

POISSON BOUNDARIES OF MONOIDAL CATEGORIES

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ABSTRACT. Given a rigid C^* -tensor category \mathcal{C} with simple unit and a probability measure μ on the set of isomorphism classes of its simple objects, we define the Poisson boundary of (\mathcal{C}, μ) . This is a new C^* -tensor category \mathcal{P} , generally with nonsimple unit, together with a unitary tensor functor $\Pi: \mathcal{C} \rightarrow \mathcal{P}$. Our main result is that if \mathcal{P} has simple unit (which is a condition on some classical random walk), then Π is a universal unitary tensor functor defining the amenable dimension function on \mathcal{C} . Corollaries of this theorem unify various results in the literature on amenability of C^* -tensor categories, quantum groups, and subfactors.

RÉSUMÉ. Étant données une C^* -catégorie tensorielle rigide \mathcal{C} dont l'objet unité est simple ainsi qu'une mesure de probabilité μ sur l'ensemble de classes d'isomorphisme des objets simples, nous définissons la frontière de Poisson de (\mathcal{C}, μ) . C'est une nouvelle C^* -catégorie tensorielle \mathcal{P} dont l'objet unité n'est pas, en général, simple, couplée avec un foncteur unitaire tensoriel $\Pi: \mathcal{C} \rightarrow \mathcal{P}$. Notre résultat principal assure que si l'objet unité de \mathcal{P} est simple (ce qui se traduit par une condition sur une certaine marche aléatoire classique), alors Π est un foncteur unitaire tensoriel universel qui définit la fonction de dimension moyennable sur \mathcal{C} . Les corollaires de ce théorème unifient différents résultats connus sur la moyennabilité des C^* -catégories tensorielles, des groupes quantiques et des sous-facteurs.

INTRODUCTION

The notion of amenability for monoidal categories first appeared in Popa's seminal work [Pop94] on classification of subfactors as a crucial condition defining a class of inclusions admitting good classification. He then gave various characterizations of this property analogous to the usual amenability conditions for discrete groups: a Kesten type condition on the norm of the principal graph, a Følner type condition on the existence of almost invariant sets, and a Shannon–McMillan–Breiman type condition on relative entropy, to name a few.

This stimulated a number of interesting developments in related fields of operator algebras. First, Longo and Roberts [LR97] developed a general theory of dimension for C^* -tensor categories, and indicated that the language of sectors/subfactors is well suited for studying amenability in this context. Then Hiai and Izumi [HI98] studied amenability for fusion algebras/hypergroups endowed with a probability measure, and obtained many characterizations of this property in terms of random walks and almost invariant vectors in the associated ℓ^p -spaces. These studies were followed by the work of Hayashi and Yamagami [HY00], who established a way to realize amenable monoidal categories as bimodule categories over the hyperfinite II_1 factor.

In addition to the subfactor theory, another source of interesting monoidal categories is the theory of quantum groups. In this framework, the amenability question concerns the existence of almost invariant vectors and invariant means for a discrete quantum group, or some property of the dimension function on the category of unitary representations of a compact quantum group [Ban99, Tom06, BCT05]. Here, one should be aware that there are two different notions of amenability involved.

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One is coamenability of compact quantum groups (equivalently, amenability of their discrete duals) considered in the regular representations, the other is amenability of representation categories. These notions coincide only for quantum groups of Kac type.

In yet another direction, Izumi [Izu02] developed a theory of noncommutative Poisson boundary for discrete quantum groups in order to study the minimality (or lack thereof) of infinite tensor product type actions of compact quantum groups. From the subsequent work [INT06, Tom07] it became increasingly clear that for coamenable compact quantum groups the Poisson boundary captures a very elaborate difference between the two amenability conditions. Later, an important result on noncommutative Poisson boundaries was obtained by De Rijdt and Vander Venet [DRVV10], who found a way to compute the boundaries through monoidal equivalences. In light of the categorical duality for compact quantum group actions recently developed in [DCY13, Nes14], this result suggests that the Poisson boundary should really be an intrinsic notion of the representation category $\text{Rep } G$ itself, rather than of the choice of a fiber functor giving a concrete realization of $\text{Rep } G$ as a category of Hilbert spaces. Starting from this observation, in this paper we define Poisson boundaries for monoidal categories.

To be more precise, our construction takes a rigid C^* -tensor category \mathcal{C} with simple unit and a probability measure μ on the set $\text{Irr}(\mathcal{C})$ of isomorphism classes of simple objects, and gives another C^* -tensor category \mathcal{P} together with a unitary tensor functor $\Pi: \mathcal{C} \rightarrow \mathcal{P}$. Although the category \mathcal{P} is defined purely categorically, there are several equivalent ways to describe it, or at least its morphism sets, that are more familiar to the operator algebraists. One is an analogue of the standard description of classical Poisson boundaries as ergodic components of the time shift. Another is in terms of relative commutants of von Neumann algebras, in the spirit of [LR97, HY00, Izu02]. For categories arising from subfactors and quantum groups, this can be made even more concrete. For subfactors, computing the Poisson boundary essentially corresponds to passing to the standard model of a subfactor [Pop94]. For quantum groups, not surprisingly as this was our initial motivation, the Poisson boundary of the representation category of G can be described in terms of the Poisson boundary of \hat{G} . The last result will be discussed in detail in a separate publication [NY14], since we also want to describe the action of \hat{G} on the boundary in categorical terms and this would lead us away from the main subject of this paper.

Our main result is that if \mathcal{P} has simple unit, which corresponds to ergodicity of the classical random walk defined by μ on $\text{Irr}(\mathcal{C})$, then $\Pi: \mathcal{C} \rightarrow \mathcal{P}$ is a universal unitary tensor functor which induces the amenable dimension function on \mathcal{C} . From this we conclude that \mathcal{C} is amenable if and only if there exists a measure μ such that Π is a monoidal equivalence. The last result is a direct generalization of the famous characterization of amenability of discrete groups in terms of their Poisson boundaries due to Furstenberg [Fur73], Kaimanovich and Vershik [KV83], and Rosenblatt [Ros81]. From this comparison it should be clear that, contrary to the usual considerations in subfactor theory, it is not enough to work only with finitely supported measures, since there are amenable groups which do not admit any finitely supported ergodic measures [KV83]. The characterization of amenability in terms of Poisson boundaries generalizes several results in [Pop94, LR97, HY00]. Our main result also allows us to describe functors that factor through Π in terms of categorical invariant means. For quantum groups this essentially reduces to the equivalence between coamenability of G and amenability of \hat{G} [Tom06, BCT05].

Although our theory gives a satisfactory unification of various amenability results, the main remarkable property of the functor $\Pi: \mathcal{C} \rightarrow \mathcal{P}$ is, in our opinion, the universality. If the category \mathcal{P} happens to have a simpler structure compared to \mathcal{C} , this universality allows one to reduce classification of functors from \mathcal{C} inducing the amenable dimension function to an easier classification problem for functors from \mathcal{P} . This idea will be used in [NY15] to classify a class of compact quantum groups.

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1. PRELIMINARIES

1.1. Monoidal categories. In this paper we study *rigid C*-tensor categories*. By now there are many materials covering the basics of this subject, see for example [Yam04, Müg10, NT13] and references therein. We mainly follow the conventions of [NT13], but for the convenience of the reader we summarize the basic definitions and facts below.

A *C*-category* is a category \mathcal{C} whose morphism sets $\mathcal{C}(U, V)$ are complex Banach spaces endowed with complex conjugate involution $\mathcal{C}(U, V) \rightarrow \mathcal{C}(V, U)$, $T \mapsto T^*$ satisfying the C*-identity. Unless said otherwise, we always assume that \mathcal{C} is closed under finite direct sums and subobjects. The latter means that any idempotent in the endomorphism ring $\mathcal{C}(X) = \mathcal{C}(X, X)$ comes from a direct summand of X .

A C*-category is said to be *semisimple* if any object is isomorphic to a direct sum of simple (that is, with the endomorphism ring \mathbb{C}) objects. We then denote the isomorphism classes of simple objects by $\text{Irr}(\mathcal{C})$ and assume that this set is at most countable. Many results admit formulations which do not require this assumption and can be proved by considering subcategories generated by countable sets of simple objects, but we leave this matter to the interested reader.

A *unitary functor*, or a *C*-functor*, is a linear functor of C*-categories $F: \mathcal{C} \rightarrow \mathcal{C}'$ satisfying $F(T^*) = F(T)^*$.

In this paper we frequently perform the following operation: starting from a C*-category \mathcal{C} , we replace the morphisms sets by some larger system $\mathcal{D}(X, Y)$ naturally containing the original $\mathcal{C}(X, Y)$. Then we perform the *idempotent completion* to construct a new category \mathcal{D} . That is, we regard the projections $p \in \mathcal{D}(X)$ as objects in the new category, and take $q\mathcal{D}(X, Y)p$ as the morphism set from the object represented by $p \in \mathcal{D}(X)$ to the one by $q \in \mathcal{D}(Y)$. Then the embeddings $\mathcal{C}(X, Y) \rightarrow \mathcal{D}(X, Y)$ can be considered as a C*-functor $\mathcal{C} \rightarrow \mathcal{D}$.

A *C*-tensor category* is a C*-category endowed with a unitary bifunctor $\otimes: \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$, a distinguished object $\mathbb{1} \in \mathcal{C}$, and natural unitary isomorphisms

$$\mathbb{1} \otimes U \simeq U \simeq U \otimes \mathbb{1}, \quad \Phi(U, V, W): (U \otimes V) \otimes W \rightarrow U \otimes (V \otimes W)$$

satisfying certain compatibility conditions.

A *unitary tensor functor*, or a *C*-tensor functor*, between two C*-tensor categories \mathcal{C} and \mathcal{C}' is given by a triple (F_0, F, F_2) , where F is a C*-functor $\mathcal{C} \rightarrow \mathcal{C}'$, F_0 is a unitary isomorphism $\mathbb{1}_{\mathcal{C}'} \rightarrow F(\mathbb{1}_{\mathcal{C}})$, and F_2 is a natural unitary isomorphism $F(U) \otimes F(V) \rightarrow F(U \otimes V)$, which are compatible with the structure morphisms of \mathcal{C} and \mathcal{C}' . As a rule, we denote tensor functors by just one symbol F .

When \mathcal{C} is a strict C*-tensor category and $U \in \mathcal{C}$, an object V is said to be a *dual object* of U if there are morphisms $R \in \mathcal{C}(\mathbb{1}, V \otimes U)$ and $\bar{R} \in \mathcal{C}(\mathbb{1}, U \otimes V)$ satisfying the conjugate equations

$$(\iota_V \otimes \bar{R}^*)(R \otimes \iota_U) = \iota_V, \quad (\iota_U \otimes R^*)(\bar{R} \otimes \iota_U) = \iota_U.$$

If any object in \mathcal{C} admits a dual, \mathcal{C} is said to be *rigid* and we denote a choice of a dual of $U \in \mathcal{C}$ by \bar{U} . We assume that rigid C*-tensor categories have simple tensor units.

A rigid C*-tensor category (with simple unit) has finite dimensional morphism spaces and hence is automatically semisimple by our assumption of existence of subobjects.

The quantity

$$d^{\mathcal{C}}(U) = \min_{(R, \bar{R})} \|R\| \|\bar{R}\|$$

is called the *intrinsic dimension* of U , where (R, \bar{R}) runs through the set of solutions of conjugate equations as above. We omit the superscript \mathcal{C} when there is no danger of confusion. A solution (R, \bar{R}) of the conjugate equations for U is called *standard* if

$$\|R\| = \|\bar{R}\| = d(U)^{1/2}.$$

Solutions of the conjugate equations for U are unique up to the transformations

$$(R, \bar{R}) \mapsto ((T^* \otimes \iota)R, (\iota \otimes T^{-1})\bar{R}).$$

Furthermore, if (R, \bar{R}) is standard, then such a transformation defines a standard solution if and only if T is unitary.

In a rigid C^* -tensor category \mathcal{C} we often fix standard solutions (R_U, \bar{R}_U) of the conjugate equations for every object U . Then \mathcal{C} becomes *spherical* in the sense that one has the equality $R_U^*(\iota \otimes T)R_U = \bar{R}_U^*(T \otimes \iota)\bar{R}_U$ for any $T \in \mathcal{C}(U)$. The normalized linear functional

$$\mathrm{tr}_U(T) = d(U)^{-1}R_U^*(\iota \otimes T)R_U = d(U)^{-1}\bar{R}_U^*(T \otimes \iota)\bar{R}_U$$

is a tracial state on the finite dimensional C^* -algebra $\mathcal{C}(U)$. It is independent of the choice of a standard solution. More generally, for any objects X, U and V we can consider the normalized partial categorical traces

$$\mathrm{tr}_X \otimes \iota: \mathcal{C}(X \otimes U, X \otimes V) \rightarrow \mathcal{C}(U, V) \quad \text{and} \quad \iota \otimes \mathrm{tr}_X: \mathcal{C}(U \otimes X, V \otimes X) \rightarrow \mathcal{C}(U, V).$$

Namely, with a standard solution (R_X, \bar{R}_X) as above, we have

$$(\mathrm{tr}_X \otimes \iota)(T) = d(X)^{-1}(R_X^* \otimes \iota)(\iota \otimes T)(R_X \otimes \iota), \quad (\iota \otimes \mathrm{tr}_X)(T) = d(X)^{-1}(\iota \otimes \bar{R}_X^*)(T \otimes \iota)(\iota \otimes \bar{R}_X).$$

Given a rigid C^* -tensor category \mathcal{C} , if $[U]$ and $[V]$ are elements of $\mathrm{Irr}(\mathcal{C})$, we can define their product in $\mathbb{Z}_+[\mathrm{Irr}(\mathcal{C})]$ by putting

$$[U] \cdot [V] = \sum_{[W] \in \mathrm{Irr}(\mathcal{C})} \dim \mathcal{C}(W, U \otimes V)[W],$$

thus getting a semiring $\mathbb{Z}_+[\mathrm{Irr}(\mathcal{C})]$. Extending this formula by bilinearity, we obtain a ring structure on $\mathbb{Z}[\mathrm{Irr}(\mathcal{C})]$. The map $[U] \mapsto d(U)$ extends to a ring homomorphism $\mathbb{Z}[\mathrm{Irr}(\mathcal{C})] \rightarrow \mathbb{R}$. The pair $(\mathbb{Z}[\mathrm{Irr}(\mathcal{C})], d)$ is called the *fusion algebra* of \mathcal{C} . In general, a ring homomorphism $d': \mathbb{Z}[\mathrm{Irr}(\mathcal{C})] \rightarrow \mathbb{R}$ satisfying $d'([U]) > 0$ and $d'([U]) = d'([\bar{U}])$ for every $[U] \in \mathrm{Irr}(\mathcal{C})$ is said to be a *dimension function* on \mathcal{C} .

For a rigid C^* -tensor category \mathcal{C} , the right multiplication by $[U] \in \mathrm{Irr}(\mathcal{C})$ on $\mathbb{Z}[\mathrm{Irr}(\mathcal{C})]$ can be considered as a densely defined operator Γ_U on $\ell^2(\mathrm{Irr}(\mathcal{C}))$. This definition extends to arbitrary objects of \mathcal{C} by the formula $\Gamma_U = \sum_{[V] \in \mathrm{Irr}(\mathcal{C})} \dim(V, U)\Gamma_V$. If d' is a dimension function on \mathcal{C} , one has the estimate

$$\|\Gamma_U\|_{B(\ell^2(\mathrm{Irr}(\mathcal{C})))} \leq d'(U).$$

If the equality holds for all objects U , then the dimension function d' is called *amenable*. Clearly, there can be at most one amenable dimension function. If the intrinsic dimension function is amenable, then \mathcal{C} itself is called amenable.

1.2. Categories of functors. Given a rigid C^* -tensor category \mathcal{C} we will consider the category of unitary tensor functors from \mathcal{C} into C^* -tensor categories. Its objects are pairs (\mathcal{A}, E) , where \mathcal{A} is a C^* -tensor category and $E: \mathcal{C} \rightarrow \mathcal{A}$ is a unitary tensor functor. The morphisms $(\mathcal{A}, E) \rightarrow (\mathcal{B}, F)$ are unitary tensor functors $G: \mathcal{A} \rightarrow \mathcal{B}$, considered up to natural unitary monoidal isomorphisms,* such that GE is naturally unitarily isomorphic to F .

A more concrete way of thinking of this category is as follows. First of all we may assume that \mathcal{C} is strict. Consider a unitary tensor functor $E: \mathcal{C} \rightarrow \mathcal{A}$. The functor E is automatically faithful by semisimplicity and existence of conjugates in \mathcal{C} . It follows that by replacing the pair (\mathcal{A}, E) by an isomorphic one, we may assume that \mathcal{A} is a strict C^* -tensor category containing \mathcal{C} and E is simply the embedding functor. Namely, define the new sets of morphisms between objects U and V in \mathcal{C} as $\mathcal{A}(E(U), E(V))$, and then complete the category we thus obtain with respect to subobjects.

Assume now that we have two strict C^* -tensor categories \mathcal{A} and \mathcal{B} containing \mathcal{C} , and let $E: \mathcal{C} \rightarrow \mathcal{A}$ and $F: \mathcal{C} \rightarrow \mathcal{B}$ be the embedding functors. Assume $[G]: (\mathcal{A}, E) \rightarrow (\mathcal{B}, F)$ is a morphism. This

*Therefore the category of functors from \mathcal{C} we consider here is different from the category $\mathcal{Tens}(\mathcal{C})$ defined in [NY14], where we wanted to distinguish between isomorphic functors and defined a more refined notion of morphisms.

means that there exist unitary isomorphisms $\eta_U: G(U) \rightarrow U$ in \mathcal{B} such that $G(T) = \eta_V^{-1}T\eta_U$ for any morphism $T \in \mathcal{C}(U, V)$, and the morphisms

$$G_2(U, V): G(U) \otimes G(V) \rightarrow G(U \otimes V)$$

defining the tensor structure of G restricted to \mathcal{C} are given by $G_2(U, V) = \eta_{U \otimes V}^{-1}(\eta_U \otimes \eta_V)$. For objects U of \mathcal{A} that are not in \mathcal{C} put $\eta_U = 1 \in \mathcal{B}(G(U))$. We can then define a new unitary tensor functor $\tilde{G}: \mathcal{A} \rightarrow \mathcal{B}$ by letting $\tilde{G}(U) = U$ for objects U in \mathcal{C} and $\tilde{G}(U) = G(U)$ for the remaining objects, $\tilde{G}(T) = \eta_V G(T) \eta_U^{-1}$ for morphisms, and $\tilde{G}_2(U, V) = \eta_{U \otimes V} G_2(U, V) (\eta_U^{-1} \otimes \eta_V^{-1})$. Then $[G] = [\tilde{G}]$ and the restriction of \tilde{G} to $\mathcal{C} \subset \mathcal{A}$ coincides with the embedding (tensor) functor $\mathcal{C} \rightarrow \mathcal{B}$.

Therefore, any unitary tensor functor $\mathcal{C} \rightarrow \mathcal{A}$ is naturally unitary isomorphic to an embedding functor, and the morphisms between two such embeddings $E: \mathcal{C} \rightarrow \mathcal{A}$ and $F: \mathcal{C} \rightarrow \mathcal{B}$ are the unitary tensor functors $G: \mathcal{A} \rightarrow \mathcal{B}$ extending F , considered up to natural unitary isomorphisms. If, furthermore, \mathcal{A} is generated by the objects of \mathcal{C} then $[G]$ is completely determined by the maps $\mathcal{A}(U, V) \rightarrow \mathcal{B}(U, V)$ extending the identity maps on $\mathcal{C}(U, V)$ for all objects U and V in \mathcal{C} .

1.3. Subfactor theory. Let $N \subset M$ be an inclusion of von Neumann algebras represented on a Hilbert space H . There is a canonical bijective correspondence between the normal semifinite faithful operator valued weights $\Phi: M \rightarrow N$ and the ones $\Psi: N' \rightarrow M'$ in terms of spatial derivatives [Con80]. Given such a Φ , one associates a unique Ψ denoted by Φ^{-1} and characterized by the equation

$$\frac{d\omega\Phi}{d\omega'} = \frac{d\omega}{d\omega'\Phi^{-1}},$$

where ω and ω' are any choices of normal semifinite faithful weights on N and M' .

