi^*(x))_\nu$. So if $S_\omega K_\phi^* = K_\phi^* T_\omega$, then taking $x = I_0$, using $K_\phi^*(I_0) = 1$ and $T_{\bar{\omega}R}(1) = \overline{\omega(1)}1$, we obtain $\nu T_\omega(a) = \omega(1)\nu(a)$. Conversely, if ν is invariant, then $T_\omega^* = T_{\bar{\omega}R}$ by Proposition 1.6, and we get $S_\omega K_\phi^* = K_\phi^* T_\omega$.

To settle (ii) first notice that $y = (K_\phi^* \otimes \iota)\hat{\Delta}(I_0)$ is a cocycle, i.e. $(\iota \otimes \hat{\Delta})(y) = (\hat{\Delta} \otimes \iota)(y)(y \otimes 1)$. This follows from coassociativity of $\hat{\Delta}$ and (3.3), which can be written as $(K_\phi^* \otimes \iota)\hat{\Delta}(x) = \hat{\Delta}K_\phi^*(x)y$. To see this, we compute

$$\begin{aligned} (\iota \otimes \hat{\Delta})(y) &= (K_\phi^* \otimes \iota \otimes \iota)(\iota \otimes \hat{\Delta})\hat{\Delta}(I_0) = ((K_\phi^* \otimes \iota)\hat{\Delta} \otimes \iota)\hat{\Delta}(I_0) \\ &= (\hat{\Delta}K_\phi^* \otimes \iota)\hat{\Delta}(I_0)(y \otimes 1) = (\hat{\Delta} \otimes \iota)(y)(y \otimes 1). \end{aligned}$$

Now for $\omega \in \mathcal{A}$, consider the operators $S_\omega = (\iota \otimes \omega)\hat{\Delta}$ and $T_\omega = (\iota \otimes \omega)\hat{\Delta}$ on $M(\hat{A})$ and A_ϕ , respectively. As in Proposition 1.6 strong right invariance of the Haar weight $\hat{\psi}$ can be expressed by the equality $S_\omega^* = S_{\bar{\omega}\hat{R}}$.

Suppose K_ϕ^* intertwines the coactions of $(\hat{A}, \hat{\Delta})$. Then for any $a \in A_\phi$ and $\omega \in \mathcal{A}$, we get

$$\begin{aligned} \nu T_\omega(a) &= \hat{\varepsilon}K_\phi^* T_\omega(a) = (K_\phi^* T_\omega(a), I_0)_{\hat{\psi}} = (S_\omega K_\phi^*(a), I_0)_{\hat{\psi}} \\ &= (K_\phi^*(a), S_{\bar{\omega}\hat{R}}(I_0))_{\hat{\psi}} = (a, K_\phi^* S_{\bar{\omega}\hat{R}}(I_0))_\nu. \end{aligned}$$

Since $\gamma_t K_\phi^* = K_\phi^* \sigma_t^{\hat{\psi}}$ and $\hat{\Delta} \sigma_t^{\hat{\psi}} = (\sigma_t^{\hat{\psi}} \otimes \hat{\tau}_{-t})\hat{\Delta}$, this equality is equivalent to property (i) in Definition 1.9. Thus ν is quasi-invariant with Radon-Nikodym cocycle $y = (K_\phi^* \otimes \iota)\hat{\Delta}(I_0)$. Moreover, as $(S_\omega K_\phi^*(a), I_0)_{\hat{\psi}} = \hat{\varepsilon}S_\omega K_\phi^*(a) = \omega K_\phi^*(a)$ and $\nu T_\omega(a) = \omega(\nu \otimes \iota)\hat{\Delta}(a)$, we have $K_\phi^*(a) = (\nu \otimes \iota)\hat{\Delta}(a)$. Note that the last argument does not use the definition of K_ϕ^* . The very fact that K_ϕ^* intertwines the coactions implies $K_\phi^* = (\hat{\varepsilon}K_\phi^* \otimes \iota)\hat{\Delta}$.

Conversely, suppose ν is quasi-invariant with Radon-Nikodym cocycle $y = (K_\phi^* \otimes \iota)\hat{\Delta}(I_0)$. For any $a, b \in A_\phi$ and $\omega \in \mathcal{A}$, Proposition 1.10(ii) yields

$$(\nu \otimes \omega)(\hat{\Delta}(a)(b \otimes 1)) = (\nu \otimes \omega \hat{S})((a \otimes 1)\hat{\Delta}(b)y).$$

Taking $b = K_\phi^*(x)$ for $x \in \hat{A}$, and using $(K_\phi^* \otimes \iota)\hat{\Delta}(x) = \hat{\Delta}K_\phi^*(x)y$, we obtain

$$(\nu \otimes \omega)(\hat{\Delta}(a)(K_\phi^*(x) \otimes 1)) = (\nu \otimes \omega \hat{S})((a \otimes 1)(K_\phi^* \otimes \iota)\hat{\Delta}(x)).$$

Replacing x by $\sigma_{-\frac{i}{2}}^{\hat{\psi}}(x^*)$, this identity can be rewritten as $(T_\omega(a), K_\phi^*(x))_\nu = (a, K_\phi^* S_{\bar{\omega}\hat{R}}(x))_\nu$. It follows that $K_\phi^* T_\omega = S_\omega K_\phi^*$ for all $\omega \in \mathcal{A}$. In other words, K_ϕ^* intertwines the right coactions. ■

If ν is an invariant state for the coaction Φ of (A, Δ) on A_ϕ , it induces a coaction of (A, Δ) on $\pi_\nu(A_\phi)$. The latter can be extended to a coaction of (M, Δ) on $\pi_\nu(A_\phi)''$. The reason for this is that there exists a unitary on $H_\varphi \otimes H_\nu$ implementing this coaction (see the proof of Proposition 1.5).

If ν is a quasi-invariant γ -KMS state with Radon-Nikodym cocycle $y = (K_{\check{\phi}} \otimes \iota)\hat{\Delta}(I_0)$ for the right coaction $\hat{\Delta}$ of $(\hat{A}, \hat{\Delta})$ on A_ϕ , then this coaction is implemented on $H_\nu \otimes H_\varphi$ by the unitary U given by

$$U(J_\nu \otimes J_\varphi)y^{\frac{1}{2}}(J_\nu \otimes J_\varphi)(a\xi_\nu \otimes \xi) = \hat{\Delta}(a)(\xi_\nu \otimes \xi) \quad \text{for } a \in A_\phi \text{ and } \xi \in \mathcal{A}\xi_\varphi.$$

To verify that U is unitary, recall that $\hat{R}(x) = J_\varphi x^* J_\varphi$ for $x \in M(\hat{A})$. So if we denote by ω_ξ the linear functional (\cdot, ξ, ξ) on $M(\hat{A})$, we get

$$\begin{aligned} \|(J_\nu \otimes J_\varphi)y^{\frac{1}{2}}(J_\nu \otimes J_\varphi)(a\xi_\nu \otimes \xi)\|^2 &= (\nu \otimes \omega_\xi \hat{R})((a^*a \otimes 1)(\gamma_{-\frac{i}{2}} \otimes \iota)(y)) \\ &= (\nu \otimes \omega_\xi \hat{S})((a^*a \otimes 1)y) \\ &= (\nu \otimes \omega_\xi)\hat{\Delta}(a^*a) = \|\hat{\Delta}(a)(\xi_\nu \otimes \xi)\|^2. \end{aligned}$$

It follows that $\hat{\Delta}$ induces a coaction of $(\hat{A}, \hat{\Delta})$ on $\pi_\nu(A_\phi)$, which can be extended to a coaction of $(\hat{M}, \hat{\Delta})$ on $\pi_\nu(A_\phi)''$. (In fact, Proposition 1.11 shows that the particular form of the cocycle y does not play any role in this argument.)

A detailed study of connections between Martin and Poisson boundaries will be given in a subsequent paper. For the computation in the next section the following result is sufficient.

