Noncommutative boundaries of q-deformations

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Introduction

The study of random walks on duals of compact quantum groups was initiated by Masaki Izumi in [I1]. The motivation was to compute the relative commutant of the fixed point algebra for product-type actions of compact quantum groups. In the classical case such actions are always minimal, that is, the commutant is trivial. For quantum groups this is not so. It turns out that the relative commutant can be interpreted as the algebra of bounded measurable functions on the Poisson boundary of the dual discrete quantum group. The general theory of noncommutative Poisson boundaries developed in [I1] was illustrated by the computation of the boundary of $\widehat{SU_q(2)}$, which was shown to be isomorphic to the quantum sphere S^2_q. In the present note we discuss the work of Izumi, Tuset and the author [INT], where we computed the Poisson boundary of the dual of $SU_q(n)$ for arbitrary $n \ge 2$.

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1 Main result

Let G be a compact quantum group [W]. In other words, we are given a unital C*-algebra $C(G)$ and a unital homomorphism $\Delta\colon C(G) \to C(G) \otimes C(G)$ such that $(\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta$ and that both $\Delta(C(G))(C(G) \otimes 1)$ and $\Delta(C(G))(1 \otimes C(G))$ are dense in $C(G) \otimes C(G)$. Then we also have a dual discrete quantum group $\hat{G}$ such that the algebra $c_0(\hat{G})$ of functions on $\hat{G}$ vanishing at infinity is the group C*-algebra $C^*(G)$ of G, so

$$c_0(\hat{G}) = \bigoplus_{s \in \mathrm{Irr}(G)} B(H_s),$$

where the sum is over the set $\mathrm{Irr}(G)$ of equivalence classes of irreducible representations of G.

Given a normal state ϕ on $\ell^\infty(\hat{G}) = W^*(G)$, consider the convolution operator P_ϕ on $\ell^\infty(\hat{G})$,

$$P_\phi(x) = (\phi \otimes \iota)\hat{\Delta}(x).$$

Then consider the space

$$H^\infty(\hat{G}, \phi) = \{x \in \ell^\infty(\hat{G}) \mid P_\phi(x) = x\}.$$

of P_ϕ-harmonic elements. A priori this is just a weakly operator closed operator system. But it has a unique von Neumann algebra structure, which is explicitly given by

$$x \cdot y = wo^* - \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} P_\phi^k(xy).$$

The algebra $H^\infty(\hat{G},\phi)$ should be thought of as the algebra of bounded measurable functions on the Poisson boundary of $\hat{G}$ defined by ϕ [I1].

This algebra has a right action of the quantum group $\hat{G}$ coming from the right action by translations of $\hat{G}$ on itself.

There exists a left adjoint action of the quantum group G on $W^*(G)$. We shall only consider states ϕ which are invariant under this action. This gives us an additional symmetry of the Poisson boundary, so $H^\infty(\hat{G},\phi)$ becomes a von Neumann algebra with a left action of G and a right action of $\hat{G}$. Notice that the action of $\hat{G}$ is always ergodic. It turns out that the right action of G is also ergodic, if the fusion algebra, or the representation ring, of G is commutative.

Consider now the group $G = SU_q(n)$, $q \in [-1,1]$. By definition the algebra $C(SU_q(n))$ is generated by n^2 elements u_{ij}, $1 \le i,j \le n$, satisfying the relations

$$u_{ik}u_{jk} = qu_{jk}u_{ik},\quad u_{ki}u_{kj} = qu_{kj}u_{ki}\ \text{ for } i<j,$$
$$u_{il}u_{jk} = u_{jk}u_{il}\ \text{ for } i<j,\ k<l,$$
$$u_{ik}u_{jl} - u_{jl}u_{ik} = (q-q^{-1})u_{jk}u_{il}\ \text{ for } i<j,\ k<l,$$
$$\det{}_q(U) = 1,$$