If E is a normal faithful conditional expectation from M to N , its *index* $\text{Ind } E$ can be defined as $E^{-1}(1)$ [Kos86]. Suppose that M and N are factors admitting conditional expectations of finite index. Then the index is a positive scalar and there is a unique choice of E which minimizes $\text{Ind } E$. This E is called the *minimal conditional expectation* of the subfactor $N \subset M$ [Hia88].

Suppose that $N \subset M$ is a subfactor endowed with a normal conditional expectation of finite index $E: M \rightarrow N$. We then obtain a von Neumann algebra M_1 called the *basic extension* of $N \subset M$ with respect to E , as follows. Taking a normal semifinite faithful weight ψ on N , the algebra $M_1 \subset B(L^2(M, \psi E))$ is generated by M and the orthogonal projection e_N , called the Jones projection, onto $L^2(N, \psi) \subset L^2(M, \psi E)$. One has the equality $M_1 = JN'J$, where J is the modular conjugation of M with respect to ψE . From the above correspondence of operator valued weights, there is a canonical conditional expectation $E_1: M_1 \rightarrow M$ which has the same index as E , namely, $E_1 = (\text{Ind } E)^{-1}JE^{-1}(J \cdot J)J$. Iterating this procedure, we obtain a tower of von Neumann algebras

$$N \subset M \subset M_1 \subset M_2 \subset \cdots.$$

The higher relative commutants

$$N' \cap M_k = \{x \in M_k \mid \forall y \in N: xy = yx\}$$

are finite dimensional C^* -algebras, with bound $\dim(N' \cap M_k) \leq (\text{Ind } E)^k$. The algebras $N' \cap M_{2k}$ ($k \in \mathbb{N}$) can be considered as the endomorphism rings of $M \otimes_N M \otimes_N \cdots \otimes_N M$ in the category of M -bimodules, and there are similar interpretations for the algebras $N' \cap M_{2k+1}$, etc., in terms of N -bimodules, M - N -modules, and N - M -modules.

1.4. Relative entropy. An important numerical invariant for inclusions of von Neumann algebras, closely related to index, is relative entropy. For this part we follow the exposition in [NS06].

When φ and ψ are positive linear functionals on a C^* -algebra M , we denote their relative entropy by $S(\varphi, \psi)$. If M is finite dimensional, it can be defined as

$$S(\varphi, \psi) = \begin{cases} \text{Tr}(Q_\varphi(\log Q_\varphi - \log Q_\psi)), & \text{if } \varphi \leq \lambda\psi \text{ for some } \lambda > 0, \\ +\infty, & \text{otherwise,} \end{cases}$$

where Tr is the canonical trace on M , which takes value 1 on every minimal projection in M , and $Q_\varphi \in M$ is the density matrix of φ , so that we have $\varphi(x) = \text{Tr}(xQ_\varphi)$. For a single positive linear functional ψ on a finite dimensional M , we also have its von Neumann entropy defined as $S(\psi) = -\text{Tr}(Q_\psi \log Q_\psi)$.

Given an inclusion of C^* -algebras $N \subset M$ and a state φ on M , the *relative entropy* $H_\varphi(M|N)$ (also called *conditional entropy* in the classical probability theory) is defined as the supremum of the quantities

$$\sum_i (S(\varphi_i, \varphi) - S(\varphi_i|_N, \varphi|_N))$$

where $(\varphi_i)_i = (\varphi_1, \dots, \varphi_k)$ runs through the tuples of positive linear functionals on M satisfying $\varphi = \sum_{i=1}^k \varphi_i$. If M is finite dimensional, this can also be written as

$$H_\varphi(M|N) = S(\varphi) - S(\varphi|_N) + \sup_{(\varphi_i)_i} \sum_i (S(\varphi_i|_N) - S(\varphi_i)),$$

where supremum is again taken over all finite decompositions of φ .

Relative entropy has the following lower semicontinuity property. Suppose that $N \subset M$ is an inclusion of von Neumann algebras and φ is a normal state on M . Suppose that $B_i \subset A_i$ ($i = 1, 2, \dots$) are increasing sequences of subalgebras $B_i \subset N$, $A_i \subset M$ such that $\bigcup_i A_i$ and $\bigcup_i B_i$ are s^* -dense in M and N , respectively. Then one has the estimate

$$H_\varphi(M|N) \leq \liminf_i H_\varphi(A_i|B_i).$$

If $N \subset M$ is an inclusion of von Neumann algebras and $E: M \rightarrow N$ is a normal conditional expectation, the relative entropy of M and N with respect to E is defined by

$$H_E(M|N) = \sup_\varphi H_\varphi(M|N),$$

where φ runs through the normal states on M satisfying $\varphi = \varphi E$ [Hia91]. If M and N are factors, then we have the estimate $H_E(M|N) \leq \log \text{Ind } E$.

2. CATEGORICAL POISSON BOUNDARY

Let \mathcal{C} be a strict rigid C^* -tensor category satisfying our standard assumptions: it is closed under finite direct sums and subobjects, the tensor unit is simple, and $\text{Irr}(\mathcal{C})$ is at most countable.

Let μ be a probability measure on $\text{Irr}(\mathcal{C})$. The Poisson boundary of (\mathcal{C}, μ) will be a new C^* -tensor category \mathcal{P} , possibly with nonsimple unit, together with a unitary tensor functor $\Pi: \mathcal{C} \rightarrow \mathcal{P}$. In this section we define (\mathcal{P}, Π) in purely categorical terms. In the next section we will give several more concrete descriptions of this construction.

For an object U consider the functor $\iota \otimes U: \mathcal{C} \rightarrow \mathcal{C}$, $X \mapsto X \otimes U$. Given two objects U and V , consider the space $\text{Nat}(\iota \otimes U, \iota \otimes V)$ of natural transformations from $\iota \otimes U$ to $\iota \otimes V$, so elements of $\text{Nat}(\iota \otimes U, \iota \otimes V)$ are collections $\eta = (\eta_X)_X$ of morphisms $\eta_X: X \otimes U \rightarrow X \otimes V$, natural in X . For every object X we can define a linear operator P_X on $\text{Nat}(\iota \otimes U, \iota \otimes V)$ by

$$P_X(\eta)_Y = (\text{tr}_X \otimes \iota)(\eta_{X \otimes Y})$$

with the partial categorical trace introduced in Section 1.1. Denote by $\hat{\mathcal{C}}(U, V) \subset \text{Nat}(\iota \otimes U, \iota \otimes V)$ the subspace of bounded natural transformations, that is, of elements η such that $\sup_Y \|\eta_Y\| < \infty$. More concretely, taking a representative U_s for each $s \in \text{Irr}(\mathcal{C})$, we can present $\hat{\mathcal{C}}(U, V)$ as

$$\hat{\mathcal{C}}(U, V) \cong \ell^\infty\text{-}\bigoplus_s \mathcal{C}(U_s \otimes U, U_s \otimes V),$$

since the natural transformations are determined by their actions on the simple objects. This is a Banach space, and the operator P_X defines a contraction on it. It is also clear that the operator P_X depends only on the isomorphism class of X .

From now on let us fix a representative U_s for every $s \in \text{Irr}(\mathcal{C})$ as above. We write tr_s instead of tr_{U_s} , P_s instead of P_{U_s} , and so on. Similarly, for a natural transformation $\eta: \iota \otimes U \rightarrow \iota \otimes V$ we will write η_s instead of η_{U_s} . Let also denote by $e \in \text{Irr}(\mathcal{C})$ the index corresponding to $\mathbb{1}$. For convenience we assume that $U_e = \mathbb{1}$. Define an involution on $\text{Irr}(\mathcal{C})$ such that $U_{\bar{s}}$ is a dual object to U_s .

Consider now the operator

$$P_\mu = \sum_s \mu(s) P_s.$$

This is a well-defined contraction on $\hat{\mathcal{C}}(U, V)$. We say that a bounded natural transformation $\eta: \iota \otimes U \rightarrow \iota \otimes V$ is P_μ -harmonic if

$$P_\mu(\eta) = \eta.$$

Any morphism $T: U \rightarrow V$ defines a bounded natural transformation $(\iota_X \otimes T)_X$, which is obviously P_μ -harmonic for every μ . When there is no ambiguity, we denote this natural transformation simply by T .

The composition of harmonic transformations is in general not harmonic. But we can define a new composition as follows.

Proposition 2.1. *Given bounded P_μ -harmonic natural transformations $\eta: \iota \otimes U \rightarrow \iota \otimes V$ and $\nu: \iota \otimes V \rightarrow \iota \otimes W$, the limit*

$$(\nu \cdot \eta)_X = \lim_{n \rightarrow \infty} P_\mu^n(\nu\eta)_X$$

exists for all objects X and defines a bounded P_μ -harmonic natural transformation $\iota \otimes U \rightarrow \iota \otimes W$. Furthermore, the composition \cdot is associative.

Note that since the spaces $\mathcal{C}(X \otimes U, X \otimes W)$ are finite dimensional by our assumptions on \mathcal{C} , the notion of a limit is unambiguous.

Proof of Proposition 2.1. This is an immediate consequence of results of Izumi [Izu12] (another proof will be given in Section 3.1). Namely, replacing U , V and W by their direct sum we may assume that $U = V = W$. Then

$$\hat{\mathcal{C}}(U) = \hat{\mathcal{C}}(U, U) \cong \ell^\infty\text{-}\bigoplus_s \mathcal{C}(U_s \otimes U)$$

is a von Neumann algebra and P_μ is a normal unital completely positive map on it. By [Izu12, Corollary 5.2] the subspace of P_μ -invariant elements is itself a von Neumann algebra with product \cdot such that $x \cdot y$ is the s^* -limit of the sequence $\{P_\mu^n(xy)\}_n$. \square

Using this product on harmonic elements we can define a new C^* -tensor category $\mathcal{P} = \mathcal{P}_{\mathcal{C}, \mu}$ and a unitary tensor functor $\Pi = \Pi_{\mathcal{C}, \mu}: \mathcal{C} \rightarrow \mathcal{P}$ as follows.

First consider the category $\tilde{\mathcal{P}}$ with the same objects as in \mathcal{C} , but define the new spaces $\tilde{\mathcal{P}}(U, V)$ of morphisms as the spaces of bounded P_μ -harmonic natural transformations $\iota \otimes U \rightarrow \iota \otimes V$. Define the composition of morphisms as in Proposition 2.1. We thus get a C^* -category, possibly without subobjects. Furthermore, the C^* -algebras $\tilde{\mathcal{P}}(U)$ are von Neumann algebras.

Next, we define the tensor product of objects in the same way as in \mathcal{C} , and define the tensor product of morphisms by

$$\nu \otimes \eta = (\nu \otimes \iota) \cdot (\iota \otimes \eta).$$

Here, given $\nu: \iota \otimes U \rightarrow \iota \otimes V$ and $\eta: \iota \otimes W \rightarrow \iota \otimes Z$, the natural transformation $\nu \otimes \iota_Z: \iota \otimes U \otimes Z \rightarrow \iota \otimes V \otimes Z$ is defined by

$$(\nu \otimes \iota_Z)_X = \nu_X \otimes \iota_Z,$$

while the natural transformation $\iota_U \otimes \eta: \iota \otimes U \otimes W \rightarrow \iota \otimes U \otimes Z$ is defined by

$$(\iota_U \otimes \eta)_X = \eta_{X \otimes U}.$$

We remark that $\nu \otimes \iota$ and $\iota \otimes \eta$ are still P_μ -harmonic due to the identities

$$P_X(\nu \otimes \iota) = P_X(\nu) \otimes \iota, \quad P_X(\iota \otimes \eta) = \iota \otimes P_X(\eta).$$

Note also that by naturality of η we have $(\nu_X \otimes \iota_Z)\eta_{X \otimes U} = \eta_{X \otimes V}(\nu_X \otimes \iota_Z)$, which implies that

$$\nu \otimes \eta = (\iota \otimes \eta) \cdot (\nu \otimes \iota).$$

This shows that $\otimes: \tilde{\mathcal{P}} \times \tilde{\mathcal{P}} \rightarrow \tilde{\mathcal{P}}$ is indeed a bifunctor. Since \mathcal{C} is strict, this bifunctor is strictly associative.

Finally, complete the category $\tilde{\mathcal{P}}$ with respect to subobjects. This is our C^* -tensor category \mathcal{P} , possibly with nonsimple unit. Since \mathcal{C} is rigid, the category \mathcal{P} is rigid as well. The unitary tensor functor $\Pi: \mathcal{C} \rightarrow \mathcal{P}$ is defined in the obvious way: it is the strict tensor functor which is the identity map on objects and $\Pi(T) = (\iota_X \otimes T)_X$ on morphisms. We will often omit Π and simply consider \mathcal{C} as a C^* -tensor subcategory of \mathcal{P} .

Definition 2.2. The pair (\mathcal{P}, Π) is called the *Poisson boundary* of (\mathcal{C}, μ) . We say that the Poisson boundary is trivial if $\Pi: \mathcal{C} \rightarrow \mathcal{P}$ is an equivalence of categories, or in other words, for all objects U and V in \mathcal{C} the only bounded P_μ -harmonic natural transformations $\iota \otimes U \rightarrow \iota \otimes V$ are the transformations of the form $\eta = (\iota_X \otimes T)_X$ for $T \in \mathcal{C}(U, V)$.

The algebra $\mathcal{P}(\mathbb{1})$ is determined by the random walk on $\text{Irr}(\mathcal{C})$ with transition probabilities

$$p_\mu(s, t) = \sum_r \mu(r) m_{rs}^t \frac{d(t)}{d(r)d(s)},$$

where $d(s) = d(U_s)$ and $m_{rs}^t = \dim \mathcal{C}(U_t, U_r \otimes U_s)$. Namely, if we identify $\hat{\mathcal{C}}(\mathbb{1})$ with

$$\ell^\infty - \bigoplus_s \mathcal{C}(U_s) = \ell^\infty(\text{Irr}(\mathcal{C})),$$

then the operator P_μ on $\hat{\mathcal{C}}(\mathbb{1})$ is the Markov operator defined by p_μ , so $(P_\mu f)(s) = \sum_t p_\mu(s, t) f(t)$. Therefore $\mathcal{P}(\mathbb{1})$ is the algebra of bounded measurable functions on the Poisson boundary, in the usual probabilistic sense, of the random walk on $\text{Irr}(\mathcal{C})$ with transition probabilities $p_\mu(s, t)$. We say that μ is *ergodic*, if this boundary is trivial, that is, the tensor unit of \mathcal{P} is simple.

We say that μ is *symmetric* if $\mu(s) = \mu(\bar{s})$ for all s , and that μ is *generating* if every simple object appears in the decomposition of $U_{s_1} \otimes \cdots \otimes U_{s_n}$ for some $s_1, \dots, s_n \in \text{supp } \mu$ and $n \geq 1$. Equivalently, μ is generating if $\bigcup_{n \geq 1} \text{supp } \mu^{*n} = \text{Irr}(\mathcal{C})$, where the convolution of probability measures on $\text{Irr}(\mathcal{C})$ is defined by

$$(\nu * \mu)(t) = \sum_{s, r} \nu(s) \mu(r) m_{sr}^t \frac{d(t)}{d(s)d(r)}.$$

We will write μ^n instead of μ^{*n} . The definition of the convolution is motivated by the identity $P_\mu P_\nu = P_{\nu * \mu}$.

We remark that a symmetric ergodic measure μ , or even an ergodic measure with symmetric support, is automatically generating. Indeed, the symmetry assumption implies that we have a well-defined equivalence relation on $\text{Irr}(\mathcal{C})$ such that $s \sim t$ if and only if t can be reached from s with nonzero probability in a finite nonzero number of steps. Then any bounded function on $\text{Irr}(\mathcal{C})$ that is constant on equivalence classes is P_μ -harmonic. Hence μ is generating by the ergodicity assumption.

Let us say that \mathcal{C} is *weakly amenable* if the fusion algebra $(\mathbb{Z}[\text{Irr}(\mathcal{C})], d)$ is weakly amenable in the sense of Hiai and Izumi [HI98], that is, there exists a left invariant mean on $\ell^\infty(\text{Irr}(\mathcal{C}))$. By definition this is a state m such that $m(P_s(f)) = m(f)$ for all $f \in \ell^\infty(\text{Irr}(\mathcal{C}))$ and $s \in \text{Irr}(\mathcal{C})$. Of course, it is also possible to define right invariant means, and by [HI98, Proposition 4.2] if there exists a left or right invariant mean, then there exists a bi-invariant mean. By the same proposition amenability implies weak amenability, as the term suggests. But as opposed to the group case, in general, the converse is not true. Using this terminology let us record the following known result.

Proposition 2.3. *An ergodic probability measure on $\text{Irr}(\mathcal{C})$ exists if and only if \mathcal{C} is weakly amenable. Furthermore, if an ergodic measure exists, then it can be chosen to be symmetric and with support equal to the entire space $\text{Irr}(\mathcal{C})$.*

Proof. If μ is an ergodic measure, then any weak* limit point of the sequence $n^{-1} \sum_{k=0}^{n-1} \mu^k$ defines a right invariant mean. For random walks on groups this implication was observed by Furstenberg. The other direction is proved in [HY00, Theorem 2.5]. It is an analogue of a result of Kaimanovich–Vershik and Rosenblatt. \square

It should be remarked that if the fusion algebra of \mathcal{C} is weakly amenable and finitely generated, in general it is not possible to find a finitely supported ergodic measure [KV83, Proposition 6.1].

To finish the section, let us show that, not surprisingly, categorical Poisson boundaries are of interest only for infinite categories.

Proposition 2.4. *Assume \mathcal{C} is finite, meaning that $\text{Irr}(\mathcal{C})$ is finite, and μ is generating. Then the Poisson boundary of (\mathcal{C}, μ) is trivial.*

Proof. The proof is similar to the proof of triviality of the Poisson boundary of a random walk on a finite set based on the maximum principle. Fix an object U in \mathcal{C} and assume that $\eta \in \hat{\mathcal{C}}(U)$ is positive and P_μ -harmonic. We claim that if $\eta \neq 0$ then there exists a positive nonzero morphism $T \in \mathcal{C}(U)$ such that $\eta \geq T$. Assuming that the claim is true, we can then choose a maximal T with this property. Applying again the claim to the element $\eta - T$, we conclude that $\eta = T$ by maximality.

In order to prove the claim observe that $\eta_e \in \mathcal{C}(U)$ is nonzero. Indeed, by assumption there exists s such that $\eta_s \neq 0$. Since the categorical traces are faithful, and therefore partial categorical traces are faithful completely positive maps, it follows that $P_s(\eta)_e \neq 0$. Since $s \in \text{supp } \mu^n$ for some $n \geq 1$, we conclude that $\eta_e = P_{\mu^n}(\eta)_e \neq 0$.

Denote the positive nonzero element $\eta_e \in \mathcal{C}(U)$ by S . Fix $s \in \text{Irr}(\mathcal{C})$. Let (R_s, \bar{R}_s) be a standard solution of the conjugate equations for U_s , and $p \in \mathcal{C}(\bar{U}_s \otimes U_s)$ be the projection defined by $p = d(s)^{-1} R_s R_s^*$. By naturality of η we then have $\eta_{\bar{U}_s \otimes U_s} \geq p \otimes S$, whence

$$P_{\bar{s}}(\eta)_s \geq (\text{tr}_{\bar{s}} \otimes \iota)(p) \otimes S = d(s)^{-2} (\iota \otimes S).$$

Using the generating property of μ and finiteness of $\text{Irr}(\mathcal{C})$, we conclude that there exists a number $\lambda > 0$ such that $\eta_s \geq \iota \otimes \lambda S$ for all s . This proves the claim. \square

3. REALIZATIONS OF THE POISSON BOUNDARY

As in the previous section, we fix a strict rigid C^* -tensor category \mathcal{C} and a probability measure μ on $\text{Irr}(\mathcal{C})$. In Sections 3.2 and 3.3 we will in addition assume that μ is generating. Let $\Pi: \mathcal{C} \rightarrow \mathcal{P}$ be the Poisson boundary of (\mathcal{C}, μ) . Our goal is to give several descriptions of the algebras $\mathcal{P}(U)$ of harmonic elements.