Proposition 3.12 *Suppose that the Martin kernel $K_{\check{\phi}}$ considered as a map from \hat{A} into A_ϕ has dense range. Then*

- (i) *the map $\omega \mapsto x_\omega$, which associates a superharmonic element to a positive linear functional on A_ϕ , is injective;*
- (ii) *if ν is the unique state on A_ϕ representing the unit, then ν is γ -KMS and the map $K_{\check{\phi}}^*: \pi_\nu(A_\phi)'' \rightarrow H^\infty(M(\hat{A}), P_\phi)$ is an isomorphism of von Neumann algebras intertwining the left coactions of (M, Δ) .*

Proof. Part (i) is obvious as x_ω by definition determines the values of ω on $K_{\check{\phi}}(\hat{A})$. It remains to prove (ii). Since there always exists a γ -KMS state representing the unit by Theorem 3.10, the state ν must be γ -KMS and, being regarded as a state on \tilde{A}_ϕ , be the weak* limit of the sequence $\{\phi^n|_{\tilde{A}_\phi}\}_{n=1}^\infty$. As the states ϕ^n are invariant with respect to the left coaction of $(\hat{A}, \hat{\Delta})$ on \tilde{A}_ϕ , the state ν is invariant as well. (Another way to see this is to note that the state $(\varphi \otimes \nu)\Phi$ also represents the unit and is invariant.) It follows that we have a well-defined coaction of (M, Δ) on $\pi_\nu(A_\phi)''$, and $K_{\check{\phi}}^*$ intertwines this coaction with the coaction on $H^\infty(M(\hat{A}), P_\phi)$. We need now only to prove that $K_{\check{\phi}}^*$ is an isomorphism. Since $K_{\check{\phi}}(\hat{A})$ is dense in $\pi_\nu(A_\phi)''$, obviously $K_{\check{\phi}}^*$ is injective. Let $x \in H^\infty(M(\hat{A}), P_\phi)$ be positive and non-zero. Suppose ω is the unique positive linear functional on A_ϕ representing x . Then the linear functional $\nu - \|x\|^{-1}\omega$ represents the positive harmonic element $1 - \|x\|^{-1}x$. Since such a functional is unique and positive, clearly $\omega \leq \|x\|\nu$. Hence $\omega = (\cdot, a)_\nu$ for some positive $a \in \pi_\nu(A_\phi)''$. Then $K_{\check{\phi}}^*(a) = x$, so $K_{\check{\phi}}^*$ is a bijection. Moreover, it maps positive elements of $\pi_\nu(A_\phi)''$ onto positive elements of $H^\infty(M(\hat{A}), P_\phi)$, so the inverse map is positive. Since $K_{\check{\phi}}^*$ is also unital and 2-positive, it is a *-isomorphism (see e.g. [8, Corollaries 2.2 and 3.2]). ■

4 Martin Boundary of the Dual of Quantum $SU(2)$

In this section (A, Δ) will denote the compact quantum group $SU_q(2)$ of Woronowicz [34]. We assume that the deformation parameter q lies in $(0, 1)$, but our results are also true for $q \in (-1, 0)$. So A is the universal unital C^* -algebra with generators α and γ satisfying the relations

$$\begin{aligned}\alpha^* \alpha + \gamma^* \gamma &= 1, & \alpha \alpha^* + q^2 \gamma^* \gamma &= 1, & \gamma^* \gamma &= \gamma \gamma^*, \\ \alpha \gamma &= q \gamma \alpha, & \alpha \gamma^* &= q \gamma^* \alpha.\end{aligned}$$

The comultiplication Δ is determined by the formulas

$$\Delta(\alpha) = \alpha \otimes \alpha - q \gamma^* \otimes \gamma, \quad \Delta(\gamma) = \gamma \otimes \alpha + \alpha^* \otimes \gamma.$$

Recall that the Haar state φ of (A, Δ) is faithful [21] (see also [3]). The characters f_{it} , $t \in \mathbb{R}$, on \mathcal{A} can be extended to bounded characters on A which we denote by the same symbols.

Consider the C^* -subalgebra B of A given by

$$B = \{a \in A \mid f_{it} * a = a \ \forall t \in \mathbb{R}\}.$$

Then Δ and $\hat{\Phi}$ (see Proposition 1.7) induce a left coaction of (A, Δ) on B and a right coaction of $(\hat{A}, \hat{\Delta})$ on B , respectively. The C^* -algebra B with the left coaction of (A, Δ) is the quantum homogeneous sphere of Podleś. Note that Podleś considers right coactions, so he deals with $\mathbb{T} \backslash SU_q(2)$, while we consider $SU_q(2)/\mathbb{T}$.

We are now in the position to state our main results in this section.

Theorem 4.1 *Let $\phi = \sum_s \lambda_s \phi_s \in \mathcal{C}$ be a generating state. Suppose $\sum_s \lambda_s \dim H_s < \infty$. Then there exists an isomorphism $A_\phi \xrightarrow{\simeq} B$ which intertwines the left coactions of (A, Δ) and the right coactions of $(\hat{A}, \hat{\Delta})$.*

Thus the Martin boundary of the dual of quantum $SU(2)$ is identified with the quantum homogeneous 2-sphere of Podleś. The result is also valid for $\|\phi\| < 1$. Under this assumption the corresponding result for ordinary $SU(2)$ was first established by Biane [7].

Theorem 4.2 *Retaining the assumptions on ϕ from Theorem 4.1, there exists a unique state ν on A_ϕ representing the unit. The dual map $K_\phi^*: \pi_\nu(A_\phi)'' \rightarrow H^\infty(M(\hat{A}), P_\phi)$ is an isomorphism intertwining the left coactions of (M, Δ) and the right coactions of $(\hat{M}, \hat{\Delta})$. The composition of K_ϕ^* with the isomorphism $B \xrightarrow{\simeq} A_\phi$ from Theorem 4.1 is given by $b \mapsto (\varphi \otimes \iota) \hat{\Phi}(b)$.*

Thus one can think of $H^\infty(M(\hat{A}), P_\phi)$ as the algebra of bounded measurable functions on the quantum homogeneous sphere.

In the case when $\text{supp } \phi$ is finite, the fact that the map $\Theta = (\varphi \otimes \iota) \hat{\Phi}$ gives an isomorphism of $\pi_\varphi(B)''$ onto $H^\infty(M(\hat{A}), P_\phi)$ was established by Izumi [12].

The rest of this section is devoted to formulating and proving more precise versions of these results.

4.1 Quantum Spheres and the Dual of Quantum $SU(2)$

First recall that \mathcal{A} is the $*$ -subalgebra of A generated by α and γ . The coinverse and the counit on \mathcal{A} are given by

$$S(\alpha) = \alpha^*, \quad S(\gamma) = -q\gamma, \quad \varepsilon(\alpha) = 1, \quad \varepsilon(\gamma) = 0.$$

The characters f_z are defined by

$$f_z(\gamma) = 0, \quad f_z(\alpha) = q^{-z}. \quad (4.1)$$

From the point of view of representation theory it is convenient to introduce the quantized universal enveloping algebra $U_q(\mathfrak{su}_2)$. It is by definition the universal unital $*$ -algebra generated by elements e, f, k, k^{-1} satisfying the relations

$$kk^{-1} = k^{-1}k = 1, \quad ke = qek, \quad kf = q^{-1}fk, \quad ef - fe = \frac{1}{q - q^{-1}}(k^2 - k^{-2}),$$

$$k^* = k, \quad e^* = f.$$

The algebra $U_q(\mathfrak{su}_2)$ is, in fact, a dense subalgebra of $\mathcal{A}' = M(\hat{\mathcal{A}})$ and the restrictions of $\hat{\Delta}$, \hat{S} and $\hat{\varepsilon}$ to $U_q(\mathfrak{su}_2)$ turn it into a Hopf $*$ -algebra. Explicitly,

$$\begin{aligned} \hat{\Delta}(k) &= k \otimes k, \quad \hat{\Delta}(e) = e \otimes k^{-1} + k \otimes e, \quad \hat{\Delta}(f) = f \otimes k^{-1} + k \otimes f, \\ \hat{S}(k) &= k^{-1}, \quad \hat{S}(e) = -q^{-1}e, \quad \hat{S}(f) = -qf, \\ \hat{\varepsilon}(k) &= 1, \quad \hat{\varepsilon}(e) = \hat{\varepsilon}(f) = 0. \end{aligned}$$

The set I of equivalence classes of irreducible representations is identified with the set $\frac{1}{2}\mathbb{Z}_+$ of non-negative half-integers. Note that $\dim H_s = 2s + 1$. The basis $\{\xi_i^s\}_{i=-s}^s$ for H_s is chosen in such a way that

$$\pi_s(k)\xi_i^s = q^{-i}\xi_i^s, \quad (4.2)$$

$$\pi_s(e)\xi_i^s = ([s+i]_q[s-i+1]_q)^{\frac{1}{2}}\xi_{i-1}^s, \quad (4.3)$$

$$\pi_s(f)\xi_i^s = ([s-i]_q[s+i+1]_q)^{\frac{1}{2}}\xi_{i+1}^s, \quad (4.4)$$

where $[n]_q$ is the q -number for n , so

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}.$$

The fundamental corepresentation corresponds to spin $s = \frac{1}{2}$. Thus

$$U^{\frac{1}{2}} = (u_{ij}^{\frac{1}{2}})_{ij} = \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix}, \quad (4.5)$$

so e.g. $u_{-\frac{1}{2}, -\frac{1}{2}}^{\frac{1}{2}} = \alpha$ and $u_{\frac{1}{2}, -\frac{1}{2}}^{\frac{1}{2}} = \gamma$. The formulas (4.2–4.4) for $s = \frac{1}{2}$ and (4.5) determine the pairing between (\mathcal{A}, Δ) and $(U_q(\mathfrak{su}_2), \hat{\Delta})$ uniquely. Since k is a character on \mathcal{A} , the formulas (4.1) and (4.5) show that

$$\rho = k^{-2}. \quad (4.6)$$

In particular, for the quantum dimension we have $d_s = [2s+1]_q$. Also the basis $\{\xi_i^s\}_i$ satisfies the conventions of Subsection 1.3 in that $\pi_s(\rho)$ is diagonal. Introducing matrix units m_{ij}^s and $n_{ij}^{\bar{s}}$ as in that subsection, we get the following identities.