where $U = (u_{ij})_{i,j}$ and $\det_q(U) = \sum_{w\in S_n}(-q)^{\ell(w)}u_{w(1)1}\dots u_{w(n)n}$, with $\ell(w)$ being the number of inversions in $w \in S_n$. The involution on $C(SU_q(n))$ is given by

$$u_{ij}^* = (-q)^{j-i}\det{}_q(U^{\hat{i}}_{\hat{j}}),$$

where $U^{\hat{i}}_{\hat{j}}$ is the matrix obtained from U by removing the ith row and the jth column. The comultiplication is given by

$$\Delta(u_{ij}) = \sum_k u_{ik}\otimes u_{kj}.$$

Denote by T the maximal torus in $SU_q(n)$ and consider the homogeneous space $SU_q(n)/T$. More explicitly, for $(t_1,\dots,t_n) \in \mathbb{T}^n$ such that $t_1\dots t_n = 1$ define an automorphism of $C(SU_q(n))$ by

$$u_{ij} \mapsto t_j u_{ij}.$$

This way we get an action of $T \cong \mathbb{T}^{n-1}$ on $C(SU_q(n))$, and $C(SU_q(n)/T)$ is the fixed point algebra for this action.

We can now formulate our main result.

Theorem *If $0<q<1$, then for any left $SU_q(n)$-invariant normal generating state on $\ell^\infty(\widehat{SU_q(n)})$, the Poisson boundary of $\widehat{SU_q(n)}$ is $SU_q(n)$- and $\widehat{SU_q(n)}$-equivariantly isomorphic to the quantum flag manifold $SU_q(n)/T$.*

Here the left action of $SU_q(n)$ on $SU_q(n)/T$ is the action by translations. The right action of $\widehat{SU_q(n)}$ comes from the right adjoint action of $\widehat{SU_q(n)}$ on $C(SU_q(n)) = C^*(\widehat{SU_q(n)})$. However, a more correct way of thinking of this action is to consider it as a quantum analogue of dressing transformations, see e.g. [KoS]. In the classical case the orbits of the action by dressing transformations are leaves of the canonical Poisson structure. The flag manifold has one open dense leave, the Schubert cell, so that the action is ergodic. This makes it easier to believe that the action of $\widehat{SU_q(n)}$ on the quantum flag manifold is ergodic, which should be the case if our result is true.

The above theorem says in particular that the Poisson boundary of $\widehat{SU_q(n)}$ does not depend on the generating state. This in fact can be shown without actually computing the boundary: if the fusion algebra of G is commutative, then the space $H^\infty(\hat{G},\phi)$ is the same for any G-invariant generating state.

2 Poisson integral and Berezin transform

For any compact quantum group G there exists a unique normal unital G- and $\hat{G}$-equivariant map Θ from $L^\infty(G)$ into $\ell^\infty(\hat{G})$. Explicitly,

$$\Theta = (\varphi \otimes \iota)\hat{\Phi},$$

where φ is the Haar state on $L^\infty(G)$ and $\hat{\Phi}\colon L^\infty(G) \to L^\infty(G) \otimes \ell^\infty(\hat{G})$ is the right adjoint action of $\hat{G}$ on G, which we discussed above. This map was introduced in [I1] to show that an isomorphism between S_q^2 and the Poisson boundary of $\widehat{SU_q(2)}$ is $\widehat{SU_q(2)}$-equivariant. It was also shown in [I1] that for any G and any normal left G-invariant state on $\ell^\infty(\hat{G})$ the image of this map is contained in $H^\infty(\hat{G}, \phi)$. Thus if we expect the Poisson boundary of $\hat{G}$ to be a homogeneous space G/H, then this map should be an isomorphism of $L^\infty(G/H)$ onto $H^\infty(\hat{G}, \phi)$. But the only thing we can say in general is that this map is completely positive. We call this map the Poisson integral.