3.1. Time shift on the categorical path space. Fix an object U . Denote by $M_U^{(0)}$ the von Neumann algebra $\hat{\mathcal{C}}(U) \cong \ell^\infty\text{-}\bigoplus_s \mathcal{C}(U_s \otimes U)$. More generally, for every $n \geq 0$ consider the von Neumann algebra

$$M_U^{(n)} = \text{End}_b(\iota_{\mathcal{C}^{n+1}} \otimes U),$$

so $M_U^{(n)}$ consists of bounded collections $\eta = (\eta_{X_n, \dots, X_0})_{X_n, \dots, X_0}$ of natural in X_n, \dots, X_0 endomorphisms of $X_n \otimes \dots \otimes X_0 \otimes U$. We consider $M_U^{(n)}$ as a subalgebra of $M_U^{(n+1)}$ using the embedding

$$(\eta_{X_n, \dots, X_0})_{X_n, \dots, X_0} \mapsto (\iota_{X_{n+1}} \otimes \eta_{X_n, X_{n-1}, \dots, X_0})_{X_{n+1}, \dots, X_0}.$$

Define a conditional expectation $E_{n+1, n}: M_U^{(n+1)} \rightarrow M_U^{(n)}$ by

$$E_{n+1, n}(\eta)_{X_n, \dots, X_0} = \sum_s \mu(s) (\text{tr}_s \otimes \iota)(\eta_{U_s, X_n, \dots, X_0}).$$

Taking compositions of such conditional expectations we get normal conditional expectations

$$E_{n, 0}: M_U^{(n)} \rightarrow M_U^{(0)}.$$

These conditional expectations are not faithful for $n \geq 1$ unless the support of μ is the entire space $\text{Irr}(\mathcal{C})$. The support of $E_{n,0}$ is a central projection, and we denote by $\mathcal{M}_U^{(n)}$ the reduction of $M_U^{(n)}$ by this projection. More concretely, we have a canonical isomorphism

$$\mathcal{M}_U^{(n)} \cong \ell^\infty\text{-}\bigoplus_{\substack{s_n, \dots, s_1 \in \text{supp } \mu \\ s_0 \in \text{Irr}(\mathcal{C})}} \mathcal{C}(U_{s_n} \otimes \cdots \otimes U_{s_0} \otimes U). \quad (3.1)$$

The conditional expectations $E_{n,0}$ define normal faithful conditional expectations $\mathcal{E}_{n,0}: \mathcal{M}_U^{(n)} \rightarrow \mathcal{M}_U^{(0)} = M_U^{(0)}$, and similarly $E_{n+1,n}$ define conditional expectations $\mathcal{E}_{n+1,n}$. Denote by \mathcal{M}_U the von Neumann algebra obtained as the inductive limit of the algebras $\mathcal{M}_U^{(n)}$ with respect to $\mathcal{E}_{n,0}$. In other words, take any faithful normal state $\phi_U^{(0)}$ on $\mathcal{M}_U^{(0)}$. By composing it with the conditional expectation $\mathcal{E}_{n,0}$ we get a state $\phi_U^{(n)}$ on $\mathcal{M}_U^{(n)}$. Together these states define a state on $\bigcup_n \mathcal{M}_U^{(n)}$. Finally, complete $\bigcup_n \mathcal{M}_U^{(n)}$ to a von Neumann algebra in the GNS-representation corresponding to this state. Denote the corresponding normal state on \mathcal{M}_U by ϕ_U .

Note that if we start with a trace on $\mathcal{M}_U^{(0)}$ which is a convex combination of the traces $\text{tr}_{U_s \otimes U}$, then the corresponding state ϕ_U on \mathcal{M}_U is tracial. Since it is faithful on $\mathcal{M}_U^{(n)}$ for every n , it is faithful on \mathcal{M}_U . This shows that \mathcal{M}_U is a finite von Neumann algebra. Furthermore, the ϕ_U -preserving normal faithful conditional expectation $\mathcal{E}_n: \mathcal{M}_U \rightarrow \mathcal{M}_U^{(n)}$ coincides with $\mathcal{E}_{n+1,n}$ on $\mathcal{M}_U^{(n+1)}$. It follows that on the dense algebra $\bigcup_m \mathcal{M}_U^{(m)}$ the conditional expectation \mathcal{E}_n is the limit, in the pointwise s^* -topology, of $\mathcal{E}_{n+1,n} \mathcal{E}_{n+2,n+1} \cdots \mathcal{E}_{m+1,m}$ as $m \rightarrow \infty$. Hence \mathcal{E}_n is independent of the choice of a faithful normal trace $\phi_U^{(0)}$ as above.

Define a unital endomorphism θ_U of $\bigcup_n \mathcal{M}_U^{(n)}$ such that $\theta_U(M_U^{(n)}) \subset M_U^{(n+1)}$ by

$$\theta_U(\eta)_{X_{n+1}, \dots, X_0} = \eta_{X_{n+1}, \dots, X_2, X_1 \otimes X_0}.$$

Considering $\mathcal{M}_U^{(k)}$ as a quotient of $M_U^{(k)}$ we get a unital endomorphism of $\bigcup_n \mathcal{M}_U^{(n)}$.

Lemma 3.1. *The endomorphism θ_U of $\bigcup_n \mathcal{M}_U^{(n)}$ extends to a normal faithful endomorphism of \mathcal{M}_U , which we continue to denote by θ_U .*

Proof. Consider the normal semifinite faithful (n.s.f.) trace $\psi_U^{(0)} = \sum_s d(s)^2 \text{tr}_{U_s \otimes U}$ on

$$\mathcal{M}_U^{(0)} \cong \ell^\infty\text{-}\bigoplus_s \mathcal{C}(U_s \otimes U)$$

and put $\psi_U = \psi_U^{(0)} \mathcal{E}_0$. Then ψ_U is an n.s.f. trace. In order to prove the lemma it suffices to show that the restriction of ψ_U to $\bigcup_n \mathcal{M}_U^{(n)}$ is θ_U -invariant. Indeed, if the invariance holds, then we can define an isometry U on $L^2(\mathcal{M}_U, \psi_U)$ by $U \Lambda_{\psi_U}(x) = \Lambda_{\psi_U}(\theta_U(x))$ for $x \in \bigcup_n \mathcal{M}_U^{(n)}$ such that $\psi_U(x^*x) < \infty$. Let $H \subset L^2(\mathcal{M}_U, \psi_U)$ be the image of U and \mathcal{M} be the von Neumann algebra generated by the image of θ_U . Then H is \mathcal{M} -invariant. We can choose $0 \leq e_i \leq 1$ such that $\theta_U(e_i) \rightarrow 1$ strongly and $\psi_U(e_i) < \infty$. Now, if $x \in \mathcal{M}_+$ is such that $x|_H = 0$, then $\psi_U(\theta_U(e_i)x\theta_U(e_i)) = 0$, and by lower semicontinuity we get $\psi_U(x) = 0$, so $x = 0$. Therefore we can define θ_U as the composition of the map $\mathcal{M}_U \rightarrow B(H)$, $x \mapsto UxU^*$, with the inverse of the map $\mathcal{M} \rightarrow \mathcal{M}|_H$.

It remains to check the invariance. By definition we have $\mathcal{E}_{n+2,n+1}\theta_U = \theta_U\mathcal{E}_{n+1,n}$ on $\mathcal{M}_U^{(n+1)}$ for all $n \geq 0$. This implies that $\mathcal{E}_{n+1}\theta_U = \theta_U\mathcal{E}_n$ on $\bigcup_k \mathcal{M}_U^{(k)}$. It follows that for any $x \in \bigcup_n \mathcal{M}_U^{(n)}$ we have

$$\psi_U\theta_U(x) = \psi_U\mathcal{E}_1\theta_U(x) = \psi_U\theta_U\mathcal{E}_0(x) = \psi_U\mathcal{E}_0\theta_U\mathcal{E}_0(x).$$

This implies that it suffices to show that $\psi_U\mathcal{E}_0\theta_U = \psi_U$ on $\mathcal{M}_U^{(0)}$. Since $\text{tr}_{U_s \otimes U} = \text{tr}_s(\iota \otimes \text{tr}_U)$, it is enough to consider the case $U = \mathbb{1}$. Note also that $\mathcal{E}_0\theta_U = P_\mu$ on $\mathcal{M}_U^{(0)}$. Thus we have to check that

$\psi_{\mathbb{1}}P_\mu = \psi_{\mathbb{1}}$ on $\mathcal{M}_{\mathbb{1}}^{(0)} \cong \ell^\infty(\text{Irr}(\mathcal{C}))$. This is equivalent to the easily verifiable identity $\mu * m = m$, where $m = \sum_s d(s)^2 \delta_s$. \square

We call the endomorphism θ_U of \mathcal{M}_U the *time shift*. Now, take $\eta \in \mathcal{M}_U^{(0)}$. Then for every $n \geq 0$ we can define an element $\eta^{(n)} \in \mathcal{M}_U^{(n)}$ by

$$\eta_{X_n, \dots, X_0}^{(n)} = \eta_{X_n \otimes \dots \otimes X_0}.$$

Consider the image of $\eta^{(n)}$ in $\mathcal{M}_U^{(n)}$ and denote it again by $\eta^{(n)}$, since this is the only element we are interested in. Then η is P_μ -harmonic if and only if $\mathcal{E}_{1,0}(\eta^{(1)}) = \eta$, and in this case $\mathcal{E}_{n+1,n}(\eta^{(n+1)}) = \eta^{(n)}$ for all n . Therefore if η is P_μ -harmonic, then the sequence $\{\eta^{(n)}\}_n$ is a martingale. Denote by $\eta^{(\infty)} \in \mathcal{M}_U$ its s^* -limit.

Proposition 3.2. *The map $\eta \mapsto \eta^{(\infty)}$ is an isomorphism between the von Neumann algebra $\mathcal{P}(U)$ of P_μ -harmonic bounded natural transformations $\iota \otimes U \rightarrow \iota \otimes U$ and the fixed point algebra $\mathcal{M}_U^{\theta_U}$. The inverse map is given by $x \mapsto \mathcal{E}_0(x)$.*

Proof. By definition we have $\eta^{(n)} = \theta_U^n(\eta)$. It follows that if η is P_μ -harmonic, so that $\eta^{(n)} \rightarrow \eta^{(\infty)}$, then the element $\eta^{(\infty)}$ is θ_U -invariant. We also clearly have $\mathcal{E}_0(\eta^{(\infty)}) = \eta$.

Conversely, take $x \in \mathcal{M}_U^{\theta_U}$. The proof of Lemma 3.1 implies that $\mathcal{E}_{n+1} \theta_U = \theta_U \mathcal{E}_n$. Hence the martingale $\{x_n = \mathcal{E}_n(x)\}_n$ has the property $x_{n+1} = \theta_U(x_n)$. As $\mathcal{E}_0 \theta_U = P_\mu$ on $\mathcal{M}_U^{(0)}$, we conclude that x_0 is P_μ -harmonic and $x_0^{(\infty)} = x$.

We have thus proved that the maps in the formulation are inverse to each other. Since they are unital completely positive, they must be isomorphisms. \square

The bijection between $\mathcal{P}(U)$ and $\mathcal{M}_U^{\theta_U}$ could be used to give an alternative proof of Proposition 2.1. Namely, we could define a product \cdot on harmonic elements by $\nu \cdot \eta = \mathcal{E}_0(\nu^{(\infty)} \eta^{(\infty)})$. Since $\nu^{(\infty)} \eta^{(\infty)}$ is the s^* -limit of the elements $\nu^{(n)} \eta^{(n)} = (\nu \eta)^{(n)}$, and $\mathcal{E}_0((\nu \eta)^{(n)}) = P_\mu^n(\nu \eta)$, it follows that $P_\mu^n(\nu \eta) \rightarrow \nu \cdot \eta$ in the s^* -topology, which is equivalent to saying that $P_\mu^n(\nu \eta)_X \rightarrow (\nu \cdot \eta)_X$ for every X .

3.2. Relative commutants: Izumi–Longo–Roberts approach. We will now modify the construction of the algebras \mathcal{M}_U to get algebras \mathcal{N}_U and an identification of $\mathcal{P}(U)$ with $\mathcal{N}'_{\mathbb{1}} \cap \mathcal{N}_U$. Conceptually, instead of considering all paths of the random walk defined by μ , we consider only paths starting at the unit object. The time shift is no longer defined on this space, but by considering a larger space we can still get a description of $\mathcal{P}(U)$ in simple von Neumann algebraic terms. For this to work we have to assume that μ is generating, so that we can reach any simple object from the unit.

This identification of harmonic elements is closely related to Izumi's description of Poisson boundaries of discrete quantum groups [Izu02]. A similar construction was also used by Longo and Roberts using sector theory [LR97]. More precisely, they worked with a somewhat limited form of μ and what we obtain is a possibly infinite von Neumann algebra for what corresponds to the finite gauge-invariant von Neumann subalgebra in their work.

We first put $V = \bigoplus_{s \in \text{supp } \mu} U_s$. In the case $\text{supp } \mu$ is infinite, this should be understood only as a suggestive notation which does not make sense inside \mathcal{C} . Given an object U , by $\mathcal{C}(V^{\otimes n} \otimes U)$ we understand the space

$$\bigoplus_{s_*, s'_* \in \text{supp } \mu^n} \mathcal{C}(U_{s_n} \otimes \dots \otimes U_{s_1} \otimes U, U_{s'_n} \otimes \dots \otimes U_{s'_1} \otimes U)$$

endowed with the obvious $*$ -algebra structure. Similarly to Section 3.1 we have completely positive maps

$$\mathcal{E}_{n+1,n} = \sum_s \mu(s)(\text{tr}_s \otimes \iota): \mathcal{C}(V^{\otimes(n+1)} \otimes U) \rightarrow \mathcal{C}(V^{\otimes n} \otimes U),$$

and taking the composition of these maps we get maps

$$\mathcal{E}_{n,0}: \mathcal{C}(V^{\otimes n} \otimes U) \rightarrow \mathcal{C}(U).$$

Then $\omega_U^{(n)} = \text{tr}_U \mathcal{E}_{n,0}$ is a state on $\mathcal{C}(V^{\otimes n} \otimes U)$. We denote by $\mathcal{N}_U^{(n)}$ the von Neumann algebra generated by $\mathcal{C}(V^{\otimes n} \otimes U)$ in the GNS-representation defined by this state. The elements of $\mathcal{N}_U^{(n)}$ are represented by certain bounded families in the direct product of the morphism sets

$$\mathcal{C}(U_{s_n} \otimes \cdots \otimes U_{s_1} \otimes U, U_{s'_n} \otimes \cdots \otimes U_{s'_1} \otimes U).$$

Since the positive elements of $\mathcal{N}_U^{(n)}$ have positive diagonal entries, the state $\omega_U^{(n)}$ is faithful on $\mathcal{N}_U^{(n)}$.

There is a natural diagonal embedding $\mathcal{N}_U^{(n)} \rightarrow \mathcal{N}_U^{(n+1)}$ defined by $T \mapsto \iota_V \otimes T$. The map $\mathcal{E}_{n+1,n}$ extends then to a normal conditional expectation $\mathcal{N}_U^{(n+1)} \rightarrow \mathcal{N}_U^{(n)}$ such that $\omega_U^{(n)} \mathcal{E}_{n+1,n} = \omega_U^{(n+1)}$. This way we obtain an inductive system $(\mathcal{N}_U^{(n)}, \omega_U^{(n)})_n$ of von Neumann algebras, and we let $(\mathcal{N}_U, \omega_U)$ be the von Neumann algebra and the faithful state obtained as the limit. As in Section 3.1, composing the conditional expectations $\mathcal{E}_{n+1,n}$ and passing to the limit we get ω_U -preserving conditional expectations $\mathcal{E}_n: \mathcal{N}_U \rightarrow \mathcal{N}_U^{(n)}$.

When $U = \mathbb{1}$, we simply write $\mathcal{N}^{(n)}$ and \mathcal{N} instead of $\mathcal{N}_{\mathbb{1}}^{(n)}$ and $\mathcal{N}_{\mathbb{1}}$. If U' and U are objects in \mathcal{C} , then the map $x \mapsto x \otimes \iota_U$ defines an embedding $\mathcal{N}_{U'} \hookrightarrow \mathcal{N}_{U' \otimes U}$. In particular, the algebra \mathcal{N} is contained in any of \mathcal{N}_U .

When η is a natural transformation in $\hat{\mathcal{C}}(U)$, the morphism

$$\eta_{V^{\otimes n}} = \bigoplus_{s_*} \eta_{U_{s_n} \otimes \cdots \otimes U_{s_1}}$$

defines an element in the diagonal part of $\mathcal{N}_U^{(n)}$, which we denote by $\eta^{[n]}$. Note that the direct summand $s_0 = e$ of (3.1) can be identified with the diagonal part of $\mathcal{N}_U^{(n)}$, and $\eta^{[n]}$ simply becomes the component of $\eta^{(n)}$ in this summand. If η is P_μ -harmonic, the sequence $\{\eta^{[n]}\}_n$ forms a martingale and defines an element $\eta^{[\infty]} \in \mathcal{N}_U^{(\infty)}$.

Proposition 3.3. *For every object U in \mathcal{C} , the map $\eta \mapsto \eta^{[\infty]}$ defines an isomorphism of von Neumann algebras $\mathcal{P}(U) \cong \mathcal{N}' \cap \mathcal{N}_U$.*

Proof. If η is a harmonic element in $\hat{\mathcal{C}}(U)$, the naturality implies that the elements $\eta_{V^{\otimes m}}$ commute with the image of $\mathcal{C}(V^{\otimes n})$ for $m \geq n$. Thus, $\eta^{[\infty]} = \lim_m \eta_{V^{\otimes m}}$ is in the relative commutant. Since μ is generating, it is also clear that the map $\eta \mapsto \eta^{[\infty]}$ is injective.

To construct the inverse map, take an element $x \in \mathcal{N}' \cap \mathcal{N}_U$. Then $x_n = \mathcal{E}_n(x)$ is an element of $(\mathcal{N}^{(n)})' \cap \mathcal{N}_U^{(n)}$. Hence, for every $n \geq 1$ and $s \in \text{supp } \mu^n$, there is a morphism $x_{n,s} \in \mathcal{C}(U_s \otimes U)$ such that x_n is the direct sum of the $x_{n,s}$ (with multiplicities). It follows that we can choose $\eta(n) \in \hat{\mathcal{C}}(U)$ such that $\|\eta(n)\| \leq \|x\|$ and $x_n = \eta(n)^{[n]}$. The elements $\eta(n)$ are not uniquely determined, only their components corresponding to $s \in \text{supp } \mu^n$ are. The identity $\mathcal{E}_{n+1,n}(x_{n+1}) = x_n$ translates into $P_\mu(\eta(n+1))_s = \eta(n)_s$ for $s \in \text{supp } \mu^n$.

We now define an element $\eta \in \hat{\mathcal{C}}(U)$ by letting

$$\eta_s = \eta(n)_s \text{ if } s \in \text{supp } \mu^n \text{ for some } n \geq 1.$$

In order to see that this definition is unambiguous, assume $s \in (\text{supp } \mu^n) \cap (\text{supp } \mu^{n+k})$ for some n and k . Then by the 0-2 law, see [NT04, Proposition 2.12], we have $\|P_\mu^m - P_\mu^{m+k}\| \rightarrow 0$ as $m \rightarrow \infty$. Since the sequence $\{\eta(m)\}_m$ is bounded and we have $\eta(n)_s = P_\mu^{m+k}(\eta(n+m+k))_s$ and $\eta(n+k)_s = P_\mu^m(\eta(n+m+k))_s$, letting $m \rightarrow \infty$ we conclude that $\eta(n)_s = \eta(n+k)_s$. Hence η is well-defined, P_μ -harmonic, and $x_n = \eta^{[n]}$. Therefore $x = \eta^{[\infty]}$.

The linear isomorphism $\mathcal{P}(U) \rightarrow \mathcal{N}' \cap \mathcal{N}_U$ and its inverse that we have constructed, are unital and completely positive, hence they are isomorphisms of von Neumann algebras. \square

As in the case of Proposition 3.2, the linear isomorphism $\mathcal{P}(U) \cong \mathcal{N}' \cap \mathcal{N}_U$ could be used to give an alternative proof of Proposition 2.1, at least for generating measures.