Lemma 4.3

- (i) $s = \bar{s}$ and $n_{ij}^s = (-1)^{i-j}m_{-i, -j}^s$;
- (ii) $\mathcal{F}(u_{ij}^s) = (-1)^{i-j}d_s^{-1}q^{-i-j}m_{-i, -j}^s$ and $(u_{ij}^s)^* = (-1)^{i-j}q^{-i+j}u_{-i, -j}^s$;
- (iii) $\alpha^*u_{jj}^s = d_s^{-1}(q^{-s+j}[s+j+1]_q u_{j+\frac{1}{2}, j+\frac{1}{2}}^{s+\frac{1}{2}} + q^{s+j+1}[s-j]_q u_{j+\frac{1}{2}, j+\frac{1}{2}}^{s-\frac{1}{2}})$;
- (iv) $u_{sj}^s = \left[\begin{matrix} 2s \\ s+j \end{matrix} \right]_{q^2}^{\frac{1}{2}} (\alpha^*)^{s+j} \gamma^{s-j}$.

Here $\begin{bmatrix} n \\ m \end{bmatrix}_r = \frac{(r; r)_n}{(r; r)_m (r; r)_{n-m}}$, where $(a; r)_n = \prod_{i=0}^{n-1} (1 - ar^i)$ for $n \geq 1$ and $(a; r)_0 = 1$.

Proof of Lemma 4.3. It is well-known that any unitary corepresentation of $SU_q(2)$ is self-conjugate. Explicitly, using (4.2–4.4) and (4.6) it is easy to check that the unitary $\tilde{J}_s: H_s \rightarrow \overline{H}_s$ given by $\tilde{J}_s \xi_j^s = (-1)^{[j]} \overline{\xi_{-j}^s}$, where $[j]$ is the integral part of j , intertwines π_s and $\pi_{\overline{U^s}}$, as

$$\pi_{\overline{U^s}}(x) \bar{\xi} = \overline{\pi_s(\rho^{-\frac{1}{2}} \hat{S}(x^*) \rho^{\frac{1}{2}}) \xi}.$$

Thus $s = \bar{s}$ and $\hat{R}(x) = J_s x^* J_s^{-1}$ for $x \in B(H_s)$, where J_s is the antilinear isometry on H_s given by $J_s \xi_j^s = (-1)^{[j]} \xi_{-j}^s$ (see Subsection 1.3). Hence $n_{ij}^s = (-1)^{i-j} m_{-i, -j}^s$.

Now (ii) follows from (i) and Lemma 1.1. Part (iii) is a particular case of the formulas for the Clebsch-Gordan coefficients [29] (see e.g. [11] for more details). The formula in (iv) is from [20, Theorem 1.8]. ■

Let $\alpha: B \rightarrow A \otimes B$ be a left coaction of (A, Δ) on a unital C^* -algebra B . For $s \in I$, consider the spectral subspace

$$B(s) = \text{span}\{(\varphi \otimes \iota)((a^* \otimes 1)\alpha(b)) \mid b \in B, a \in \text{span}\{u_{ij}^s\}_{i,j}\}.$$

Podleś [23] classified the coactions satisfying three additional properties: $B(0) = \mathbb{C}1$, $B(1)$ has multiplicity one (equivalently, $\dim B(1) = \dim H_1 = 3$), and B is generated as a C^* -algebra by $B(1)$. Such pairs (B, α) he called quantum spheres. They are classified by a parameter $c \in \{-(q^n + q^{-n})^{-2} \mid n = 2, 3, \dots\} \cup [0, \infty]$. The corresponding algebra B is denoted by $C(S_{q,c}^2)$. We are not interested in $C(S_{q,\infty}^2)$. For $c \neq \infty$, the algebra $C(S_{q,c}^2)$ is the universal unital C^* -algebra generated by elements X_{-1} , X_0 and X_1 satisfying the relations

$$X_{-1}^* = -qX_1, \quad X_0^* = X_0,$$

$$X_0^2 + X_{-1}^* X_{-1} + X_1^* X_1 = 1 + (q + q^{-1})^2 c,$$

$$qX_1 X_0 - q^{-1} X_0 X_1 = (q^{-1} - q) X_1, \tag{4.7}$$

$$(q^{-1} - q) X_0^2 + X_{-1} X_1 - X_1 X_{-1} = -(q^{-1} - q) X_0. \tag{4.8}$$

The coaction α is uniquely defined by the identity

$$\alpha(X_j) = \sum_{k=-1}^1 u_{jk}^1 \otimes X_k, \quad j = -1, 0, 1.$$

For $c = -(q^{2s+1} + q^{-2s-1})^{-2}$, $s \in \frac{1}{2}\mathbb{N}$, the quantum sphere $(C(S_{q,c}^2), \alpha)$ is nothing else than $(B(H_s), \alpha_s)$. Thus there exist uniquely determined generators $X_j^s \in B(H_s)$, $j = -1, 0, 1$, to be displayed in the result below.

Lemma 4.4 *Set $\chi_{-1} = -qfk$, $\chi_0 = \frac{ef - q^2 fe}{\sqrt{[2]_q}}$ and $\chi_1 = qek$. Then $X_j^s = \lambda_s^{-1} \pi_s(\chi_j)$, where $\lambda_s = \frac{q(q^{2s+1} + q^{-2s-1})}{(q - q^{-1})\sqrt{[2]_q}}$.*

Proof. See Lemma 4.1 and Lemma 5.7 in [12]. For the sake of convenience we sketch a proof.

The fact that $\tilde{X}_j^s = \pi_s(\chi_j)$, $j = -1, 0, 1$, is a basis in the spectral subspace of $B(H_s)$ corresponding to spin 1, that is

$$\Phi(\tilde{X}_j^s) = \sum_{k=-1}^1 u_{jk}^1 \otimes \tilde{X}_k^s, \quad (4.9)$$

is deduced using the adjoint action. Namely, if we set $\text{ad } X = (X \otimes \iota)\Phi$, the mapping $X \mapsto \text{ad } X$ is an antirepresentation (i.e. linear but antimultiplicative) of $U_q(\mathfrak{su}_2)$ on \hat{A} . Thus, to check (4.9) it suffices to verify the equality

$$(\text{ad } X)(\tilde{X}_j^s) = \sum_{k=-1}^1 \pi_1(X)_{jk} \otimes \tilde{X}_k^s$$

for $X = e, f, k, k^{-1}$. The coefficients $\pi_1(X)_{jk}$ can be read off (4.2–4.4). On the other hand, the adjoint action can be computed explicitly. Namely, if $\hat{\Delta}(X) = \sum_i X_i \otimes Y_i$, then $(\text{ad } X)(x) = \sum_i \hat{S}(X_i)xY_i$. Thus

$$(\text{ad } k)(x) = k^{-1}xk, \quad (\text{ad } e)(x) = -q^{-1}exk^{-1} + k^{-1}xe, \quad (\text{ad } f)(x) = -qfxxk^{-1} + k^{-1}xf. \quad (4.10)$$

Once we know that \tilde{X}_j^s , $j = -1, 0, 1$, satisfy (4.9), then $\tilde{X}_j^s = \lambda_s X_j^s$ for some $\lambda_s \in \mathbb{C}$. This follows immediately from irreducibility of U^1 , the fact that $B(H_s) = \bigoplus_{n=0}^{2s} B(H_s)(n)$ and that each spectral subspace $B(H_s)(n)$ has multiplicity one. The constant λ_s is uniquely determined by (4.7) or (4.8). So a direct computation yields λ_s as in the formulation of Lemma. ■

Define $\lambda \in M(\hat{A})$ by requiring $\pi_s(\lambda) = \lambda_s I_s$. It is a straightforward computation to verify that

$$\lambda = \frac{q^2 - 1}{\sqrt{[2]_q}} \left(C + \frac{2}{(q - q^{-1})^2} \right), \quad \text{where } C = fe + \left(\frac{q^{\frac{1}{2}}k - q^{-\frac{1}{2}}k^{-1}}{q - q^{-1}} \right)^2$$

is the Casimir element. So λ belongs to the center of $U_q(\mathfrak{su}_2)$.