Recall that given von Neumann algebras N_1 and N_2, normal faithful states ν_1 and ν_2 on N_1 and N_2, respectively, and a normal unital completely positive map $\theta\colon N_1 \to N_2$ such that $\nu_2\theta = \nu_1$ and $\sigma_t^{\nu_2}\theta = \theta\sigma_t^{\nu_1}$, there exists a normal unital completely positive map $\theta^*\colon N_2 \to N_1$ such that

$$\nu_2(\theta(x_1)x_2) = \nu_1(x_1\theta^*(x_2)) \text{ for } x_1 \in N_1,\ x_2 \in N_2.$$

Then by [P] an element $x \in N_1$ is in the multiplicative domain of θ if and only if $(\theta^*\theta)(x) = x$.

We want to apply this criterion to our map Θ. For this we need to compute Θ^*. Let $\Theta_s\colon L^\infty(G) \to B(H_s)$, $s \in \mathrm{Irr}(G)$, be the components of Θ. Explicitly,

$$\Theta_s(a) = (\varphi \otimes \iota)(U^s(a \otimes 1)U^{s*}),$$

where $U^s \in C(G) \otimes B(H_s)$ is a fixed representative of the equivalence class $s \in \mathrm{Irr}(G)$ of irreducible representations of G. The representation U^s defines two adjoint actions of G on $B(H_s)$. There exist a unique left-invariant state ϕ_s and a unique right-invariant state ω_s on $B(H_s)$ (in the classical case they both coincide with the normalized trace). Then it is not difficult to check that the adjoint Θ_s^* for $\Theta_s\colon (L^\infty(G), \varphi) \mapsto (B(H_s), \phi_s)$ is given by

$$\Theta_s^*(x) = (\iota \otimes \omega_s)(U^{s*}(1 \otimes x)U^s).$$

On $H^\infty(\hat{G}, \phi)$ there is a canonical state ν_0 given by the restriction of the counit $\hat{\varepsilon}$, that is, by "the evaluation at the unit of $\hat{G}$". Then it can be shown that the adjoint Θ^* to $\Theta\colon (L^\infty(G), \varphi) \to (H^\infty(\hat{G}, \phi), \nu_0)$ is

$$\Theta^*(x) = s^* - \lim_{n\to\infty} \sum_{s\in\mathrm{Irr}(G)} \phi^n(I_s)\Theta_s^*(x),$$

where I_s is the unit in $B(H_s)$ and ϕ^n is the nth convolution power of ϕ. Thus, if we denote $\Theta_s^*\Theta_s$ by B_s, an element $a \in L^\infty(G)$ lies in the multiplicative domain of the Poisson integral if and only if

$$\sum_{s\in\mathrm{Irr}(G)} \phi^n(I_s)B_s(a) \to a.$$

It turns out that the operators Θ_s and Θ_s^* are analogues of well-known classical constructions [Be, Per]. Let for the moment G be a compact Lie group, $U\colon G \to B(H)$ a finite dimensional unitary representation. Fix a vector $\xi \in H$, $\|\xi\| = 1$. Let $T \subset G$ be the stabilizer of the line $\mathbb{C}\xi$. For an operator $S \in B(H)$, its covariant Berezin symbol $\sigma(S)$ is defined by $\sigma(S)(g) = (SU_g\xi, U_g\xi)$. The covariant symbol σ is a G-equivariant map from $B(H)$ into $C(G/T)$. Consider the inner products on $C(G/T)$ and $B(H)$ given by the G-invariant probability measure and the normalized trace, respectively. Then there exists an adjoint $\breve{\sigma}\colon C(G/T) \to B(H)$ of σ. Explicitly,

$$\breve{\sigma}(f) = d\int f(g)U_gP_\xi U_g^* dg,$$

where $d = \dim H$ and P_ξ is the projection onto $\mathbb{C}\xi$. A function f is called a contravariant Berezin symbol of $\breve{\sigma}(f)$. The map $B = \sigma\breve{\sigma}$ is called the Berezin transform.