Applying Proposition 3.3 to $U = \mathbf{1}$ we get the following.

Corollary 3.4. *The von Neumann algebra \mathcal{N} is a factor if and only if μ is ergodic.*

Under a mildly stronger assumption on the measure we can prove a better result than Proposition 3.3, which will be important later.

Proposition 3.5. *Assume that for any $s, t \in \text{Irr}(\mathcal{C})$ there exists $n \geq 0$ such that*

$$\text{supp}(\mu^n * \delta_s) \cap \text{supp}(\mu^n * \delta_t) \neq \emptyset.$$

Then for any objects U and U' in \mathcal{C} , the map $\eta \mapsto (\iota_{U'} \otimes \eta)^{[\infty]}$ defines an isomorphism of von Neumann algebras $\mathcal{P}(U) \cong \mathcal{N}'_{U'} \cap \mathcal{N}_{U' \otimes U}$.

Proof. That we get a map $\mathcal{P}(U) \rightarrow \mathcal{N}'_{U'} \cap \mathcal{N}_{U' \otimes U}$ does not require any assumptions on μ and is easy to see: if η is a harmonic element in $\hat{\mathcal{C}}(U)$, the naturality implies that the elements $\eta_{V^{\otimes m} \otimes U'}$ commute with $\mathcal{C}(V^{\otimes n} \otimes U')$ for $m \geq n$, and hence $(\iota_{U'} \otimes \eta)^{[\infty]} = \lim_m \eta_{V^{\otimes m} \otimes U'}$ lies in $\mathcal{N}'_{U'} \cap \mathcal{N}_{U' \otimes U}$.

To construct the inverse map assume first $U' = U_t$ for some t . Take $x \in \mathcal{N}'_{U'} \cap \mathcal{N}_{U' \otimes U}$. Similarly to the proof of Proposition 3.3 we can find elements $\eta(n) \in \hat{\mathcal{C}}(U)$ such that $\|\eta(n)\| \leq \|x\|$ and $\mathcal{E}_n(x) = (\iota_{U'} \otimes \eta(n))^{[n]}$. The identity $\mathcal{E}_{n+1,n}(x_{n+1}) = x_n$ means now that $P_\mu(\eta(n+1))_s = \eta(n)_s$ for $s \in \text{supp}(\mu^n * \delta_t)$. We want to define an element $\eta \in \hat{\mathcal{C}}(U)$ by

$$\eta_s = \eta(n)_s \text{ if } s \in \text{supp}(\mu^n * \delta_t) \text{ for some } n \geq 1.$$

As in the proof of Proposition 3.3, in order to see that η is well-defined, it suffices to show that if $s \in \text{supp}(\mu^n * \delta_t) \cap \text{supp}(\mu^{n+k} * \delta_t)$ for some n and k , then $\|P_\mu^m - P_\mu^{m+k}\| \rightarrow 0$ as $m \rightarrow \infty$. Since μ is assumed to be generating, there exists l such that $t \in \text{supp} \mu^l$. But then

$$s \in (\text{supp} \mu^{n+l}) \cap (\text{supp} \mu^{n+l+k}),$$

so the convergence $\|P_\mu^m - P_\mu^{m+k}\| \rightarrow 0$ indeed holds by the 0-2 law. This finishes the proof of the proposition for $U' = U_t$, and we see that no assumption in addition to the generating property of μ is needed in this case.

Consider now an arbitrary U' . Decompose U' into a direct sum of simple objects:

$$U \cong U_{s_1} \oplus \cdots \oplus U_{s_n}.$$

Denote by $p_i \in \mathcal{C}(U')$ the corresponding projections. Then the inclusion $p_i \mathcal{N}_{U'} p_i \subset p_i \mathcal{N}_{U' \otimes U} p_i$ can be identified with $\mathcal{N}_{U_{s_i}} \subset \mathcal{N}_{U_{s_i} \otimes U}$.

Take $x \in \mathcal{N}'_{U'} \cap \mathcal{N}_{U' \otimes U}$. Then x commutes with p_i . Since the element $x p_i$ lies in $\mathcal{N}'_{U_{s_i}} \cap \mathcal{N}_{U_{s_i} \otimes U}$, it is defined by a P_μ -harmonic element $\eta(i) \in \hat{\mathcal{C}}(U)$. In terms of these elements the condition that $\mathcal{E}_n(x)$ commutes with $\mathcal{C}(V^{\otimes n} \otimes U')$ means that $\eta(i)_s = \eta(j)_s$ whenever $s \in \text{supp}(\mu^n * \delta_{s_i}) \cap \text{supp}(\mu^n * \delta_{s_j})$, while to finish the proof we need the equality $\eta(i) = \eta(j)$.

Fix $s \in \text{Irr}(\mathcal{C})$ and indices i and j . By assumption there exists $t \in \text{supp}(\mu^n * \delta_{s_i}) \cap \text{supp}(\mu^n * \delta_{s_j})$ for some n . Since μ is generating, there exists m such that $s \in \text{supp}(\mu^m * \delta_t)$. Then

$$s \in \text{supp}(\mu^{m+n} * \delta_{s_i}) \cap \text{supp}(\mu^{m+n} * \delta_{s_j}),$$

and therefore $\eta(i)_s = \eta(j)_s$. □

Note that the proof shows that the additional assumption on the measure is not only sufficient but also necessary for the result to be true. Even for symmetric ergodic measures this condition does not always hold: take the random walk on \mathbb{Z} defined by the measure $\mu = 2^{-1}(\delta_{-1} + \delta_1)$. At the same time this condition is satisfied, for example, for any generating measure μ with $\mu(e) > 0$. Indeed, for such a measure we can find n such that $s \in \text{supp}(\mu^n * \delta_t)$, and then $s \in \text{supp}(\mu^n * \delta_s) \cap \text{supp}(\mu^n * \delta_t)$.

Applying the proposition to $U = \mathbf{1}$ we get the following result.

Corollary 3.6. *Assume μ is ergodic and satisfies the assumption of Proposition 3.5. Then \mathcal{N}_U is a factor for every object U in \mathcal{C} .*

Remark 3.7. It is sometimes convenient to consider slightly more general constructions allowing multiplicities. Namely, instead of $V = \bigoplus_{s \in \text{supp } \mu} U_s$ we could take $V = \bigoplus_{i \in I} U_{s_i}$, where $(s_i)_{i \in I}$ is any finite or countable collection of elements running through $\text{supp } \mu$. For the state on $\mathcal{C}(V)$ we could take $\mathcal{C}(U_{s_i}, U_{s_j}) \ni T \mapsto \delta_{ij} \lambda_i \text{tr}_{s_i}(T)$, where $\lambda_i > 0$ are any numbers such that $\sum_{i: s_i=s} \lambda_i = \mu(s)$ for all $s \in \text{supp } \mu$. All the above results would remain true, with essentially identical proofs.

3.3. Relative commutants: Hayashi–Yamagami approach. We will now explain a modification of the Izumi–Longo–Roberts construction due to Hayashi and Yamagami [HY00]. Its advantage is that, at the expense of introducing an extra variable in a II_1 factor, we can stay in the framework of finite von Neumann algebras.

We continue to assume that μ is generating. We will use a slightly different notation compared [HY00] to be more consistent with the previous sections.

Let \mathcal{R} be the hyperfinite II_1 -factor and τ be the unique normal tracial state on \mathcal{R} . Choose a partition of unity by projections $(e_s)_{s \in \text{supp } \mu}$ in \mathcal{R} which satisfy

$$\tau(e_s) = \frac{\mu(s)}{cd(s)}, \quad \text{where } c = \sum_{s \in \text{supp } \mu} \frac{\mu(s)}{d(s)}.$$

When $(s_n, \dots, s_1) \in (\text{supp } \mu)^n$, we write $e_{s_*} = e_{s_n} \otimes \dots \otimes e_{s_1} \in \mathcal{R}^{\otimes n}$. As in Section 3.2, put $V = \bigoplus_{s \in \text{supp } \mu} U_s$. Now, for a fixed object U in \mathcal{C} , instead of the algebra $\mathcal{C}(V^{\otimes n} \otimes U)$ used there, consider the algebra

$$\tilde{\mathcal{C}}(V^{\otimes n} \otimes U) = \bigoplus_{s_*, s'_* \in (\text{supp } \mu)^n} \mathcal{C}(U_{s_n} \otimes \dots \otimes U_{s_1} \otimes U, U_{s'_n} \otimes \dots \otimes U_{s'_1} \otimes U) \otimes e_{s'_*} \mathcal{R}^{\otimes n} e_{s_*}.$$

It carries a tracial state $\tau_U^{(n)}$ defined by

$$\tau_U^{(n)}(T \otimes x) = \delta_{s_*, s'_*} c^n d(s_1) \dots d(s_n) \text{tr}_{U_{s_n} \otimes \dots \otimes U_{s_1} \otimes U}(T) \tau^{\otimes n}(x)$$

for $T \otimes x \in \mathcal{C}(U_{s_n} \otimes \dots \otimes U_{s_1} \otimes U, U_{s'_n} \otimes \dots \otimes U_{s'_1} \otimes U) \otimes e_{s'_*} \mathcal{R}^{\otimes n} e_{s_*}$. Let $\mathcal{A}_U^{(n)}$ be the von Neumann algebra generated by $\tilde{\mathcal{C}}(V^{\otimes n} \otimes U)$ in the GNS-representation defined by $\tau_U^{(n)}$. These algebras form an inductive system under the embeddings

$$\mathcal{A}_U^{(n)} \hookrightarrow \mathcal{A}_U^{(n+1)}, \quad T \otimes x \mapsto \sum_{s \in \text{supp } \mu} (\iota_s \otimes T) \otimes (e_s \otimes x).$$

Passing to the limit we get a von Neumann algebra \mathcal{A}_U equipped with a faithful tracial state τ_U . We write \mathcal{A} for \mathcal{A}_1 .

Given $\eta \in \hat{\mathcal{C}}(U)$, consider the elements

$$\eta^{\{n\}} = \sum_{s_* \in (\text{supp } \mu)^n} \eta_{U_{s_n} \otimes \dots \otimes U_{s_1}} \otimes e_{s_*} \in \mathcal{A}_U^{(n)}.$$

If η is P_μ -harmonic, then the sequence $\{\eta^{\{n\}}\}_n$ forms a martingale with respect to the τ_U -preserving conditional expectations $\mathcal{E}_n: \mathcal{A}_U \rightarrow \mathcal{A}_U^{(n)}$. Denote its limit by $\eta^{\{\infty\}}$. Then we get the following analogues of Propositions 3.3 and 3.5, with almost identical proofs, which we omit.

Proposition 3.8. *For every object U in \mathcal{C} , the map $\eta \mapsto \eta^{\{\infty\}}$ defines an isomorphism of von Neumann algebras $\mathcal{P}(U) \cong \mathcal{A}' \cap \mathcal{A}_U$. If in addition to the generating property the measure μ satisfies the assumption of Proposition 3.5, then also the map $\eta \mapsto (\iota_{U'} \otimes \eta)^{\{\infty\}}$ defines an isomorphism of von Neumann algebras $\mathcal{P}(U) \cong \mathcal{A}'_{U'} \cap \mathcal{A}_{U' \otimes U}$ for any object U' .*

The work of Hayashi and Yamagami contains much more than the construction of the algebras \mathcal{A}_U and, in fact, allows us to describe, under mild additional assumptions on μ , not only the morphisms but the entire Poisson boundary $\Pi: \mathcal{C} \rightarrow \mathcal{P}$ in terms of Hilbert bimodules over \mathcal{A} .

For objects X and Y consider their direct sum $X \oplus Y$, and denote by $p_X, p_Y \in \mathcal{C}(X \oplus Y)$ the corresponding projections. We can consider p_X and p_Y as projections in $\mathcal{A}_{X \oplus Y}$, then $p_X(\mathcal{A}_{X \oplus Y})p_X \cong \mathcal{A}_X$ and $p_Y(\mathcal{A}_{X \oplus Y})p_Y \cong \mathcal{A}_Y$. Put

$$\mathcal{A}_{X,Y} = p_Y(\mathcal{A}_{X \oplus Y})p_X.$$

The \mathcal{A}_Y - \mathcal{A}_X -module $\mathcal{A}_{X,Y}$ can be described as an inductive limit of completions of the spaces

$$\tilde{\mathcal{C}}(V^{\otimes n} \otimes X, V^{\otimes n} \otimes Y) = \bigoplus_{s_*, s'_* \in (\text{supp } \mu)^n} \mathcal{C}(U_{s_n} \otimes \cdots \otimes U_{s_1} \otimes X, U_{s'_n} \otimes \cdots \otimes U_{s'_1} \otimes Y) \otimes e_{s'_*} \mathcal{R}^{\otimes n} e_{s_*}.$$

Denote by \mathcal{H}_X the Hilbert space completion of $\mathcal{A}_{\mathbb{1},X}$ with respect to the scalar product

$$(x, y) = \tau_{\mathbb{1}}(y^* x).$$

Then \mathcal{H}_X is a Hilbert \mathcal{A}_X - \mathcal{A} -module (it is denoted by X_∞ in [HY00]). Viewing \mathcal{H}_X as a Hilbert bimodule over \mathcal{A} , we get a unitary functor F from \mathcal{C} into the category $\text{Hilb}_{\mathcal{A}}$ of Hilbert bimodules over \mathcal{A} such that $F(U) = \mathcal{H}_U$ on objects and in the obvious way on morphisms in \mathcal{C} . We want to make F into a tensor functor. By the computation on pp. 40–41 of [HY00] the map

$$\begin{aligned} \tilde{\mathcal{C}}(V^{\otimes n}, V^{\otimes n} \otimes X) \otimes \tilde{\mathcal{C}}(V^{\otimes n}, V^{\otimes n} \otimes Y) &\rightarrow \tilde{\mathcal{C}}(V^{\otimes n}, V^{\otimes n} \otimes X \otimes Y), \\ (S \otimes a) \otimes (T \otimes b) &\mapsto (S \otimes \iota_Y)T \otimes ab, \end{aligned}$$

defines an isometry

$$F_2(X, Y): \mathcal{H}_X \otimes_{\mathcal{A}} \mathcal{H}_Y \rightarrow \mathcal{H}_{X \otimes Y}.$$

Lemma 3.9. *Assume that for every $s \in \text{Irr}(\mathcal{C})$ we have*

$$(\mu^n * \delta_s)(\text{supp } \mu^n) \rightarrow 1 \quad \text{as } n \rightarrow \infty.$$

Then the maps $F_2(X, Y)$ are unitary.

Proof. It suffices to prove the lemma for simple objects. Assume $X = U_s$ for some s . For every $n \geq 1$ and $s_* \in (\text{supp } \mu)^n$, let $p_{s_*}^{(n)} \in \mathcal{C}(U_{s_n} \otimes \cdots \otimes U_{s_1} \otimes X)$ be the projection onto the direct sum of the isotypic components corresponding to U_t for some $t \in \text{supp } \mu^n$. Put

$$p^{(n)} = \sum_{s_* \in (\text{supp } \mu)^n} p_{s_*}^{(n)} \otimes e_{s_*} \in \mathcal{A}_{U_s}^{(n)}.$$

Then $\tau_X(p^{(n)}) = (\mu^n * \delta_s)(\text{supp } \mu^n)$. Therefore by assumption $p^{(n)} \rightarrow 1$ in the s^* -topology. It follows that to prove the lemma it suffices to show that if

$$T \otimes x \in \mathcal{C}(U_{s_n} \otimes \cdots \otimes U_{s_1}, U_{s'_n} \otimes \cdots \otimes U_{s'_1} \otimes X \otimes Y) \otimes e_{s'_*} \mathcal{R}^{\otimes n} e_{s_*}$$

is such that $p^{(n)}(T \otimes x) = T \otimes x$, then $T \otimes x$ is in the image of $F_2(X, Y)$. The assumption on T means that the simple objects appearing in the decomposition of $U_{s'_n} \otimes \cdots \otimes U_{s'_1} \otimes X$ appear also in the decomposition of $U_{t_n} \otimes \cdots \otimes U_{t_1}$ for $t_* \in (\text{supp } \mu)^n$. This implies that T can be written as a finite direct sum of morphisms of the form $(S \otimes \iota_Y)R$, with $R \in \mathcal{C}(U_{s_n} \otimes \cdots \otimes U_{s_1}, U_{t_n} \otimes \cdots \otimes U_{t_1} \otimes Y)$ and $S \in \mathcal{C}(U_{t_n} \otimes \cdots \otimes U_{t_1}, U_{s'_n} \otimes \cdots \otimes U_{s'_1} \otimes X)$. Since we also have density of $e_{s'_*} \mathcal{R} e_{t_*} \mathcal{R} e_{s_*}$ in $e_{s'_*} \mathcal{R} e_{s_*}$, this proves the lemma. \square

We remark that the assumption of the lemma is obviously satisfied if $\text{supp } \mu = \text{Irr}(\mathcal{C})$. It is also satisfied if μ is ergodic and $\mu(e) > 0$, since then $\|\mu^n * \delta_s - \mu^n\|_1 \rightarrow 0$ by [HI98, Proposition 3.3].

Once the maps $F_2(X, Y)$ are unitary, it is easy to see that (F, F_2) is a unitary tensor functor $\mathcal{C} \rightarrow \text{Hilb}_{\mathcal{A}}$.

Proposition 3.10. *Assume the measure μ satisfies the assumption of Lemma 3.9. Let \mathcal{B} be the full C^* -tensor subcategory of $\text{Hilb}_{\mathcal{A}}$ generated by the image of $F: \mathcal{C} \rightarrow \text{Hilb}_{\mathcal{A}}$. Then the Poisson boundary $\Pi: \mathcal{C} \rightarrow \mathcal{P}$ of (\mathcal{C}, μ) is isomorphic to $F: \mathcal{C} \rightarrow \mathcal{B}$.*

Proof. The functor F extends to the full subcategory $\tilde{\mathcal{P}}$ of \mathcal{P} formed by the objects of \mathcal{C} using the isomorphisms $\mathcal{P}(U) \cong \mathcal{A}' \cap \mathcal{A}_U$. It follows immediately by definition that this way we get a unitary tensor functor $E: \tilde{\mathcal{P}} \rightarrow \mathcal{B}$ if we put $E_2(X, Y) = F_2(X, Y)$. We then extend this functor to a unitary tensor functor $\mathcal{P} \rightarrow \mathcal{B}$, which we continue to denote by E . To prove the proposition it remains to show that E is fully faithful. In other words, we have to show that the left action of \mathcal{A}_U on \mathcal{H}_U defines an isomorphism $\mathcal{A}' \cap \mathcal{A}_U \cong \text{End}_{\mathcal{A}\text{-}\mathcal{A}}(\mathcal{H}_U)$.

Let us check the stronger statement that the left action defines an isomorphism $\mathcal{A}_U \cong \text{End}_{\mathcal{A}}(\mathcal{H}_U)$. Recalling how \mathcal{H}_U was constructed using complementary projections in $\mathcal{A}_{\mathbb{1} \oplus U}$, it becomes clear that the map $\mathcal{A}_U \rightarrow \text{End}_{\mathcal{A}}(\mathcal{H}_U)$ is always surjective, and it is injective if and only if the projection $p_{\mathbb{1}} \in \mathcal{A}_{\mathbb{1} \oplus U}$ has central support 1. Using the Frobenius reciprocity isomorphism

$$\mathcal{C}(V^{\otimes n} \otimes U) \cong \mathcal{C}(V^{\otimes n}, V^{\otimes n} \otimes U \otimes \bar{U}),$$

it is easy to check that $\mathcal{H}_{U \otimes \bar{U}} \cong L^2(\mathcal{A}_U, \tau_U)$ as a Hilbert \mathcal{A}_U - \mathcal{A} -module. Hence the representation of \mathcal{A}_U on $\mathcal{H}_{U \otimes \bar{U}}$ is faithful. Since $\mathcal{H}_{U \otimes \bar{U}} \cong \mathcal{H}_U \otimes_{\mathcal{A}} \mathcal{H}_{\bar{U}}$, it follows that the representation of \mathcal{A}_U on \mathcal{H}_U is faithful as well. \square

A similar result could also be proved using the algebras \mathcal{N}_U from Section 3.2 instead of \mathcal{A}_U . The situation would be marginally more complicated, since in dealing with the Connes fusion tensor product $\otimes_{\mathcal{N}}$ we would have to take into account the modular group of ω_U . We are not going to pursue this topic here, although it could provide a somewhat alternative route to Proposition 5.2 below.