The sphere $(C(S_{q,0}^2), \alpha)$ is a distinguished one. If X_{-1}, X_0, X_1 are its canonical generators, then the map $X_j \mapsto -u_{j0}^1$ extends to an embedding of $C(S_{q,0}^2)$ into A , which intertwines the coaction on $C(S_{q,0}^2)$ with the left coaction Δ of (A, Δ) on A . Under this embedding $C(S_{q,0}^2)$ is identified with the subalgebra

$$B = \{a \in A \mid f_{it} * a = a \ \forall t \in \mathbb{R}\}$$

of A introduced at the beginning of this section. In particular, the algebra $C(S_{q,0}^2)$ also carries a right coaction of $(\hat{A}, \hat{\Delta})$. We shall give one more description of $C(S_{q,0}^2)$, which will be pertinent in the sequel.

Let Ψ be the unital C^* -subalgebra of $M(\hat{A})$ generated by \hat{A} and the elements $\lambda^{-1}\chi_j$, $j = -1, 0, 1$.

Proposition 4.5

- (i) *The algebra Ψ has the properties $\Phi(\Psi) \subset A \otimes \Psi$ and $\hat{\Delta}(\Psi) \subset M(\Psi \otimes \hat{A})$, so we have well-defined coactions of (A, Δ) and $(\hat{A}, \hat{\Delta})$ on Ψ and the quotient C^* -algebra Ψ/\hat{A} .*
- (ii) *There exists a unique isomorphism $\sigma: C(S_{q,0}^2) \cong \Psi/\hat{A}$, which maps X_j to $\lambda^{-1}\chi_j \pmod{\hat{A}}$ and intertwines the left coactions of (A, Δ) and the right coactions of $(\hat{A}, \hat{\Delta})$.*

Proof. Since $\Phi(\lambda^{-1}\chi_j) = \sum_{k=-1}^1 u_{jk}^1 \otimes (\lambda^{-1}\chi_k)$ by Lemma 4.4, we obviously have $\Phi(\Psi) \subset A \otimes \Psi$. Furthermore, the homomorphism Φ is a coaction of (A, Δ) on Ψ by Corollary 1.4(i). As $\pi_s(\lambda^{-1}\chi_j)$,

$j = -1, 0, 1$, are the canonical generators of $C(S_{q,c(s)}^2)$, where $c(s) = -(q^{2s+1} + q^{-2s-1})^{-2}$, and because $c(s) \rightarrow 0$ as $s \rightarrow \infty$, the elements $\lambda^{-1}\chi_j \bmod \hat{A}$, $j = -1, 0, 1$, satisfy the same relations as the generators of $C(S_{q,0}^2)$. Thus there exists a surjective $*$ -homomorphism $\sigma: C(S_{q,0}^2) \rightarrow \Psi/\hat{A}$, which maps X_j to $\lambda^{-1}\chi_j$ (modulo \hat{A}) and intertwines the left coactions of (A, Δ) . Since Ψ/\hat{A} is non-zero, by Podleś' classification, this homomorphism must be an isomorphism.

The assertions that $\hat{\Delta}(\Psi) \subset M(\Psi \otimes \hat{A})$ and that the isomorphism σ intertwines the right coactions of $(\hat{A}, \hat{\Delta})$, will be proved in Subsection 4.3 below. ■

The elements χ_j , $j = -1, 0, 1$, and $\lambda - \lambda_0 1$ span the 4-dimensional quantum Lie algebra associated to the $4D_+$ -bicovariant calculus of Woronowicz [35, 17], so $da = \sum_j (\chi_j * a)\omega_j + ((\lambda - \lambda_0 1) * a)\omega$ in that context. Thus we can think of χ_j , $j = -1, 0, 1$, and $\lambda - \lambda_0 1$ as left-invariant first order differential operators, and Ψ as the C^* -algebra of left-invariant pseudodifferential operators of order zero on $SU_q(2)$. Then the composition $\Psi \rightarrow \Psi/\hat{A} \xrightarrow{\sigma^{-1}} C(S_{q,0}^2)$ should be thought of as the principal symbol.

Before we embark on proving Theorem 4.1, we end this subsection by giving some heuristic reasons for why this result should be true.

The restriction of P_ϕ to the center, which is isomorphic to $c_0(\mathbb{Z}_+)$, is not given by a convolution operator on \mathbb{Z} . However, it is not far from being such an operator. Thus the theory of random walks on \mathbb{Z} suggests that the Martin compactification of the center is obtained by adding one point at infinity. Now let H be a copy of H_1 in \hat{A} , that is, there exists a basis \tilde{X}_j , $j = -1, 0, 1$, in H such that $\Phi(\tilde{X}_j) = \sum_{k=-1}^1 u_{jk}^1 \otimes \tilde{X}_k$. Since $K_{\check{\phi}}$ commutes with Φ , the elements $K_{\check{\phi}}(\tilde{X}_j)$, $j = -1, 0, 1$, have the same property as the elements \tilde{X}_j . Hence, for each $s \in \frac{1}{2}\mathbb{N}$, there exists a constant c_s such that $K_{\check{\phi}}(\tilde{X}_j)I_s = c_s \lambda^{-1} \chi_j I_s$. The function $\frac{1}{2}\mathbb{N} \ni s \mapsto c_s$ (which is easily seen to be bounded) embodies certain properties of the random walk on the center, so it is natural to assume that it extends to a continuous function on the Martin compactification. Then the fact that the boundary consists of one point means that $c_s \rightarrow c \in \mathbb{C}$ as $s \rightarrow \infty$. So, modulo \hat{A} , we have $K_{\check{\phi}}(\tilde{X}_j) = c \lambda^{-1} \chi_j$, and therefore $K_{\check{\phi}}(H) \subset \Psi$. Moreover, if $c \neq 0$, then $\Psi \subset \tilde{A}_\phi$. This argument can be repeated for all spectral subspaces (recall that both $C(S_{q,0}^2)$ and \hat{A} have non-zero spectral subspaces for each integer spin), so we get $K_{\check{\phi}}(\hat{A}) \subset \Psi$. Thus $\tilde{A}_\phi \subset \Psi$, and finally $\tilde{A}_\phi = \Psi$ if at least one of the functions $s \mapsto c_s$ has a non-zero limit at infinity.

4.2 Random Walk on the Center

The aim of this subsection is to describe the asymptotic behavior of the Martin kernel on the center. But let us first make some remarks.

Since $q < 1$, we have $n < [n]_q$ for any $n > 1$. Hence by Theorem 2.6, any state $\phi \in \mathcal{C}$ with $\text{supp } \phi \neq \{0\}$ is transient. It is worthwhile to note that this result is also valid for ordinary $SU(2)$. This can be deduced from the existence of potential kernels for *recurrent* random walks on \mathbb{Z} (see e.g. [28]). Nothing like the estimate in Theorem 2.6 is, however, available. For example, if we consider the random walk corresponding to the fundamental corepresentation (so $q = 1$ and $\phi = \phi_{\frac{1}{2}}$), then the probability of return to 0 at the n th step is given by the semicircular law, that is $p_{\phi^n}(0, 0) = \frac{2}{\pi} \int_{-1}^1 t^n \sqrt{1 - t^2} dt$.

It is known that $U^s \times U^t \simeq \sum_{r=|s-t|}^{s+t} U^r$. It follows that ϕ is generating if and only if $(\text{supp } \phi) \cap (\frac{1}{2} + \mathbb{Z}_+) \neq \emptyset$.

Since any unitary corepresentation is self-conjugate, we also have $\phi = \check{\phi}$ for any positive $\phi \in \mathcal{C}$.

Let now $\phi \in \mathcal{C}$ be a generating positive functional with $\|\phi\| \leq 1$. Set $g_\phi(s, t) = \sum_{n=0}^\infty p_{\phi^n}(s, t)$, so $G_\phi(I_t)I_s = g_\phi(s, t)I_s$. We want to describe the behavior of the function $g_\phi(s, t)$ as $s \rightarrow \infty$. For

this we could apply the results of Biane [7] for ordinary $SU(2)$. Namely, let us for a moment write the superscript 'cl' for the states on the dual of $SU(2)$. If $\phi = \sum_s \lambda_s \phi_s$, we set $\phi^{cl} = \sum_s \lambda_s \frac{\dim H_s}{d_s} \phi_s^{cl}$. Then using Lemma 2.4 and the facts that the fusion coefficients N_{rs}^t are independent of the deformation parameter $q \in (0, 1]$, it is easy to see that $p_\phi(s, t) = \frac{\dim H_s}{d_s} p_{\phi^{cl}}(s, t) \frac{d_t}{\dim H_t}$. In other words, the element $\sum_s \frac{\dim H_s}{d_s} I_s$ is an eigenvector for P_ϕ with eigenvalue $\sum_s \lambda_s \frac{\dim H_s}{d_s}$, and if we consider the Doob transformation of $P_\phi|_{Z(\hat{A})}$ with respect to this eigenfunction, we get an operator corresponding to the deformation parameter $q = 1$. Then we can apply the results of Biane to ϕ^{cl} (note that even if $\phi(1) = 1$, we have $\phi^{cl}(1) < 1$). We will instead give a slightly different but more direct proof.

