Maximal abelian subalgebras of the hyperfinite factor, entropy and ergodic theory

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Abstract

Given a free ergodic action of a discrete abelian group G on a measure space (X, μ), the crossed product $L^\infty(X, \mu) \rtimes G$ contains two distinguished maximal abelian subalgebras. We discuss what kind of information about the action can be extracted from the positions of these two subalgebras inside the crossed product algebra.

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1 Introduction

Via the crossed product construction it is well known that ergodic theory is closely related to the study of maximal abelian subalgebras, called masas in the sequal, of the hyperfinite II_1-factor R. In the present note we shall discuss work of the authors [NS2] on this relationship, in particular how positions of some canonically defined masas determine the ergodic transformation completely. Furthermore, we shall see that entropy is a local property of these masas and the generating unitary.

To be more specific let us fix notation. Let (X, μ) be a Lebesgue space with μ a probability measure, G a countable discrete abelian group and $g \to T_g$ a free ergodic measure preserving action of G on X. This action defines an action α of automorphisms of $L^\infty(X, \mu)$ by $\alpha_g(f) = f \circ T_g^{-1}$. $L^\infty(X, \mu)$ acts on $L^2(X, \mu)$ by multiplication. Define representations π and u of $L^\infty(X, \mu)$ and G on $L^2(G, L^2(X, \mu))$ by

$$\begin{aligned}(\pi(f)\xi)(g) &= \alpha_{-g}(f)\xi(g)\ , \qquad \xi \in L^2(G, L^2(X, \mu)), \quad f \in L^\infty(X, \mu)\\ (u_h\xi)(g) &= \xi(g-h)\ .\end{aligned}$$

Then u is a unitary representation, and

$$u_g \pi(f) u_g^* = \pi(\alpha_g(f))\ .$$

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Let R be the von Neumann algebra generated by $\pi(f)$, u_g, $f \in L^\infty(X,\mu)$, $g \in G$. Then R is the crossed product $L^\infty(X,\mu) \times_T G$ and is isomorphic to the hyperfinite II_1-factor with trace τ given by

$$\tau(\pi(f)) = \int f\, d\mu\ , \quad \tau(\pi(u_g)) = 0 \quad \text{if} \;\; g \neq 0\ .$$

We shall denote by S_T the (abelian) von Neumann algebra generated by u_g, $g \in G$, and denote $L^\infty(X,\mu)$ by $L^\infty(T)$. Each element x in R has a unique Fourier series expansion as a sum

$$x = \sum f_g u_g\ , \qquad f_g \in L^\infty(T)\ ,$$

which converges in the $\| \, \|_2$-norm defined by τ. From the uniqueness of the series expansion it is easy to see that S_T is a masa in R, and by ergodicity of T is also easy to see that $L^\infty(T)$ is a masa in R.

The problem discussed in this note is: How much information on the action T can we read out of what we know about the masa S_T, or the pair $(S_T, L^\infty(T))$?

2 The normalizer

A natural invariant first studied by Dixmier [D] is the normalizer $\mathcal{N}(S_T)$ of S_T. By definition, if A is a masa in R then its normalizer

$$\mathcal{N}(A) = \{u : u \text{ is unitary in } R,\, uAu^* = A\}\ .$$

$\mathcal{N}(A)$ is a group; if it generates R we say A is a *Cartan algebra* (or regular), and if $\mathcal{N}(A) \subset A$, A is called *singular*. Since $u_g \in \mathcal{N}(L^\infty(T))$ it is clear that $L^\infty(T)$ is a Cartan algebra. If T is weakly mixing, equivalently α has no eigenfunctions in $L^\infty(T)$ other than the constants, then S_T is singular [N]. More generally one can show, see [P], [H].

Proposition 2.1 *Let $L_0^\infty(X)$ be the subalgebra of $L^\infty(T)$ generated by the eigenfunctions of α. Then the von Neumann algebra $\mathcal{N}(S_T)''$ generated by $\mathcal{N}(S_T)$ is $L_0^\infty(X) \times_T G$.*

Even though $\mathcal{N}(L^\infty(T))'' = R$, the normalizer itself is much smaller. With G ergodic and freely acting as before it is rather straightforward to show that

$$(*) \qquad S_T \cap \mathcal{N}(L^\infty(T)) = \{zu_g : z \in \mathbb{C}, |z| = 1, g \in G\}\ .$$

Having this identity it follows that the pair $(S_T, L^\infty(T))$ determines T completely. Indeed, we have