4. A UNIVERSAL PROPERTY OF THE POISSON BOUNDARY

Let \mathcal{C} be a weakly amenable strict C^* -tensor category. Fix an ergodic probability measure μ on $\text{Irr}(\mathcal{C})$. Recall that such a measure exists by Proposition 2.3. Let $\Pi: \mathcal{C} \rightarrow \mathcal{P}$ be the Poisson boundary of (\mathcal{C}, μ) .

For an object U in \mathcal{C} define

$$d_{\min}^{\mathcal{C}}(U) = \inf d^{\mathcal{A}}(F(U)),$$

where the infimum is taken over all unitary tensor functors $F: \mathcal{C} \rightarrow \mathcal{A}$ from \mathcal{C} into rigid C^* -tensor categories \mathcal{A} . We will show in the next section that $d_{\min}^{\mathcal{C}}$ is the amenable dimension function on \mathcal{C} . The goal of the present section is to prove the following.

Theorem 4.1. *The Poisson boundary $\Pi: \mathcal{C} \rightarrow \mathcal{P}$ is a universal unitary tensor functor such that $d_{\min}^{\mathcal{C}} = d^{\mathcal{P}}\Pi$.*

In other words, $d_{\min}^{\mathcal{C}} = d^{\mathcal{P}}\Pi$ and for any unitary tensor functor $F: \mathcal{C} \rightarrow \mathcal{A}$ such that $d_{\min}^{\mathcal{C}} = d^{\mathcal{A}}F$ there exists a unique, up to a natural unitary monoidal isomorphism, unitary tensor functor $\Lambda: \mathcal{P} \rightarrow \mathcal{A}$ such that $\Lambda\Pi \cong F$.

For a rigid C^* -tensor category \mathcal{A} , consider a unitary tensor functor $F: \mathcal{C} \rightarrow \mathcal{A}$, with no restriction on the dimension function. As we discussed in Section 1.2, we may assume that \mathcal{A} is strict, \mathcal{C} is a C^* -tensor subcategory of \mathcal{A} and F is the embedding functor. Motivated by Izumi's Poisson integral [Izu02] we will define linear maps

$$\Theta_{U,V}: \mathcal{A}(U, V) \rightarrow \mathcal{P}(U, V).$$

We will write Θ_U for $\Theta_{U,U}$ and often omit the subscripts altogether, if there is no danger of confusion. The proof of the theorem will be based on analysis of the multiplicative domain of Θ .

For every object U in \mathcal{C} fix a standard solution (R_U, \bar{R}_U) of the conjugate equations in \mathcal{C} . Define a faithful state ψ_U on $\mathcal{A}(U)$ by

$$\psi_U(T) = d^{\mathcal{C}}(X)^{-1} \bar{R}_U^*(T \otimes \iota) \bar{R}_U.$$

Since any other standard solution has the form $((u \otimes \iota)R_U, (\iota \otimes u)\bar{R}_U)$ for a unitary u , this definition is independent of any choices. More generally, we can define in a similar way ‘‘slice maps’’

$$\iota \otimes \psi_V: \mathcal{A}(U \otimes V) \rightarrow \mathcal{A}(U).$$

Then, since $((\iota \otimes R_U \otimes \iota)R_V, (\iota \otimes \bar{R}_V \otimes \iota)\bar{R}_U)$ is a standard solution for $U \otimes V$, we get

$$\psi_{U \otimes V} = \psi_U(\iota \otimes \psi_V). \quad (4.1)$$

By definition the state ψ_U extends the trace tr_U on $\mathcal{C}(U)$.

Lemma 4.2. *The subalgebra $\mathcal{C}(U) \subset \mathcal{A}(U)$ is contained in the centralizer of the state ψ_U .*

Proof. If u is a unitary in $\mathcal{C}(U)$, then the state $\psi_U(u \cdot u^*)$ is defined similarly to ψ_U , but using the solution $((\iota \otimes u^*)R_U, (u^* \otimes \iota)\bar{R}_U)$ of the conjugate equations for U . Since ψ_U is independent of the choice of standard solutions, it follows that $\psi_U(u \cdot u^*) = \psi_U$. But this exactly means that $\mathcal{C}(U)$ is contained in the centralizer of ψ_U . \square

It follows that there exists a unique ψ_U -preserving conditional expectation $E_U: \mathcal{A}(U) \rightarrow \mathcal{C}(U)$. For objects U and V we can consider $\mathcal{A}(U, V)$ as a subspace of $\mathcal{A}(U \oplus V)$. Then $E_{U \oplus V}$ defines a linear map

$$E_{U,V}: \mathcal{A}(U, V) \rightarrow \mathcal{C}(U, V).$$

Again, we omit the subscripts when convenient.

Lemma 4.3. *The maps $E_{U,V}$ satisfy the following properties:*

- (i) $E_{U,V}(T)^* = E_{V,U}(T^*)$;
- (ii) if $T \in \mathcal{A}(U, V)$ and $S \in \mathcal{C}(V, W)$, then $E_{U,W}(ST) = SE_{U,V}(T)$;
- (iii) for any object X in \mathcal{C} we have $E_{U \otimes X, V \otimes X}(T \otimes \iota_X) = E_{U,V}(T) \otimes \iota_X$.

Proof. Properties (i) and (ii) follows immediately from the corresponding properties of conditional expectations. To prove (iii), it suffices to consider the case $U = V$. Take $S \in \mathcal{C}(U \otimes X)$. Then we have to check that

$$\psi_{U \otimes X}(S(T \otimes \iota)) = \psi_{U \otimes X}(S(E(T) \otimes \iota)).$$

This follows from (4.1) and the fact that by definition we have $(\iota \otimes \psi_X)(S) \in \mathcal{C}(U)$. \square

Now, given a morphism $T \in \mathcal{A}(U, V)$, define a bounded natural transformation $\Theta_{U,V}(T): \iota \otimes U \rightarrow \iota \otimes V$ of functors on \mathcal{C} by

$$\Theta_{U,V}(T)_X = E_{X \otimes U, X \otimes V}(\iota_X \otimes T).$$

Lemma 4.4. *The natural transformation $\Theta_{U,V}(T)$ is P_X -harmonic for any object X in \mathcal{C} .*

Proof. It suffices to consider the case $U = V$. We claim that

$$(\text{tr}_X \otimes \iota)E(\iota_X \otimes T) = E(T).$$

Indeed, for any $S \in \mathcal{C}(U)$ we have

$$\text{tr}_U(S(\text{tr}_X \otimes \iota)E(\iota_X \otimes T)) = \text{tr}_{X \otimes U}(E(\iota_X \otimes ST)) = \psi_{X \otimes U}(\iota \otimes ST) = \psi_U(ST) = \text{tr}_U(SE(T)),$$

where in the third equality we used (4.1). This proves the claim.

We now compute:

$$P_X(\Theta(T))_Y = (\text{tr}_X \otimes \iota)(\Theta(T)_{X \otimes Y}) = (\text{tr}_X \otimes \iota_Y \otimes \iota_U)(E(\iota_X \otimes \iota_Y \otimes T)) = E(\iota_Y \otimes T) = \Theta(T)_Y,$$

so $\Theta(T)$ is P_X -harmonic. \square

It follows that $\Theta_{U,V}$ is a well-defined linear map $\mathcal{A}(U, V) \rightarrow \mathcal{P}(U, V)$.

Lemma 4.5. *The maps $\Theta_{U,V}$ satisfy the following properties:*

- (i) $\Theta_{U,V}(T)^* = \Theta_{V,U}(T^*)$;
- (ii) if $T \in \mathcal{A}(U, V)$ and $S \in \mathcal{C}(V, W)$, then $\Theta_{U,W}(ST) = S\Theta_{U,V}(T)$;
- (iii) for any object X in \mathcal{C} we have $\Theta_{U \otimes X, V \otimes X}(T \otimes \iota_X) = \Theta_{U,V}(T) \otimes \iota_X$ and $\Theta_{X \otimes U, X \otimes V}(\iota_X \otimes T) = \iota_X \otimes \Theta_{U,V}(T)$;
- (iv) the maps $\Theta_U: \mathcal{A}(U) \rightarrow \mathcal{P}(U)$ are unital, completely positive and faithful.

Proof. All these properties are immediate consequences of the definitions and the properties of the maps $E_{U,V}$ given in Lemma 4.3. We would like only to point out that the property $\Theta(\iota \otimes T) = \iota \otimes \Theta(T)$ follows from the definition of the tensor product in \mathcal{P} , the corresponding property for the maps E is neither satisfied nor needed. \square

Our goal now is to understand the multiplicative domains of the maps $\Theta_U: \mathcal{A}(U) \rightarrow \mathcal{P}(U)$. We will first show that these domains cannot be very large. More precisely, assume we have an intermediate C^* -tensor category $\mathcal{C} \subset \mathcal{B} \subset \mathcal{A}$ such that $d^{\mathcal{A}} = d^{\mathcal{B}}$ on \mathcal{C} . For an object U in \mathcal{C} denote by $E_U^{\mathcal{B}}: \mathcal{A}(U) \rightarrow \mathcal{B}(U)$ the conditional expectation preserving the categorical trace on \mathcal{A} . Then we have the following result inspired by [Tom07, Lemma 4.5].

Lemma 4.6. *We have $\Theta_U = \Theta_U E_U^{\mathcal{B}}$.*

Proof. We will first show that a similar property holds for the maps E , so $E_U = E_U E_U^{\mathcal{B}}$.

Consider the normalized categorical trace $\text{tr}_U^{\mathcal{A}}$ on $\mathcal{A}(U)$. We have $\psi_U = \text{tr}_U^{\mathcal{A}}(\cdot Q)$ for some $Q \in \mathcal{A}(U)$. The identity $E_U = E_U E_U^{\mathcal{B}}$ holds if and only if the conditional expectation $E_U^{\mathcal{B}}$ is ψ_U -preserving, or equivalently, $Q \in \mathcal{B}(U)$.

By assumption we have $d^{\mathcal{A}}(U) = d^{\mathcal{B}}(U)$ for every object U in \mathcal{C} . It follows that a standard solution $(R_U^{\mathcal{B}}, \bar{R}_U^{\mathcal{B}})$ of the conjugate equations for U and \bar{U} in \mathcal{B} remains standard in \mathcal{A} . We have $\bar{R}_U = (T \otimes \iota) \bar{R}_U^{\mathcal{B}}$ for a uniquely defined $T \in \mathcal{B}(U)$. Then $Q = \frac{d_{\mathcal{B}}(U)}{d_{\mathcal{A}}(U)} T T^* \in \mathcal{B}(U)$.

We also need the simple property $E_{X \otimes U}^{\mathcal{B}}(\iota_X \otimes T) = \iota_X \otimes E_U^{\mathcal{B}}(T)$. This is proved similarly to Lemma 4.3(iii), using that $\text{tr}_{X \otimes U}^{\mathcal{A}} = \text{tr}_U^{\mathcal{A}}(\text{tr}_X^{\mathcal{A}} \otimes \iota)$ and the fact that $\text{tr}^{\mathcal{A}}$ is defined using standard solutions in \mathcal{B} , so that $(\text{tr}_X^{\mathcal{A}} \otimes \iota)(\mathcal{B}(X \otimes U)) \subset \mathcal{B}(U)$.

The equality $\Theta_U E_U^{\mathcal{B}} = \Theta_U$ is now immediate:

$$\Theta E^{\mathcal{B}}(T)_X = E(\iota_X \otimes E^{\mathcal{B}}(T)) = E E^{\mathcal{B}}(\iota_X \otimes T) = E(\iota_X \otimes T) = \Theta(T)_X.$$

This proves the assertion. \square

Since the completely positive map Θ_U is faithful, the multiplicative domain of $\Theta_U = \Theta_U E_U^{\mathcal{B}}$ is contained in that of $E_U^{\mathcal{B}}$, which is exactly $\mathcal{B}(U)$. Therefore to find this domain we have to consider the smallest possible subcategory that contains \mathcal{C} and still defines the same dimension function as \mathcal{A} .

Lemma 4.7. *For every object U in \mathcal{C} there exists a unique positive invertible element $a_U \in \mathcal{A}(U)$ such that*

$$(\iota \otimes a_U^{1/2})R_U \quad \text{and} \quad (a_U^{-1/2} \otimes \iota)\bar{R}_U$$

form a standard solution of the conjugate equations for U in \mathcal{A} .

Proof. We can find an invertible element $T \in \mathcal{A}(U)$ such that $(\iota \otimes T)R_U$ and $((T^*)^{-1} \otimes \iota)\bar{R}_U$ form a standard solution in \mathcal{A} . Then we can take $a_U = T^*T$, since $T a_U^{-1/2}$ is unitary and hence the morphisms $(\iota \otimes a_U^{1/2})R_U$ and $(a_U^{-1/2} \otimes \iota)\bar{R}_U$ still form a standard solution.

Any other standard solution for U and \bar{U} has the form $(\iota \otimes v a_U^{1/2})R_U$, $(v a_U^{-1/2} \otimes \iota)\bar{R}_U$ for a unitary $v \in \mathcal{A}(U)$. By uniqueness of the polar decomposition the element $v a_U^{1/2}$ is positive only if $v = 1$. \square

Note that if we replace (R_U, \bar{R}_U) by $((\iota \otimes u)R_U, (u \otimes \iota)\bar{R}_U)$ for a unitary $u \in \mathcal{C}(U)$, then a_U gets replaced by $u a_U u^*$.

Lemma 4.8. *For every object U in \mathcal{C} we have $d^{\mathcal{P}}(U) \leq d^{\mathcal{A}}(U)$, and if the equality holds, then we have $\Theta_U(a_U)^{-1} = \Theta_U(a_U^{-1})$.*

Proof. As usual, we omit the subscript U in the computations. Consider the solution

$$r = (\iota \otimes \Theta(a)^{1/2})R, \quad \bar{r} = (\Theta(a)^{-1/2} \otimes \iota)\bar{R}$$

of the conjugate equations for U in \mathcal{P} . Then from the equality

$$r^*r = R^*(\iota \otimes \Theta(a))R = \Theta(R^*(\iota \otimes a)R),$$

we have $\|r\| = d^{\mathcal{A}}(U)^{1/2}$. On the other hand, we also have

$$\bar{r}^*\bar{r} = \bar{R}^*(\Theta(a)^{-1} \otimes \iota)\bar{R}.$$

By Jensen's inequality for positive maps and the fact that the function $t \mapsto t^{-1}$ on $(0, +\infty)$ is operator convex (see, e.g., [NS06, B.2]), we have $\Theta(a)^{-1} \leq \Theta(a^{-1})$. Hence we have the estimate

$$\bar{r}^*\bar{r} \leq \bar{R}^*(\Theta(a^{-1}) \otimes \iota)\bar{R} = \Theta(\bar{R}^*(a^{-1} \otimes \iota)\bar{R}),$$

and we conclude that $\|\bar{r}\| \leq d^{\mathcal{A}}(U)^{1/2}$. Hence $d^{\mathcal{P}}(U) \leq d^{\mathcal{A}}(U)$, and if the equality holds, then we have $\|\bar{r}\| = d^{\mathcal{A}}(U)^{1/2}$ and

$$\bar{R}^*(\Theta(a)^{-1} \otimes \iota)\bar{R} = \bar{R}^*(\Theta(a^{-1}) \otimes \iota)\bar{R}.$$

Since $T \mapsto \bar{R}^*(T \otimes \iota)\bar{R}$ is a faithful positive linear functional on $\mathcal{P}(U)$, this is equivalent to $\Theta(a)^{-1} = \Theta(a^{-1})$. \square

If we have $d^{\mathcal{P}}(U) = d^{\mathcal{A}}(U)$, we can then apply the following general result, which is surely well-known.

Lemma 4.9. *Assume $\theta: A \rightarrow B$ is a unital completely positive map of C^* -algebras and $a \in A$ is a positive invertible element such that $\theta(a)^{-1} = \theta(a^{-1})$. Then a lies in the multiplicative domain of θ .*

Proof. It suffices to show that $a^{1/2}$ lies in the multiplicative domain. This, in turn, is equivalent to the equality $\theta(a)^{1/2} = \theta(a^{1/2})$.

Using Jensen's inequality and operator convexity of the functions $t \mapsto -t^{1/2}$ and $t \mapsto t^{-1}$, we have

$$\theta(a)^{1/2} \geq \theta(a^{1/2}), \quad \theta(a^{-1})^{1/2} \geq \theta(a^{-1/2}), \quad \text{and} \quad \theta(a^{-1/2})^{-1} \leq \theta(a^{1/2}).$$

The second and the third inequalities imply

$$\theta(a^{-1})^{-1/2} \leq \theta(a^{1/2}).$$

Since $\theta(a^{-1}) = \theta(a)^{-1}$, this gives $\theta(a)^{1/2} \leq \theta(a^{1/2})$. Hence $\theta(a)^{1/2} = \theta(a^{1/2})$. \square

To finish the preparation for the proof of Theorem 4.1 we consider the maps Θ for $\mathcal{A} = \mathcal{P}$.

Lemma 4.10. *The maps $\Theta_{U,V}: \mathcal{P}(U,V) \mapsto \mathcal{P}(U,V)$ defined by the functor $\Pi: \mathcal{C} \rightarrow \mathcal{P}$ are the identity maps.*

Proof. It suffices to consider $U = V$. Take $\eta \in \mathcal{P}(U)$. Let us show first that $\Theta(\eta)_{\mathbb{1}} = \eta_{\mathbb{1}}$, that is, $E(\eta) = \eta_{\mathbb{1}}$. In other words, we have to check that for any $S \in \mathcal{C}(U)$ we have

$$\psi_U(S\eta) = \text{tr}_U(S\eta_{\mathbb{1}}).$$

This follows immediately by definition, since

$$(\bar{R}_U^*(S\eta \otimes \iota)\bar{R}_U)_{\mathbb{1}} = \bar{R}_U^*(S\eta_{\mathbb{1}} \otimes \iota)\bar{R}_U.$$

Now, for any object X in \mathcal{C} , we have

$$\Theta(\eta)_X = E(\iota_X \otimes \eta) = \Theta(\iota_X \otimes \eta)_{\mathbb{1}} = (\iota_X \otimes \eta)_{\mathbb{1}} = \eta_X.$$

Therefore we have $\Theta(\eta) = \eta$. \square

Proof of Theorem 4.1. The equality $d_{\min}^{\mathcal{C}}(U) = d^{\mathcal{P}}(U)$ for objects U in \mathcal{C} follows from Lemma 4.8.

Let $F: \mathcal{C} \rightarrow \mathcal{A}$ be a unitary tensor functor such that $d_{\min}^{\mathcal{C}} = d^{\mathcal{A}}F$. As above, we assume that F is simply an embedding functor. Consider the minimal subcategory $\tilde{\mathcal{B}} \subset \mathcal{A}$ containing $\mathcal{C} \subset \mathcal{A}$ and the morphisms $\iota_V \otimes a_U \otimes \iota_W$ for all objects V, U and W in \mathcal{C} , where $a_U \in \mathcal{A}(U)$ are the morphisms defined in Lemma 4.7. This is a C^* -tensor subcategory, in general without subobjects. Complete $\tilde{\mathcal{B}}$ with respect to subobjects to get a C^* -tensor category \mathcal{B} . By adding more objects to \mathcal{A} we may assume without loss of generality that $\mathcal{B} \subset \mathcal{A}$. Lemmas 4.5, 4.8 and 4.9 imply that the maps $\Theta_{U,V}$ define a strict unitary tensor functor $\tilde{\mathcal{B}} \rightarrow \mathcal{P}$. Thus $\tilde{\mathcal{B}}$ is unitary monoidally equivalent to a C^* -tensor subcategory $\tilde{\mathcal{P}} \subset \mathcal{P}$, possibly without subobjects. Completing $\tilde{\mathcal{P}}$ with respect to subobjects we get a C^* -tensor subcategory $\mathcal{P}' \subset \mathcal{P}$, which is unitarily monoidally equivalent to \mathcal{B} .