Consider the von Neumann algebra $L(\mathbb{T}) \cong l^\infty(\mathbb{Z})$ of the circle group \mathbb{T} . Since ρ is an operator with pure point spectrum $q^{\mathbb{Z}}$, we have an embedding of $L(\mathbb{T})$ into $M(\hat{A})$ given by $e^{it} \mapsto u_t = \rho^{-\frac{it}{\log q}}$. Since $\hat{\Delta}(\rho) = \rho \otimes \rho$, this is an embedding of Hopf algebras. It follows that $P = P_\phi|_{l^\infty(\mathbb{Z})}$ is the operator of convolution with the measure $\phi|_{l^\infty(\mathbb{Z})}$. Let $\{e_n\}_{n \in \mathbb{Z}}$ be the canonical projections in $l^\infty(\mathbb{Z})$, so

$$e_n = \frac{1}{2\pi} \int_0^{2\pi} e^{-int} u_t dt = \sum_{s=\frac{|n|}{2}}^{\infty} m_{-\frac{n}{2}, -\frac{n}{2}}^s.$$

If we set $p(n) = \phi(e_n)$, then $P(e_m)e_n = p(m-n)e_n$.

Lemma 4.6 *We have $p_\phi(s, 0) = \frac{1}{d_s}(q^{2s}p(-2s) - q^{2s+2}p(-2s-2))$.*

Proof. It is enough to check this for $\phi = \phi_t$. In this case $P_\phi(I_0) = d_t^{-2}I_t$. So $p_\phi(t, 0) = d_t^{-2}$ and $p_\phi(s, 0) = 0$ for $s \neq t$. On the other hand, $p(n) = d_t^{-1}q^n$ if $n \in \{-2t, -2t+2, \dots, 2t\}$, and $p(n) = 0$ otherwise. ■

There exists a unique $\delta_\phi \geq 0$ such that $\phi(\rho^{-\delta_\phi}) = 1$ (note that the function $f(t) = \phi(\rho^{-t})$ is convex and $f(-2-t) = f(t)$ as $\phi_s(\rho^{-t}) = d_s^{-1} \text{Tr} \pi_s(\rho^{-1-t})$, and $f(0) = \phi(1) \leq 1$). Since $\rho^{-\delta_\phi} e_n = q^{n\delta_\phi} e_n$, this is the same as to require $\sum_{n \in \mathbb{Z}} q^{n\delta_\phi} p(n) = 1$. Suppose $\phi = \sum_s \lambda_s \phi_s$. From this point onwards we assume that

$$\sum_{s \in \frac{1}{2}\mathbb{Z}_+} s q^{-2s\delta_\phi} \lambda_s < \infty. \quad (4.11)$$

Equivalently, $\sum_{n \in \mathbb{Z}} |n| q^{n\delta_\phi} p(n) < \infty$. Then we set

$$\lambda_\phi = \sum_{s \in \frac{1}{2}\mathbb{Z}_+} \frac{\lambda_s}{d_s} \sum_{j=-s}^s 2j q^{2j(1+\delta_\phi)} = \sum_{n \in \mathbb{Z}} n q^{n\delta_\phi} p(n).$$

Note that $\lambda_\phi < 0$ as $q < 1$.

Proposition 4.7 *With notation as above we have*

$$g_\phi(s, 0) \sim -\lambda_\phi^{-1} (1 - q^{2+2\delta_\phi}) \frac{q^{2s(1+\delta_\phi)}}{d_s} \quad \text{as } s \rightarrow \infty.$$

In particular, $\frac{g_\phi(s + \frac{1}{2}, 0)}{g_\phi(s, 0)} \rightarrow q^{2+\delta_\phi}$ as $s \rightarrow \infty$.

The precise values of the constants are unimportant for the computation of the Martin boundary. What matters is that $\frac{g_\phi(s+\frac{1}{2},0)}{g_\phi(s,0)}$ converges to some number in the interval $(0,1)$, and that for states this number is independent of the particular choice of ϕ .

Proof of Proposition 4.7. Set $\tilde{p}(n) = q^{n\delta_\phi}p(n)$ and $\tilde{g}(n) = \sum_{k=0}^{\infty}(\tilde{p}^{*k})(n)$. Then by the renewal theorem for \mathbb{Z} (see [28, Proposition 24.6]), the sequence $\{\tilde{g}(n)\}_n$ tends to $-\lambda_\phi^{-1}$ as $n \rightarrow -\infty$. On the other hand, if $g(n) = \sum_{k=0}^{\infty}(p^{*k})(n)$, then $\tilde{g}(n) = q^{n\delta_\phi}g(n)$. As $g_\phi(s,0) = \frac{q^{2s}}{d_s}(g(-2s) - q^2g(-2s-2))$ by Lemma 4.6, we get the desired result (note also that $d_s d_{s+\frac{1}{2}}^{-1} \rightarrow q$ as $s \rightarrow \infty$). ■

Corollary 4.8 *The Martin compactification of $\frac{1}{2}\mathbb{Z}_+$ with respect to $P_\phi|_{Z(\hat{A})}$ is obtained by adding one point at infinity. In particular, if ϕ is a state, then the constants are the only central harmonic elements with respect to P_ϕ .*

Proof. Since $U^t \times U^r \simeq \sum_{s=|r-t|}^{r+t} U^s$, by Lemma 2.4(iv) we get $P_{\phi_t}(I_r) = \sum_{s=|r-t|}^{r+t} \frac{d_r}{d_t d_s} I_s$. As the algebra \mathcal{C} is commutative in our case, we get

$$G_\phi(I_t) = d_t^2 G_\phi P_{\phi_t}(I_0) = d_t^2 P_{\phi_t} G_\phi(I_0) = d_t^2 \sum_{r \in \frac{1}{2}\mathbb{Z}_+} g_\phi(r,0) P_{\phi_t}(I_r),$$

whence $g_\phi(s,t) = \sum_{r=|s-t|}^{s+t} g_\phi(r,0) \frac{d_t d_r}{d_s}$. By Proposition 4.7 we conclude that for any $t \in \frac{1}{2}\mathbb{Z}_+$,

there exists a constant c_t (depending on ϕ) such that $g_\phi(s,t) \sim c_t q^{2s(2+\delta_\phi)}$ as $s \rightarrow \infty$. Thus

$$K_\phi(I_t) = \sum_s \frac{g_\phi(s,t)}{g_\phi(s,0)} I_s = \frac{c_t}{c_0} 1 \pmod{Z(\hat{A})}.$$
■

4.3 Martin Boundary

Here we will prove Theorem 4.1 and complete the proof of Proposition 4.5.

First, analogously to [7], we compute the action of the Martin kernel on certain elements of $\hat{\mathcal{A}}$.

Proposition 4.9 *Let $\phi \in \mathcal{C}$ be a generating positive functional which has norm not greater than one and satisfies condition (4.11). Set $\tilde{\lambda} = \sum_s q^{-2s} I_s$. Then $K_\phi \mathcal{F}((\alpha^*)^n) = p_n(\tilde{\lambda}^{-1} k^2) \pmod{\hat{A}}$ for $n \geq 0$, where p_n is the polynomial of degree n defined by the recurrence relation*

$$p_{n+1}(x) = c_\phi p_n(x)x - c_\phi^{-1} p_n(q^{-2}x)(x-1), \quad p_0 = 1,$$

where $c_\phi = q^{2+\delta_\phi}$.

Proof. We shall prove by induction on n that there exist constants $a_n(s,m)$, $n \geq 0$, $0 \leq m \leq n$, $s \in \frac{1}{2}\mathbb{Z}_+$, such that

$$G_\phi \mathcal{F}((\alpha^*)^n) = \sum_s \sum_{m=0}^n a_n(s,m) q^{2ms} k^{2m} I_s,$$

and obtain a recurrence relation for $a_n(s,m)$.