If we consider U as a corepresentation of $C(G)$, then the definition of σ can be written as

$$\sigma(S) = (\iota \otimes \omega_\xi)(U^*(1 \otimes S)U),$$

where $\omega_\xi = (\cdot\, \xi, \xi)$ is the vector-state defined by ξ. Thus we see that our operator Θ_s^* is just σ with ω_ξ replaced by ω_s. Then $\Theta_s = (\Theta_s^*)^*$ is an analogue of $\breve{\sigma}$.

Suppose now that G is a semisimple Lie group, $U = U^\lambda \colon G \to B(H_\lambda)$ an irreducible representation with highest weight λ, $\xi = \xi_\lambda$ a highest weight vector. Let B_λ be the corresponding Berezin transform. It is proved in [D] that the sequence $\{B_{n\lambda}\}_{n=1}^\infty$ converges to the identity on $C(G/T)$ as $n \to \infty$. This is a key step to show that the full matrix algebras $B(H_{n\lambda})$, $n \in \mathbb{N}$, provide a quantization of $C(G/T)$, see [L, R]. In other words, in the classical case the Berezin transforms converge to the identity operator on the flag manifold along any ray in the Weyl chamber. While to prove multiplicativity of the Poisson integral we have to show that certain convex combinations of quantum Berezin transforms defined by a random walk on the Weyl chamber converge to the identity operator on the quantum flag manifold.

The proof in the classical case is based on the observation that the operators $B_{n\lambda}$ are given by convolution with measures which are absolutely continuous with respect to the Haar measure and such that the Radon-Nikodym derivatives are, up to normalization, powers of a single function h such that $h(g) = 1$ for $g \in T$ and $h(g) < 1$ for $g \notin T$. The proof of our q-analogue will be based on the study of ergodic properties of an auxiliary operator.

Since the maps B_s are G-equivariant, it suffices to check the convergence "at the unit element of G". More precisely, assuming that the counit ε is well-defined on $C(G)$, that is, the group G is coamenable, a subalgebra $C(G/H)$ is in the multiplicative domain of Θ if and only if

$$\sum_s \phi^n(I_s)(\varepsilon B_s)(a) \to \varepsilon(a)$$

for any $a \in C(G/H)$. We have

$$(\varepsilon B_s)(a) = (\omega_s \Theta_s)(a) = (\varphi \otimes \omega_s)(U^s(a \otimes 1)U^{s*}).$$

Consider the operator $A_{\omega_s} \colon C(G) \to C(G)$ defined by

$$A_{\omega_s}(a) = (\iota \otimes \omega_s)(U^s(a \otimes 1)U^{s*}).$$

Then $\varepsilon B_s = \varphi A_{\omega_s}$, and we arrive at the following criterion.

Proposition *Assume G is coamenable. For a normal left G-invariant state ϕ consider the corresponding right G-invariant state $\omega = \sum_s \phi(I_s)\omega_s$ and the operator $A_\omega \colon C(G) \to C(G)$,*

$$A_\omega = \sum_s \phi(I_s) A_{\omega_s}.$$

Then the states φA_ω^n converge to a state on $C(G)$ as $n \to \infty$. For a subgroup $H \subset G$ the algebra $C(G/H)$ is in the multiplicative domain of $\Theta \colon L^\infty(G) \to H^\infty(\hat{G}, \phi)$ if and only if the limit state coincides with the counit ε on $C(G/H)$.

Consider now $G = SU_q(2)$. Then

$$U = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} = \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix},$$

and the relations can be written as

$$\alpha^*\alpha+\gamma^*\gamma=1,\quad \alpha\alpha^*+q^2\gamma^*\gamma=1,\quad \gamma^*\gamma=\gamma\gamma^*,\quad \alpha\gamma=q\gamma\alpha,\quad \alpha\gamma^*=q\gamma^*\alpha.$$

The comultiplication Δ is determined by the formulas

$$\Delta(\alpha)=\alpha\otimes\alpha-q\gamma^*\otimes\gamma,\quad \Delta(\gamma)=\gamma\otimes\alpha+\alpha^*\otimes\gamma.$$