We claim that the embedding functor $\mathcal{P}' \rightarrow \mathcal{P}$ is a unitary monoidal equivalence. Indeed, by construction we have $d^{\mathcal{P}'}(U) = d_{\min}^{\mathcal{C}}(U)$ for every object U in \mathcal{C} . By Lemmas 4.6 and 4.10 it follows then that the identity maps $\mathcal{P}(U) \rightarrow \mathcal{P}'(U)$ factor through the conditional expectations $E_U^{\mathcal{P}'}: \mathcal{P}(U) \rightarrow \mathcal{P}'(U)$. Hence $\mathcal{P}(U) = \mathcal{P}'(U)$. Since the objects of \mathcal{C} generate \mathcal{P} , this implies that the embedding functor $\mathcal{P}' \rightarrow \mathcal{P}$ is a unitary monoidal equivalence.

We have therefore shown that \mathcal{P} and \mathcal{B} are unitarily monoidally equivalent, and furthermore, by properties of the maps Θ such an equivalence $\Lambda: \mathcal{P} \rightarrow \mathcal{B}$ can be chosen to be the identity tensor functor on \mathcal{C} . Considered as a functor $\mathcal{P} \rightarrow \mathcal{A}$, the unitary tensor functor Λ gives the required factorization of $F: \mathcal{C} \rightarrow \mathcal{A}$.

It remains to prove uniqueness. Denote by $\rho_U \in \mathcal{P}(U)$ the elements a_U constructed in Lemma 4.7 for the category \mathcal{P} . By the uniqueness part of that lemma, it is clear that any unitary tensor functor $\Lambda: \mathcal{P} \rightarrow \mathcal{A}$ extending the embedding functor $\mathcal{C} \rightarrow \mathcal{A}$ must map $\rho_U \in \mathcal{P}(U)$ into $a_U \in \mathcal{A}(U)$. But this completely determines Λ up to a unitary monoidal equivalence, since by the above considerations the category \mathcal{P} is obtained from \mathcal{C} by adding the morphisms ρ_U and then completing the new category with respect to subobjects. \square

We finish the section with a couple of corollaries.

The universality of the Poisson boundary implies that up to an isomorphism the boundary does not depend on the choice of an ergodic measure. But the proof shows that a stronger result is true.

Corollary 4.11. *Let \mathcal{C} be a weakly amenable C^* -tensor category and μ be an ergodic probability measure on $\text{Irr}(\mathcal{C})$. Then any bounded P_μ -harmonic natural transformation is P_s -harmonic for every $s \in \text{Irr}(\mathcal{C})$, so the Poisson boundary $\Pi: \mathcal{C} \rightarrow \mathcal{P}$ of (\mathcal{C}, μ) does not depend on the choice of an ergodic measure.*

Proof. By Lemma 4.10 the maps $\Theta_{U,V}: \mathcal{P}(U,V) \rightarrow \mathcal{P}(U,V)$ are the identity maps, while by Lemma 4.4 their images consist of elements that are P_s -harmonic for all s . \square

When \mathcal{C} is amenable, then $d_{\min}^{\mathcal{C}} = d^{\mathcal{C}}$ and we get the following.

Corollary 4.12. *If \mathcal{C} is an amenable C^* -tensor category, then its Poisson boundary with respect to any ergodic probability measure on $\text{Irr}(\mathcal{C})$ is trivial. In other words, any bounded natural transformation $\iota \otimes U \rightarrow \iota \otimes V$ which is P_s -harmonic for all $s \in \text{Irr}(\mathcal{C})$, is defined by a morphism in $\mathcal{C}(U, V)$.*

Proof. The identity functor $\mathcal{C} \rightarrow \mathcal{C}$ is already universal, so it is isomorphic to the Poisson boundary. \square

We remark that if we were interested only in proving this corollary, a majority of the above arguments, being applied to the functor $\mathcal{C} \rightarrow \mathcal{P}$, would become either trivial or unnecessary. Namely, in this case a standard solution of the conjugate equations in \mathcal{C} remains standard in \mathcal{P} , so we have $E_U = E_U^{\mathcal{C}}$, and the key parts of the proof are contained in Lemmas 4.6 and 4.10. The first lemma shows that given $\eta \in \mathcal{P}(U)$ we have $E(\iota_X \otimes \eta) = E(\iota_X \otimes E(\eta))$, while the second shows that $E(\iota_X \otimes \eta) = \eta_X$. Since $E(\iota_X \otimes E(\eta)) = \iota_X \otimes E(\eta)$, we therefore see that η coincides with $E(\eta) \in \mathcal{C}(U)$.

Corollary 4.12 is more or less known: in view of Proposition 3.8, for measures considered in [HY00] it is equivalent to [HY00, Theorem 7.6]. For an even more restrictive class of measures the result also follows from [LR97, Theorem 5.16].

5. AMENABILITY OF THE MINIMAL DIMENSION FUNCTION

As in the previous section, let \mathcal{C} be a weakly amenable strict C^* -tensor category. We defined the dimension function $d_{\min}^{\mathcal{C}}$ on \mathcal{C} as the infimum of dimension functions under all possible embeddings of \mathcal{C} into rigid C^* -tensor categories, and showed that it is indeed a dimension function realized by the Poisson boundary of \mathcal{C} with respect to any ergodic measure. The goal of this section is to prove the following.

Theorem 5.1. *The dimension function $d_{\min}^{\mathcal{C}}$ is amenable, that is, $d_{\min}^{\mathcal{C}}(U) = \|\Gamma_U\|$ holds for every object U in \mathcal{C} .*

We remark that already the fact the fusion algebra of a weakly amenable C^* -tensor category admits an amenable dimension function is nontrivial. We do not know whether this is true for weakly amenable dimension functions on fusion algebras that are not of categorical origin. If the fusion algebra is commutative, this is true by a result of Yamagami [Yam99].

Let μ be an ergodic probability measure μ on $\text{Irr}(\mathcal{C})$ and consider the corresponding Poisson boundary $\Pi: \mathcal{C} \rightarrow \mathcal{P}$. By Theorem 4.1 we already know that $d_{\min}^{\mathcal{C}} = d^{\mathcal{P}}\Pi$. Therefore Theorem 5.1 is equivalent to saying that $d^{\mathcal{P}}\Pi$ is the amenable dimension function on \mathcal{C} .

We will use the realization of harmonic transformations as elements of $\mathcal{N}' \cap \mathcal{N}_U$ given in Section 3.2. It will also be important to work with factors. Therefore we assume that in addition to being ergodic the measure μ is generating and satisfies the assumption of Proposition 3.5 (recall that for the latter it suffices to require $\mu(e) > 0$). Recall once again that by Proposition 2.3 such a measure exists. We also remind that by Corollary 4.11 the Poisson boundary does not depend on the ergodic measure, but its realization in terms of relative commutants does. We then have the following expected (in view of Proposition 3.10 and the discussion following it), but crucial, result.

Proposition 5.2. *For every object U in \mathcal{C} , we have $d^{\mathcal{P}}(U) = [\mathcal{N}_U: \mathcal{N}]_0^{1/2}$, where $[\mathcal{N}_U: \mathcal{N}]_0$ is the minimal index of the subfactor $\mathcal{N} \subset \mathcal{N}_U$.*

Before we turn to the proof, recall the construction of \mathcal{N}_U . Consider $V = \bigoplus_{s \in \text{supp } \mu} U_s$. We will work with V as with a well-defined object. If $\text{supp } \mu$ is infinite, to be rigorous, in what follows we have to replace V by finite sums of objects U_s , $s \in \text{supp } \mu$, and then pass to the limit, but we will omit this repetitive simple argument. With this understanding, \mathcal{N}_U is the inductive limit of the algebras $\mathcal{N}_U^{(n)} = \mathcal{C}(V^{\otimes n} \otimes U)$ equipped with the faithful states $\omega_U^{(n)}$.

Given another object U' , the partial trace $\iota \otimes \text{tr}_U$ defines, for each n , a conditional expectation $\mathcal{N}_{U' \otimes U}^{(n)} \rightarrow \mathcal{N}_{U'}^{(n)}$ which preserves the state $\omega_{U' \otimes U}^{(n)}$. The conditional expectation $\mathcal{N}_{U' \otimes U} \rightarrow \mathcal{N}_{U'}$ which we get in the limit, is denoted by $E_{U', U}$, or simply by E_U if there is no danger of confusion. Fix a standard solution (R_U, \bar{R}_U) of the conjugate equations for U in \mathcal{C} .

Lemma 5.3. *The index of the conditional expectation $E_U: \mathcal{N}_U \rightarrow \mathcal{N}$ equals $d^{\mathcal{C}}(U)^2$, the corresponding basic extension is $\mathcal{N}_U \subset \mathcal{N}_{U \otimes \bar{U}}$, with the Jones projection $e_U = d^{\mathcal{C}}(U)^{-1} \bar{R}_U \bar{R}_U^* \in \mathcal{N}_{U \otimes \bar{U}}^{(0)} \subset \mathcal{N}_{U \otimes \bar{U}}$ and the conditional expectation $E_{\bar{U}}: \mathcal{N}_{U \otimes \bar{U}} \rightarrow \mathcal{N}_U$.*

Proof. By the abstract characterization of the basic extension [HK93, Theorem 8] it suffices to check the following three properties: $E_{\bar{U}}(e_U) = d^{\mathcal{C}}(U)^{-2} 1$, $E_{\bar{U}}(xe_U)e_U = d^{\mathcal{C}}(U)^{-2} xe_U$ for all $x \in \mathcal{N}_{U \otimes \bar{U}}$, and $e_U xe_U = E_U(x)e_U$ for all $x \in \mathcal{N}_U$. The first and the third properties are immediate by definition. To prove the second, it is enough to show that for all $x \in \mathcal{C}(X \otimes U \otimes \bar{U})$ we have

$$d^{\mathcal{C}}(U)((\iota \otimes \text{tr}_{\bar{U}})(x(\iota_X \otimes \bar{R}_U \bar{R}_U^*)) \otimes \iota_{\bar{U}})(\iota_X \otimes \bar{R}_U \bar{R}_U^*) = x(\iota_X \otimes \bar{R}_U \bar{R}_U^*).$$

The left hand side equals

$$\begin{aligned}
& (\iota_X \otimes \iota_U \otimes R_U^* \otimes \iota_{\bar{U}})(x \otimes \iota_U \otimes \iota_{\bar{U}})(\iota_X \otimes \bar{R}_U \bar{R}_U^* \otimes \iota_U \otimes \iota_{\bar{U}})(\iota_X \otimes \iota_U \otimes R_U \otimes \iota_{\bar{U}})(\iota_X \otimes \bar{R}_U \bar{R}_U^*) \\
&= (\iota_X \otimes \iota_U \otimes R_U^* \otimes \iota_{\bar{U}})(x \otimes \iota_U \otimes \iota_{\bar{U}})(\iota_X \otimes \bar{R}_U \otimes \iota_U \otimes \iota_{\bar{U}})(\iota_X \otimes \bar{R}_U \bar{R}_U^*) \\
&= (\iota_X \otimes \iota_U \otimes R_U^* \otimes \iota_{\bar{U}})(x \otimes \iota_U \otimes \iota_{\bar{U}})(\iota_X \otimes \bar{R}_U \otimes \bar{R}_U)(\iota_X \otimes \bar{R}_U^*) \\
&= x(\iota_X \otimes \bar{R}_U \bar{R}_U^*),
\end{aligned}$$

which proves the lemma. \square

This lemma implies in particular that there exists a unique representation

$$\pi: \mathcal{N}_{U \otimes \bar{U}} \rightarrow B(L^2(\mathcal{N}_U, \omega_U))$$

that extends the representation of \mathcal{N}_U and is such that $\pi(e_U)$ is the projection onto the closure of $\Lambda_{\omega_U}(\mathcal{N}) \subset L^2(\mathcal{N}_U, \omega_U)$.

Lemma 5.4. *The representation $\pi: \mathcal{N}_{U \otimes \bar{U}} \rightarrow B(L^2(\mathcal{N}_U, \omega_U))$ is given by*

$$\pi(x)\Lambda_{\omega_U}(y) = \Lambda_{\omega_U}((\iota \otimes R_U^*)(x \otimes \iota_U)(y \otimes \iota_{\bar{U}} \otimes \iota_U)(\iota \otimes \bar{R}_U \otimes \iota_U))$$

for $x \in \bigcup_n \mathcal{N}_{U \otimes \bar{U}}^{(n)}$ and $y \in \bigcup_n \mathcal{N}_U^{(n)}$.

Proof. Let us write $\tilde{\pi}(x)$ for the operators in the formulation of the lemma. The origin of the formula for $\tilde{\pi}$ is the Frobenius reciprocity isomorphism

$$\mathcal{C}(V^{\otimes n} \otimes U) \cong \mathcal{C}(V^{\otimes n}, V^{\otimes n} \otimes U \otimes \bar{U}), \quad T \mapsto (T \otimes \iota_{\bar{U}})(\iota_{V^{\otimes n}} \otimes \bar{R}_U),$$

with inverse $S \mapsto (\iota \otimes R_U^*)(S \otimes \iota_U)$. Up to scalar factors these isomorphisms become unitary once we equip both spaces with scalar products defined by the states $\omega_U^{(n)}$ and $\omega_{\mathbb{1}}^{(n)}$, respectively. The algebra $\mathcal{C}(V^{\otimes n} \otimes U \otimes \bar{U})$ is represented on $\mathcal{C}(V^{\otimes n}, V^{\otimes n} \otimes U \otimes \bar{U})$ by the operators of multiplication on the left. Being written on the space $\mathcal{C}(V^{\otimes n} \otimes U)$, this representation is exactly $\tilde{\pi}$. Therefore $\tilde{\pi}$ certainly defines a representation of the $*$ -algebra $\bigcup_n \mathcal{N}_{U \otimes \bar{U}}^{(n)}$ on the dense subspace $\bigcup_n L^2(\mathcal{N}_U^{(n)}, \omega_U^{(n)})$ of $L^2(\mathcal{N}_U, \omega_U)$. In order to see that this representation extends to a normal representation of $\mathcal{N}_{U \otimes \bar{U}}$, observe that the vector $\Lambda_{\omega_U}(1)$ is cyclic and

$$(\tilde{\pi}(x)\Lambda_{\omega_U}(1), \Lambda_{\omega_U}(1)) = d^{\mathcal{C}}(U)^2 \omega_{U \otimes \bar{U}}(e_U x e_U),$$

since for every $z \in \mathcal{C}(U \otimes \bar{U})$ we have

$$\begin{aligned}
\text{tr}_U((\iota_U \otimes R_U^*)(z \otimes \iota_U)(\bar{R}_U \otimes \iota_U)) &= d^{\mathcal{C}}(U)^{-1} \bar{R}_U^*(\iota_U \otimes R_U^* \otimes \iota_{\bar{U}})(z \otimes \iota_U \otimes \iota_{\bar{U}})(\bar{R}_U \otimes \iota_U \otimes \iota_{\bar{U}}) \bar{R}_U \\
&= d^{\mathcal{C}}(U)^{-1} \bar{R}_U^* z \bar{R}_U = d^{\mathcal{C}}(U) \text{tr}_{U \otimes \bar{U}}(z \bar{R}_U^* \bar{R}_U) \\
&= d^{\mathcal{C}}(U)^2 \text{tr}_{U \otimes \bar{U}}(z e_U) = d^{\mathcal{C}}(U)^2 \text{tr}_{U \otimes \bar{U}}(e_U z e_U).
\end{aligned}$$

It is clear that $\tilde{\pi}(x)\Lambda_{\omega_U}(y) = \Lambda_{\omega_U}(xy)$ for $x \in \bigcup_n \mathcal{N}_U^{(n)}$, so $\tilde{\pi}$ extends the representation of \mathcal{N}_U on $L^2(\mathcal{N}_U, \omega_U)$. Therefore to prove that $\pi = \tilde{\pi}$ it remains to show that $\tilde{\pi}(e_U)$ is the projection onto $\overline{\Lambda_{\omega_U}(\mathcal{N})}$, that is,

$$\tilde{\pi}(e_U)\Lambda_{\omega_U}(y) = \Lambda_{\omega_U}(E_U(y)) \quad \text{for } y \in \bigcup_n \mathcal{N}_U^{(n)}.$$

But this is obvious, as $(\iota_U \otimes R_U^*)(\bar{R}_U \bar{R}_U^* \otimes \iota_U) = \bar{R}_U^* \otimes \iota_U$. \square

It is easy to describe the modular group σ^{ω_U} of ω_U . For $s_* = (s_1, \dots, s_n) \in (\text{supp } \mu)^n$, let us put

$$\delta_{s_*} = \frac{\mu(s_1) \cdots \mu(s_n)}{d^{\mathcal{C}}(U_{s_1}) \cdots d^{\mathcal{C}}(U_{s_n})}.$$

Then

$$\sigma_t^{\omega_U}(x) = \left(\frac{\delta_{s'_*}}{\delta_{s_*}} \right)^{it} x \quad \text{for } x \in \mathcal{C}(U_{s_n} \otimes \cdots \otimes U_{s_n} \otimes U, U_{s'_n} \otimes \cdots \otimes U_{s'_1} \otimes U).$$

What matters for us is that since the automorphisms $\sigma_t^{\omega_U}$ are approximately implemented by unitaries in \mathcal{N} , the relative commutant $\mathcal{N}' \cap \mathcal{N}_U$ is contained in the centralizer of the state ω_U .

Consider the modular conjugation $J = J_{\omega_U}$ on $L^2(\mathcal{N}_U, \omega_U)$. By Lemma 5.3 and definition of the basic extension we have

$$J\mathcal{N}'J = \pi(\mathcal{N}'_{U \otimes \bar{U}}).$$

Therefore the map $x \mapsto Jx^*J$ defines a $*$ -anti-isomorphism

$$\mathcal{N}' \cap \mathcal{N}_U \cong \mathcal{N}'_{U \otimes \bar{U}} \cap \mathcal{N}_{U \otimes \bar{U}}.$$

Identifying these relative commutants with $\mathcal{P}(U)$ and $\mathcal{P}(\bar{U})$, respectively, we get a $*$ -anti-isomorphism $\mathcal{P}(U) \cong \mathcal{P}(\bar{U})$, which we denote by $\eta \mapsto \eta^\vee$.

