Poisson boundary of the dual of $SU_q(n)$

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Abstract

We prove that for any non-trivial product-type action α of $SU_q(n)$ $(0 < q < 1)$ on an ITPFI factor N, the relative commutant $(N^\alpha)' \cap N$ is isomorphic to the algebra $L^\infty(SU_q(n)/\mathbb{T}^{n-1})$ of bounded measurable functions on the quantum flag manifold $SU_q(n)/\mathbb{T}^{n-1}$. This is equivalent to the computation of the Poisson boundary of the dual discrete quantum group $\widehat{SU_q(n)}$. The proof relies on a connection between the Poisson integral and the Berezin transform. Our main technical result says that a sequence of Berezin transforms defined by a random walk on the dominant weights of $SU(n)$ converges to the identity on the quantum flag manifold.

Introduction

The study of random walks on duals of compact groups, by which is meant the study of convolution operators on group von Neumann algebras of compact groups, was initiated by Biane in [B1], who developed a theory which parallels the theory of random walks on discrete abelian groups [B2, B3]. The study of random walks in the more general setting of duals of compact quantum groups [W] was undertaken by the first author [I1], who was motivated by product-type actions of such groups on infinite tensor products of factors of type I (ITPFI). In the classical case such actions are always minimal. For quantum groups this is not so. In fact, the relative commutant of the fixed point algebra can be interpreted as the algebra of bounded measurable functions on the Poisson boundary of the dual discrete quantum group. The general theory of noncommutative Poisson boundaries developed in [I1] was illustrated by the computation of the boundary of $\widehat{SU_q(2)}$, which was shown to be isomorphic to the quantum sphere S^2_q. Later the second and the third author [NT1] computed the Martin boundary of $\widehat{SU_q(2)}$, that is, described all (unbounded) harmonic functions, thus generalizing the results in [B3] and establishing a connection between the results in [B3] and [I1].

There are important differences between the classical boundary theory (or even that for duals of compact groups as considered by Biane) and the boundary theory for discrete quantum groups, which we want to stress. In the classical setting, if one forgets about the action of a group on its boundary, the computation of the Poisson boundary reduces to the question whether it is trivial or not. In the quantum case just the description of the (noncommutative) algebra of functions on the Poisson boundary is a highly nontrivial problem. On the other hand, such a central question within the classical theory as the description of minimal harmonic functions, becomes of peripheral interest in the quantum setting. We refer the reader to the survey articles [I2] and [NT2] for further discussions of noncommutative boundaries.

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The purpose of the present paper is to compute the Poisson boundary of $\widehat{SU_q(n)}$. As we already said, this problem can be formulated without any reference to random walks. Suppose one is given a product-type action α of $SU_q(n)$ on N. By restriction we get an action of $SU_q(n)$ on $(N^\alpha)'\cap N$. If the action α is faithful, then $(N^\alpha)'\cap N$ is anti-isomorphic to $N'\cap(SU_q(n)\ltimes N)$. In this case the dual action of $\widehat{SU_q(n)}$ on $N'\cap(SU_q(n)\ltimes N)$ induces an action on $(N^\alpha)'\cap N$. The problem is to compute $(N^\alpha)'\cap N$ together with the two above actions of $SU_q(n)$ and $\widehat{SU_q(n)}$. Our main result can be stated in this setting as follows.

Theorem A *For any non-trivial product-type action α of $SU_q(n)$ $(0<q<1)$ on an ITPFI factor N, the relative commutant $(N^\alpha)'\cap N$ is $SU_q(n)$-equivariantly isomorphic to the algebra $L^\infty(SU_q(n)/\mathbb{T}^{n-1})$ of bounded measurable functions on the quantum flag manifold $SU_q(n)/\mathbb{T}^{n-1}$. When the action of $\widehat{SU_q(n)}$ on $(N^\alpha)'\cap N$ is well-defined, the isomorphism is also $\widehat{SU_q(n)}$-equivariant.*

As for the boundary interpretation of $(N^\alpha)'\cap N$, the action of $\widehat{SU_q(n)}$ corresponds to the usual action by translations of a group on its boundary, while the action of $SU_q(n)$ can be thought of as coming from the symmetry group of the measure. Then an equivalent form of Theorem A is the following.

Theorem B *Let G be the q-deformation $(0<q<1)$ of the compact group $SU(n)$, or of its quotient by a normal subgroup. Let $T\subset G$ be the maximal torus. Then for any G-invariant normal generating state on $l^\infty(\hat G)$, the corresponding Poisson boundary of $\hat G$ is G- and $\hat G$-equivariantly isomorphic to the quantum flag manifold G/T.*

There are several heuristic reasons why such a result should be true. For example, by the results of Biane [B2] and Collins [C] the minimal Martin boundary of the dual of $SU(n)$ is the sphere in the dual of the Lie algebra, and the action of $SU(n)$ on the boundary is just the coadjoint action. Thus one can expect that a part of the Martin boundary of $\widehat{SU_q(n)}$ carrying the canonical harmonic state is a certain quantization of the sphere. By [H] the action of $SU_q(n)$ on the Poisson boundary of $\widehat{SU_q(n)}$ is ergodic. This corresponds in the classical case to the fact that the measure is supported on some orbit, and indeed, the typical coadjoint orbit is the flag manifold.

Since, however, the Martin boundary seems difficult to compute, we shall not follow the approach suggested above. Our computation of the Poisson boundary is based on a careful study of the completely positive map Θ from $L^\infty(SU_q(n))$ into the algebra of harmonic functions introduced in [I1]. We call this map the Poisson integral. We show that multiplicativity of Θ restricted to $L^\infty(SU_q(n)/\mathbb{T}^{n-1})$ is equivalent to convergence to the identity of a certain sequence of operators on $L^\infty(SU_q(n)/\mathbb{T}^{n-1})$. These operators are analogues of Berezin transforms. In the classical case it is known [D] that Berezin transforms converge to the identity on the flag manifold along any ray in the Weyl chamber. We prove that our operators converge to the identity on the quantum flag manifold along almost every path in the Weyl chamber. It is worth noticing that though we benefited from these analogies, our proof bears no relation to the classical proof. As a matter of fact, our operators are not the most straightforward analogues of Berezin transforms. In particular, in the classical limit they give operators mapping everything to the scalars. Our proof of convergence invokes yet another auxiliary operator, which in the classical limit yields the identity operator, whereas in the quantum case it is uniquely ergodic on $C(SU_q(n)/\mathbb{T}^{n-1})$.

Multiplicativity of Θ on $L^\infty(SU_q(n)/\mathbb{T}^{n-1})$ implies injectivity. This can be interpreted as the existence of a surjective map from the Poisson boundary onto the quantum flag manifold. To show that this is an isomorphism we then use a counting argument relying on the already mentioned ergodicity of the action of $SU_q(n)$ on the Poisson boundary.

Finally note that our approach gives a less computational proof of the result for $SU_q(2)$ than in [I1] and [NT1]. For example, we do not need any knowledge of the Clebsch-Gordan coefficients.

1 Preliminaries

We will use the same conventions for quantum groups as in [NT1]. We will, however, change the notation slightly to hopefully make it more transparent. So a compact quantum group G is defined by a unital C*-algebra $C(G)$ with comultiplication $\Delta\colon C(G)\to C(G)\otimes C(G)$, which is a unital $*$-homomorphism such that $(\Delta\otimes\iota)\Delta=(\iota\otimes\Delta)\Delta$ and both $\Delta(C(G))(C(G)\otimes 1)$ and $\Delta(C(G))(1\otimes C(G))$ are dense in $C(G)\otimes C(G)$. We will always work in the reduced setting, that is, we assume that the Haar state φ on $C(G)$ is faithful. In cases when it is more convenient to deal with von Neumann algebras, we shall use the notation $L^\infty(G)$ for $\pi_\varphi(C(G))''$.

By a unitary representation of G on a Hilbert space H we mean a unitary corepresentation of $(C(G),\Delta)$, that is, a unitary $U\in M(C(G)\otimes B_0(H))$ such that $(\Delta\otimes\iota)(U)=U_{13}U_{23}$. Here $B_0(H)$ denotes the algebra of compact operators on H. Let $I=\mathrm{Irr}(G)$ be the set of equivalence classes of irreducible unitary representations of G. For each $s\in I$, we fix a representative $U^s\in C(G)\otimes B(H_s)$. Then the algebra $c_0(\hat G)$ of functions on the dual discrete quantum group $\hat G$ vanishing at infinity is defined as the C*-direct sum of $B(H_s)$, $s\in I$. The algebra $l^\infty(\hat G)$ is defined as the W*-direct sum of $B(H_s)$, $s\in I$.

Let $\mathcal{A}(G)$ be the $*$-subalgebra of $C(G)$ generated by the matrix coefficients of finite dimensional unitary representations of G. Let also $\mathcal{A}(\hat G)\subset c_0(\hat G)$ be the algebraic direct sum of $B(H_s)$, $s\in I$. There is a pairing between $\mathcal{A}(G)$ and $\mathcal{A}(\hat G)$. If we fix matrix units $\{m^s_{ij}\}_{i,j}$ in $B(H_s)$, and denote by $\{u^s_{ij}\}_{i,j}$ the corresponding matrix coefficients of U^s, then the pairing is given by $(u^s_{ij},m^t_{kl})=\delta_{st}\delta_{ik}\delta_{jl}$. For a unitary representation U of G on H, the corresponding representation $\pi_U\colon l^\infty(\hat G)\to B(H)$ is given by

$$\pi_U(\omega)=(\omega\otimes\iota)(U).$$

In particular, if $W\in B(H_\varphi\otimes H_\varphi)$ is the multiplicative unitary for G,

$$W^*(\xi\otimes a\xi_\varphi)=\Delta(a)(\xi\otimes\xi_\varphi),\quad \xi\in H_\varphi,\quad a\in C(G),$$

then we get a faithful representation $\pi_W\colon l^\infty(\hat G)\to B(H_\varphi)$. Thus we can (and often will) think of $L^\infty(G)$ and $l^\infty(\hat G)$ as subalgebras of $B(H_\varphi)$. Note that the comultiplications $\Delta\colon L^\infty(G)\to L^\infty(G)\otimes L^\infty(G)$ and $\hat\Delta\colon l^\infty(\hat G)\to l^\infty(\hat G)\otimes l^\infty(\hat G)$ are then given by

$$\Delta(a)=W^*(1\otimes a)W,\quad \hat\Delta(x)=W(x\otimes 1)W^*.$$

Given a unitary representation U of G on H, we define a left and a right action of G on $B(H)$ by

$$\alpha_{U,l}\colon B(H)\to L^\infty(G)\otimes B(H),\quad \alpha_{U,l}(x)=U^*(1\otimes x)U,$$
$$\alpha_{U,r}\colon B(H)\to B(H)\otimes L^\infty(G),\quad \alpha_{U,r}(x)=U_{21}(x\otimes 1)U^*_{21}.$$

As $l^\infty(\hat G)=\oplus_s B(H_s)$, we get in particular a left and a right action of G on $l^\infty(\hat G)$, which we denote by Φ_l and Φ_r, respectively. Thinking of $l^\infty(\hat G)$ as the group von Neumann algebra of G, they are analogues of the adjoint action. If the representation U is finite dimensional, then the actions of G on $B(H)$ have canonical invariant states

$$\phi_U=\frac{1}{d_U}\mathrm{Tr}(\cdot\,\pi_U(\rho^{-1})),\quad (\iota\otimes\phi_U)\alpha_{U,l}=\phi_U(\cdot)1,$$

$$\omega_U=\frac{1}{d_U}\mathrm{Tr}(\cdot\,\pi_U(\rho)),\quad (\omega_U\otimes\iota)\alpha_{U,r}=\omega_U(\cdot)1,$$

where

$$\rho=f_1$$

is the Woronowicz character and $d_U = \mathrm{Tr}(\pi_U(\rho^{-1})) = \mathrm{Tr}(\pi_U(\rho))$ is the quantum dimension of U. We will write ϕ_s and ω_s instead of ϕ_{U^s} and ω_{U^s}, respectively. In general, all normal left (resp. right) G-invariant states on $B(H)$ are given by $\mathrm{Tr}(\cdot a\pi_U(\rho^{-1}))$ (resp. by $\mathrm{Tr}(\cdot a\pi_U(\rho))$), where $a \in \pi_U(l^\infty(\hat{G}))'$ is a positive element such that $\mathrm{Tr}(a\pi_U(\rho^{-1})) = 1$ (resp. $\mathrm{Tr}(a\pi_U(\rho)) = 1$).

Denote by $\mathcal{C}_l(\hat{G})$ (resp. by $\mathcal{C}_r(\hat{G})$) the space of normal left (resp. right) G-invariant functionals on $l^\infty(\hat{G})$. Then $\mathcal{C}_l(\hat{G})$ (resp. $\mathcal{C}_r(\hat{G})$) is the closed linear span of ϕ_s (resp. ω_s), $s \in I$. The space $\mathcal{C}_l(\hat{G})$ (as well as $\mathcal{C}_r(\hat{G})$) is an algebra with product $\phi_1\phi_2 = (\phi_1 \otimes \phi_2)\hat{\Delta}$. Alternatively, one can define a fusion algebra structure on $R(G) = \oplus_{s\in I}\mathbb{Z}$ in the sense of [HI]. Then the quantum dimension function on $R(G)$ defines a convolution algebra $l^1(I)$, which is isomorphic to $\mathcal{C}_l(\hat{G})$ and $\mathcal{C}_r(\hat{G})$. The algebra $\mathcal{C}_l(\hat{G})$ has an anti-linear involution $\phi \mapsto \check{\phi}$ such that $\check{\phi}_U = \phi_{\bar{U}}$, where $\overline{U}$ is the conjugate unitary representation. Let also $s \mapsto \bar{s}$ be the involution on I, so $U^{\bar{s}} \cong \overline{U^s}$.