To do this, for $a \in \mathcal{A}$, consider the operator Q_a on $M(\hat{A})$ of left convolution with a , i.e. $Q_a = (\iota \otimes a)\hat{\Delta}$. Then $P_\phi Q_a = Q_a P_\phi$. As $\mathcal{F}(ab) = Q_a \mathcal{F}(b)$, we have $P_\phi \mathcal{F}(ab) = Q_a P_\phi \mathcal{F}(b)$ for any $a, b \in \mathcal{A}$, so $G_\phi \mathcal{F}(ab) = Q_a G_\phi \mathcal{F}(b)$. Hence

$$G_\phi \mathcal{F}((\alpha^*)^{n+1}) = Q_{\alpha^*} G_\phi \mathcal{F}((\alpha^*)^n) = \sum_s \sum_{m=0}^n a_n(s,m) q^{2ms} Q_{\alpha^*}(k^{2m} I_s). \quad (4.12)$$

By Lemma 4.3(ii) and formula (4.2), we have

$$k^{2m}I_s = \sum_{j=-s}^s q^{-2mj} m_{jj}^s = d_s \sum_{j=-s}^s q^{-2j-2mj} \mathcal{F}(u_{-j,-j}^s). \quad (4.13)$$

Finally, by Lemma 4.3(iii),

$$\begin{aligned} Q_{\alpha^*} \mathcal{F}(u_{-j,-j}^s) &= \mathcal{F}(\alpha^* u_{-j,-j}^s) \\ &= d_s^{-1} (q^{-s-j} [s-j+1]_q \mathcal{F}(u_{-j+\frac{1}{2},-j+\frac{1}{2}}^{s+\frac{1}{2}}) + q^{s-j+1} [s+j]_q \mathcal{F}(u_{-j+\frac{1}{2},-j+\frac{1}{2}}^{s-\frac{1}{2}})) \\ &= \frac{q^{-s+j-1}}{d_s d_{s+\frac{1}{2}}} [s-j+1]_q m_{j-\frac{1}{2},j-\frac{1}{2}}^{s+\frac{1}{2}} + \frac{q^{s+j}}{d_s d_{s-\frac{1}{2}}} [s+j]_q m_{j-\frac{1}{2},j-\frac{1}{2}}^{s-\frac{1}{2}}. \end{aligned} \quad (4.14)$$

Using (4.12)–(4.14) a tedious computation yields

$$\begin{aligned} a_{n+1}(s, m) &= \frac{q^{-2s-1}}{q^{-2s-1} - q^{2s+1}} \left(-q^{-2(m-1)} a_n(s - \frac{1}{2}, m-1) \right. \\ &\quad \left. + q^{-2m} a_n(s - \frac{1}{2}, m) + a_n(s + \frac{1}{2}, m-1) - q^{4s+2} a_n(s + \frac{1}{2}, m) \right). \end{aligned}$$

We also have $a_0(s, 0) = g_\phi(s, 0)$. Since $\frac{g_\phi(s + \frac{1}{2}, 0)}{g_\phi(s, 0)} \rightarrow c_\phi = q^{2+\delta_\phi}$ as $s \rightarrow \infty$ by Proposition 4.7, we see that for any n and m , $0 \leq m \leq n$, there exists a finite limit $a_n(m) = \lim_{s \rightarrow \infty} \frac{a_n(s, m)}{g_\phi(s, 0)}$, and these limits satisfy the recurrence relation

$$a_{n+1}(m) = -c_\phi^{-1} q^{-2(m-1)} a_n(m-1) + c_\phi^{-1} q^{-2m} a_n(m) + c_\phi a_n(m-1). \quad (4.15)$$

Thus if p_n is the polynomial with the coefficients $a_n(m)$, so $p_n(x) = \sum_{m=0}^n a_n(m) x^m$, then as $\tilde{\lambda}^{-1} k^2 \in M(\hat{A})$ and

$$K_\phi \mathcal{F}((\alpha^*)^n) = \sum_s \sum_{m=0}^n \frac{a_n(s, m)}{g_\phi(s, 0)} (\tilde{\lambda}^{-1} k^2)^m I_s,$$

we conclude that $K_\phi \mathcal{F}((\alpha^*)^n) = p_n(\tilde{\lambda}^{-1} k^2) \bmod \hat{A}$. Condition (4.15) can be written as $p_{n+1}(x) = c_\phi p_n(x)x - c_\phi^{-1} p_n(q^{-2}x)(x-1)$. The leading coefficient of p_n equals

$$(c_\phi - c_\phi^{-1} q^{-2(n-1)})(c_\phi - c_\phi^{-1} q^{-2(n-2)}) \dots (c_\phi - c_\phi^{-1}) = (-1)^n q^{-n(n-1)} c_\phi^{-n} (c_\phi^2; q^2)_n.$$

As $c_\phi < 1$, it is indeed non-zero, so p_n is a polynomial of degree n . ■

Theorem 4.10 *Let $\phi \in \mathcal{C}$ be a generating positive functional which has norm not greater than one and satisfies condition (4.11). Then*

- (i) *the Martin compactification \tilde{A}_ϕ coincides with Ψ and, moreover, the linear subspace $K_\phi(\hat{A}) + \hat{A}$ is dense in Ψ ;*
- (ii) *if ϕ is in addition a state, then the Martin kernel K_ϕ , regarded as a map $\hat{A} \rightarrow \Psi/\hat{A}$, is independent of the particular choice of ϕ .*

Proof. By Lemma 4.3 we have $u_{ss}^s = (\alpha^*)^{2s}$ and $\mathcal{F}(u_{ss}^s) = \frac{q^{-2s}}{d_s} m_{-s,-s}^s$. Thus Proposition 4.9 together with the fact that $\lambda \tilde{\lambda}^{-1} \in \mathbb{C}1 + \hat{A}$, imply

$$\text{span}\{K_\phi(m_{-s,-s}^s) \mid s \in \frac{1}{2}\mathbb{Z}_+\} + \hat{A} = \mathbb{C}[\lambda^{-1} k^2] + \hat{A},$$

where $\mathbb{C}[\lambda^{-1}k^2]$ is the unital algebra generated by $\lambda^{-1}k^2$. On the other hand, it is easily checked that

$$\chi_0 = -\lambda + \frac{q\sqrt{[2]_q}}{q - q^{-1}}k^2. \quad (4.16)$$

Thus, we get $\text{span}\{K_\phi(m_{-s,-s}^s) \mid s \in \frac{1}{2}\mathbb{Z}_+\} + \hat{A} = \mathbb{C}[\lambda^{-1}\chi_0] + \hat{A} \subset \Psi$.

Note that the minimal A -invariant subspace of \hat{A} containing $m_{-s,-s}^s$, that is, the subspace spanned by the elements $(\omega \otimes \iota)\Phi(m_{-s,-s}^s)$, $\omega \in A^*$, coincides with $B(H_s)$. Indeed, this amounts to saying that $m_{-s,-s}^s$ is a cyclic vector for the adjoint action of $U_q(\mathfrak{su}_2)$ on $B(H_s)$, which in turn can easily be verified using (4.2)–(4.4) and the formulas (4.10) for the adjoint action. As $\Phi K_\phi = (\iota \otimes K_\phi)\Phi$ by (3.2), we conclude that $K_\phi(\hat{A}) \subset \Psi$. Furthermore, the linear space $K_\phi(\hat{A}) + \hat{A}$ coincides with the minimal A -invariant subspace of Ψ containing both $\mathbb{C}[\lambda^{-1}\chi_0]$ and \hat{A} .

Similarly, if X_{-1}, X_0, X_1 are the canonical generators of $B = C(S_{q,0}^2)$, then the minimal A -invariant subspace of B containing X_0^n , for all $n \in \mathbb{Z}_+$, coincides with the $*$ -subalgebra \mathcal{B} of B generated by X_j , $j = -1, 0, 1$. Indeed, Podleś proved [23] that $\mathcal{B} = \bigoplus_{n=0}^{\infty} B(n)$, each spectral subspace $B(n)$ has multiplicity one, and $(\mathbb{C}1 + B(1))^n \subset \tilde{B}(n) = \bigoplus_{m=0}^n B(m)$. Thus to prove the claim it is enough to check that $X_0^n \in \tilde{B}(n) \setminus \tilde{B}(n-1)$ for any $n \in \mathbb{N}$. To this end consider the automorphism $T = (k \otimes \iota)\alpha$ of \mathcal{B} . As ξ_0^s is the only $\pi_s(k)$ -invariant vector in H_s by formula (4.2), the space of T -invariant vectors in $\tilde{B}(m)$ is $(m+1)$ -dimensional. Since $T(X_0) = X_0$, we conclude that if $X_0^n \in \tilde{B}(n-1)$, then the elements $1, X_0, \dots, X_0^n$ are linearly dependent. This is a contradiction (because e.g. $X_0 = \lambda^{-1}\chi_0 \bmod \hat{A}$).