Lemma 5.5. *For every $\eta \in \mathcal{P}(U)$ we have*

$$\eta^\vee = (R_U^* \otimes \iota_{\bar{U}})(\iota_{\bar{U}} \otimes \eta \otimes \iota_{\bar{U}})(\iota_{\bar{U}} \otimes \bar{R}_U).$$

Proof. Consider the element $\tilde{\eta} = (R_U^* \otimes \iota_{\bar{U}})(\iota_{\bar{U}} \otimes \eta \otimes \iota_{\bar{U}})(\iota_{\bar{U}} \otimes \bar{R}_U)$. In terms of families of morphisms this means that

$$\tilde{\eta}_X = (\iota_X \otimes R_U^* \otimes \iota_{\bar{U}})(\eta_{X \otimes \bar{U}} \otimes \iota_{\bar{U}})(\iota_X \otimes \iota_{\bar{U}} \otimes \bar{R}_U),$$

or equivalently,

$$(\iota_X \otimes R_U^*)(\tilde{\eta}_X \otimes \iota_U) = (\iota_X \otimes R_U^*)\eta_{X \otimes \bar{U}}. \quad (5.1)$$

For every n consider the projection $p_n: L^2(\mathcal{N}_U, \omega_U) \rightarrow L^2(\mathcal{N}_U^{(n)}, \omega_U^{(n)})$. Let $x \in \mathcal{N}' \cap \mathcal{N}_U$ be the element corresponding to η , and $\tilde{x} \in \mathcal{N}'_{U \otimes \bar{U}} \cap \mathcal{N}_{U \otimes \bar{U}}$ be the element corresponding to $\tilde{\eta}$. By Lemma 5.4 and the way we represent $\tilde{\eta}$ by \tilde{x} , for every $y \in \mathcal{N}_U^{(n)}$ we have

$$p_n \pi(\tilde{x}) \Lambda_{\omega_U}(y) = \Lambda_{\omega_U}((\iota \otimes R_U^*)(\tilde{\eta}_{V \otimes n \otimes U} \otimes \iota_U)(y \otimes \iota_{\bar{U}} \otimes \iota_U)(\iota \otimes \bar{R}_U \otimes \iota_U)). \quad (5.2)$$

On the other hand, since x is contained in the centralizer of ω_U , we have

$$\begin{aligned} p_n Jx^*J \Lambda_{\omega_U}(y) &= p_n \Lambda_{\omega_U}(yx) = \Lambda_{\omega_U}(y \eta_{V \otimes n}) \\ &= \Lambda_{\omega_U}((\iota \otimes R_U^*)(y \otimes \iota_{\bar{U}} \otimes \iota_U)(\iota \otimes \bar{R}_U \otimes \iota_U) \eta_{V \otimes n}) \\ &= \Lambda_{\omega_U}((\iota \otimes R_U^*) \eta_{V \otimes n \otimes U \otimes \bar{U}}(y \otimes \iota_{\bar{U}} \otimes \iota_U)(\iota \otimes \bar{R}_U \otimes \iota_U)). \end{aligned}$$

By (5.1) the last expression equals (5.2), so

$$p_n \pi(\tilde{x}) \Lambda_{\omega_U}(y) = p_n Jx^*J \Lambda_{\omega_U}(y).$$

Since this is true for all n and $y \in \mathcal{N}_U^{(n)}$, we conclude that $\pi(\tilde{x}) = Jx^*J$. \square

Proof of Proposition 5.2. The operator valued weights from \mathcal{N}_U to \mathcal{N} are parametrized by the positive elements $a \in \mathcal{N}' \cap \mathcal{N}_U$ by $a \mapsto E^a$, where E^a is defined by $E^a(x) = E_U(a^{1/2} x a^{1/2})$. The map E^a is a conditional expectation if and only if the normalization condition $E_U(a) = 1$ holds. Moreover, by the proof of [Hia88, Theorem 1], $(E^a)^{-1}$ is given by $x \mapsto E_U^{-1}(a^{-1/2} x a^{-1/2})$. Therefore we have

$$[\mathcal{N}_U: \mathcal{N}]_0 = \min_{\substack{a \in \mathcal{N}' \cap \mathcal{N}_U, \\ a > 0}} E_U(a) E_U^{-1}(a^{-1}) = \min_{\substack{a \in \mathcal{N}' \cap \mathcal{N}_U, \\ a > 0}} d^{\mathcal{C}}(U)^2 E_U(a) \tilde{E}_U(Ja^{-1}J),$$

where $\tilde{E}_U = d^{\mathcal{C}}(U)^{-2} J E_U^{-1}(J \cdot J) J: \pi(\mathcal{N}_{U \otimes \bar{U}}) \rightarrow \mathcal{N}_U$.

If $a \in \mathcal{N}' \cap \mathcal{N}_U$ corresponds to $\eta \in \mathcal{P}(U)$, we have

$$E_U(a) = d^{\mathcal{C}}(U)^{-1} \bar{R}_U(\eta \otimes \iota) \bar{R}_U^*$$

By Lemma 5.3 we have $\tilde{E}_U(\pi(x)) = \pi(E_{\bar{U}}(x))$ for $x \in \mathcal{N}_{U \otimes \bar{U}}$. Hence by Lemma 5.5 we get

$$\begin{aligned} \tilde{E}_U(Ja^{-1}J) &= d^{\mathcal{C}}(U)^{-1} R_U^*((\eta^{-1})^\vee \otimes \iota) R_U \\ &= d^{\mathcal{C}}(U)^{-1} R_U^*(R_U^* \otimes \iota_{\bar{U}} \otimes \iota_U)(\iota_{\bar{U}} \otimes \eta^{-1} \otimes \iota_{\bar{U}} \otimes \iota_U)(\iota_{\bar{U}} \otimes \bar{R}_U \otimes \iota_U) R_U \\ &= d^{\mathcal{C}}(U)^{-1} R_U^*(\iota_{\bar{U}} \otimes \eta^{-1}) R_U. \end{aligned}$$

We thus conclude that $[\mathcal{N}_U: \mathcal{N}]_0$ is the minimum of the products of the scalars

$$\bar{R}_U(\eta \otimes \iota)\bar{R}_U^* \text{ and } R_U^*(\iota \otimes \eta^{-1})R_U$$

over all positive invertible $\eta \in \mathcal{P}(U)$. This is exactly $d^{\mathcal{P}}(U)^2$. \square

Proof of Theorem 5.1. The estimate $\|\Gamma_U\| \leq d^{\mathcal{P}}(U)$ comes for free. We thus need to prove the opposite inequality.

Let $E_U^{\mathcal{P}}: \mathcal{N}_U \rightarrow \mathcal{N}$ be the minimal conditional expectation. Let us first assume that \mathcal{N}_U (and hence \mathcal{N}) is infinite. Then by Proposition 5.2 and [Hia91, Corollary 7.2] we have the equalities

$$2 \log d^{\mathcal{P}}(U) = \log \text{Ind } E_U^{\mathcal{P}} = H_{E_U^{\mathcal{P}}}(\mathcal{N}_U | \mathcal{N}).$$

Let $\epsilon > 0$ and ψ be a normal state on \mathcal{N}_U such that

$$H_{\psi}(\mathcal{N}_U | \mathcal{N}) \geq 2 \log d^{\mathcal{P}}(U) - \epsilon.$$

When A is a finite subset of $\text{supp } \mu$, consider the projection $p_A = \bigoplus_{s \in A} \iota_s$ in $\mathcal{N}^{(1)}$. If A_1, \dots, A_n are finite subsets of $\text{supp } \mu$, then $p_{A_*} = p_{A_n} \otimes \dots \otimes p_{A_1}$ is a projection in $\mathcal{N}^{(n)}$, and we consider the corresponding corner

$$\mathcal{N}_U^{A_*} = p_{A_*} \mathcal{N}_U^{(n)} p_{A_*} = \bigoplus_{\substack{s_i, s'_i \in A_i \\ i=1, \dots, n}} \mathcal{C}(U_{s_n} \otimes \dots \otimes U_{s_1} \otimes U, U_{s'_n} \otimes \dots \otimes U_{s'_1} \otimes U)$$

in $\mathcal{N}_U^{(n)}$ and the similarly defined corner \mathcal{N}^{A_*} in $\mathcal{N}^{(n)}$. When $\psi(p_{A_*}) \neq 0$, define also a state ψ_{A_*} on $\mathcal{N}_U^{A_*}$ by $\psi_{A_*} = \psi(p_{A_*})^{-1} \psi(p_{A_*} \cdot p_{A_*})$. By the lower semicontinuity of relative entropy, we can find n and finite sets A_1, \dots, A_n such that

$$H_{\psi_{A_*}}(\mathcal{N}_U^{A_*} | \mathcal{N}^{A_*}) \geq H_{\psi}(\mathcal{N}_U | \mathcal{N}) - \epsilon.$$

By Proposition A.3, the inclusion matrix $\Gamma_{A_*, U}$ of $\mathcal{N}^{A_*} \subset \mathcal{N}_U^{A_*}$ satisfies

$$2 \log \|\Gamma_{A_*, U}\| \geq H_{\psi_{A_*}}(\mathcal{N}_U^{A_*} | \mathcal{N}^{A_*}).$$

Therefore we have the estimate

$$\log \|\Gamma_{A_*, U}\| \geq \log d^{\mathcal{P}}(U) - \epsilon.$$

But the transpose of the matrix $\Gamma_{A_*, U}$ is obtained from Γ_U by considering only columns that correspond to the simple objects appearing in the decomposition of $U_{s_n} \otimes \dots \otimes U_{s_1}$ for $s_i \in A_i$, and then removing the zero rows. Hence

$$\|\Gamma_U\| \geq \|\Gamma_{A_*, U}\|.$$

Since ϵ was arbitrary, we thus get $\|\Gamma_U\| \geq d^{\mathcal{P}}(U)$.

If \mathcal{N}_U is finite, we consider the inclusion $\mathcal{N} \bar{\otimes} M \subset \mathcal{N}_U \bar{\otimes} M$ for some infinite hyperfinite von Neumann algebra M with a prescribed strongly operator dense increasing sequence $M_{n_k}(\mathbb{C}) \subset M$; for example, we could take a Powers factor R_λ with the usual copies of $M_2(\mathbb{C})^{\otimes k}$ in it. Then the minimal conditional expectation $\mathcal{N}_U \bar{\otimes} M \rightarrow \mathcal{N} \bar{\otimes} M$ is given by $E_U^{\mathcal{P}} \otimes \iota$, and its index equals that of $E_U^{\mathcal{P}}$. Since the inclusion matrix of $\mathcal{N}^{A_*} \otimes M_{n_k}(\mathbb{C}) \subset \mathcal{N}_U^{A_*} \otimes M_{n_k}(\mathbb{C})$ is the same as that of $\mathcal{N}^{A_*} \subset \mathcal{N}_U^{A_*}$, we can then argue in the same way as above. \square

Since amenability of dimension functions is preserved under homomorphisms of fusion algebras by [HI98, Proposition 7.4], we get the following corollary.

Corollary 5.6. *Let $\Pi: \mathcal{C} \rightarrow \mathcal{P}$ be the Poisson boundary of a rigid C^* -tensor category with respect to an ergodic probability measure on $\text{Irr}(\mathcal{C})$. Then \mathcal{P} is an amenable C^* -tensor category.*

Combining this with Corollary 4.12 we get the following categorical version of the Furstenberg–Kaimanovich–Vershik–Rosenblatt characterization of amenability.

Theorem 5.7. *A rigid C^* -tensor category \mathcal{C} is amenable if and only if there is a probability measure μ on $\text{Irr}(\mathcal{C})$ such that the Poisson boundary of (\mathcal{C}, μ) is trivial. Furthermore, the Poisson boundary of an amenable C^* -tensor category is trivial for any ergodic probability measure.*

Therefore we can say that while weak amenability can be detected by studying classical Poisson boundaries of random walks on the fusion algebra, for amenability we have to consider noncommutative, or categorical, random walks. We can also say that nontriviality of the Poisson boundary $\Pi: \mathcal{C} \rightarrow \mathcal{P}$ with respect to an ergodic measure shows how far a weakly amenable category \mathcal{C} is from being amenable.

6. AMENABLE FUNCTORS

In this section we will give another characterization of amenability in terms of invariant means. We know that on the level of fusion algebras existence of invariant means is not enough for amenability. Therefore we need a more refined categorical notion.

Definition 6.1. Let \mathcal{C} be a C^* -tensor category and $F: \mathcal{C} \rightarrow \mathcal{A}$ be a unitary tensor functor into a C^* -tensor category \mathcal{A} with possibly nonsimple unit. A *right invariant mean* for F is a collection $m = (m_{U,V})_{U,V}$ of linear maps

$$m_{U,V}: \hat{\mathcal{C}}(U, V) \rightarrow \mathcal{A}(F(U), F(V))$$

that are natural in U and V and satisfy the following properties:

- (i) the maps $m_U = m_{U,U}: \hat{\mathcal{C}}(U) \rightarrow \mathcal{A}(F(U))$ are unital and positive;
- (ii) for any $\eta \in \hat{\mathcal{C}}(U, V)$ and any object Y in \mathcal{C} we have

$$m_{U \otimes Y, V \otimes Y}(\eta \otimes \iota_Y) = F_2(m_{U,V}(\eta) \otimes \iota_{F(Y)});$$

- (iii) for any $\eta \in \hat{\mathcal{C}}(U, V)$ and any object Y in \mathcal{C} we have

$$m_{Y \otimes U, Y \otimes V}(\iota_Y \otimes \eta) = F_2(\iota_{F(Y)} \otimes m_{U,V}(\eta)).$$

If a right invariant mean for F exists, we say that F is *amenable*.

Note that naturality of $m_{U,V}$ and property (i) in the above definition easily imply that the maps m_U are completely positive, and $m_{U,V}(\eta)^* = m_{V,U}(\eta^*)$. As usual, we omit subscripts and simply write m instead of $m_{U,V}$ when there is no confusion.

The relevance of this notion for categorical random walks is explained by the following simple observation, similar to the easy part of Proposition 2.3.

Proposition 6.2. *Let \mathcal{C} be a rigid C^* -tensor category, μ be a probability measure on $\text{Irr}(\mathcal{C})$, and $\Pi: \mathcal{C} \rightarrow \mathcal{P}$ be the Poisson boundary of (\mathcal{C}, μ) . Then the functor $\Pi: \mathcal{C} \rightarrow \mathcal{P}$ is amenable.*

Proof. Fix a free ultrafilter ω on \mathbb{N} , and then define

$$m(\eta)_X = \lim_{n \rightarrow \omega} \frac{1}{n} \sum_{k=0}^{n-1} P_\mu^k(\eta)_X.$$

All the required properties of a right invariant mean follow immediately by definition. For example, property (iii) in the definition follows from the identity $P_X(\iota_Y \otimes \eta) = \iota_Y \otimes P_X(\eta)$. \square

For functors into categories with nonsimple units we do not have much insight into the meaning of amenability. But if we fall back to our standard assumption of simplicity of tensor units, we have the following result.

Theorem 6.3. *Let \mathcal{A} and \mathcal{C} be rigid C^* -tensor categories with simple units and $F: \mathcal{C} \rightarrow \mathcal{A}$ be a unitary tensor functor. Then F is amenable if and only if \mathcal{C} is weakly amenable and $d^{\mathcal{A}}F$ is the amenable dimension function on \mathcal{C} .*

Let $F: \mathcal{C} \rightarrow \mathcal{A}$ be an amenable unitary tensor functor with a right invariant mean m . For simplicity we assume as usual that \mathcal{C} and \mathcal{A} are strict and F is an embedding functor. Let us start by showing that existence of F implies weak amenability.

Lemma 6.4. *The linear functional $m_{\mathbb{1}}: \hat{\mathcal{C}}(\mathbb{1}) \cong \ell^\infty(\text{Irr}(\mathcal{C})) \rightarrow \mathcal{A}(\mathbb{1}) \cong \mathbb{C}$ is a right invariant mean on the fusion algebra of \mathcal{C} equipped with the dimension function $d^{\mathcal{C}}$.*

Proof. In addition to the operators P_X on $\hat{\mathcal{C}}(\mathbb{1})$ we normally use, we also have the operators Q_X given by

$$Q_X(\eta)_Y = d^{\mathcal{C}}(X)^{-1}(\iota_Y \otimes \text{tr}_X)(\eta_{Y \otimes X}) = d^{\mathcal{C}}(X)^{-1}(\iota_Y \otimes \bar{R}_X^*)(\eta_{Y \otimes X} \otimes \iota_{\bar{X}})(\iota_Y \otimes \bar{R}_X),$$

where (R_X, \bar{R}_X) is a standard solution of the conjugate equations for X in \mathcal{C} . Since

$$\eta_{Y \otimes X} \otimes \iota_{\bar{X}} = (\iota_X \otimes \eta \otimes \iota_{\bar{X}})_Y,$$

we can write this as

$$Q_X(\eta) = d^{\mathcal{C}}(X)^{-1} \bar{R}_X^*(\iota_X \otimes \eta \otimes \iota_{\bar{X}}) \bar{R}_X.$$

Applying the invariant mean we get

$$m(Q_X(\eta)) = d^{\mathcal{C}}(X)^{-1} \bar{R}_X^*(\iota_X \otimes m(\eta) \otimes \iota_{\bar{X}}) \bar{R}_X = m(\eta).$$

Thus $m_{\mathbb{1}}$ is a right invariant mean on $\ell^\infty(\text{Irr}(\mathcal{C}))$. \square

Since \mathcal{C} is weakly amenable, we can choose an ergodic probability measure and consider the corresponding Poisson boundary $\Pi: \mathcal{C} \rightarrow \mathcal{P}$. We then have the following result, which has its origin in Tomatsu's considerations in [Tom07, Section 4].

Lemma 6.5. *For every object U in \mathcal{C} the map $\Lambda_U: \mathcal{P}(U) \rightarrow \mathcal{A}(U)$ obtained by restricting m_U to $\mathcal{P}(U)$ is multiplicative.*

Proof. Recall that in Section 4 we constructed faithful unital completely positive maps $\Theta_U: \mathcal{A}(U) \rightarrow \mathcal{P}(U)$, $\Theta_U(T)_X = E_{X \otimes U}(\iota \otimes T)$. By faithfulness of Θ_U , the multiplicative domain of $\Theta_U \Lambda_U$ is contained in that of Λ_U . Therefore in order to prove the lemma it suffices to show that $\Theta_U \Lambda_U$ is the identity map.

Let us show first that for any $\eta \in \mathcal{P}(U)$ we have $\Theta \Lambda(\eta)_{\mathbb{1}} = \eta_{\mathbb{1}}$, that is,

$$E(m(\eta)) = \eta_{\mathbb{1}}.$$

Take $S \in \mathcal{C}(U)$. Then we have

$$\text{tr}_U(SE(m(\eta))) = \psi_U(m(S\eta)) = d^{\mathcal{C}}(U)^{-1} \bar{R}_U^*(m(S\eta) \otimes \iota) \bar{R}_U = d^{\mathcal{C}}(U)^{-1} m(\bar{R}_U^*(S\eta \otimes \iota) \bar{R}_U).$$

Since the element $\bar{R}_U^*(S\eta \otimes \iota) \bar{R}_U$ lies in $\mathcal{P}(\mathbb{1})$, it is scalar. This scalar must be equal to

$$\bar{R}_U^*(S\eta_{\mathbb{1}} \otimes \iota) \bar{R}_U = \text{tr}_U(S\eta_{\mathbb{1}}).$$

Hence we obtain

$$\text{tr}_U(SE(m(\eta))) = \text{tr}_U(S\eta_{\mathbb{1}}),$$

and since this is true for all S , we get $\Theta \Lambda(\eta)_{\mathbb{1}} = \eta_{\mathbb{1}}$.

Now, for any object X in \mathcal{C} , we use the above equality for $\iota_X \otimes \eta$ instead of η and get

$$\Theta \Lambda(\eta)_X = E(\iota_X \otimes m(\eta)) = E(m(\iota_X \otimes \eta)) = \Theta \Lambda(\iota_X \otimes \eta)_{\mathbb{1}} = (\iota_X \otimes \eta)_{\mathbb{1}} = \eta_X,$$

which implies the desired equality $\Theta \Lambda(\eta) = \eta$. \square

Proof of Theorem 6.3. Consider an amenable unitary tensor functor $F: \mathcal{C} \rightarrow \mathcal{A}$. By Lemma 6.4 we know that \mathcal{C} is weakly amenable. By Lemma 6.5 and the definition of invariant means, any right invariant mean for F defines a strict unitary tensor functor $\Lambda: \tilde{\mathcal{P}} \rightarrow \mathcal{A}$, where $\tilde{\mathcal{P}} \subset \mathcal{P}$ is the full subcategory consisting of objects in \mathcal{C} . Extend this functor to a unitary tensor functor $\Lambda: \mathcal{P} \rightarrow \mathcal{A}$. Then $d^{\mathcal{A}}(U) \leq d^{\mathcal{P}}(U)$ for any object U in \mathcal{C} , but since by Theorem 5.1 the dimension function $d^{\mathcal{P}} \Pi$ on \mathcal{C} is amenable, we conclude that $d^{\mathcal{A}}(U) = d^{\mathcal{P}}(U) = \|\Gamma_U\|$.

Conversely, assume \mathcal{C} is weakly amenable and $F: \mathcal{C} \rightarrow \mathcal{A}$ is a unitary tensor functor such that $d^{\mathcal{A}}F$ is the amenable dimension function. Then by Theorem 4.1 there exists a unitary tensor functor $\Lambda: \mathcal{P} \rightarrow \mathcal{A}$ such that $\Lambda\Pi \cong F$. By Proposition 6.2 there exists a right invariant mean for the functor $\Pi: \mathcal{C} \rightarrow \mathcal{P}$. Composing it with the functor Λ we get a right invariant mean for $\Lambda\Pi$, from which we get a right invariant mean for F . \square

Applying Theorem 6.3 to the identity functor we get a characterization of amenability of tensor categories in terms of invariant means.