Given a normal state $\phi \in l^\infty(\hat{G})_*$, define the convolution operator $P_\phi = (\phi \otimes \iota)\hat{\Delta}$ on $l^\infty(\hat{G})$, and set

$$H^\infty(\hat{G}, \phi) = \{x \in l^\infty(\hat{G}) \,|\, P_\phi(x) = x\}.$$

Then $H^\infty(\hat{G}, \phi)$ is (the algebra of bounded measurable functions on) the Poisson boundary of $\hat{G}$ with respect to ϕ. According to [I1] the algebra structure on $H^\infty(\hat{G}, \phi)$ is given by

$$x \cdot y = s^* - \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} P_\phi^k(xy).$$

We will only be interested in the case when $\phi \in \mathcal{C}_l(\hat{G})$. Denote by $\mathrm{supp}\,\phi \subset I$ the set of $s \in I$ such that $\phi(I_s) > 0$, where I_s is the unit in $B(H_s) \subset l^\infty(\hat{G})$. A state ϕ is called generating if $\cup_{n\in\mathbb{N}}\mathrm{supp}\,\phi^n = I$. If ϕ is not generating, but $\mathrm{supp}\,\phi$ is symmetric, then the norm closure of the linear span of the matrix coefficients u^s_{ij} for $s \in \cup_{n\in\mathbb{N}}\mathrm{supp}\,\phi^n$ is the algebra $C(H)$ for a compact quantum group H. Thus H is a quotient of G, and by the orthogonality relations we have $\mathrm{Irr}(H) = \cup_{n\in\mathbb{N}}\mathrm{supp}\,\phi^n$. The symmetry assumption for $\mathrm{supp}\,\phi$ is needed to ensure that $C(H)$ is a $*$-algebra. In the case of q-deformations of compact connected semisimple Lie groups [KoS] this assumption is redundant. Indeed, it is well-known that any compact connected semisimple Lie group G has the property that if U is a faithful unitary representation of G, then any irreducible representation of G appears as a subrepresentation of the tensor power $U^{\times n}$ of U for some $n \in \mathbb{N}$. It follows that $\cup_{n\in\mathbb{N}}\mathrm{supp}\,\phi^n$ is always symmetric, more precisely, $\cup_{n\in\mathbb{N}}\mathrm{supp}\,\phi^n = \mathrm{Irr}(H)$ with $H = G/(\cap_{s\in\mathrm{supp}\,\phi}\mathrm{Ker}\,U^s)$. As the fusion algebra is independent of the deformation parameter, we conclude that for the q-deformation G of any compact connected semisimple Lie group, the set $\cup_{n\in\mathbb{N}}\mathrm{supp}\,\phi^n$ is symmetric, so it corresponds to a certain quotient of G.

Note that the Poisson boundary in general depends on the generating state. However, in good situations, in particular for duals of q-deformations, it does not.

Proposition 1.1 *Assume that the fusion algebra $R(G)$ of G is commutative. Then for any generating state $\phi \in \mathcal{C}_l(\hat{G})$ we have*

$$H^\infty(\hat{G}, \phi) = \{x \in l^\infty(\hat{G}) \,|\, P_{\phi_s}(x) = x \text{ for all } s \in I\}.$$

Proof. The inclusion $\supset$ is obvious as P_ϕ is a (possibly infinite) convex combination of the operators P_{ϕ_s}, $s \in I$. Conversely, fix $s \in I$. Since ϕ is generating, there exists $n \in \mathbb{N}$ such that if we write $\phi^n = \sum_{t\in I} \lambda_t\phi_t$, we have $\lambda_s \neq 0$. Then

$$P_\phi^n = \lambda_s P_{\phi_s} + (1 - \lambda_s)P_\psi,$$

where $\psi = (1-\lambda_s)^{-1}\sum_{t\ne s}\lambda_t\phi_t$. Since $P_{\phi_s}P_\psi = P_{\psi\phi_s}$ and $R(G)$ is commutative, P_{ϕ_s} and P_ψ are commuting contractions on the Banach space $l^\infty(\hat G)$. Hence by Lemma 1.1 in Chapter 5 of [Re], if $P^n_\phi(x)=x$ then $P_{\phi_s}(x)=x$.

■

Recall now the connection between Poisson boundaries and product-type actions [I1]. Let U be a unitary representation of G on H, and $\tilde\phi \in B(H)_*$ a normal faithful $\alpha_{U,l}$-invariant state. Set

$$(N,\nu) = \mathop{\otimes}_{-\infty}^{-1}(B(H),\tilde\phi),$$

and $N_n = \ldots\otimes 1\otimes B(H)^{\otimes n}\subset N$. The actions $\alpha_{U^{\times n},l}$ of G on $B(H^{\otimes n})\cong N_n$ define a left action α of G on N, where $U^{\times n} = U_{12}\ldots U_{1,n+1}$. To describe the relative commutant $(N^\alpha)'\cap N$, consider $\phi = \tilde\phi\pi_U \in \mathcal{C}_l(\hat G)$. Let $E_n\colon N\to N_n$ be the ν-preserving conditional expectation, and $j_n\colon l^\infty(\hat G)\to N$ the homomorphism defined by $j_n(x) = \ldots\otimes 1\otimes\pi_{U^{\times n}}(x)$. Then

$$E_n j_{n+1} = j_n P_\phi.$$

The kernel of j_n is $\oplus_{s\notin \mathrm{supp}\,\phi^n}B(H_s)$. So if $F_n\colon l^\infty(\hat G)\to\oplus_{s\in\mathrm{supp}\,\phi^n}B(H_s)$ is the canonical projection, we have $j_nF_n = j_n$, and j_n is faithful on $F_n(l^\infty(\hat G))$. As the image of $(N^\alpha)'\cap N$ under E_n is contained in the relative commutant

$$(B(H^{\otimes n})^{\alpha_{U^{\times n},l}})'\cap B(H^{\otimes n}),$$

which coincides with the image of $\pi_{U^{\times n}}$, we conclude that the elements of $(N^\alpha)'\cap N$ are in one-to-one correspondence with bounded sequences $\{x_n\}_{n=1}^\infty$ such that $x_n\in F_n(l^\infty(\hat G))$ and

$$x_n = F_nP_\phi(x_{n+1}).$$

Such sequences are called P_ϕ-harmonic. It turns out that if ϕ is generating, then any bounded harmonic sequence $\{x_n\}_n$ defines an element $x\in H^\infty(\hat G,\phi)$ such that $x_n = F_n(x)$, and vice versa. Thus we get an isomorphism

$$j_\infty\colon H^\infty(\hat G,\phi)\xrightarrow{\sim}(N^\alpha)'\cap N,\quad j_\infty(x) = s^*-\lim_{n\to\infty} j_n(x).$$

By restricting the left action Φ_l of G and the right action $\hat\Delta$ of $\hat G$ on $l^\infty(\hat G)$ to $H^\infty(\hat G,\phi)$, we get actions of G and $\hat G$ on the Poisson boundary. When ϕ is generating and we identify $H^\infty(\hat G,\phi)$ with $(N^\alpha)'\cap N$, the action of G is just the restriction of α to the relative commutant, since the homomorphisms j_n are obviously G-equivariant. As was conjectured in [I1], the action of $\hat G$ is related to the dual action of $\hat G$ on $N'\cap(G\ltimes N)$. In the rest of this section we want to clarify this point. This will not be used in the subsequent sections.

Consider first a more general situation. Suppose we are given a left action of G on a von Neumann algebra N,

$$\alpha\colon N\to L^\infty(G)\otimes N.$$

The crossed product $G\ltimes N$ is by definition the W*-subalgebra of $B(H_\varphi)\otimes N$ generated by $l^\infty(\hat G)\otimes 1$ and $\alpha(N)$. Let ν be a normal faithful G-invariant state on N. Then the map

$$v\colon H_\nu\to H_\varphi\otimes H_\nu,\quad a\xi_\nu\mapsto\alpha(a)(\xi_\varphi\otimes\xi_\nu)\ \text{ for } a\in N,$$

is an isometry. As $\alpha(N)(l^\infty(\hat G)\otimes 1)$ is dense in $G\ltimes N$ and $x\xi_\varphi = \hat\varepsilon(x)\xi_\varphi$ for $x\in l^\infty(\hat G)$, where $\hat\varepsilon$ is the counit on $l^\infty(\hat G)$, we see that the image of v is a $(G\ltimes N)$-invariant subspace of $H_\varphi\otimes H_\nu$. Let V be the canonical representation of G on H_ν implementing the action,

$$V^*(\xi\otimes a\xi_\nu) = \alpha(a)(\xi\otimes\xi_\nu),\ \xi\in H_\varphi,\ a\in N.$$

Since

$$\begin{aligned}(1\otimes v)V^*(\xi\otimes a\xi_\nu) &= (1\otimes v)\alpha(a)(\xi\otimes\xi_\nu) = (\iota\otimes\alpha)\alpha(a)(\xi\otimes\xi_\varphi\otimes\xi_\nu)\\ &= (\Delta\otimes\iota)\alpha(a)(\xi\otimes\xi_\varphi\otimes\xi_\nu) = (W^*\otimes 1)(1\otimes\alpha(a))(\xi\otimes\xi_\varphi\otimes\xi_\nu)\\ &= (W^*\otimes 1)(1\otimes v)(\xi\otimes a\xi_\nu),\end{aligned}$$

$V^* = (S\otimes\iota)(V)$ and $W^* = (S\otimes\iota)(W)$, where S is the coinverse for G, we get

$$v\pi_V(x) = (x\otimes 1)v \quad \text{for } x\in l^\infty(\hat G).$$

We also have $v^*\alpha(a)v = a$ for $a\in N$. Thus

$$v^*(x\otimes 1)\alpha(a)v = \pi_V(x)a \quad \text{for } x\in l^\infty(\hat G),\ a\in N. \tag{1.1}$$

Noting that $(J_\varphi\otimes J_\nu)v = vJ_\nu$, and recalling that $J_\varphi x^* J_\varphi = \hat R(x)$ for $x\in l^\infty(\hat G)$, where $\hat R$ is the unitary antipode on $l^\infty(\hat G)$, we see that $J_\nu\pi_V(l^\infty(\hat G))J_\nu = \pi_V(l^\infty(\hat G))$. Thus the map $x\mapsto J_\nu x^* J_\nu$ defines an anti-isomorphism of $N'\cap(N\vee\pi_V(l^\infty(\hat G)))$ onto

$$N\cap(N'\vee\pi_V(l^\infty(\hat G))) = N\cap(N\cap\pi_V(l^\infty(\hat G))')' = N\cap(N^\alpha)'.$$

It follows that the map

$$\theta\colon N'\cap(G\ltimes N)\to(N^\alpha)'\cap N,\quad x\mapsto J_\nu v^* x^* v J_\nu,$$

is a normal surjective $*$-anti-homomorphism.

The dual action $\hat\alpha\colon G\ltimes N\to l^\infty(\hat G)\otimes(G\ltimes N)$ is defined by

$$\hat\alpha(x) = \tilde W(1\otimes x)\tilde W^*,\quad x\in G\ltimes N,$$

where $\tilde W = ((J_\varphi\otimes J_\varphi)W_{21}(J_\varphi\otimes J_\varphi))\otimes 1$, so that

$$\hat\alpha((x\otimes 1)\alpha(a)) = (\hat\Delta(x)\otimes 1)(1\otimes\alpha(a)) \quad \text{for } a\in N,\ x\in l^\infty(\hat G).$$

The left action $\hat\alpha$ of $\hat G$ on $G\ltimes N$ induces a right action $\hat\alpha^{op}$ of $\hat G$ on $(G\ltimes N)^{op}$,

$$\hat\alpha^{op}(x) = (\iota\otimes\hat R)(\alpha(x)_{21}).$$

In the simplest case when $N = B(H)$ all these constructions can be made more explicit.

Lemma 1.2 *Let U be a unitary representation of G on H, $N = B(H)$, $\alpha = \alpha_{U,l}$, ν a normal faithful G-invariant state on $B(H)$, $\phi = \nu\pi_U$. Then the map*

$$\theta_0\colon l^\infty(\hat G)\otimes N\to G\ltimes N,\quad \theta_0(x) = U^* x U,$$

is an isomorphism with the following properties:
(i) $(\iota\otimes\theta_0)(\hat\Delta\otimes\iota)(x) = \hat\alpha\theta_0(x)$ *for* $x\in l^\infty(\hat G)\otimes N$;
(ii) *for the conditional expectation* $E_0 = \iota\otimes\nu\colon G\ltimes N\to l^\infty(\hat G)$, *we have* $E_0\theta_0(x\otimes 1) = \hat R P_\phi\hat R(x)$ *for* $x\in l^\infty(\hat G)$;
(iii) $\theta\theta_0(x\otimes 1) = \pi_U\hat R(x)$ *for* $x\in l^\infty(\hat G)$.

Proof. Since $(\iota\otimes\pi_U)\hat\Delta(l^\infty(\hat G))$ and $1\otimes N$ generate $l^\infty(\hat G)\otimes N$, the fact that θ_0 is an isomorphism follows from the identities

$$U^*(1\otimes a)U=\alpha(a)\ \text{ for } a\in N,\ \ U^*(\iota\otimes\pi_U)\hat\Delta(x)U=x\otimes1\ \text{ for } x\in l^\infty(\hat G),$$

where we have used that $U=(\iota\otimes\pi_U)(W)$.