Since the Poisson boundary does not depend on the generating state, it suffices to consider the state ϕ corresponding to the fundamental representation U. So

$$\phi=\frac{1}{q+q^{-1}}\mathrm{Tr}\left(\cdot\begin{pmatrix} q & 0\\ 0 & q^{-1}\end{pmatrix}\right),\quad \omega=\frac{1}{q+q^{-1}}\mathrm{Tr}\left(\cdot\begin{pmatrix} q^{-1} & 0\\ 0 & q\end{pmatrix}\right),$$

and thus

$$\begin{aligned} A_\omega(a) &= \frac{1}{q+q^{-1}}\mathrm{Tr}\left(\begin{pmatrix}\alpha & -q\gamma^*\\ \gamma & \alpha^*\end{pmatrix}\begin{pmatrix} a & 0\\ 0 & a\end{pmatrix}\begin{pmatrix}\alpha^* & \gamma^*\\ -q\gamma & \alpha\end{pmatrix}\begin{pmatrix} q^{-1} & 0\\ 0 & q\end{pmatrix}\right)\\ &= \frac{1}{q+q^{-1}}(q^{-1}(\alpha a\alpha^*+q^2\gamma^*a\gamma)+q(\gamma a\gamma^*+\alpha^*a\alpha)).\end{aligned}$$

We see in particular that A_ω commutes with the left and the right actions of $T\cong\mathbb{T}$ given by $\alpha\mapsto z\alpha$, $\gamma\mapsto\bar z\gamma$ and $\alpha\mapsto z\alpha$, $\gamma\mapsto z\gamma$, respectively. It follows that the limit of the states φA_ω^n is also invariant with respect to these actions. Though the counit is not invariant on the whole group, notice that in general ε is invariant with respect to the left action of H on G/H, as well as with respect to the right action of H on $H\backslash G$. It follows that to prove multiplicativity of the Poisson integral on $C(SU_q(2)/T)$ it suffices to prove that the limit state coincides with ε on $C(T\backslash SU_q(2)/T)$. The latter algebra is generated by $\gamma^*\gamma$. It is known that the spectrum of $\gamma^*\gamma$ is the set $I_{q^2}=\{0\}\cup\{q^{2n}\}_{n=0}^\infty$. Thus we can identify $C(T\backslash SU_q(2)/T)$ with the algebra $C(I_{q^2})$ of continuous functions on I_{q^2}. Since the counit is a character such that $\alpha\mapsto 1$ and $\gamma\mapsto 0$, under this identification it is given by the evaluation at $0\in I_{q^2}$. Furthermore, using the identities

$$\alpha^*(\gamma^*\gamma)^k\alpha=q^{-2k}(\gamma^*\gamma)^k(1-\gamma^*\gamma),\quad \alpha(\gamma^*\gamma)^k\alpha^*=q^{2k}(\gamma^*\gamma)^k(1-q^2\gamma^*\gamma),$$

we see that the action of A_ω on the functions on I_{q^2} is given by

$$(A_\omega f)(t)=\frac{1}{q+q^{-1}}\Big(q^{-1}\Big((1-q^2t)f(q^2t)+q^2tf(t)\Big)+q\Big(tf(t)+(1-t)f(q^{-2}t)\Big)\Big).$$

In other words, A_ω is the Markov operator with transition probabilities $p(0,0)=1$,

$$\begin{aligned} p(q^{2n},q^{2(n-1)}) &= \frac{q-q^{2n+1}}{q+q^{-1}},\\ p(q^{2n},q^{2n}) &= \frac{2q^{2n+1}}{q+q^{-1}},\\ p(q^{2n},q^{2(n+1)}) &= \frac{q^{-1}-q^{2n+1}}{q+q^{-1}}.\end{aligned}$$

It is not difficult to show that the restriction of this random walk to $I_{q^2}\backslash\{0\}$ is transient. In particular, $(A_\omega^n\delta_x)(y)\to 0$ for any $x,y\in I_{q^2}\backslash\{0\}$. Hence for any $f\in C(I_{q^2})$ the functions A_ω^nf converge pointwise to the constant $f(0)$. This completes the proof of multiplicativity of the Poisson integral on $C(SU_q(2)/T)$.