It follows that modulo \hat{A} the space $K_\phi(\hat{A})$ coincides with the $*$ -algebra generated by $\lambda^{-1}\chi_j$, $j = -1, 0, 1$. This proves part (i) of Theorem. To show (ii), note that by Proposition 4.9 the element $K_\phi(m_{-s,-s}^s) \bmod \hat{A}$ is independent of the choice of the state ϕ . As we explained above, any element in \hat{A} is a linear combination of elements $(\omega \otimes \iota)\Phi(m_{-s,-s}^s)$. Since $K_\phi(\omega \otimes \iota)\Phi(m_{-s,-s}^s) = (\omega \otimes \iota)\Phi K_\phi(m_{-s,-s}^s)$, we see that K_ϕ , regarded as a map from \hat{A} to Ψ/\hat{A} , is independent of the state ϕ . ■

Theorem 4.1 would now obviously follow from Theorem 4.10 and Proposition 4.5. However, the proof of Proposition 4.5 is not yet complete. We shall now focus on how Theorem 4.10 can be used to complete the proof of this proposition. Due to the fact that $\hat{\Delta}(\tilde{A}_\phi) \subset M(\tilde{A}_\phi \otimes \hat{A})$, we conclude that $\hat{\Delta}(\Psi) \subset M(\Psi \otimes \hat{A})$. Thus there are well-defined right coactions of $(\hat{A}, \hat{\Delta})$ on Ψ and Ψ/\hat{A} . It remains to show that the isomorphism $\sigma: C(S_{q,0}^2) \cong \Psi/\hat{A}$ intertwines the coactions of $(\hat{A}, \hat{\Delta})$.

Let K be the map from \hat{A} to Ψ/\hat{A} defined by $K(x) = K_\phi(x) \bmod \hat{A}$, where ϕ is any generating state in \mathcal{C} satisfying condition (4.11). Let ν be the state on $\Psi/\hat{A} = A_\phi$ representing the unit, that is $\nu K = \hat{\psi}$ on \hat{A} . By Proposition 3.12(ii) and Theorem 4.10(i) such a state is unique and Φ -invariant. Since $C(S_{q,0}^2) \subset A$, the state $\varphi|_{C(S_{q,0}^2)}$ is the only invariant state on $C(S_{q,0}^2)$. Hence $\nu\sigma = \varphi|_{C(S_{q,0}^2)}$. By Proposition 1.12(ii), the state φ on A is quasi-invariant with respect to the right coaction of $(\hat{A}, \hat{\Delta})$ with Radon-Nikodym cocycle $y = W(1 \otimes \rho^{-2})W^*$. Note that $y \in M(C(S_{q,0}^2) \odot \hat{A})$ as

$$y = \sum_s \sum_{i,j,l} f_{-2}(u_{il}^s) u_{il}^s (u_{jl}^s)^* \otimes m_{ij}^s$$

and $f_z * (u_{il}^s (u_{jl}^s)^*) = u_{il}^s (u_{jl}^s)^*$. Thus $\varphi|_{C(S_{q,0}^2)}$ is quasi-invariant with Radon-Nikodym cocycle y . The results of Subsection 3.3 suggest that $(K \otimes \iota)\hat{\Delta}(I_0) \in M(\Psi/\hat{A} \odot \hat{A})$ is the Radon-Nikodym cocycle for ν . Thus if $\sigma: C(S_{q,0}^2) \rightarrow \Psi/\hat{A}$ intertwines the right coactions of $(\hat{A}, \hat{\Delta})$, we should expect $(\sigma \otimes \iota)(y) = (K \otimes \iota)\hat{\Delta}(I_0)$.

Proposition 4.11 *We have $(\sigma \otimes \iota)(y) = (K \otimes \iota)\hat{\Delta}(I_0)$.*

Proof. Set $y_{ij}^s = \sum_{l=-s}^s f_{-2}(u_{il}^s)u_{il}^s(u_{jl}^s)^*$. By Lemma 1.2

$$(K \otimes \iota)\hat{\Delta}(I_0) = \sum_s \sum_{i,j=-s}^s K\mathcal{F}(u_{ij}^s) \otimes m_{ij}^s.$$

Thus we must prove that $\sigma(y_{ij}^s) = K\mathcal{F}(u_{ij}^s)$. We shall first verify this identity for $i = j = s$. As already remarked in the proof of Theorem 4.10, we have $u_{ss}^s = (\alpha^*)^{2s}$. By Proposition 4.9 it is enough to check that $y_{ss}^s = p_{2s}(a)$, where $a \in C(S_{q,0}^2)$ is determined by $\sigma(a) = \tilde{\lambda}^{-1}k^2 \pmod{\hat{A}}$, and the polynomials p_n , $n \geq 0$, are defined according to the recurrence relation

$$p_{n+1}(x) = q^2 p_n(x)x - q^{-2} p_n(q^{-2}x)(x-1), \quad p_0 = 1.$$

To find the element a , note that by definition of λ (Lemma 4.4) and $\tilde{\lambda}$, we have

$$\tilde{\lambda}^{-1} = \frac{1}{(q - q^{-1})\sqrt{[2]_q}} \lambda^{-1} \pmod{\hat{A}},$$

so (4.16) and the definition of $\sigma: C(S_{q,0}^2) \rightarrow \Psi/\hat{A}$ yield

$$\sigma(X_0) = \lambda^{-1}\chi_0 \pmod{\hat{A}} = -1 + \frac{q\sqrt{[2]_q}}{q - q^{-1}} \lambda^{-1}k^2 \pmod{\hat{A}} = -1 + (1 + q^2)\tilde{\lambda}^{-1}k^2 \pmod{\hat{A}}.$$

Since $X_0 = -u_{00}^1$, the element X_0 can be found by applying Lemma 4.3(iii) to $s = \frac{1}{2}$, $j = -\frac{1}{2}$. As $u_{-\frac{1}{2}, -\frac{1}{2}}^{\frac{1}{2}} = \alpha$, $u_{00}^0 = 1$ and $\alpha^*\alpha = 1 - \gamma^*\gamma$, we get $X_0 = -1 + (1 + q^2)\gamma^*\gamma$ (though we don't need them, the two other generators are given by $X_{-1} = \sqrt{q[2]_q}\alpha\gamma^*$ and $X_1 = -\sqrt{q[2]_q}\alpha^*\gamma$). Hence $a = \gamma^*\gamma$.

By Lemma 4.3(iv) we have

$$y_{ss}^s = \sum_{j=-s}^s f_{-2}(u_{jj}^s)u_{sj}^s(u_{sj}^s)^* = \sum_{j=-s}^s q^{-4j} \begin{bmatrix} 2s \\ s+j \end{bmatrix}_{q^2} (\alpha^*)^{s+j}(\gamma^*\gamma)^{s-j}\alpha^{s+j}.$$

To prove that $y_{ss}^s = p_{2s}(\gamma^*\gamma)$, we introduce the polynomials $f_{n,m}$, $n \geq 0$, $m \geq 0$, by

$$f_{n,m}(x) = q^{-2mn}x^n(1-x)(1-q^{-2}x)\dots(1-q^{-2(m-1)}x) = q^{-2mn}x^n(x; q^{-2})_m.$$

We assert that $(\alpha^*)^m(\gamma^*\gamma)^n\alpha^m = f_{n,m}(\gamma^*\gamma)$. As $\alpha^*\gamma^*\gamma = q^{-2}\gamma^*\gamma\alpha^*$, we need only to show that $(\alpha^*)^m\alpha^m = (1 - \gamma^*\gamma)\dots(1 - q^{-2(m-1)}\gamma^*\gamma)$. Since $\alpha^*(1 - q^{-2l}\gamma^*\gamma) = (1 - q^{-2(l+1)}\gamma^*\gamma)\alpha^*$, this is easily verified by induction on m . It follows that $y_{ss}^s = \tilde{p}_{2s}(\gamma^*\gamma)$, where

$$\tilde{p}_{2s}(x) = \sum_{j=-s}^s q^{-4j} \begin{bmatrix} 2s \\ s+j \end{bmatrix}_{q^2} f_{s-j, s+j}(x).$$

It remains to prove that $\tilde{p}_n = p_n$, or equivalently, that the polynomials \tilde{p}_n , $n \geq 0$, satisfy the relation

$$\tilde{p}_{2s}(x) = q^2 \tilde{p}_{2(s-\frac{1}{2})}(x)x - q^{-2} \tilde{p}_{2(s-\frac{1}{2})}(q^{-2}x)(x-1).$$