The equality in (i) obviously holds for $x=1\otimes a$, $a\in N$. Thus it is enough to consider $x=(\iota\otimes\pi_U)\hat\Delta(y)$. Then $\theta_0(x)=y\otimes1$, so $\hat\alpha\theta_0(x)=\hat\Delta(y)\otimes1$. On the other hand,

$$\begin{aligned}(\iota\otimes\theta_0)(\hat\Delta\otimes\iota)(x) &= (\iota\otimes\theta_0)(\iota\otimes\iota\otimes\pi_U)(\hat\Delta\otimes\iota)\hat\Delta(y)\\ &= (\iota\otimes\theta_0)(\iota\otimes\iota\otimes\pi_U)(\iota\otimes\hat\Delta)\hat\Delta(y)=\hat\Delta(y)\otimes1.\end{aligned}$$

Thus (i) is proved.

To prove (ii), note that for any $y,z\in l^\infty(\hat G)$ we have

$$E_0\theta_0(\iota\otimes\pi_U)(\hat\Delta(y)(1\otimes z))=E_0((y\otimes1)\alpha\pi_U(z))=\nu\pi_U(z)y=\phi(z)y.$$

In other words, if we define a map $r\colon\mathcal{A}(\hat G)\otimes\mathcal{A}(\hat G)\to\mathcal{A}(\hat G)\otimes\mathcal{A}(\hat G)$ by $r(y\otimes z)=\hat\Delta(y)(1\otimes z)$, then

$$E_0\theta_0(\iota\otimes\pi_U)r=\iota\otimes\phi.$$

It is well-known (see e.g. [NT1, Proposition 1.3]) that the map r is bijective with inverse s given by $s(y\otimes z)=(\iota\otimes\hat S)((1\otimes\hat S^{-1}(z))\hat\Delta(y))$. Hence for any $y,z\in\mathcal{A}(\hat G)$ we get

$$E_0\theta_0(y\otimes\pi_U(z))=(\iota\otimes\phi)s(y\otimes z)=(\iota\otimes\phi\hat S)((1\otimes\hat S^{-1}(z))\hat\Delta(y)).$$

As the state ϕ is G-invariant and $\hat S=\rho^{\frac12}\hat R(\cdot)\rho^{-\frac12}$, we have $\phi\hat S=\phi\hat R$. Choosing a net $\{z_i\}_i\subset\mathcal{A}(\hat G)$ of central projections converging strongly to the unit, we thus get

$$E_0\theta_0(y\otimes1)=(\iota\otimes\phi\hat R)\hat\Delta(y)=\hat R(\iota\otimes\phi)(\hat R\otimes\hat R)\hat\Delta(y)=\hat R(\phi\otimes\iota)\hat\Delta\hat R(y)=\hat RP_\phi\hat R(y),$$

where we have used that $(\hat R\otimes\hat R)\hat\Delta=\hat\Delta^{op}\hat R$. This proves (ii).

To prove (iii) we identify H_ν with the space $HS(H)$ of Hilbert-Schmidt operators on H, so that $\xi_\nu=T_0^{\frac12}$ if $T_0\in B(H)$ is the operator defining ν, $\nu=\mathrm{Tr}(\cdot T_0)$. If we further identify $HS(H)$ with $\overline{H}\otimes H$, we have $J_\nu(\bar\xi\otimes\zeta)=\bar\zeta\otimes\xi$, and $a(\bar\xi\otimes\zeta)=\bar\xi\otimes a\zeta$ for $a\in N=B(H)$. Thus we have to prove that $\theta\theta_0(x\otimes1)=1\otimes\pi_U\hat R(x)\in B(\overline{H}\otimes H)$ for $x\in l^\infty(\hat G)$.

To compute θ we will check first that $V=\overline{U}\times U$. The unitary $V\in M(C(G)\otimes B_0(H_\nu))$ is characterized by the properties that it implements the action, $V^*(1\otimes a)V=\alpha(a)$ for $a\in N$, and that $V(\xi\otimes\xi_\nu)=\xi\otimes\xi_\nu$ for $\xi\in H_\varphi$. The first property is obviously satisfied by $\overline{U}\times U$. To check the second one, recall that by definition $\overline{U}=(R\otimes j)(U)\in M(C(G)\otimes B_0(\overline{H}))$, where R is the unitary antipode on $C(G)$ and $j(x)\bar\xi=\overline{x^*\xi}$. If we consider $C(G)\otimes HS(H)$ as a left module over $M(C(G)\otimes B_0(\overline{H}\otimes H))$, it is then easy to check that

$$(\overline{U}\times U)(1\otimes T)=(R\otimes\iota)((R\otimes\iota)(U)(1\otimes T)U)\ \text{ for } T\in HS(H).$$

We have to prove that $(\overline{U}\times U)(1\otimes T_0^{\frac12})=1\otimes T_0^{\frac12}$. This follows from the identities

$$(R\otimes\iota)(U)=(1\otimes\pi_U(\rho^{-\frac12}))(S\otimes\iota)(U)(1\otimes\pi_U(\rho^{\frac12}))=(1\otimes\pi_U(\rho^{-\frac12}))U^*(1\otimes\pi_U(\rho^{\frac12}))$$

and

$$(1\otimes T_0^{\frac12})U=(1\otimes\pi_U(\rho^{-\frac12}))U(1\otimes\pi_U(\rho^{\frac12}))(1\otimes T_0^{\frac12}),$$

where we have used that $\pi_U(\rho)T_0 \in \pi_U(l^\infty(\hat{G}))'$. Thus $V = \overline{U} \times U$.

By virtue of (1.1), for any $y, z \in l^\infty(\hat{G})$ we then have

$$\begin{aligned} v^*\theta_0((\iota \otimes \pi_U)(\hat{\Delta}(y)(1 \otimes z)))v &= v^*(y \otimes 1)\alpha\pi_U(z)v = \pi_V(y)(1 \otimes \pi_U(z)) \\ &= (\pi_{\bar{U}} \otimes \pi_U)\hat{\Delta}(y)(1 \otimes \pi_U(z)) = (\pi_{\bar{U}} \otimes \pi_U)(\hat{\Delta}(y)(1 \otimes z)). \end{aligned}$$

Since $\hat{\Delta}(l^\infty(\hat{G}))(1 \otimes l^\infty(\hat{G}))$ is dense in $l^\infty(\hat{G}) \otimes l^\infty(\hat{G})$, for any $x \in l^\infty(\hat{G})$ we thus get

$$v^*\theta_0(x \otimes 1)v = (\pi_{\bar{U}} \otimes \pi_U)(x \otimes 1) = \pi_{\bar{U}}(x) \otimes 1,$$

whence

$$\theta\theta_0(x \otimes 1) = J_\nu v^*\theta_0(x^* \otimes 1)vJ_\nu = J_\nu(\pi_{\bar{U}}(x^*) \otimes 1)J_\nu = 1 \otimes \overline{\pi_{\bar{U}}(x^*)},$$

where $\overline{\pi_{\bar{U}}(x^*)}\xi = \overline{\pi_{\bar{U}}(x^*)\bar{\xi}}$. By definition of the conjugate representation we have

$$\pi_{\bar{U}}(x^*)\bar{\xi} = \overline{\pi_U\hat{R}(x)\xi}.$$

It follows that $\theta\theta_0(x \otimes 1) = 1 \otimes \pi_U\hat{R}(x)$.

■

Return now to the study of $N' \cap (G \ltimes N)$ for a product-type action defined by a representation U of G on H and a G-invariant state $\tilde{\phi} \in B(H)_*$. The relative commutant can be described in a way similar to that for $(N^\alpha)' \cap N$, see [V]. The conditional expectation $E_n: N \to N_n$ extends to the conditional expectation $\iota \otimes E_n: G \ltimes N \to G \ltimes N_n$. Let $y \in N' \cap (G \ltimes N)$. Then $(\iota \otimes E_n)(y) \in N_n' \cap (G \ltimes N_n)$, so by Lemma 1.2

$$(\iota \otimes E_n)(y) = (U^{\times n})^*(y_n \otimes 1)U^{\times n}$$

for a unique element $y_n \in l^\infty(\hat{G})$. As $E_nE_{n+1} = E_n$, we have

$$\begin{aligned} (U^{\times n})^*(y_n \otimes 1)U^{\times n} &= (\iota \otimes E_n)\Big((U^{\times(n+1)})^*(y_{n+1} \otimes 1)U^{\times(n+1)}\Big) \\ &= (U^{\times n})^*((\iota \otimes \tilde{\phi})(U^*(y_{n+1} \otimes 1)U) \otimes 1)U^{\times n}, \end{aligned}$$

whence by Lemma 1.2(ii)

$$y_n = (\iota \otimes \tilde{\phi})(U^*(y_{n+1} \otimes 1)U) = \hat{R}P_\phi\hat{R}(y_{n+1}),$$

where $\phi = \tilde{\phi}\pi_U$. Taking into account parts (i) and (iii) of Lemma 1.2 as well as the description of $(N^\alpha)' \cap N$ in terms of harmonic sequences, we get the following result.

Proposition 1.3 *For the product-type action α of a compact quantum group G defined by a representation U of G on H and a G-invariant normal faithful state $\tilde{\phi}$ on $B(H)$, there exists a $\hat{G}$-equivariant complete order isometry between $N' \cap (G \ltimes N)$ and the space of bounded sequences $\{y_n\}_{n=1}^\infty \subset l^\infty(\hat{G})$ such that $\hat{R}(y_n) = P_\phi\hat{R}(y_{n+1})$, where $\phi = \tilde{\phi}\pi_U$.*

The surjective anti-homomorphism $\theta: N' \cap (G \ltimes N) \to (N^\alpha)' \cap N$ maps the element defined by a sequence $\{y_n\}_{n=1}^\infty$ to the element defined by the P_ϕ-harmonic sequence $\{F_n\hat{R}(y_n)\}_{n=1}^\infty$. ■

Note that even if ϕ is generating, the homomorphism $(N' \cap (G \ltimes N))^{op} \to (N^\alpha)' \cap N$ need not be injective nor $\hat{G}$-equivariant (where we consider $(N' \cap (G \ltimes N))^{op}$ with the right action $\hat{\alpha}^{op}$ of $\hat{G}$, and identify $(N^\alpha)' \cap N$ with $H^\infty(\hat{G}, \phi)$ to get a right action of $\hat{G}$ on $(N^\alpha)' \cap N$).

Example 1.4 Let $G = SU(2)$ and U be the fundamental representation of G. Identify $\mathrm{Irr}(G)$ with $\frac{1}{2}\mathbb{Z}_+$ as usual. Set $p = \sum_{n=1}^{\infty} I_{n-\frac{1}{2}}$. Then

$$\hat{\Delta}(p) = p \otimes (1-p) + (1-p) \otimes p.$$

In particular, $P_\phi(p) = 1-p$. Consider the sequence $\{y_n\}_{n=1}^{\infty}$ defined by $y_{2k+1} = 1-p$, $y_{2k} = p$, and let y be the corresponding element of $N' \cap (SU(2) \ltimes N)$. Then y is in the kernel of θ. Moreover, as

$$\hat{\alpha}(y) = p \otimes (1-y) + (1-p) \otimes y,$$

the $\hat{G}$-equivariance also does not hold.

It is clear, however, what goes wrong in the previous example. Though the representation U was faithful, $\overline{U} \times U$ was not, so we had in fact an action of $SO(3)$. More generally, assume G is the q-deformation of a compact connected semisimple Lie group. Let H be the quotient of G defined by $\overline{U} \times U$, that is, $C(H)$ is the C*-subalgebra of $C(G)$ generated by the matrix coefficients of $\overline{U} \times U$. Then $\mathrm{Irr}(H) = \cup_{n\in\mathbb{N}} \mathrm{supp}\,(\check{\phi}\phi)^n$. Choose also $k \in \mathbb{N}$ such that $0 \in \mathrm{supp}\,\phi^k$, that is, $U^{\times k}$ contains the trivial representation. Since $0 \in \mathrm{supp}\,\phi^k$, we have $\mathrm{supp}\,\phi^k \subset \mathrm{supp}\,(\check{\phi}^k\phi^k) = \mathrm{supp}\,(\check{\phi}\phi)^k \subset \mathrm{Irr}(H)$. We have thus shown that by replacing G by its quotient and U by some power $U^{\times k}$, in the study of product-type actions we can always assume that $\overline{U} \times U$ is faithful, that is, $C(G)$ is generated by the matrix coefficients of $\overline{U} \times U$.

Assume now that $\overline{U} \times U$ is faithful. Then $\mathrm{Irr}(G) = \cup_{n\in\mathbb{N}} \mathrm{supp}\,\phi^{kn}$ for any $k \in \mathbb{N}$. Indeed, as we discussed earlier, the set $\cup_{n\in\mathbb{N}} \mathrm{supp}\,\phi^{kn}$ is always symmetric. Hence it contains $\mathrm{supp}\,(\check{\phi}\phi)^{kn}$ for any $n \in \mathbb{N}$. Since $0 \in \mathrm{supp}\,(\check{\phi}\phi)$, we have $\mathrm{supp}\,(\check{\phi}\phi)^m \subset \mathrm{supp}\,(\check{\phi}\phi)^{m+1}$ for any m. Thus

$$\mathrm{Irr}(G) = \cup_{n\in\mathbb{N}} \mathrm{supp}\,(\check{\phi}\phi)^n = \cup_{n\in\mathbb{N}} \mathrm{supp}\,(\check{\phi}\phi)^{kn} \subset \cup_{n\in\mathbb{N}} \mathrm{supp}\,\phi^{kn}.$$

In particular, we can find k and n such that $0 \in \mathrm{supp}\,\phi^k$ and $\mathrm{supp}\,\phi \cap \mathrm{supp}\,\phi^{kn} \neq \emptyset$. As $0 \in \mathrm{supp}\,\phi^{kn}$, we have

$$\mathrm{supp}\,\phi \cap \mathrm{supp}\,\phi^{kn} \subset \mathrm{supp}\,\phi^{kn} \cap \mathrm{supp}\,\phi^{kn+1}.$$

By the 0-2 law (see e.g [NT1, Proposition 2.12]) we can conclude that $\|P_\phi^m - P_\phi^{m+1}\| \to 0$ as $m \to \infty$. It follows that if $\{x_n\}_{n=1}^{\infty}$ is a bounded sequence in $l^\infty(\hat{G})$ such that $x_n = P_\phi(x_{n+1})$, then the sequence is constant, $x_n = x_{n+1}$. Note also that if $y \in N' \cap (G \ltimes N)$ is the element defined by the sequence $\{\hat{R}(x_n)\}_{n=1}^{\infty}$, then by Lemma 1.2(ii), for the conditional expectation $E_0 = \iota \otimes \nu\colon G \ltimes N \to l^\infty(\hat{G})$ we have $E_0(y) = \hat{R}P_\phi(x_1)$. We have thus proved the following result.