Proposition 2.2 *Let $g \to T_g^{(i)}$ be a free measure preserving action of a countable abelian group G_i on a Lebesgue space (X_i, μ_i), $i = 1, 2$. Suppose there exists an isomorphism $\gamma : L^\infty(T^{(1)}) \times_{T^{(1)}} G_1 \to L^\infty(T^{(2)}) \times_{T^{(2)}} G_2$ such that $\gamma(S_{T^{(1)}}) = S_{T^{(2)}}$ and $\gamma(L^\infty(T^{(1)})) = L^\infty(T^{(2)})$. Then there exist an isomorphism $S : (X_1, \mu) \to (X_2, \mu_2)$ of measure spaces and a group isomorphism $\beta : G_2 \to G_1$ such that*

$$T_g^{(2)} = S\, T_{\beta(g)}^{(1)} S^{-1} \qquad \textit{for} \;\; g \in G_2\ .$$

PROOF. If γ is as in the proposition it maps $S_{T^{(1)}} \cap \mathcal{N}(L^\infty(T^{(1)}))$ onto the corresponding set for $T^{(2)}$. Thus there exist by $(*)$ an isomorphism $\beta : G_2 \to G_1$ and a character $\mathcal{X} \in \widehat{G}_2$ such that

$$\gamma(u^{(1)}_{\beta(g)}) = \langle \mathcal{X}, g \rangle u^{(2)}_g \ , \qquad g \in G_2 \ ,$$

in obvious notation. Then for $x \in L^\infty(T^{(1)})$ and $g \in G_2$ we have

$$\gamma(\alpha^{(1)}_{\beta(g)}(x)) = \gamma(u^{(1)}_{\beta(g)} x u^{(1)*}_{\beta(g)}) = u^{(2)}_g \gamma(x) u^{(2)*}_g = \alpha^{(2)}_g(\gamma(x)) \ .$$

So for S we can take the transformation which implements the isomorphism γ of $L^\infty(T^{(1)})$ onto $L^\infty(T^{(2)})$. □

3 The masa S_T

From the result in section 2 it is natural to ask: How much information is contained in the masa S_T itself? If the spectrum is purely discrete, then S_T is a Cartan algebra, and such masas are all conjugate [CFW], i.e. there is an automorphism γ of R carrying one masa onto the other, so in that case we get no information.

Conjecture 1 For a weakly mixing system the masa S_T determines the system completely; in other words, the assumption $\gamma(L^\infty(T^{(1)})) = L^\infty(T^{(2)})$ in Proposition 2.2 is redundant.

Our main result is the following generalization of Proposition 2.2. Recall that two masas A and B of R are *inner conjugate* if there is an inner automorphism $\operatorname{Ad} w$ such that $wAw^* = B$.

Theorem 3.1 *With the notation of Proposition 2.2 suppose $\gamma(S_{T^{(1)}}) = S_{T^{(2)}}$ and that the masas $\gamma(L^\infty(T^{(1)}))$ and $L^\infty(T^{(2)})$ are inner conjugate. Then the conclusions of Proposition 2.2 hold.*

It follows from a result of Popa [Po] that if for two Cartan algebras A and B of R we have

$$\sup \Big\{ \|u - E_B u\|_2 : u \text{ unitary in } A \Big\} < 1$$

where E_B is the canonical conditional expectation of R on B, then A and B are inner conjugate. Hence by Theorem 3.1 if $\gamma(L^\infty(T^{(1)}))$ is sufficiently close to $L^\infty(T^{(2)})$ then $T^{(1)}$ and $T^{(2)}$ are isomorphic.

The proof of the theorem is a consequence of the classification of the class of automorphisms described in the theorem. In other words let T and G be as before with T weakly mixing. Let $\operatorname{Aut}(R, S_T \mid L^\infty(T))$ consist of all automorphisms γ of R such that $\gamma(S_T) = S_T$, and $\gamma(L^\infty(T))$ and $L^\infty(T)$ are inner conjugate.

It was shown by Feldman and Moore [FM] that any automorphism S of the orbit equivalence relation defined by the action α of G, extends canonically to an automorphism α_S of R. α_S leaves S_T globally invariant if and only if there exists an automorphism β of G such that $T_g S = S\,T_{\beta(g)}$. Denote by $I(T)$ the group of all such transformations. For $S \in I(T)$, α_S is defined by the identities,

$$\alpha_S(f) = f \circ S^{-1}\,, \qquad f \in L^\infty(T)$$
$$\alpha_S(u_g) = u_{\beta^{-1}(g)}\,, \qquad g \in G$$

Let σ be the dual action of $\widehat{G}$, so that for $\mathcal{X} \in \widehat{G}$,

$$\sigma_{\mathcal{X}}(f) = f\,, \qquad f \in L^\infty(T)$$
$$\sigma_{\mathcal{X}}(u_g) = \langle \mathcal{X}, -g\rangle u_g\,, \qquad g \in G\,.$$

Theorem 3.2 *The group* $\mathrm{Aut}(R, S_T \mid L^\infty(T))$ *of automorphisms* γ *of* R *for which* $\gamma(S_T) = S_T$*, and* $\gamma(L^\infty(T))$ *and* $L^\infty(T)$ *are inner conjugate, consists of automorphisms of the form* $\mathrm{Ad}\,w \circ \sigma_{\mathcal{X}} \circ \alpha_S$*, where* w *is a unitary in* S_T*,* $\mathcal{X} \in \widehat{G}$*,* $S \in I(T)$*.*

For the proof we refer to [NS2].