For $n > 2$ we prove that ε is the only A_ω-invariant state on $C(SU_q(n)/T)$. The proof is by induction on n, and it turns out that for the induction step it suffices to check that ε is the only A_ω-invariant state on

$$C(S(U_q(n-1)\times\mathbb{T})\backslash SU_q(n)/S(U_q(n-1)\times\mathbb{T}))\cong C(T\backslash SU_q(2)/T),$$

which is already established.

3 Random walk on the center

In the previous section we sketched an argument for multiplicativity of the Poisson integral. Since the Haar state is faithful, we also automatically have injectivity of the Poisson integral on its multiplicative domain. We shall next discuss surjectivity.

We need an estimate on the dimensions of the spectral subspaces of $H^\infty(\hat{G},\phi)$. By a result of Hayashi [H], which we have already mentioned, if the fusion algebra of a group G is commutative, the action of G on the Poisson boundary is ergodic. This already implies that the spectral subspaces of $H^\infty(\hat{G},\phi)$ are finite dimensional.

To obtain a bound on their multiplicities, note that ergodicity of the action of G on $H^\infty(\hat{G},\phi)$ is equivalent to triviality of the Poisson boundary of the restriction of P_ϕ to the center of $\ell^\infty(\hat{G})$, since the center is exactly the fixed point algebra $\ell^\infty(\hat{G})^G$. Identify the center with $\ell^\infty(I)$, where $I=\mathrm{Irr}(G)$. For a fixed generating state ϕ, let $\{p(s,t)\}_{s,t\in I}$ be the transition probabilities defined by the restriction of P_ϕ to $\ell^\infty(I)$, so $P_\phi(I_t)I_s=p(s,t)I_s$. Let $(\Omega,\mathbb{P}_0)$ be the path space of the corresponding random walk,

$$\Omega=\prod_{n=1}^{\infty}I,\quad \mathbb{P}_0(\{\underline{s}\,|\,s_1=t_1,\dots,s_n=t_n\})=p(0,t_1)p(t_1,t_2)\dots p(t_{n-1},t_n).$$

Denote by π_n the projection $\Omega\to I$ onto the nth factor. Let $E\colon\ell^\infty(\hat{G})\to\ell^\infty(I)$ be the unique G-equivariant conditional expectation, $E(x)(s)=\phi_s(x)$. If $x\in\ell^\infty(\hat{G})$ is P_ϕ-harmonic, then

$$x^*x=P_\phi(x)^*P_\phi(x)\le P_\phi(x^*x),$$

whence $E(x^*x)\le P_\phi(E(x^*x))$. It follows that the sequence $\{\pi_n(E(x^*x))\}_n$ is a submartingale. Hence it converges a.e. But since the Poisson boundary of the center is trivial, the limit must be a constant. This constant is $\nu_0(x^*\cdot x)$. Thus we get the following result.

Proposition *Let ϕ be a normal left G-invariant generating state on $\ell^\infty(\hat{G})$. Assume that the Poisson boundary of the center is trivial. Then for any $x,y\in H^\infty(\hat{G},\phi)$ and almost every path $\underline{s}\in\Omega$, we have $\phi_{s_n}(xy)\to\nu_0(x\cdot y)$ as $n\to\infty$. In other words, the completely positive maps $(H^\infty(\hat{G},\phi),\nu_0)\to(B(H_{s_n}),\phi_{s_n})$ are asymptotically isometric in L^2-norm.*

In particular, for any irreducible representation V of G the multiplicity of V in $H^\infty(\hat{G},\phi)$ is not larger than the supremum of the multiplicities of V in $\bar{U}\times U$ for all irreducible representations U of G.