Corollary 1.5 *Let G be the q-deformation of a compact connected semisimple Lie group, α the product-type action of G on (N,ν) defined by a representation U of G on H and a G-invariant normal faithful state $\tilde{\phi}$ on $B(H)$. Assume that the representation $\overline{U} \times U$ is faithful. Then we have isomorphisms*

$$(N' \cap (G \ltimes N))^{op} \underset{\theta}{\xrightarrow{\simeq}} (N^\alpha)' \cap N \underset{j_\infty}{\xleftarrow{\simeq}} H^\infty(\hat{G}, \phi),$$

where $\phi = \tilde{\phi}\pi_U$. Moreover, the homomorphism $j_\infty^{-1}\theta$ is $\hat{G}$-equivariant and coincides with the map $\hat{R} \otimes \nu$. ■

For a more general group G and a generating state $\phi \in \mathcal{C}_l(\hat{G})$, the homomorphism $j_\infty^{-1}\theta$ is a $\hat{G}$-equivariant isomorphism if any bounded sequence $\{x_n\}_{n=1}^{\infty} \subset l^\infty(\hat{G})$ such that $P_\phi(x_{n+1}) = x_n$ is constant, that is, $x_{n+1} = x_n$. For this it is enough to require that $\mathrm{supp}\,\phi^m \cap \mathrm{supp}\,\phi^{m+1} \neq \emptyset$ for some m.

2 Poisson integral and Berezin transform

Let G be a compact quantum group and $\phi \in \mathcal{C}_l(\hat{G})$ be a normal left G-invariant state on $l^\infty(\hat{G})$. Consider the right action $\hat{\Phi}$ of $\hat{G}$ on $L^\infty(G)$,

$$\hat{\Phi}: L^\infty(G) \to L^\infty(G) \otimes l^\infty(\hat{G}), \quad \hat{\Phi}(a) = W(a \otimes 1)W^*.$$

It was proved in [I1] that

$$\Theta = (\varphi \otimes \iota)\hat{\Phi}$$

maps $L^\infty(G)$ into $H^\infty(\hat{G}, \phi)$. In fact, this is the only normal unital G- and $\hat{G}$-equivariant map from $L^\infty(G)$ into $l^\infty(\hat{G})$ (we consider $L^\infty(G)$ and $l^\infty(\hat{G})$ with the left actions of G given by Δ and Φ_l, respectively, and with the right actions of $\hat{G}$ given by $\hat{\Phi}$ and $\hat{\Delta}$, respectively). Indeed, assume T is such a map. Since the counit $\hat{\varepsilon}$ on $l^\infty(\hat{G})$ is G-invariant, $\hat{\varepsilon}T$ is a G-invariant normal linear functional on $L^\infty(G)$ such that $\hat{\varepsilon}T(1) = 1$. Hence $\hat{\varepsilon}T = \varphi$. Then by $\hat{G}$-equivariance of T, we get

$$T = (\hat{\varepsilon} \otimes \iota)\hat{\Delta}T = (\hat{\varepsilon} \otimes \iota)(T \otimes \iota)\hat{\Phi} = (\varphi \otimes \iota)\hat{\Phi} = \Theta.$$

Thus if the Poisson boundary is G- and $\hat{G}$-equivariantly isomorphic to a homogeneous space G/H of G, the map Θ must be an isomorphism of $L^\infty(G/H)$ onto $H^\infty(\hat{G}, \phi)$. We call the map Θ the Poisson integral.

To show multiplicativity of Θ we will use the following criterion.

Lemma 2.1 *Let N_i be a von Neumann algebra, ν_i a normal faithful state on N_i, $i = 1, 2$, $\theta: N_1 \to N_2$ a normal ucp map such that $\nu_2\theta = \nu_1$ and $\sigma_t^{\nu_2}\theta = \theta\sigma_t^{\nu_1}$. Then there exists a normal ucp map $\theta^*: N_2 \to N_1$ such that*

$$\nu_2(\theta(x_1)x_2) = \nu_1(x_1\theta^*(x_2)) \quad \text{for } x_1 \in N_1,\ x_2 \in N_2.$$

For any $x \in N_1$, the following conditions are then equivalent:
(i) *x is in the multiplicative domain of θ;*
(ii) $||\theta(x)||_2 = ||x||_2$*;*
(iii) $||\theta(x^*)||_2 = ||x^*||_2$*;*
(iv) $\theta^*\theta(x) = x$*.*

Proof. Since $\sigma_t^{\nu_2}\theta = \theta\sigma_t^{\nu_1}$, the map θ^* can equivalently be defined by the condition

$$(\theta(x_1)J_{\nu_2}x_2^*\xi_{\nu_2}, \xi_{\nu_2}) = \nu_2(\theta(x_1)\sigma^{\nu_2}_{-\frac{i}{2}}(x_2)) = \nu_1(x_1\sigma^{\nu_1}_{-\frac{i}{2}}\theta^*(x_2)) = (x_1J_{\nu_1}\theta^*(x_2^*)\xi_{\nu_1}, \xi_{\nu_1}).$$

Hence θ^* exists [AC]. A more general result than the equivalence of the conditions (i)-(iv) can be found in [P]. We will give a proof of our particular case for the reader's convenience.

Note that for a contraction T on a Hilbert space and a vector ξ, the equality $\|T\xi\| = \|\xi\|$ holds if and only if $T^*T\xi = \xi$. This shows that (ii) is equivalent to (iv). Clearly, then (iii) also is equivalent to (iv).

For any $x \in N_1$, we have $\theta(x^*)\theta(x) \le \theta(x^*x)$ by Schwarz inequality. Since ν_2 is faithful, it follows that the equality $||\theta(x)||_2 = ||x||_2$ holds if and only if $\theta(x^*)\theta(x) = \theta(x^*x)$. It is well-known that (i) is equivalent to the conditions $\theta(x^*)\theta(x) = \theta(x^*x)$ and $\theta(x)\theta(x^*) = \theta(xx^*)$. Thus (i) implies (ii) and (iii), and (ii) and (iii) together imply (i). Since we already know that (ii) and (iii) are equivalent, we conclude that all four conditions are equivalent.

■

For $s \in I = \mathrm{Irr}(G)$, consider the map $\Theta_s: L^\infty(G) \to B(H_s)$,

$$\Theta_s(a) = (\varphi \otimes \iota)(U^s(a \otimes 1)U^{s*}) = \Theta(a)I_s.$$

The following lemma shows that there exists a map $\Theta_s^*\colon B(H_s) \to L^\infty(G)$ such that

$$\phi_s(\Theta_s(a)x) = \varphi(a\Theta_s^*(x)), \quad a \in L^\infty(G), x \in B(H_s).$$

Lemma 2.2 *We have*

$$\Theta_s^*(x) = (\iota \otimes \omega_s)\Phi_l(x) = (\iota \otimes \omega_s)(U^{s*}(1 \otimes x)U^s).$$

Proof. Recall that for the modular group σ_t^φ we have

$$(\sigma_t^\varphi \otimes \iota)(W) = (1 \otimes \rho^{it})W(1 \otimes \rho^{it}). \tag{2.1}$$

On the other hand, $\sigma_t^{\phi_s}(x) = \rho^{-it}x\rho^{it}$ for $x \in B(H_s)$. Thus $(\sigma_t^\varphi \otimes \sigma_t^{\phi_s})(U^s) = U^s(1 \otimes \rho^{2it})$. Hence

$$\begin{aligned}
\phi_s(\Theta_s(a)x) &= (\varphi \otimes \phi_s)(U^s(a \otimes 1)U^{s*}(1 \otimes x)) \\
&= (\varphi \otimes \phi_s)((a \otimes 1)U^{s*}(1 \otimes x)(\sigma_{-i}^\varphi \otimes \sigma_{-i}^{\phi_s})(U^s)) \\
&= (\varphi \otimes \phi_s(\cdot\rho^2))((a \otimes 1)U^{s*}(1 \otimes x)U^s),
\end{aligned}$$

whence $\Theta_s^*(x) = (\iota \otimes \omega_s)(U^{s*}(1 \otimes x)U^s)$, because $\omega_s = \phi_s(\cdot\rho^2)$.

■

As a byproduct we get $\phi_s\Theta_s = \varphi$. Note also that both maps Θ_s and Θ_s^* are G-equivariant. Now we can compute Θ^*.

Lemma 2.3 *Let $\phi \in \mathcal{C}_l(\hat{G})$ be a generating state. Set $\nu_0 = \hat{\varepsilon}|_{H^\infty(G,\phi)}$. Then for the Poisson integral $\Theta\colon (L^\infty(G), \varphi) \to (H^\infty(\hat{G}, \phi), \nu_0)$, we have*

$$\Theta^*(x) = s^* - \lim_{n\to\infty} \sum_{s\in I} \phi^n(I_s)\Theta_s^*(x).$$

Proof. First note that under the identification of $H^\infty(\hat{G}, \phi)$ with the relative commutant for a product-type action, the state ν_0 coincides with the product-state. It follows that ν_0 is faithful, and its modular group is given by the restriction of the modular group $\sigma_t^{\hat{\psi}} = \operatorname{Ad}\rho^{-it}$ of the right-invariant Haar weight $\hat{\psi}$ on $l^\infty(\hat{G})$ to $H^\infty(\hat{G}, \phi)$. It is then easy to see that $\nu_0\Theta = \varphi$ and using (2.1) that $\sigma_t^{\nu_0}\Theta = \Theta\sigma_t^\varphi$. Hence Θ^* indeed exists.

Recall [I1, Theorem 3.6(2)] that as ϕ is generating, the product on $H^\infty(\hat{G}, \phi)$ is given by

$$x \cdot y = s^* - \lim_{n\to\infty} P_\phi^n(xy).$$

Thus for any $a \in L^\infty(G)$ and $x \in H^\infty(\hat{G}, \phi)$ we have

$$\begin{aligned}
\nu_0(\Theta(a) \cdot x) &= \lim_{n\to\infty} \hat{\varepsilon}P_\phi^n(\Theta(a)x) = \lim_{n\to\infty} \phi^n(\Theta(a)x) \\
&= \lim_{n\to\infty} \sum_{s\in I} \phi^n(I_s)\phi_s(\Theta_s(a)x) \\
&= \lim_{n\to\infty} \sum_{s\in I} \phi^n(I_s)\varphi(a\Theta_s^*(x)),
\end{aligned}$$

so $\sum_{s\in I} \phi^n(I_s)\Theta_s^*(x) \to \Theta^*(x)$ in weak operator topology. Note, however, that by G-equivariance of Θ_s^* if x is in the spectral subspace of $H^\infty(\hat{G}, \phi)$ corresponding to an irreducible representation of G, then $\Theta_s^*(x)$ is in the spectral subspace of $L^\infty(G)$, which is finite dimensional. It follows that the convergence is in norm on a dense $*$-subalgebra of $H^\infty(\hat{G},\phi)$. Since

$$\varphi\left(\sum_s \phi^n(I_s)\Theta_s^*(x)\right) = \sum_s \phi^n(I_s)\phi_s(x) = \phi^n(x) = \nu_0(x)$$

for $x \in H^\infty(\hat{G},\phi)$, the operators $\sum_s \phi^n(I_s)\Theta_s^*$ are contractions with respect to the L^2-norms. Hence we have s^*-convergence on the whole space $H^\infty(\hat{G},\phi)$.

■

From now onwards we assume that the counit ε is bounded on $C(G)$. This is the case for q-deformations of compact connected semisimple Lie groups.

Since the map $\Theta^*\Theta$ is G-equivariant, it maps the spectral subspaces of $L^\infty(G)$ into themselves. The same is true for $\Theta_s^*\Theta_s$. It follows that for any $a \in C(G)$ the sequence $\{\sum_s \phi^n(I_s)\Theta_s^*\Theta_s(a)\}_n$ is in $C(G)$, and it converges in norm to $\Theta^*\Theta(a)$. Note now that G-equivariance of $\Theta^*\Theta$ implies that

$$\Theta^*\Theta(a) = (\iota\otimes\varepsilon)\Delta\Theta^*\Theta(a) = (\iota\otimes\varepsilon\Theta^*\Theta)\Delta(a).$$

So to prove that $\Theta^*\Theta(a) = a$ for an element $a \in C(G)$, it is enough to show that $\varepsilon\Theta^*\Theta(b) = \varepsilon(b)$ for any element b of the form $(\omega\otimes\iota)\Delta(a)$, $\omega \in C(G)^*$. As $\varepsilon\Theta_s^* = \omega_s$ and

$$\sum_s \phi^n(I_s)\omega_s = \sum_s \phi^n(I_s)\phi_s(\cdot\rho^2) = \phi^n(\cdot\rho^2) = \omega^n,$$

where $\omega = \phi(\cdot\rho^2) \in \mathcal{C}_r(\hat{G})$, we have

$$\varepsilon\Theta^*\Theta(b) = \lim_{n\to\infty}\omega^n\Theta(b).$$

Thus using equivalence of (i) and (iv) in Lemma 2.1 we get the following criterion for multiplicativity of Θ.