4 Entropy

Since its introduction in 1958 by Kolmogoroff [K] entropy has been the most celebrated invariant in ergodic theory. In 1975 Connes and Størmer [CS] extended the definition to finite von Neumann algebras, and some years later it was shown, see [V, GN] that in the notation of the previous sections the entropy $H(T)$ of the transformation T on (X, μ) with now $G = \mathbb{Z}$, equals the CS-entropy $H(\mathrm{Ad}\,u_T)$ of the inner automorphism $\mathrm{Ad}\,u_T$ of R (where $u_T = u_1$ is the generator for the group $\{u_n = (u_T)^n : n \in \mathbb{Z}\}$), i.e. $H(T) = H(\mathrm{Ad}\,u_T)$. It was shown in [S] that if S_T is Cartan in R then $H(\mathrm{Ad}\,u_T) = 0$, a result which also follows from Proposition 2.1. A weaker problem than that of the previous section can be phrased as follows.

Conjecture 2. If T is ergodic then the entropy $H(T)$ of T is determined by the conjugacy class of the masa S_T.

We shall show a local result which implies a partial solution to the conjecture. We start with a more general situation. If M is a von Neumann algebra with a von Neumann subalgebra N we denote by $\mathrm{Aut}(M, N)$ the subgroup of $\mathrm{Aut}(M)$ consisting of the automorphisms of M which leave N globally invariant. If $\alpha \in \mathrm{Aut}(M)$ we denote by M^α the fixed point algebra for α in M.

Lemma 4.1 *Let* M *be an injective von Neumann algebra with a faithful normal tracial state* τ*. Let* $\alpha \in \mathrm{Aut}(M)$ *be* τ*-invariant with* $H(\alpha) < \infty$*. Let* $G_\alpha = \{\beta \in \mathrm{Aut}(M, M^\alpha) : \beta \circ \alpha = \alpha \circ \beta, \tau \circ \beta = \tau\}$*. Then the function* τ_α *defined on the projections* $p \in M^\alpha$*,* $p \neq 0$*, by*

$$\tau_\alpha(p) = \tau(p) H(\alpha \mid pMp)\,,$$

extends to a finite normal G_α*-invariant trace on* M^α*.*

Note that the entropy $H(\alpha \mid pMp)$ is taken with respect to the tracial state $\tau_p = \tau(p)^{-1}\tau$ on pMp.

Proof. The proof of the fact that τ_α is countably additive on orthogonal families of projections in M^α can be found in [NS2]. The crucial part is to show finite additivity. The idea is as follows. Let $p_1, \dots, p_n$ be pairwise orthogonal nonzero projections in M^α, and let $p = \sum_{i=1}^{n} p_i$. Let

$$B = p_1 M p_1 + \dots + p_n M p_n$$

By affinity of entropy

$$H(\alpha \mid B) = \sum_{i=1}^{n} \frac{\tau(p_i)}{\tau(p)} H(\alpha \mid p_i M p_i) = \tau(p)^{-1} \sum_{i=1}^{n} \tau_\alpha(p_i)$$

Thus finite additivity follows if we can show $H(\alpha \mid pMp) = H(\alpha \mid B)$. The trace preserving conditional expectation $E_B : pMp \to B$ is given by

$$E_B(x) = \sum_{i=1}^{n} p_i x p_i \ .$$

E_B has finite index, indeed $E_B(x) \geq \frac{1}{n} x$ for $x \geq 0$ in pMp. Thus by [NS1], the assertion $H(\alpha \mid pMp) = H(\alpha \mid B)$ follows.

Let $\beta \in G_\alpha$. Then the systems $(pMp, \tau_p, \alpha \mid pMp)$ and $(\beta(p)M\beta(p), \tau_{\beta(p)}, \alpha \mid \beta(p)M\beta(p))$ are isomorphic, hence have equal entropies. Thus τ_α is G_α-invariant on projections. By [C, Y], see also the arguments in [KR, Ch. 8], τ_α extends to a normal trace on M^α unless possibly M has a type I_2 direct summand. In the latter case τ_α extends at least to a trace on the center Z of M^α. If p is an abelian projection in the I_2-summand let c_p be its central support. Then there is a unitary $u \in M^\alpha$ such that $upu^* = c_p(1-p)$, hence $\tau_\alpha(p) = \tau_\alpha(c_p(1-p))$. Thus

$$\begin{aligned} \tau_\alpha(p) &= \tfrac{1}{2}(\tau_\alpha(p) + \tau_\alpha(c_p(1-p))) = \tfrac{1}{2}\tau_\alpha(c_p) \\ &= \tfrac{1}{2}\tau(h\, c_p) = \tau(h\, p) \ , \end{aligned}$$

where h is the Radon-Nikodym derivative of $\tau_\alpha\big|_Z$ with respect to $\tau|_Z$. It follows that $\tau(h\,\cdot)$ is the desired extension of τ_α. □