For the q-deformation G of a compact connected semisimple Lie group the last estimate is optimal, see e.g. [Ž, §131]. For example, for $SU_q(2)$ if we take the spin $s\in\frac{1}{2}\mathbb{Z}_+$ representation U^s then $\bar{U}^s\cong U^s$, and $U^s\times U^s$ is isomorphic to the direct sum of U^t, $t=0,1,\dots,2s$. On the other hand we know that only integer spin representations appear in $C(SU_q(2)/T)$, each with multiplicity one.

Thus if G is the q-deformation of a compact connected semisimple Lie group, $T\subset G$ the maximal torus, ϕ a generating state, then the Poisson integral $\Theta\colon L^\infty(G/T)\to H^\infty(\hat{G},\phi)$ is an isomorphism as soon as it is multiplicative.

4 Conclusion

A significant part of our considerations is valid for q-deformations of arbitrary compact connected semisimple Lie groups. The only point where we used that the group was $SU_q(n)$ is the induction step in the proof of multiplicativity of the Poisson integral. It is based on a quantum analogue of the fact that $S(U(n-1)\times\mathbb{T})\subset SU(n)$ is a Riemannian symmetric pair of rank one. Hence there is hope that similar considerations could work for $SO(n)$, $Sp(n)$ and F_4, see [He, Ch. X, Table V]. Nevertheless for the exceptional groups E_6, E_7, E_8 and G_2 this approach definitely requires more computations than was the case for A_n. So it would be desirable to find a universal method. A more difficult and interesting problem is to compute the Martin boundary.

The Poisson boundary of a noncompact semisimple Lie group G was computed by Furstenberg [Fu], who showed that it is isomorphic to the flag manifold $B=G/P$, where P is the minimal parabolic subgroup of G. The Martin boundary of G, or of the corresponding symmetric space $S=G/K$, was computed by Olshanetsky, who announced the result more than thirty years ago [Ol1], but a detailed proof appeared only recently [Ol2, GJT]. The result says that the Martin compactification is the minimal compactification which dominates both the visibility compactification and the Furstenberg compactification. The latter is defined as the closure of S in $M_1(B)$, the space of probability measures on $B=G/P$, under the map $gx_0\mapsto gm$, where m is the unique K-invariant probability measure on B (note that quite confusingly the boundary of S in this compactification does not always coincide with the Furstenberg boundary B, but in the rank one case they both do coincide with the sphere at infinity, which is the boundary in the visibility compactification). More concretely the Furstenberg compactification can be obtained by embedding S in the projective space of matrices using certain irreducible representations of G. The results of Furstenberg were extended to arbitrary (connected) Lie groups by Raugi [R]. On the other hand, the problem of computing Martin boundaries even for solvable Lie groups remains unsettled.

The rank one case considered in [NT1] is not sufficient to formulate a plausible conjecture about the Martin boundary of the dual of the q-deformation of a compact semisimple Lie group. The works of Biane [B] and Collins [C] suggest that the minimal boundary is a quantization of the sphere in the dual of the Lie algebra. Since we now know that the Poisson boundary is the quantization of the Furstenberg boundary, it is natural to conjecture that to compute the whole Martin boundary one should look for quantizations of classical Martin boundaries.

The basis for the above conjecture is that the Poisson boundary of $\widehat{SU_q(n)}$ is the quantization of the Poisson boundary of $SL_n(\mathbb{C})$. For the moment we do not have a good explanation of this phenomenon. To really understand it we need a better understanding of what happens when $q=1$. The Poisson boundary of $\widehat{SU(n)}$, and in fact of the dual of any compact group, is trivial. However, the limit $q\to 1$ should rather be considered in the category of Hopf-Poisson algebras. In other words, the classical limit of $\widehat{SU_q(n)}$ is not the Pontryagin dual $\widehat{SU(n)}$, but the Poisson dual of $SU(n)$. Thus the question is whether Furstenberg's results have dual Poisson analogues. This could also clarify the appearance of the Berezin transform in our work [INT], since to our knowledge it has not played any role in the classical picture.

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