Proposition 2.4 *Let $\phi \in \mathcal{C}_l(\hat{G})$ be a generating state. Set $\omega = \phi(\cdot\rho^2)$. Then the sequence $\{\omega^n\Theta\}_{n=1}^\infty$ of states on $C(G)$ is w^*-convergent. For a G-invariant subspace X of $C(G)$ (that is, $\Delta(X) \subset C(G)\otimes X$), the limit state coincides with the counit ε on X if and only if X is in the multiplicative domain of the Poisson integral $\Theta\colon L^\infty(G)\to H^\infty(\hat{G},\phi)$.* ■

The operators Θ_s and Θ_s^* are analogues of well-known classical constructions [Be, Per]. Let for the moment G be a compact Lie group, $U\colon G\to B(H)$ a finite dimensional unitary representation. Fix a vector $\xi \in H$, $\|\xi\| = 1$. Let $T \subset G$ be the stabilizer of the line $\mathbb{C}\xi$. For an operator $S \in B(H)$, its covariant Berezin symbol $\sigma(S)$ is defined by $\sigma(S)(g) = (SU_g\xi, U_g\xi)$. The covariant symbol σ is a G-equivariant map from $B(H)$ into $C(G/T)$. Consider the inner products on $C(G/T)$ and $B(H)$ given by the G-invariant probability measure and the normalized trace, respectively. Then there exists an adjoint $\breve{\sigma}\colon C(G/T)\to B(H)$ of σ. Explicitly,

$$\breve{\sigma}(f) = d\int f(g)U_gP_\xi U_g^*dg,$$

where $d = \dim H$ and P_ξ is the projection onto $\mathbb{C}\xi$. A function f is called a contravariant Berezin symbol of $\breve{\sigma}(f)$. The map $B = \sigma\breve{\sigma}$ is called the Berezin transform.

If we consider U as a corepresentation of $C(G)$, then the definition of σ can be written as

$$\sigma(S) = (\iota\otimes\omega_\xi)(U^*(1\otimes S)U),$$

where $\omega_\xi = (\cdot\xi, \xi)$ is the vector-state defined by ξ. Thus we see that our operator Θ_s^* is just σ with ω_ξ replaced by ω_s. Then $\Theta_s = (\Theta_s^*)^*$ is an analogue of $\breve{\sigma}$.

Suppose now that G is a semisimple Lie group, $U = U^\lambda\colon G \to B(H_\lambda)$ an irreducible representation with highest weight λ, $\xi = \xi_\lambda$ a highest weight vector. Let B_λ be the corresponding Berezin transform. Note that $\omega^n_{\xi_\lambda} = \omega_{\xi_{n\lambda}}$. It is proved in [D] that the sequence $\{B_{n\lambda}\}_{n=1}^\infty$ converges to the identity on $C(G/T)$ as $n \to \infty$. This is a key step to show that the full matrix algebras $B(H_{n\lambda})$, $n \in \mathbb{N}$, provide a quantization of $C(G/T)$, see [L, R]. In view of the G-equivariance of the Berezin transform, it is enough to prove the convergence at the unit of G. The proof is based on the observation that the states $\varepsilon B_{n\lambda}$ are given by measures which are absolutely continuous with respect to the Haar measure and such that the Radon-Nikodym derivatives, up to normalization, are powers of a single function h such that $h(g) = 1$ for $g \in T$ and $h(g) < 1$ for $g \notin T$. The proof of our q-analogue will be based on the study of ergodic properties of an auxiliary operator.

For a normal linear functional ω on $l^\infty(\hat{G})$ define a linear operator $A_\omega\colon C(G) \to C(G)$ by

$$A_\omega(a) = (\iota \otimes \omega)\hat{\Phi}(a) = (\iota \otimes \omega)(W(a \otimes 1)W^*).$$

Then $\omega\Theta = \varphi A_\omega$. Since $\hat{\Phi}$ is a right action of $\hat{G}$, so that $(\iota \otimes \hat{\Delta})\hat{\Phi} = (\hat{\Phi} \otimes \iota)\hat{\Phi}$, we have $A_{\omega_1\omega_2} = A_{\omega_1}A_{\omega_2}$ for any $\omega_1, \omega_2 \in l^\infty(\hat{G})_*$. Thus $\omega^n\Theta = \varphi A^n_\omega$, and by Proposition 2.4 to show multiplicativity of Θ on the quantum flag manifold, it is enough to prove the following result.

Theorem 2.5 *Let $G = SU_q(n)$ $(0 < q < 1)$, $T \subset SU_q(n)$ the maximal torus, and $\omega \in \mathcal{C}_r(\hat{G})$ a normal right G-invariant state, $\omega \ne \hat{\varepsilon}$. Then the counit ε is the only A_ω-invariant state on $C(G/T)$.*

We will prove the result by induction. For this we first have to establish functorial properties of the operators A_ω.

Let H be a closed subgroup of G. By this we mean that H a compact quantum group and that we are given a surjective unital $*$-homomorphism $\pi\colon C(G) \to C(H)$ which respects comultiplication. We can define a left and a right action of H on $C(G)$ by the homomorphisms

$$C(G) \to C(H) \otimes C(G), \quad a \to (\pi \otimes \iota)\Delta,$$

and

$$C(G) \to C(G) \otimes C(H), \quad a \to (\iota \otimes \pi)\Delta,$$

respectively. The corresponding fixed point algebras are denoted by $C(H\backslash G)$ and $C(G/H)$.

By considering the elements of $l^\infty(\hat{G})$ as linear functionals on $\mathcal{A}(G)$ we can define a dual homomorphism $\hat{\pi}\colon l^\infty(\hat{H}) \to l^\infty(\hat{G})$. Equivalently, one can consider the unitary corepresentation $U = (\pi \otimes \iota)(W)$ of $C(H)$, where W is the multiplicative unitary for G, and set $\hat{\pi} = \pi_U$.

Lemma 2.6 *Let H be a closed subgroup of G defined by $\pi\colon C(G) \to C(H)$. Then*
(i) $\pi A_\omega = A_{\omega\hat{\pi}}\pi$ *for any* $\omega \in l^\infty(\hat{G})_*$;
(ii) $A_\omega(C(G/H)) \subset C(G/H)$ *for any* $\omega \in l^\infty(\hat{G})_*$;
(iii) $A_\omega(C(H\backslash G)) \subset C(H\backslash G)$ *for any* $\omega \in \mathcal{C}_r(\hat{G})$.

Proof. Let W and W_0 be the multiplicative unitaries for G and H, respectively. Set $U = (\pi \otimes \iota)(W)$ as above, so that $\hat{\pi} = \pi_U$. As $U = (\iota \otimes \pi_U)(W_0) = (\iota \otimes \hat{\pi})(W_0)$, we get

$$\pi A_\omega(a) = (\iota \otimes \omega)(U(\pi(a) \otimes 1)U^*) = (\iota \otimes \omega\hat{\pi})(W_0(\pi(a) \otimes 1)W_0^*) = A_{\omega\hat{\pi}}\pi(a),$$

which proves (i).

Let $a \in C(G/H)$, so that $(\iota\otimes\pi)\Delta(a) = a\otimes 1$. Then using the pentagon equation $W_{12}W_{13}W_{23} = W_{23}W_{12}$ we get

$$\begin{aligned}(\iota\otimes\pi)\Delta A_\omega(a) &= (\iota\otimes\pi\otimes\omega)(W_{12}^*W_{23}(1\otimes a\otimes 1)W_{23}^*W_{12})\\ &= (\iota\otimes\pi\otimes\omega)(W_{13}W_{23}W_{12}^*(1\otimes a\otimes 1)W_{12}W_{23}^*W_{13}^*)\\ &= (\iota\otimes\pi\otimes\omega)(W_{13}W_{23}(\Delta(a)\otimes 1)W_{23}^*W_{13}^*)\\ &= (\iota\otimes\pi\otimes\omega)(W_{13}W_{23}(a\otimes 1\otimes 1)W_{23}^*W_{13}^*)\\ &= A_\omega(a)\otimes 1,\end{aligned}$$

which shows (ii).

Suppose now that $a \in C(H\backslash G)$. Then

$$\begin{aligned}(\pi\otimes\iota)\Delta A_\omega(a) &= (\pi\otimes\iota\otimes\omega)(W_{12}^*W_{23}(1\otimes a\otimes 1)W_{23}^*W_{12})\\ &= (\pi\otimes\iota\otimes\omega)(W_{13}W_{23}W_{12}^*(1\otimes a\otimes 1)W_{12}W_{23}^*W_{13}^*)\\ &= (\pi\otimes\iota\otimes\omega)(W_{13}W_{23}(\Delta(a)\otimes 1)W_{23}^*W_{13}^*)\\ &= (\pi\otimes\iota\otimes\omega)(W_{13}W_{23}(1\otimes a\otimes 1)W_{23}^*W_{13}^*)\\ &= (\pi\otimes\iota\otimes\omega)(W_{23}(1\otimes a\otimes 1)W_{23}^*)\\ &= 1\otimes A_\omega(a),\end{aligned}$$

where in the next to last equality we used right G-invariance of ω. This proves (iii).

■

Lemma 2.7 *For any $\omega \in \mathcal{C}_r(\hat{G})$, we have $\omega\hat{\pi} \in \mathcal{C}_r(\hat{H})$.*

An equivalent way of saying this is that if $\rho_0 = f_1$ is the Woronowicz character for H, then $\hat{\pi}(\rho_0)\rho^{-1}$ commutes with $\hat{\pi}(l^\infty(\hat{H}))$. Yet another equivalent statement is that the homomorphism π intertwines the scaling groups of $C(G)$ and $C(H)$.

Proof of Lemma 2.7. Keeping the notation of the proof of the previous lemma, consider the right actions $\alpha = \alpha_{W,r}$ and $\alpha_0 = \alpha_{U,r}$ of G and H, respectively, on $B(L^2(G))$. Extend ω to a normal G-invariant functional $\tilde{\omega}$ on $B(L^2(G))$. Since $(\pi\otimes\iota)(W) = U$, we have $(\iota\otimes\pi)\alpha(x) = \alpha_0(x)$ for any $x \in B(L^2(G))$ (note that the expression $(\iota\otimes\pi)\alpha(x)$ makes sense as we have a well-defined homomorphism $\iota\otimes\pi\colon M(B_0(L^2(G))\otimes C(G)) \to M(B_0(L^2(G))\otimes C(H))$). Hence $\tilde{\omega}$ is H-invariant, so $\omega\hat{\pi} = \tilde{\omega}\pi_U \in \mathcal{C}_r(\hat{H})$.

■

We can now lay the foundation for our induction argument.

Lemma 2.8 *Suppose η is a state on $C(G)$ such that $\eta = \varepsilon$ on $C(H\backslash G/H) = C(H\backslash G)\cap C(G/H)$. Then there exists a state η_0 on $C(H)$ such that $\eta_0\pi = \eta$.*

Proof. Suppose $a \geq 0$, $\pi(a) = 0$. We have to prove that $\eta(a) = 0$. Let φ_0 be the Haar state on $C(H)$. We have

$$(\varphi_0\pi\otimes\iota\otimes\varphi_0\pi)\Delta^2(a) \in C(H\backslash G/H),$$

where $\Delta^2 = (\Delta\otimes\iota)\Delta = (\iota\otimes\Delta)\Delta$. Hence

$$\begin{aligned}(\varphi_0\pi\otimes\eta\otimes\varphi_0\pi)\Delta^2(a) &= (\varphi_0\pi\otimes\varepsilon\otimes\varphi_0\pi)\Delta^2(a) = (\varphi_0\pi\otimes\varphi_0\pi)\Delta(a)\\ &= (\varphi_0\otimes\varphi_0)\Delta_0\pi(a) = 0,\end{aligned}$$

where Δ_0 is the comultiplication on $C(H)$. Since the state $\varphi_0 \otimes \varphi_0$ is faithful by our assumptions on quantum groups, we conclude that

$$(\pi \otimes \eta \otimes \pi)\Delta^2(a) = 0.$$

Applying $\varepsilon_0 \otimes \iota \otimes \varepsilon_0$, where ε_0 is the counit on $C(H)$, and using $\varepsilon_0\pi = \varepsilon$ we get $\eta(a) = 0$.

■

Corollary 2.9 *Let $T \subset H \subset G$ be compact quantum groups, and $\pi\colon C(G) \to C(H)$ the homomorphism defining the inclusion $H \hookrightarrow G$. Let ω be a state in $\mathcal{C}_r(\hat{G})$. Assume that*
(i) *the counit ε on $C(G)$ is the only A_ω-invariant state on $C(H\backslash G/H)$;*
(ii) *the counit ε_0 on $C(H)$ is the only $A_{\omega\hat{\pi}}$-invariant state on $C(H/T)$.*
Then ε is the only A_ω-invariant state on $C(G/T)$.

Proof. If η is A_ω-invariant, then $\eta = \varepsilon$ on $C(H\backslash G/H)$, so by the previous lemma $\eta = \eta_0\pi$ for some state η_0 on $C(H)$. By Lemma 2.6(i) and surjectivity of π, the state η_0 is $A_{\omega\hat{\pi}}$-invariant. Hence $\eta_0 = \varepsilon_0$ on $C(H/T)$. As $\pi(C(G/T)) \subset C(H/T)$, we have $\eta = \eta_0\pi = \varepsilon_0\pi = \varepsilon$ on $C(G/T)$.

■

Let us now recall the definition of $SU_q(n)$, see e.g. [KS].