Theorem 4.2 *In the notation of Lemma 4.1 assume G_α acts ergodically on the center of M^α. Then*

$$H(\alpha \mid pMp) = H(\alpha) \quad \text{for all nonzero projections } p \in M^\alpha \ .$$

Proof. By Lemma 4.1 there is a positive self-adjoint G_α-invariant operator h affiliated with the center of M^α such that

$$\tau_\alpha(x) = \tau(h\, x) \quad \text{for } x \in M^\alpha \ .$$

Since G_α acts ergodically on the center, h is a scalar operator, hence there is $c > 0$ such that

$$\tau_\alpha(x) = c\,\tau(x)\;, \qquad x \in M^\alpha\;.$$

Then

$$c = \tau_\alpha(1) = \tau(1)H(\alpha) = H(\alpha)\;.$$

Hence we have for a projection $p \in M^\alpha$,

$$H(\alpha \mid pMp) = \tau(p)^{-1}\tau_\alpha(p) = c = H(\alpha)\;.$$

□

Corollary 4.3 *Let M, α, τ be as above. If M^α is a factor then $H(\alpha \mid pMp) = H(\alpha)$ for all nonzero projections $p \in M^\alpha$.*

Corollary 4.4 *Let $R = L^\infty(T) \times_T \mathbb{Z}$ with T ergodic. Then $H(\operatorname{Ad} u_T \mid pRp) = H(\operatorname{Ad} u_T)$ for all nonzero projections $p \in S_T$.*

Proof. The dual automorphisms are all in G_α and act ergodically on $S_T = R^{\operatorname{Ad} u_T}$. □

We next show that classical entropy is a local property in the setting of the crossed product.

Theorem 4.5 *Let $T^{(i)}$ be an ergodic measure preserving transformations of a Lebesgue space (X_i, μ_i), and let $u_{T^{(i)}}$ be the generator for $S_{T^{(i)}}$ implementing $T^{(i)}$, $i = 1, 2$. Suppose there are nonzero projections $p_i \in S_{T^{(i)}}$ and an isomorphism $\gamma : p_1R_1p_1 \to p_2R_2p_2$ such that $\gamma(p_1u_{T^{(1)}}) = \lambda p_2u_{T^{(2)}}$, $\lambda \in \mathbb{C}$, where $R_i = L^\infty(T^{(i)}) \times_{T^{(i)}} \mathbb{Z}$. Then $H(T^{(1)}) = H(T^{(2)})$.*

Proof. We have $H(T^{(i)}) = H(\operatorname{Ad} u_{T^{(i)}})$. By assumption

$$\operatorname{Ad}\gamma(p_1u_{T^{(1)}}) \mid p_2Rp_2 = \operatorname{Ad} p_2u_{T^{(2)}} \mid p_2Rp_2\;.$$

Thus by Corollary 4.4

$$\begin{aligned} H(T^{(1)}) &= H(\operatorname{Ad} u_{T^{(1)}}) = H(\operatorname{Ad} u_{(T^{(1)}} \mid p_1Rp_1) \\ &= H(\operatorname{Ad}\gamma(p_1u_{T^{(1)}}) \mid p_2Rp_2) = (\operatorname{Ad} u_{T^{(2)}} \mid p_2Rp_2) \\ &= H(T^{(2)})\;. \end{aligned}$$

□

The next result is a partial result towards Conjecture 2.

Corollary 4.6 *In the notation of Theorem 4.5 if there is an isomorphism $\gamma : L^\infty(T^{(1)}) \times_{T^{(1)}} \mathbb{Z} \to L^\infty(T^{(2)}) \times_{T^{(2)}} \mathbb{Z}$ such that $\gamma(S_{T^{(1)}}) = S_{T^{(2)}}$ and such that the unitary $\gamma(u_{T^{(1)}})u^*_{T^{(2)}}$ has an eigenvalue, then $H(T^{(1)}) = H(T^{(2)})$.*

Proof. If λ is an eigenvalue then there is a nonzero projection $p \in S_{T^{(2)}}$ such that $\gamma(u_{T^{(1)}})u^*_{T^{(2)}}p = \lambda p$, or $\gamma(u_{T^{(1)}})p = \lambda u_{T^{(2)}}p$. Thus the result follows from Theorem 4.5. □

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