Using the identities

$$\begin{bmatrix} 2s \\ s+j \end{bmatrix}_{q^2} = q^{2(s+j)} \begin{bmatrix} 2s-1 \\ s+j \end{bmatrix}_{q^2} + \begin{bmatrix} 2s-1 \\ s+j-1 \end{bmatrix}_{q^2},$$

and $q^{2(s+j)}f_{s-j,s+j}(x) = f_{s-j-1,s+j}(x)x$ and $f_{s-j,s+j}(x) = f_{s-j,s+j-1}(q^{-2}x)(1-x)$, we see that the polynomial \tilde{p}_{2s} equals

$$\sum_{j=-s}^{s-1} q^{-4j} \begin{bmatrix} 2s-1 \\ s+j \end{bmatrix}_{q^2} f_{s-j-1,s+j}(x)x + \sum_{j=-s+1}^s q^{-4j} \begin{bmatrix} 2s-1 \\ s+j-1 \end{bmatrix}_{q^2} f_{s-j,s+j-1}(q^{-2}x)(1-x).$$

The first summand is $q^2 \sum_{j=-s+\frac{1}{2}}^{s-\frac{1}{2}} q^{-4j} \begin{bmatrix} 2(s-\frac{1}{2}) \\ s-\frac{1}{2}+j \end{bmatrix}_{q^2} f_{s-\frac{1}{2}-j,s-\frac{1}{2}+j}(x)x = q^2 \tilde{p}_{2(s-\frac{1}{2})}(x)x$, whereas the second one is $q^{-2} \tilde{p}_{2(s-\frac{1}{2})}(q^{-2}x)(1-x)$. Thus $\tilde{p}_n = p_n$ and we have demonstrated the equality $\sigma(y_{ss}^s) = K\mathcal{F}(u_{ss}^s)$.

Consider now the linear maps $T_1: B(H_s) \rightarrow C(S_{q,0}^2)$ and $T_2: B(H_s) \rightarrow \Psi/\hat{A}$ given by

$$T_1(m_{ij}^s) = (-1)^{i-j} q^{-i-j} d_s y_{-i,-j}^s \quad \text{and} \quad T_2(m_{ij}^s) = (-1)^{i-j} q^{-i-j} d_s K\mathcal{F}(u_{-i,-j}^s).$$

By Lemma 4.3(ii) the map T_2 is just $K|_{B(H_s)}$, so it intertwines the left coactions of (A, Δ) , i.e. $\Phi T_2 = (\iota \otimes T_2)\Phi$. By the same lemma we also have

$$(-1)^{i-j} q^{-i-j} d_s y_{-i,-j}^s = (-1)^{i-j} q^{-i-j} d_s \sum_l q^{4l} u_{-i,-l}^s (u_{-j,-l}^s)^* = d_s \sum_l q^{2l} (u_{il}^s)^* u_{jl}^s.$$

Since

$$\Delta((u_{il}^s)^* u_{jl}^s) = \sum_{n,m} (u_{in}^s)^* u_{jm}^s \otimes (u_{nl}^s)^* u_{ml}^s \quad \text{and} \quad \Phi(m_{ij}^s) = U^{s*} (1 \otimes m_{ij}^s) U^s = \sum_{n,m} (u_{in}^s)^* u_{jm}^s \otimes m_{nm}^s,$$

we see that T_1 also intertwines the left coactions of (A, Δ) . Therefore the maps σT_1 and T_2 intertwine the coactions of (A, Δ) and coincide on the element $m_{-s,-s}^s$. Since the minimal A -invariant subspace of $B(H_s)$ containing $m_{-s,-s}^s$ is the whole of $B(H_s)$, we get $\sigma T_1 = T_2$. Hence $\sigma(y_{ij}^s) = K\mathcal{F}(u_{ij}^s)$ for all i and j . ■

The following corollary completes the proof of Proposition 4.5, and thus also of Theorem 4.1.

Corollary 4.12 *The isomorphism $\sigma: C(S_{q,0}^2) \simeq \Psi/\hat{A}$ intertwines the right coactions of $(\hat{A}, \hat{\Delta})$.*

Proof. The elements $y = W(1 \otimes \rho^{-2})W^* \in M(C(S_{q,0}^2) \odot \hat{\mathcal{A}})$ and $\tilde{y} = (K \otimes \iota)\hat{\Delta}(I_0) \in M(\Psi/\hat{A} \odot \hat{\mathcal{A}})$ are invertible cocycles for the right coactions of $(\hat{A}, \hat{\Delta})$. As $(\sigma \otimes \iota)(y) = \tilde{y}$ we get

$$(\sigma \otimes \iota \otimes \iota)(\hat{\Phi} \otimes \iota)(y) = (\sigma \otimes \iota \otimes \iota)((\iota \otimes \hat{\Delta})(y)(y^{-1} \otimes 1)) = (\iota \otimes \hat{\Delta})(\tilde{y})(\tilde{y}^{-1} \otimes 1) = (\hat{\Delta} \otimes \iota)(\tilde{y}).$$

Thus $(\sigma \otimes \iota)\hat{\Phi}(a) = \hat{\Delta}\sigma(a)$ for any element a of the form $(\iota \otimes \omega)(y)$, $\omega \in \mathcal{A}$, and therefore also for any element in the C^* -algebra generated by $(\iota \otimes \omega)(y)$, $\omega \in \mathcal{A}$. Since $(\iota \otimes \omega)(\tilde{y}) = K(\iota \otimes \omega)\hat{\Delta}(I_0)$, the set of elements $(\iota \otimes \omega)(\tilde{y})$, $\omega \in \mathcal{A}$, is identical to $K(\hat{\mathcal{A}})$. The set $K(\hat{\mathcal{A}})$ is dense in Ψ/\hat{A} due to Theorem 4.10(i). Since σ is an isomorphism, the set of elements $(\iota \otimes \omega)(y)$, $\omega \in \mathcal{A}$, is dense in $C(S_{q,0}^2)$ as well. So σ intertwines the right coactions of $(\hat{A}, \hat{\Delta})$. ■

Now since $\nu\sigma = \varphi$, the state $\varphi|_{C(S_{q,0}^2)}$ is quasi-invariant with Radon-Nikodym cocycle $y = W(1 \otimes \rho^{-2})W^*$ and $(\sigma \otimes \iota)(y) = (K \otimes \iota)\hat{\Delta}(I_0)$, we obtain the following result.

Corollary 4.13 *The state ν on Ψ/\hat{A} is quasi-invariant with respect to the right coaction of $(\hat{A}, \hat{\Delta})$ with Radon-Nikodym cocycle $(K \otimes \iota)\hat{\Delta}(I_0)$.* ■

4.4 Poisson Boundary

Having the computations of the previous subsection behind us, it is now an easy matter to determine the Poisson boundary. By Theorem 4.10 we can identify the Martin boundary with Ψ/\hat{A} . Moreover, we know already that the state $\nu = \varphi\sigma^{-1}$ on Ψ/\hat{A} represents the unit, $K = K_\phi \bmod \hat{A}$ is independent of ϕ , and $K(\hat{A})$ is dense in Ψ/\hat{A} . By Proposition 3.12(ii) the map $K^*: \pi_\nu(\Psi/\hat{A})'' \rightarrow M(\hat{A})$ defined by $(K(x), a)_\nu = (x, K^*(a))_{\hat{\psi}}$ for $x \in \hat{A}$ and $a \in \pi_\nu(\Psi/\hat{A})''$, is an isomorphism of $\pi_\nu(\Psi/\hat{A})''$ onto $H^\infty(M(\hat{A}), P_\phi)$ which intertwines the left coactions of (M, Δ) . Since ν is quasi-invariant with respect to the right coaction of $(\hat{A}, \hat{\Delta})$, by the discussion following the proof of Proposition 3.11, the right coaction of $(\hat{A}, \hat{\Delta})$ extends to a right coaction of $(\hat{M}, \hat{\Delta})$ on the von Neumann algebra $\pi_\nu(\Psi/\hat{A})''$. Then by Proposition 3.11(ii) and Corollary 4.13, the map K^* intertwines the right coactions of $(\hat{M}, \hat{\Delta})$, and $K^* = (\nu \otimes \iota)\hat{\Delta}$. We summarize this discussion in the following theorem which makes the statement of Theorem 4.2 more precise.

Theorem 4.14

- (i) For any generating state $\phi = \sum_s \lambda_s \phi_s \in \mathcal{C}$ such that $\sum_s s \lambda_s < \infty$, the map $K^*: \pi_\nu(\Psi/\hat{A})'' \rightarrow M(\hat{A})$ gives an isomorphism of $\pi_\nu(\Psi/\hat{A})''$ onto $H^\infty(M(\hat{A}), P_\phi)$ intertwining the left coactions of (M, Δ) and the right coactions of $(\hat{M}, \hat{\Delta})$. In particular, $H^\infty(M(\hat{A}), P_\phi) \subset M(\hat{A})$ is independent of ϕ .
- (ii) We have $K^* = (\nu \otimes \iota)\hat{\Delta}$, so $K^*\sigma = (\varphi \otimes \iota)\hat{\Phi}$ on $C(S_{q,0}^2) \subset A$. ■

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