The algebra $C(U_q(n))$ of continuous functions on the compact quantum group $U_q(n)$ is generated by n^2+1 elements u_{ij}, $1 \le i, j \le n$, t satisfying the relations

$$u_{ik}u_{jk} = qu_{jk}u_{ik}, \quad u_{ki}u_{kj} = qu_{kj}u_{ki} \;\text{ for } i < j,$$

$$u_{il}u_{jk} = u_{jk}u_{il} \;\text{ for } i < j,\ k < l,$$

$$u_{ik}u_{jl} - u_{jl}u_{ik} = (q - q^{-1})u_{jk}u_{il} \;\text{ for } i < j,\ k < l,$$

$$\det_q(U)t = t\det_q(U) = 1, \quad u_{ij}t = tu_{ij} \;\text{ for any } i, j,$$

where $U = (u_{ij})_{i,j}$ and $\det_q(U) = \sum_{w\in S_n}(-q)^{\ell(w)}u_{w(1)1}\dots u_{w(n)n}$, with $\ell(w)$ being the number of inversions in $w \in S_n$. The involution on $C(U_q(n))$ is given by

$$t^* = \det_q(U), \quad u^*_{ij} = (-q)^{j-i}\det_q(U^{\hat{i}}_{\hat{j}})t,$$

where $U^{\hat{i}}_{\hat{j}}$ is the matrix obtained from U by removing the ith row and the jth column. Taking the quotient of $C(U_q(n))$ by the ideal generated by $\det_q(U) - 1$, we obtain the algebra $C(SU_q(n))$.

If $m < n$, then $U_q(m) \times \mathbb{T}^{n-m}$ is a subgroup of $U_q(n)$. Namely, if u'_{ij}, $1 \le i, j \le m$, and t' are the generators of $C(U_q(m))$, and $z_1, \dots, z_{n-m}$ are the canonical generators of $C(\mathbb{T}^{n-m})$, then the homomorphism $C(U_q(n)) \to C(U_q(m) \times \mathbb{T}^{n-m})$ is given by

$$u_{ij} \mapsto \begin{cases} u'_{ij} & \text{if } 1 \le i, j \le m, \\ z_{i-m} & \text{if } i = j > m, \\ 0 & \text{otherwise,} \end{cases}$$

and thus $t \mapsto t'z_1^{-1}\dots z_{n-m}^{-1}$. Taking the intersection with $SU_q(n)$, in other words, taking the quotient of $C(U_q(m) \times \mathbb{T}^{n-m})$ by the ideal generated by $1 - t'z_1^{-1}\dots z_{n-m}^{-1}$, we get a subgroup $S(U_q(m) \times \mathbb{T}^{n-m}) \cong U_q(m) \times \mathbb{T}^{n-m-1}$ of $SU_q(n)$. The subgroup $T = S(\mathbb{T}^n) \cong \mathbb{T}^{n-1}$ we call the maximal torus of $SU_q(n)$.

Consider now the filtration $T = G_1 \subset G_2 \subset \ldots \subset G_n = SU_q(n)$ of $SU_q(n)$, where $G_m = S(U_q(m) \times \mathbb{T}^{n-m})$, and let $\pi_m\colon C(SU_q(n)) \to C(G_m)$ be the corresponding homomorphisms. We will prove by induction that the counit on $C(G_m)$ is the only $A_{\omega\hat{\pi}_m}$-invariant state on $C(G_m/T)$.

For $m = 1$ there is nothing to prove as $G_1 = T$. Thus by Corollary 2.9 we just have to show that the counit is the only $A_{\omega\hat{\pi}_{m+1}}$-invariant state on $C(G_m\backslash G_{m+1}/G_m)$.

Let $SU_q(2) \hookrightarrow SU_q(n)$ be the embedding corresponding to the right lower corner of $SU_q(m+1)$. In other words, if u'_{ij}, $1 \le i, j \le 2$, are the generators of $C(SU_q(2))$, we define a homomorphism $\theta_m\colon C(SU_q(n)) \to C(SU_q(2))$ by

$$u_{ij} \mapsto \begin{cases} u'_{i-m+1,j-m+1} & \text{if } m \le i, j \le m+1, \\ \delta_{ij} & \text{otherwise.} \end{cases}$$

Note that θ_m factorizes through $\pi_{m+1}\colon C(SU_q(n)) \to C(G_{m+1})$, so $\theta_m = \theta'_m \pi_{m+1}$ for some homomorphism $\theta'_m\colon C(G_{m+1}) \to C(SU_q(2))$.

Lemma 2.10 *For each m, $1 \le m \le n-1$, the homomorphism θ'_m maps $C(G_m\backslash G_{m+1}/G_m)$ isomorphically onto $C(\mathbb{T}\backslash SU_q(2)/\mathbb{T})$.*

Proof. Note that $G_m\backslash G_{m+1}/G_m \cong (U_q(m)\times\mathbb{T})\backslash U_q(m+1)/(U_q(m)\times\mathbb{T})$. Then the result can be deduced from Theorem 4.7 in [NYM], which implies that $C((U_q(m)\times\mathbb{T})\backslash U_q(m+1)/(U_q(m)\times\mathbb{T}))$ is generated by $u_{m+1,m+1}u^*_{m+1,m+1}$.

Another possibility is to use the classification of irreducible representations of algebras of functions on homogeneous spaces, see e.g. [PV, DS]. Namely, let either $G = U_q(n)$, $H = U_q(n-1)\times\mathbb{T}$ and $T = \mathbb{T}^n$, or $G = SU_q(n)$, $H = S(U_q(n-1)\times\mathbb{T})$ and $T = S(\mathbb{T}^n) \cong \mathbb{T}^{n-1}$. Then the irreducible representations of $C(G)$ can be described as follows [KoS]. Consider the homomorphisms $\theta_k\colon C(G) \to C(SU_q(2))$, $1 \le k \le n-1$, and $\pi_1\colon C(G) \to C(T)$ as above. There exists a canonical irreducible infinite dimensional representation π of $C(SU_q(2))$. Then for a character χ of T and an element $w \in S_n$ with a reduced decomposition $w = \tau_{i_1}\dots\tau_{i_k}$, where $\tau_i = (i, i+1)$, $1 \le i \le n-1$, we set

$$\pi_{w,\chi} = (\pi\theta_{i_1}) \times \dots \times (\pi\theta_{i_k}) \times (\chi\pi_1) = ((\pi\theta_{i_1}) \otimes \dots \otimes (\pi\theta_{i_k}) \otimes (\chi\pi_1))\Delta^k.$$

Up to equivalence the representation $\pi_{w,\chi}$ is independent of the reduced decomposition of w, and $\{\pi_{w,\chi}\}_{w\in S_n,\chi\in\hat{T}}$ is a complete set of irreducible representations of $C(G)$. If $w \in S_{n-1}$, then the representation $\pi_{w,\chi}$ factorizes through $C(H)$, so its restrictions to $C(G/H)$ and $C(H\backslash G)$ are given by the counit. If $w \in S_n\backslash S_{n-1}$, then w can be written as $w_1\tau_{n-1}w_2$ with $\ell(w) = \ell(w_1)+\ell(w_2)+1$ and $w_1, w_2 \in S_{n-1}$. Since the representations $\pi_{w_1,0}$ and $\pi_{w_2,\chi}$ factorize through $C(H)$, and

$$\Delta^2(C(H\backslash G/H)) \subset C(H\backslash G) \otimes C(G) \otimes C(G/H),$$

we see that $\pi_{w,\chi}(a) = 1 \otimes (\pi\theta_{n-1})(a) \otimes 1$ for any $a \in C(H\backslash G/H)$. Thus the restriction of any irreducible representation of $C(G)$ to $C(H\backslash G/H)$ factorizes through $\theta_{n-1}\colon C(G) \to C(SU_q(2))$. Hence $\theta_{n-1}\colon C(H\backslash G/H) \to C(SU_q(2))$ is injective. As $T \subset H$, the image is obviously contained in $C(\mathbb{T}\backslash SU_q(2)/\mathbb{T})$. In fact, it coincides with $C(\mathbb{T}\backslash SU_q(2)/\mathbb{T})$, since e.g. it is easy to check that $\theta_{n-1}(u_{nn}u^*_{nn})$ generates $C(\mathbb{T}\backslash SU_q(2)/\mathbb{T})$.

■

Now to prove Theorem 2.5 we just have to show that the counit on $C(SU_q(2))$ is the only $A_{\omega\hat{\theta}_m}$-invariant state on $C(\mathbb{T}\backslash SU_q(2)/\mathbb{T})$ for $1 \le m \le n-1$. Note that as $SU(n)$ is a simple Lie group, the restriction of a non-trivial representation of $SU(n)$ to a non-discrete subgroup is non-trivial. It follows that if ω is non-trivial on $l^\infty(\widehat{SU_q(n)})$, that is, $\omega \ne \hat{\varepsilon}$, then $\omega\hat{\theta}_m$ is non-trivial on $l^\infty(\widehat{SU_q(2)})$. Thus we can reduce the proof of Theorem 2.5 to the case of $SU_q(2)$, moreover, in this case it suffices to prove that the counit is the only A_ω-invariant state on $C(\mathbb{T}\backslash SU_q(2)/\mathbb{T})$. For this we could use the results of [I1, NT1] saying that the Poisson integral is a homomorphism on $C(SU_q(2)/\mathbb{T})$. We will instead give a self-contained probabilistic proof.

Let $\{u_{ij}\}_{1\le i,j\le 2}$ be the generators of $C(SU_q(2))$. Set $\alpha = u_{11}$ and $\gamma = u_{21}$. Then

$$U = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} = \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix},$$

and the relations can be written as

$$\alpha^*\alpha + \gamma^*\gamma = 1, \quad \alpha\alpha^* + q^2\gamma^*\gamma = 1, \quad \gamma^*\gamma = \gamma\gamma^*, \quad \alpha\gamma = q\gamma\alpha, \quad \alpha\gamma^* = q\gamma^*\alpha.$$

The comultiplication Δ is determined by the formulas

$$\Delta(\alpha) = u_{11}\otimes u_{11} + u_{12}\otimes u_{21} = \alpha\otimes\alpha - q\gamma^*\otimes\gamma, \quad \Delta(\gamma) = \gamma\otimes\alpha + \alpha^*\otimes\gamma.$$

The homomorphism $C(SU_q(2)) \to C(\mathbb{T})$ is given by $\alpha \mapsto z$, $\gamma \mapsto 0$. The monomials $\alpha^k(\gamma^*)^l\gamma^m$ and $(\alpha^*)^k(\gamma^*)^l\gamma^m$, $k,l,m \ge 0$, span a dense $*$-subalgebra of $C(SU_q(2))$. It is then easy to see that $C(\mathbb{T}\backslash SU_q(2)/\mathbb{T})$ is generated by $\gamma^*\gamma$. The spectrum of $\gamma^*\gamma$ is the set $I_{q^2} = \{0\} \cup \{q^{2n}\}_{n=0}^\infty$. Thus we can identify $C(\mathbb{T}\backslash SU_q(2)/\mathbb{T})$ with the algebra $C(I_{q^2})$ of continuous functions on I_{q^2}. Under this identification the counit is given by the evaluation at $0 \in I_{q^2}$. The Markov operator A_ω defines a random walk on $I_{q^2}\backslash\{0\}$. If this random walk is transient, then $\nu A_\omega^n \to \varepsilon$ as $n \to \infty$ for any state ν on $C(\mathbb{T}\backslash SU_q(2)/\mathbb{T})$. In particular, the counit ε is the only A_ω-invariant state. As was remarked in [NT1], transience of a random walk on a non-Kac discrete quantum group follows easily from the fact that the Markov operator has a positive eigenvector with eigenvalue strictly smaller than 1. It is natural to expect that the operator A_ω acting on the dual side has the same property.

Proposition 2.11 *Consider the function f on I_{q^2} defined by $f(0) = 0$, $f(q^{2k}) = a_k$, where $\{a_k\}_{k=0}^\infty$ is the sequence defined by the recurrence relation*

$$a_0 = 1,$$
$$2(1-q^{2k+1})a_k = q^{-1}(1-q^{2k+2})a_{k+1} + q(1-q^{2k})a_{k-1}.$$

Then
(i) *$f \in C(I_{q^2})$ and $f(q^{2k}) > 0$ for any $k \ge 0$;*
(ii) *for any normal right $SU_q(2)$-invariant state $\omega = \sum_s \lambda_s\omega_s \in \mathcal{C}_r(\widehat{SU_q(2)})$, the element f is an eigenvector for A_ω with eigenvalue $\displaystyle\sum_s \lambda_s \frac{2s+1}{[2s+1]_q}$.*

As usual, we identify the set $\mathrm{Irr}(SU_q(2))$ with $\frac{1}{2}\mathbb{Z}_+$. Then $[2s+1]_q = \dfrac{q^{2s+1} - q^{-2s-1}}{q - q^{-1}}$ is the quantum dimension of the representation with spin $s \in \frac{1}{2}\mathbb{Z}_+$.

Proof of Proposition 2.11. The proof of (i) is analogous to that of [I1, Lemma 5.4]. To see that $a_k > 0$, rewrite the recurrence relation as

$$q^{-1}(1-q^{2k+2})(a_{k+1} - qa_k) = (1-q^{2k})(a_k - qa_{k-1}) + q^{2k}(1-q)^2a_k.$$

It follows by induction that $a_{k+1} - qa_k \ge 0$, so $a_k \ge q^k$. It remains to show that $a_k \to 0$. It is clear that the sequence $\{a_k\}_{k=0}^\infty$ cannot grow faster than a geometric progression. Hence the generating function

$$g(z) = \sum_{k=0}^\infty a_k z^k$$

is analytic in a neighborhood of zero. The recurrence relation can then be written as

$$2(g(z)-qg(q^2z)) = q^{-1}z^{-1}(g(z)-g(q^2z)) + qz(g(z)-q^2g(q^2z)),$$

that is,

$$g(z) = \left(\frac{1-q^2z}{1-qz}\right)^2 g(q^2z).$$

We see that g extends to a meromorphic function with poles at $z=q^{-2k-1}$, $k\geq 0$. In particular, the series $\sum_k a_k z^k$ converges for $|z|<q^{-1}$, whence $a_k\to 0$. In fact, since

$$\lim_{z\to q^{-1}}(1-qz)^2g(z) = \prod_{k=0}^{\infty}\left(\frac{1-q^{2k+1}}{1-q^{2k+2}}\right)^2 = \frac{(q;q^2)_\infty^2}{(q^2;q^2)_\infty^2},$$

we have $a_k \sim kq^k\dfrac{(q;q^2)_\infty^2}{(q^2;q^2)_\infty^2}$.

To prove (ii), first consider the case $\omega=\omega_{\frac12}$, so that

$$\omega = \frac{1}{[2]_q}\mathrm{Tr}\left(\cdot\begin{pmatrix} q^{-1} & 0\\ 0 & q\end{pmatrix}\right)$$

as $U^{\frac12}=U$. Then we have

$$\begin{aligned} A_\omega(x) &= \frac{1}{[2]_q}\mathrm{Tr}\left(\begin{pmatrix}\alpha & -q\gamma^*\\ \gamma & \alpha^*\end{pmatrix}\begin{pmatrix} x & 0\\ 0 & x\end{pmatrix}\begin{pmatrix}\alpha^* & \gamma^*\\ -q\gamma & \alpha\end{pmatrix}\begin{pmatrix} q^{-1} & 0\\ 0 & q\end{pmatrix}\right)\\ &= \frac{1}{[2]_q}(q^{-1}(\alpha x\alpha^* + q^2\gamma^*x\gamma) + q(\gamma x\gamma^* + \alpha^*x\alpha)). \end{aligned}$$

Identifying $C(\mathbb{T}\backslash SU_q(2)/\mathbb{T}) = C^*(\gamma^*\gamma)$ with $C(I_{q^2})$, and using the identities

$$\alpha^*(\gamma^*\gamma)^k\alpha = q^{-2k}(\gamma^*\gamma)^k(1-\gamma^*\gamma),\quad \alpha(\gamma^*\gamma)^k\alpha^* = q^{2k}(\gamma^*\gamma)^k(1-q^2\gamma^*\gamma),$$

we see that the action of A_ω on the functions on I_{q^2} is given by

$$(A_\omega h)(t) = \frac{1}{[2]_q}\Big(q^{-1}\Big((1-q^2t)h(q^2t) + q^2th(t)\Big) + q\Big(th(t) + (1-t)h(q^{-2}t)\Big)\Big).$$

Then the definition of $\{a_k\}_k$ shows that $A_\omega f = \frac{2}{[2]_q}f$. To prove that f is an eigenvalue for A_{ω_s} for any s, recall from [I1, Section 6] that the identity

$$\omega_s\omega_{\frac12} = \frac{d_{s-\frac12}}{d_sd_{\frac12}}\omega_{s-\frac12} + \frac{d_{s+\frac12}}{d_sd_{\frac12}}\omega_{s+\frac12},$$

where $d_s=[2s+1]_q$, implies that there exists a polynomial p_{2s} of degree $2s$ such that $p_{2s}(\omega_{\frac12})=\omega_s$. Then $A_{\omega_s}=p_{2s}(A_{\omega_{\frac12}})$, so f is an eigenvector for A_{ω_s} with eigenvalue $p_{2s}(\frac{2}{[2]_q})$. As $\frac{2s+1}{[2s+1]_q}=\omega_s(\rho^{-1})$, and ρ is group-like, we have

$$p_{2s}\left(\frac{2}{[2]_q}\right) = p_{2s}(\omega_{\frac12}(\rho^{-1})) = p_{2s}(\omega_{\frac12})(\rho^{-1}) = \omega_s(\rho^{-1}) = \frac{2s+1}{[2s+1]_q},$$

which finishes the proof of Proposition.

■

It follows that the random walk defined by A_ω on $I_{q^2}\backslash\{0\}$ is transient for any normal right-invariant state $\omega \ne \hat\varepsilon$. More precisely, if $A_\omega f = \lambda f$, then the probability of visiting a point t_2 from a point t_1 at the nth step is not larger than $f(t_1)f(t_2)^{-1}\lambda^n$. Hence $\nu A_\omega^n \to \delta_0 = \varepsilon$ as $n \to \infty$ for any state ν on $C(I_{q^2})$. Note that to see that ε is the only A_ω-invariant state is even easier. Indeed, if ν is A_ω-invariant, we have $\nu(f) = \lambda\nu(f)$, so $\nu(f) = 0$ and $\nu = \delta_0$. This completes the proof of Theorem 2.5.

3 Random walk on the center

To prove surjectivity of the Poisson integral, we will obtain an estimate on the dimensions of the spectral subspaces of $H^\infty(\hat G, \phi)$. By a result of Hayashi [H], if the fusion algebra of a group G is commutative, then any central harmonic element is a scalar. Equivalently, the action of G on the Poisson boundary is ergodic. This already implies that the spectral subspaces of $H^\infty(\hat G, \phi)$ are finite dimensional, more precisely, the dimension of the spectral subspace corresponding to an irreducible representation U is not larger than the square of the quantum dimension of U [B, HLS]. This estimate is clearly not sufficient for our purposes. We will show that in our situation ergodicity of the action provides a better estimate.

The result of Hayashi is in fact more general. It was obtained as a consequence of an analogue of double ergodicity of the Poisson boundary, see e.g. [K]. Since in our situation the proof can be made more concrete, we will present a detailed argument.

For Markov operators P and Q on a von Neumann algebra, we say that an element x is (P, Q)-harmonic if $P(x) = Q(x) = x$. Let now ϕ and ω be generating normal states on $l^\infty(\hat G)$ with the same support. Set

$$(N, \nu) = \Big(\mathop{\otimes}_{-\infty}^{-1}(l^\infty(\hat G), \phi)\Big) \bigotimes \Big(\mathop{\otimes}_{0}^{+\infty}(l^\infty(\hat G), \omega)\Big),$$

and let γ be the shift to the right on N. For any finite interval $I = [n, m] \subset \mathbb{Z}$ we have a normal homomorphism $j_I\colon l^\infty(\hat G) \to N$ defined by $\hat\Delta^{(m-n)}$. Then the space of (P_ϕ, Q_ω)-harmonic elements, where $P_\phi = (\phi\otimes\iota)\hat\Delta$ and $Q_\omega = (\iota\otimes\omega)\hat\Delta$, can be embedded in the space N^γ of γ-invariant elements by the homomorphism $j_{\mathbb{Z}}$,

$$j_{\mathbb{Z}}(x) = s^* - \lim_{\substack{n\to-\infty\\ m\to+\infty}} j_{[n,m]}(x).$$

Note that if $\phi \ne \omega$, then the automorphism γ is never ergodic [K, Lemma 4]. Nevertheless the following result holds.

Proposition 3.1 *[H, Proposition 3.4] Let $\phi \in \mathcal{C}_l(\hat G)$ be a generating normal left G-invariant state on $l^\infty(\hat G)$, $\omega = \phi(\cdot\rho^2) \in \mathcal{C}_r(\hat G)$ the corresponding right-invariant state. Then the space of central (P_ϕ, Q_ω)-harmonic elements consists of the scalars.*

Proof. Note that by definition $\phi^n = \omega^n$ on the center of $l^\infty(\hat G)$. Consider the operator $Q_\phi = (\iota\otimes\phi)\hat\Delta$. Although in general $Q_\phi \ne Q_\omega$, for any left-invariant state ϕ' we have

$$\phi' Q_\phi^n = \phi^n P_{\phi'} = \omega^n P_{\phi'} = \phi' Q_\omega^n$$

on the center. Similarly, for any right-invariant state ω' we have $\omega' P_\omega^n = \omega' P_\phi^n$ on the center. It follows that for any central elements z_1 and z_2 and any intervals $I_1 \subset I_2$, $I_1 = [n, m]$, $I_2 = [n-k, m+l]$, we have

$$\nu(j_{I_1}(z_1)j_{I_2}(z_2)) = \phi^{m-n+1}(z_1 Q_\omega^l P_\phi^k(z_2)).$$

Indeed, e.g. in the case when $m+l<0$ we get

$$\nu(j_{I_1}(z_1)j_{I_2}(z_2)) = \phi^{m-n+1}(z_1Q^l_\phi P^k_\phi(z_2)) = \phi^{m-n+1}(z_1Q^l_\omega P^k_\phi(z_2)).$$

Hence for any central z_1 and z_2 and any finite intervals $I_1\subset I_2$ we have

$$||j_{I_1}(z_1) - j_{I_2}(z_2)||_2 = ||\gamma j_{I_1}(z_1) - \gamma j_{I_2}(z_2)||_2$$

Thus if z is a central (P_ϕ, Q_ω)-harmonic element, then the distance $||j_{\mathbb{Z}}(z) - \gamma^n j_I(z)||_2$ is independent of n. Since on the one hand this distance goes to zero as $I\nearrow\mathbb{Z}$, and on the other hand $\gamma^n j_I(z)$ converges in weak operator topology to a scalar as $n\to+\infty$, we conclude that $j_{\mathbb{Z}}(z)$ is a scalar.

■

Corollary 3.2 *[H, Corollary 3.5] Assume that the fusion algebra $R(G)$ of the group G is commutative. Then for any generating state $\phi\in\mathcal{C}_l(\hat G)$, the scalars are the only central P_ϕ-harmonic elements.*

Proof. Commutativity of the fusion algebra means that $P_\phi = Q_\omega$ on the center. Thus any central P_ϕ-harmonic element is (P_ϕ, Q_ω)-harmonic, and we can apply the previous proposition.

■

Consider now the random walk on the center in more detail. Identify the center of $l^\infty(\hat G)$ with $l^\infty(I)$, where $I=\mathrm{Irr}(G)$. For a fixed generating state $\phi\in\mathcal{C}_l(\hat G)$, let $\{p(s,t)\}_{s,t\in I}$ be the transition probabilities defined by the restriction of P_ϕ to $l^\infty(I)$, so $P_\phi(I_t)I_s = p(s,t)I_s$. Let $(\Omega, \mathbb{P}_0)$ be the path space of the corresponding random walk,

$$\Omega = \prod_{n=1}^\infty I,\ \ \mathbb{P}_0(\{\underline{s}\,|\,s_1=t_1,\dots,s_n=t_n\}) = p(0,t_1)p(t_1,t_2)\dots p(t_{n-1},t_n).$$

Denote by π_n the projection $\Omega\to I$ onto the nth factor.

Similarly to Section 1, set

$$(N,\nu) = \overset{-1}{\underset{-\infty}{\otimes}}(l^\infty(\hat G),\phi),$$

$j_n(x) = \ldots\otimes 1\otimes\hat\Delta^{n-1}(x)$ for $x\in l^\infty(\hat G)$, and $j_\infty(x) = s^*-\lim_n j_n(x)$ for $x\in H^\infty(\hat G,\phi)$. (In Section 1 we embedded $F_n(l^\infty(\hat G))$ into $B(H)$ for some H and extended ϕ to a G-invariant normal faithful state $\tilde\phi$ on $B(H)$, which we don't do now as the relative commutant interpretation of the Poisson boundary will not be important.) As was remarked in [NT1], there is an embedding

$$j^\infty\colon (L^\infty(\Omega,\mathbb{P}_0),\mathbb{P}_0)\hookrightarrow(N,\nu)$$

such that $f\pi_n\mapsto j_n(f)$ for any $f\in l^\infty(I)\subset l^\infty(\hat G)$. If $f\in l^\infty(I)$ is harmonic, then the sequence $\{f\pi_n\}_{n=1}^\infty$ is a martingale, so it converges almost everywhere (a.e.) to a function $f_\infty\in L^\infty(\Omega,\mathbb{P}_0)$. Then $j_\infty(f) = j^\infty(f_\infty)$.

Denote by $H^\infty(I,\phi)$ the space $l^\infty(I)\cap H^\infty(\hat G,\phi)$ of central harmonic elements. Let $E\colon l^\infty(\hat G)\to l^\infty(I)$ be the unique G-equivariant conditional expectation,

$$E(x) = (\varphi\otimes\iota)\Phi_l(x) = \sum_{s\in I}\phi_s(x)I_s.$$

By restricting E to $H^\infty(\hat G,\phi)$ we get a conditional expectation $H^\infty(\hat G,\phi)\to H^\infty(I,\phi)$.

Proposition 3.3 *Let $x, y \in H^\infty(\hat{G}, \phi)$. Then the sequence $\{f_n\}_{n=1}^\infty$ of functions on Ω defined by*

$$f_n(\underline{s}) = \phi_{s_n}(xy)$$

converges a.e. to $E(x \cdot y)_\infty \in L^\infty(\Omega, \mathbb{P}_0)$.

Since $f_n = E(xy)\pi_n$, one can equivalently state that $\{E(xy)\pi_n\}_n$ and $\{E(x \cdot y)\pi_n\}_n$ converge a.e. to the same limit.

Proof of Proposition 3.3. Let α be the product-type action of G on N, and $\tilde{E} = (\varphi \otimes \iota)\alpha \colon N \to N^\alpha$ the ν-preserving conditional expectation. Since $j_n(x) \to j_\infty(x)$ and $j_n(y) \to j_\infty(y)$ in s^*-topology, we have

$$j_n(xy) = j_n(x)j_n(y) \to j_\infty(x)j_\infty(y) = j_\infty(x \cdot y)$$

in s^*-topology. Hence $\tilde{E}j_n(xy) \to \tilde{E}j_\infty(x \cdot y)$, and as $\tilde{E}j_n = j_nE$, we get $j_nE(xy) \to j_\infty E(x \cdot y)$ in strong* operator topology. Using that $f_n = E(xy)\pi_n$, $j_nE(xy) = j^\infty(E(xy)\pi_n)$ and $j_\infty E(x \cdot y) = j^\infty(E(x \cdot y)_\infty)$, we conclude that $f_n \to E(x \cdot y)_\infty$ in measure. It remains to show that the sequence $\{f_n\}_n$ is a.e. convergent.

It is enough to consider the case $x = y^*$. Let $L^\infty(I^n, \mathbb{P}_0^{(n)})$ be the subalgebra of $L^\infty(\Omega, \mathbb{P}_0)$ consisting of the functions depending only on the first n coordinates, $E_n \colon L^\infty(\Omega, \mathbb{P}_0) \to L^\infty(I^n, \mathbb{P}_0^{(n)})$ the $\mathbb{P}_0$-preserving conditional expectation. For any $f \in l^\infty(I)$ we have $E_n(f\pi_{n+1}) = P_\phi(f)\pi_n$. As

$$y^*y = P_\phi(y)^*P_\phi(y) \le P_\phi(y^*y)$$

by Schwarz inequality, we have $E(y^*y) \le EP_\phi(y^*y) = P_\phi E(y^*y)$, whence

$$f_n = E(y^*y)\pi_n \le P_\phi(E(y^*y))\pi_n = E_n(E(y^*y)\pi_{n+1}) = E_n(f_{n+1}).$$

Thus the sequence $\{f_n\}_{n=1}^\infty$ is a bounded submartingale. By Doob's theorem, see e.g. [KSK], it must converge a.e.

■

Corollary 3.4 *Let $\phi \in \mathcal{C}_l(\hat{G})$ be a generating state. Assume that the Poisson boundary of the center is trivial, i.e. $H^\infty(I, \phi) = \mathbb{C}1$. Then for any $x, y \in H^\infty(\hat{G}, \phi)$ and almost every path $\underline{s} \in \Omega$, we have $\phi_{s_n}(xy) \to \hat{\varepsilon}(x \cdot y)$ as $n \to \infty$.* ■

Corollary 3.5 *Let $\phi \in \mathcal{C}_l(\hat{G})$ be a generating state, V an irreducible representation of G. Assume that the Poisson boundary of the center is trivial. Then the multiplicity of V in $H^\infty(\hat{G}, \phi)$ is not larger than the supremum of the multiplicities of V in $\overline{U} \times U$ for all irreducible representations U of G.*

Proof. By the previous corollary, for any finite dimensional subspace X of $H^\infty(\hat{G}, \phi)$ and almost every path $\underline{s} \in \Omega$, the restrictions of the irreducible representations $l^\infty(\hat{G}) \to B(H_{s_n})$ to X are asymptotically isometric in L^2-norm as $n \to \infty$. In particular, these restrictions are eventually injective. Since the maps $H^\infty(\hat{G}, \phi) \to B(H_s)$ are G-equivariant, it follows that the multiplicity of V in $H^\infty(\hat{G}, \phi)$ is not larger than the supremum of the multiplicities of V in $B(H_U)$ for all irreducible representations U of G on H_U. It remains to note that the G-module $B(H_U)$, or more precisely, the $\mathcal{A}(\hat{G})$-module such that $\omega x = (\hat{S}(\omega) \otimes \iota)\alpha_{U,l}(x)$ for $\omega \in \mathcal{A}(\hat{G})$ and $x \in B(H_U)$, is isomorphic to $\overline{H}_U \otimes H_U$.

■

For the q-deformation G of a compact connected semisimple Lie group the last estimate is optimal. Indeed, let $T \subset G$ be the maximal torus. Then for an irreducible representation V of G

on H, the multiplicity of V in $L^\infty(G/T)$ is equal to the dimension of the space of zero weight vectors in H, that is, the space of T-invariant vectors. On the other hand, by the Frobenius reciprocity the multiplicity of V in $\overline{U}\times U$ is the same as the multiplicity of U in $U\times V$. Both the multiplicity $N^U_{U,V}$ of U in $U\times V$ and the dimension $m_0(V)$ of the space of zero weight vectors are known to be independent of the deformation parameter. Hence $N^U_{U,V}\le m_0(V)$, see e.g. [Ž, §131]. It follows that the spectral subspaces of $H^\infty(\hat G,\phi)$ are not larger than the spectral subspaces of $L^\infty(G/T)$. Note also that as $\hat\varepsilon\Theta=\varphi$ and φ is faithful, the Poisson integral is injective on its multiplicative domain. Thus we get the following result.

Theorem 3.6 *Let G be the q-deformation of a compact connected semisimple Lie group, $T\subset G$ the maximal torus, $\phi\in\mathcal{C}_l(\hat G)$ a generating state. Assume that the Poisson integral $\Theta\colon L^\infty(G/T)\to H^\infty(\hat G,\phi)$ is a homomorphism. Then it is an isomorphism.* ■

Theorems A and B now follow from Proposition 2.4, Theorem 2.5 and Theorem 3.6. Indeed, it follows immediately that if $\phi\in\mathcal{C}_l(l^\infty(\widehat{SU_q(n)}))$ is a generating state, then the Poisson integral $\Theta\colon L^\infty(SU_q(n)/\mathbb{T}^{n-1})\to H^\infty(\widehat{SU_q(n)},\phi)$ is a $SU_q(n)$- and $\widehat{SU_q(n)}$-equivariant isomorphism, where $\mathbb{T}^{n-1}=S(\mathbb{T}^n)$ is the maximal torus in $SU_q(n)$. If ϕ is not generating, then $\cup_k\operatorname{supp}\phi^k$ corresponds to a quotient $SU(n)/\Gamma$ of $SU(n)$ and to the quotient $G=SU_q(n)/\Gamma$ of $SU_q(n)$, which we call the q-deformation of $SU(n)/\Gamma$. More explicitly, if Γ is the group of roots of unity of order m, $m|n$, then $C(G)$ is the subalgebra of $C(SU_q(n))$ generated by the matrix coefficients of $U^{\times m}$, where U is the fundamental representation of $SU_q(n)$. Set $T=\mathbb{T}^{n-1}/\Gamma$, so $C(T)$ is the image of $C(G)$ under the homomorphism $C(SU_q(n))\to C(\mathbb{T}^{n-1})$. Then $L^\infty(G/T)=L^\infty(SU_q(n)/\mathbb{T}^{n-1})\subset L^\infty(SU_q(n))$. Since the assumptions of Theorem 2.5 don't require $\omega=\phi(\cdot\rho^2)$ to be generating on $\widehat{SU_q(n)}$, we again conclude that the Poisson integral $\Theta\colon L^\infty(G/T)\to H^\infty(\hat G,\phi)$ is a G- and $\hat G$-equivariant isomorphism. This completes the proof of Theorem B. To prove Theorem A, note that the fixed point algebra is independent of whether we consider the action of $SU_q(n)$, or the action of its quotient G such that $\operatorname{Irr}(G)=\cup_k\operatorname{supp}\phi^k$. Thus $(N^\alpha)'\cap N$ is G-equivariantly isomorphic to $L^\infty(G/T)=L^\infty(SU_q(n)/\mathbb{T}^{n-1})$. Clearly, the isomorphism is $SU_q(n)$-equivariant. If we identify $(N^\alpha)'\cap N$ with $H^\infty(\hat G,\phi)$, so we get an action of $\hat G$ on $(N^\alpha)'\cap N$, then the isomorphism is also $\hat G$-equivariant.

4 Concluding remarks

For any non-trivial product-type action of $SU_q(n)$ on N the fixed point algebra $N^{SU_q(n)}$ is obviously strictly contained in the fixed point algebra N^T for the action of the maximal torus $T\subset SU_q(n)$ (contrary to what is claimed in [S2]). Moreover, by our results

$$(N^{SU_q(n)})'\cap N^T\cong L^\infty(T\backslash SU_q(n)/T).$$

Consider the product-type action defined by the fundamental representation of $SU_q(n)$ on $\mathbb{C}^n$. Let $\mathcal{H}_\infty(q)$ be the Hecke algebra, that is, the algebra with generators $g_1,g_2,\dots$ and relations

$$g_i^2=(q-q^{-1})g_i+1,\quad g_ig_{i+1}g_i=g_{i+1}g_ig_{i+1},\quad g_ig_j=g_jg_i\ \text{ for }\ |i-j|\ge 2.$$

Then $N^{SU_q(n)}$ is the weak operator closure of the image of $\mathcal{H}_\infty(q)$ under the homomorphism $\pi\colon\mathcal{H}_\infty(q)\to N$ defined by

$$\pi(g_1)=\ldots\otimes 1\otimes\left(q\sum_i m_{ii}\otimes m_{ii}+(q-q^{-1})\sum_{i<j}m_{ii}\otimes m_{jj}+\sum_{i\ne j}m_{ij}\otimes m_{ji}\right),$$

$\pi(g_n) = \gamma^{n-1}\pi(g_1)$, where $\gamma\colon N \to N$ is the shift to the left, see e.g. [KS, Proposition 8.40]. The homomorphism π should not be confused with the homomorphisms π_+ and π_- defined by

$$\pi_\pm(g_1) = \ldots \otimes 1 \otimes \left(q\sum_i m_{ii} \otimes m_{ii} + (q-q^{-1})\sum_{i>j} m_{ii}\otimes m_{jj} \pm \sum_{i\neq j} m_{ij}\otimes m_{ji} \right),$$

$\pi_\pm(g_n) = \gamma^{n-1}\pi_\pm(g_1)$. Using that the unique left $SU_q(n)$-invariant state on $B(\mathbb{C}^n)$ is defined by the density matrix

$$\frac{1-q^2}{1-q^{2n}} \begin{pmatrix} q^{2(n-1)} & 0 & \ldots & 0 \\ 0 & q^{2(n-2)} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \end{pmatrix},$$

we see that if $E\colon N \to \gamma(N)$ is the ν-preserving conditional expectation, then $E\pi_+(g_1)$ and $E\pi_-(g_1)$ are scalars, while $E\pi(g_1)$ is not. According to [PP, S1], one has $\pi_+(\mathcal{H}_\infty(q))'' = \pi_-(\mathcal{H}_\infty(q))'' = N^T$.

As we showed in Section 2, if an element $a \in C(G)$ is in the multiplicative domain of Θ, then

$$a = \lim_{n\to\infty} \sum_{s\in I} \phi^n(I_s)\Theta_s^*\Theta_s(a).$$

If in addition the Poisson boundary of the center is trivial, then we have a stronger convergence result:

$$a = \lim_{n\to\infty} \Theta_{s_n}^*\Theta_{s_n}(a) \tag{4.1}$$

for almost every path $\underline{s} \in \Omega$. Indeed, first note that as we assume that $l^\infty(\hat{G})$ has a generating state, the space $C(G)$ is separable. Hence by Corollary 3.4 we have $\phi_{s_n}(\Theta(a)\Theta(b)) \to \hat{\varepsilon}(\Theta(a)\cdot\Theta(b))$ for almost every path $\underline{s} \in \Omega$ and any $b \in C(G)$. Since

$$\phi_{s_n}(\Theta(a)\Theta(b)) = \varphi(\Theta_{s_n}^*\Theta_{s_n}(a)b) \;\text{ and }\; \hat{\varepsilon}(\Theta(a)\cdot\Theta(b)) = \hat{\varepsilon}(\Theta(ab)) = \varphi(ab),$$

we see that convergence (4.1) holds in weak operator topology for almost every path $\underline{s} \in \Omega$. As in Section 2, by G-equivariance of $\Theta_{s_n}^*\Theta_{s_n}$ we conclude that the convergence is in norm.

Let now $G = SU_q(n)$. Identify $I = \mathrm{Irr}(SU_q(n))$ with the set of dominant weights of $SU(n)$. Let V_λ be an irreducible representation of $SU_q(n)$ with highest weight λ. What we used in Section 3, is that the multiplicity of V_λ in $B(H_s)$ is not larger than the multiplicity of V_λ in $C(SU_q(n)/T)$. In fact, the multiplicities are equal as soon as s is sufficiently large, see [Ž]. It is e.g. enough to require $s + w\lambda$ to be dominant for any element w of the Weyl group. Recall also that in the classical case Berezin transforms converge to the identity on the flag manifold along any ray in the Weyl chamber. Thus it is natural to conjecture that convergence (4.1) holds for every sequence $\underline{s} = \{s_n\}_{n=1}^\infty$ such that the distance from s_n to the walls of the Weyl chamber goes to infinity.

A significant part of our results is valid for q-deformations of arbitrary compact connected semisimple Lie groups. The point where we crucially used that the group was $SU_q(n)$, was Lemma 2.10, which allowed us to reduce the proof of Theorem 2.5 to the study of a one-dimensional random walk. Lemma 2.10 is also valid for $q = 1$, in which case it is an immediate consequence of the fact that $S(U(m)\times\mathbb{T}) \subset SU(m+1)$ is a Riemannian symmetric pair of rank one. Hence there is hope that similar considerations could work for $SO(n)$, $Sp(n)$ and F_4, see [He, Ch. X, Table V]. This will be discussed in detail elsewhere. For the exceptional groups E_6, E_7, E_8 and G_2, however, our reduction procedure leads us to consider random walks of higher dimensions. The ultimate goal would of course be to find a unified proof. For this it could be instructive to understand the origin of the eigenvector constructed in Proposition 2.11, since to prove that ε is the only A_ω-invariant state on $C(G/T)$, it is enough to find a strictly positive eigenvector for A_ω in the kernel of ε on $C(G/T)$ with eigenvalue less than 1. Remark also that by using commutativity of the fusion algebra as in the proof of Proposition 1.1, one can show that if G is the q-deformation of a compact connected simple Lie group and ε is the only A_ω-invariant state on $C(G/T)$ for some $\omega \neq \hat{\varepsilon}$, then ε is the only invariant state for any $\omega \neq \hat{\varepsilon}$. This, however, does not simplify our considerations in Section 2.

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