Theorem 6.6. *A rigid C^* -tensor category \mathcal{C} is amenable if and only if the identity functor $\mathcal{C} \rightarrow \mathcal{C}$ is amenable.*

Note that by the proof of Theorem 6.3, given an amenable C^* -tensor category \mathcal{C} , we can construct a right invariant mean for the identity functor as follows. Choose an ergodic probability measure μ on $\text{Irr}(\mathcal{C})$ and a free ultrafilter ω on \mathbb{N} . Then we can define

$$m(\eta) = \lim_{n \rightarrow \omega} \frac{1}{n} \sum_{k=0}^{n-1} P_{\mu}^k(\eta) \mathbf{1}.$$

On the other hand, the construction of a right invariant mean for a functor $F: \mathcal{C} \rightarrow \mathcal{A}$ such that \mathcal{C} is weakly amenable, but not amenable, and $d^{\mathcal{A}}F$ is the amenable dimension function, is more elusive, as it relies on the existence of a factorization of F through the Poisson boundary $\mathcal{C} \rightarrow \mathcal{P}$.

7. AMENABILITY OF QUANTUM GROUPS AND SUBFACTORS

In this section we apply some of our results to categories considered in the theory of compact quantum groups and in subfactor theory.

7.1. Quantum groups. Let G be a compact quantum group. We follow the conventions of [NT13]. In particular, the algebra $\mathbb{C}[G]$ of regular functions on G is a Hopf $*$ -algebra, and by a finite dimensional unitary representation of G we mean a unitary element $U \in B(H_U) \otimes \mathbb{C}[G]$, where H_U is a finite dimensional Hilbert space, such that $(\iota \otimes \Delta)(U) = U_{12}U_{13}$. Finite dimensional unitary representations form a rigid C^* -tensor category $\text{Rep } G$, with the tensor product of U and V defined by $U_{13}V_{23} \in B(H_U) \otimes B(H_V) \otimes \mathbb{C}[G]$. The categorical dimension of U is equal to the quantum dimension, given by the trace $\text{Tr}(\rho_U)$ of the Woronowicz character.

Recall that G is called *coamenable* if $\|\Gamma_U\| = \dim H_U$ for every finite dimensional unitary representation U . There are a number of equivalent conditions, but using this definition as our starting point we immediately get that

$$\text{Rep } G \text{ is amenable} \Leftrightarrow G \text{ is coamenable, and of Kac type.}$$

Coamenability of G is known to be equivalent to amenability of the dual discrete quantum group \hat{G} . Recall that the algebra of bounded functions on \hat{G} is defined by $\ell^\infty(\hat{G}) = \ell^\infty\text{-}\bigoplus_{s \in \text{Irr}(G)} B(H_s)$, and the coproduct $\hat{\Delta}: \ell^\infty(\hat{G}) \rightarrow \ell^\infty(\hat{G}) \bar{\otimes} \ell^\infty(\hat{G})$ is defined by duality from the product on $\mathbb{C}[G]$, if we view $\ell^\infty(\hat{G})$ as a subspace of $\mathbb{C}[G]^*$ by associating to a functional $\omega \in \mathbb{C}[G]^*$ the collection of operators $\pi_s(\omega) = (\iota \otimes \omega)(U_s) \in B(H_s)$, $s \in \text{Irr}(G)$. The quantum group \hat{G} is called *amenable*, if there exists a right invariant mean on \hat{G} , that is, a state m on $\ell^\infty(\hat{G})$ such that

$$m(\iota \otimes \phi)\hat{\Delta} = \phi(\cdot)\mathbf{1} \text{ for any normal linear functional } \phi \text{ on } \ell^\infty(\hat{G}).$$

The restriction of such an invariant mean to $Z(\ell^\infty(\hat{G})) \cong \ell^\infty(\text{Irr}(G))$ defines a right invariant mean on the fusion algebra of $\text{Rep } G$ equipped with the quantum dimension function. Therefore

$$G \text{ is coamenable} \Leftrightarrow \hat{G} \text{ is amenable} \Rightarrow \text{Rep } G \text{ is weakly amenable.}$$

Among various known characterizations of coamenability the implication (\hat{G} is amenable $\Rightarrow G$ is coamenable) is probably the most nontrivial. This was proved independently in [Tom06, Theorem 3.8] and [BCT05, Corollary 9.6]. We will show now that our results on amenable functors are generalizations of this.

Theorem 7.1. *If \hat{G} is amenable, then the forgetful functor $F: \text{Rep } G \rightarrow \text{Hilb}_f$ is amenable, and therefore G is coamenable.*

Proof. We will only consider the case when $\text{Irr}(G)$ is at most countable, so that $\text{Rep } G$ satisfies our standing assumptions, the general case can be easily deduced from this.

As discussed in [NY14, Section 4.1], the space $\hat{\mathcal{C}}(U, V)$ can be identified with the space of elements

$$\eta \in \ell^\infty(\hat{G}) \otimes B(H_U, H_V) \quad \text{such that} \quad V_{31}^*(\alpha \otimes \iota)(\eta)U_{31} = 1 \otimes \eta,$$

where $\alpha: \ell^\infty(\hat{G}) \rightarrow L^\infty(G) \bar{\otimes} \ell^\infty(\hat{G})$ is the left adjoint action of G . Under this identification we have

$$\iota_Y \otimes \eta = (\iota \otimes \pi_Y \otimes \iota)(\hat{\Delta} \otimes \iota)(\eta),$$

where $\pi_Y: \ell^\infty(\hat{G}) \rightarrow B(H_Y)$ is the representation defined by Y , while the element $\eta \otimes \iota_Y$ has the obvious meaning. From this we immediately see that if m is a right invariant mean on \hat{G} , then the maps $m \otimes \iota: \ell^\infty(\hat{G}) \otimes B(H_U, H_V) \rightarrow B(H_U, H_V)$ define a right invariant mean for F . Thus F is amenable. By Theorem 6.3 we conclude that $\|\Gamma_U\| = \dim F(U) = \dim H_U$ for every U , so G is coamenable. \square

As for the Poisson boundary of $\text{Rep } G$, from the universal property of the Poisson boundary it is easy to deduce that if G is coamenable (and so $\text{Rep } G$ is weakly amenable), then the Poisson boundary of $\text{Rep } G$ with respect to any ergodic measure is the forgetful functor $\text{Rep } G \rightarrow \text{Rep } K$, where $K \subset G$ is the maximal quantum subgroup of G of Kac type. This will be discussed in detail in [NY15].

7.2. Subfactor theory. Let $N \subset M$ be a finite index inclusion of II_1 -factors. Denote by τ the tracial state on M , and by E the trace-preserving conditional expectation $M \rightarrow N$. We denote $[M : N] = \text{Ind } E$, and the minimal index of $N \subset M$ by $[M : N]_0$. Put $M_{-1} = N$, $M_0 = M$, and choose a tunnel

$$\cdots \subset M_{-3} \subset M_{-2} \subset M_{-1} \subset M_0,$$

so that M_{-n+1} is the basic extension of $M_{-n-1} \subset M_{-n}$ for all $n \geq 1$. For every $j \leq 1$ denote by $M_j^{\text{st}} \subset M_j$ the s^* -closure of $\bigcup_{n \geq 1} (M'_{j-n} \cap M_j)$ with respect to the restriction of τ . The inclusion $N^{\text{st}} \subset M^{\text{st}}$ of finite von Neumann algebras is called a standard model of $N \subset M$ [Pop94].

Let $\mathcal{B}_N(M)$ be the full C^* -tensor subcategory of the category Hilb_N of Hilbert bimodules over N generated by $L^2(M)$. Let M_1 be the basic extension of $N \subset M$, so that $\text{End}_{N-N}(L^2(M)) \cong N' \cap M_1$. The embedding $N \rightarrow M_1$ induces a morphism $L^2(N) \rightarrow L^2(M) \otimes_N L^2(M)$ in $\mathcal{B}_N(M)$, which defines a solution of the conjugate equations for $L^2(M)$ up to a scalar normalization. Moreover, it can be shown (compare with Proposition 5.2) that the categorical trace corresponds to the minimal conditional expectation $M_1 \rightarrow N$, and consequently $d(L^2(M)) = [M_1 : N]_0^{1/2} = [M : N]_0$. It is also known, see Proposition B.1, that the inductive system of the algebras $\text{End}_{N-N}(L^2(M)^{\otimes N^n})$, with respect to the embeddings $T \mapsto \iota_{L^2(M)} \otimes T$, can be identified with $(M'_{-2n+1} \cap M_1)_{n \geq 1}$ in such a way that the shift endomorphism $T \mapsto T \otimes \iota_{L^2(M)}$ of $\bigcup_{n \geq 1} \text{End}_{N-N}(L^2(M)^{\otimes N^n})$ corresponds to the endomorphism γ^{-1} of $\bigcup_{n \geq 1} (M'_{-2n+1} \cap M_1)$, where γ is the canonical shift.

The normalized categorical trace on $\text{End}_{N-N}(L^2(M))$ defines a probability measure μ_{st} on the set of isomorphism classes of simple submodules of $L^2(M)$. More explicitly, it can be shown that the value of the normalized categorical trace on any minimal projection $p \in N' \cap M_1$ equals

$$(\tau(p)\tau'(p))^{1/2} \frac{[M : N]}{[M : N]_0},$$

where τ' is the unique tracial state on $N' \subset B(L^2(M))$. See [Hia88, Section 2] and [Pop94, Section 1.3.6] for related results. Then the measure μ_{st} is defined by

$$\mu_{\text{st}}([pL^2(M)]) = m_p(\tau(p)\tau'(p))^{1/2} \frac{[M : N]}{[M : N]_0},$$

where m_p is the multiplicity of $pL^2(M)$ in $L^2(M)$.

Recall that an inclusion for which $[M : N] = [M : N]_0$, is called extremal. From the above considerations, unless $N \subset M$ is extremal, we see that the categorical trace defines a tracial state of $\bigcup_{n \geq 1} (M'_{-2n+1} \cap M_1)$ that is different from τ .

Let us first review what our results say about $(\mathcal{B}_N(M), \mu_{\text{st}})$ for extremal inclusions. From the identification of $\bigcup_{n \geq 1} \text{End}_{N-N}(L^2(M)^{\otimes N^n})$ with $\bigcup_{n \geq 1} (M'_{-2n+1} \cap M_1)$ we conclude that the von Neumann algebra $\mathcal{N}_{L^2(M)}$ constructed in Section 3.2 is isomorphic to M_1^{st} . More precisely, we take $V = L^2(M)$ for the construction of $\mathcal{N}_{L^2(M)}$, so unless $N' \cap M_1$ is abelian, we apply the modification of our construction of the algebras \mathcal{N}_U discussed in Remark 3.7. The subalgebra $\mathcal{N} \subset \mathcal{N}_{L^2(M)}$ corresponds then to $N^{\text{st}} = \gamma^{-1}(M_1^{\text{st}}) \subset M_1^{\text{st}}$. In particular, N^{st} is a factor if and only if μ_{st} is ergodic. Proposition 3.3 translates now into the following statement, which is closely related to a result of Izumi [Izu04].

Proposition 7.2. *Let $\Pi: \mathcal{B}_N(M) \rightarrow \mathcal{P}$ be the Poisson boundary of $(\mathcal{B}_N(M), \mu_{\text{st}})$. Then, assuming that $N \subset M$ is extremal, we have*

$$\mathcal{P}(L^2(M)) \cong (N^{\text{st}})' \cap M_1^{\text{st}}.$$

More generally, by the same argument we have $\mathcal{P}(L^2(M)^{\otimes N^n}) \cong (M_{-2n+1}^{\text{st}})' \cap M_1^{\text{st}}$. Since $L^2(M)$ contains a copy of the unit object induced by the inclusion $N \rightarrow M$, we have $\mu_{\text{st}}(e) > 0$. Hence the supports of μ_{st}^n are increasing, and therefore the isomorphisms $\mathcal{P}(L^2(M)^{\otimes N^n}) \cong (M_{-2n+1}^{\text{st}})' \cap M_1^{\text{st}}$ completely describe the morphisms in the category \mathcal{P} . In fact, recalling that M_1^{st} is the basic extension of $N^{\text{st}} \subset M^{\text{st}}$, see [Pop94, Section 1.4.3], we may conclude that \mathcal{P} can be identified with $\mathcal{B}_{N^{\text{st}}}(M^{\text{st}})$. We leave it to the interested reader to find a good description of the functor $\Pi: \mathcal{B}_N(M) \rightarrow \mathcal{B}_{N^{\text{st}}}(M^{\text{st}})$.

Consider the principal graph $\Gamma_{N,M}$ of $N \subset M$. Then $\Gamma_{L^2(M)}$ can be identified with $\Gamma_{M,M_1} \Gamma_{M,M_1}^t$. Recall also that we have the equality $\|\Gamma_{N,M}\| = \|\Gamma_{M,M_1}\|$ by [Pop94, Section 1.3.5].

Turning now to Theorem 5.1 and Proposition 5.2, we get the following result (again, to be more precise we use the modification of the construction of \mathcal{N}_U described in Remark 3.7).

Theorem 7.3. *Assume $N \subset M$ is extremal and N^{st} is a factor. Then we have*

$$\|\Gamma_{N,M}\|^4 = [M_1^{\text{st}} : N^{\text{st}}]_0.$$

If M^{st} is also a factor, this can of course be formulated as $\|\Gamma_{N,M}\|^2 = [M^{\text{st}} : N^{\text{st}}]_0$.

Applying Theorem 5.7 we get the following result, which recovers part of Popa's characterization of extremal subfactors with strongly amenable standard invariant [Pop94, Theorem 5.3.1].

Theorem 7.4. *Assume $N \subset M$ is extremal. The following conditions are equivalent:*

- (i) N^{st} is a factor and $\|\Gamma_{N,M}\|^2 = [M : N]_0$;
- (ii) $(M_{-2n+1}^{\text{st}})' \cap M_1^{\text{st}} = M'_{-2n+1} \cap M_1$ for all $n \geq 1$.

Proof. As we already observed, the condition that N^{st} is a factor in (i) means exactly that the measure μ_{st} is ergodic. The condition $\|\Gamma_{N,M}\|^2 = [M : N]_0$ means that $\|\Gamma_{L^2(M)}\| = d(L^2(M))$. Since the module $L^2(M)$ is self-dual and generates $\mathcal{B}_N(M)$, this condition is equivalent to amenability of $\mathcal{B}_N(M)$.

On the other hand, by Proposition 7.2 and its extension to the modules $L^2(M)^{\otimes N^n}$ discussed above, condition (ii) is equivalent to triviality of the Poisson boundary of $(\mathcal{B}_N(M), \mu_{\text{st}})$.

This shows that the equivalence of (i) and (ii) is indeed a consequence of Theorem 5.7. \square

If we write the proof of the implication (ii) \Rightarrow (i) in terms of the algebras $M'_{-2n+1} \cap M_1$ instead of $\text{End}_{N-N}(L^2(M)^{\otimes N^n})$, we get an argument similar to Popa's proof based on [PP91], which was our inspiration. On the other hand, our proof of (i) \Rightarrow (ii) seems to be very different from Popa's arguments.

Next, let us comment on the nonextremal case. One possibility is to consider the completion of $\bigcup_{n \geq 1} (M'_{j-n} \cap M_j)$ with respect to the trace induced by the minimal conditional expectation (that is, the categorical trace) instead of τ . Then all the above statements continue to hold if we replace N^{st} and M^{st} by the corresponding new von Neumann algebras. Note that the inclusion $N^{\text{st}} \subset M^{\text{st}}$ defined this way is the standard model in the conventions of [Pop95]. Then, for example, the implication (i) \Rightarrow (ii) in Theorem 7.4 corresponds to [Pop95, Lemma 5.2].

But some results, notably Theorem 7.3, continue to hold for the inclusion $N^{\text{st}} \subset M^{\text{st}}$ defined with respect to τ in the nonextremal case also. The proof goes in basically the same way as in the extremal case, by noting that the proof of the inequality $\|\Gamma_U\|^2 \geq [\mathcal{N}_U : \mathcal{N}]_0$ in Theorem 5.1 did not depend on how exactly the inductive limit of the algebras $\mathcal{C}(V^{\otimes n} \otimes U)$ was completed to get the factors $\mathcal{N} \subset \mathcal{N}_U$. Therefore we have

$$\|\Gamma_{N,M}\|^4 \geq [M_1^{\text{st}} : N^{\text{st}}]_0.$$

The opposite inequality can be proved either by realizing that the dimension function on $\mathcal{B}_{N^{\text{st}}}(M_1^{\text{st}})$ defines a dimension function on $\mathcal{B}_N(M_1)$, or by the following string of (in)equalities:

$$\|\Gamma_{N,M}\|^4 = \|\Gamma_{N,M_1}\|^2 \leq \|\Gamma_{N^{\text{st}},M_1^{\text{st}}}\|^2 \leq [M_1^{\text{st}} : N^{\text{st}}]_0,$$

compare with [Pop94, p. 235]. We remark that from this one can easily obtain the implication (ii) \Rightarrow (vii) in [Pop94, Theorem 5.3.2] promised in [Pop94].

APPENDIX A. ESTIMATING RELATIVE ENTROPY

In this appendix we estimate the relative entropy for embeddings of finite dimensional C^* -algebras.

Let $N \subset M$ be a unital inclusion of finite dimensional C^* -algebras, $\{z_k\}_{k \in K}$ be the minimal central projections of N , and $\{w_l\}_{l \in L}$ be the minimal central projections of M . Let $A = (a_{kl})_{k,l}$ be the multiplicity matrix of the inclusion $N \subset M$, so that $a_{kl} = 0$ if $z_k w_l = 0$ and $(N z_k w_l)' \cap (z_k M w_l z_k) \cong \text{Mat}_{a_{kl}}(\mathbb{C})$ otherwise, and $\{n_k\}_k$ be the dimension function of N , so $N z_k \cong \text{Mat}_{n_k}(\mathbb{C})$. The following proposition generalizes results of Pimsner and Popa for tracial states in [PP86, Section 6].

Proposition A.1. *For any state φ on M we have*

$$H_\varphi(M|N) \leq \sum_{k,l} \varphi(z_k w_l) \log \frac{\varphi(z_k) \varphi(w_l) a_{kl} \min\{a_{kl}, n_k\}}{\varphi(z_k w_l)^2},$$

and the equality holds if φ is tracial.

The proof follows closely the proof for tracial states given in [NS06, Theorem 10.1.4]. The key part is the following estimate.

Lemma A.2. *For any positive linear functional ψ on M we have*

$$-\sum_{k,l} \psi(z_k w_l) \log \frac{\psi(w_l) \min\{a_{kl}, n_k\}}{\psi(z_k w_l)} \leq S(\psi) - S(\psi|_N) \leq \sum_{k,l} \psi(z_k w_l) \log \frac{\psi(z_k) a_{kl}}{\psi(z_k w_l)}.$$

Proof. Put $M_{kl} = z_k M w_l z_k$ and $N_{kl} = N z_k w_l$. For $\psi(z_k w_l) \neq 0$ consider the state

$$\psi_{kl} = \psi(z_k w_l)^{-1} \psi|_{M_{kl}}$$

on M_{kl} . As usual in entropy theory, it is convenient to define a function η by $\eta(t) = -t \log t$ for $t \geq 0$. We will constantly use the obvious equality

$$S(\omega) = \omega(1) S(\omega(1)^{-1} \omega) + \eta(\omega(1))$$

for positive linear functionals ω . Recall that the von Neumann entropy is defined by $S(\omega) = \text{Tr}(\eta(Q_\omega))$.

Let us start by estimating $S(\psi)$. We have

$$S(\psi) = \sum_l S(\psi|_{Mw_l}) = \sum_l \psi(w_l) S(\psi(w_l)^{-1} \psi|_{Mw_l}) + \sum_l \eta(\psi(w_l)).$$

By [NS06, Lemma 2.2.4] applied to the projections $z_k w_l$ in Mw_l we have

$$S(\psi(w_l)^{-1} \psi|_{Mw_l}) \geq \sum_k S(\psi(w_l)^{-1} \psi|_{M_{kl}}) - \sum_k \eta\left(\frac{\psi(z_k w_l)}{\psi(w_l)}\right) = \sum_k \frac{\psi(z_k w_l)}{\psi(w_l)} S(\psi_{kl}).$$

It follows that

$$S(\psi) \geq \sum_{k,l} \psi(z_k w_l) S(\psi_{kl}) + \sum_l \eta(\psi(w_l)). \quad (\text{A.1})$$

On the other hand, since $\bigoplus_k M_{kl}$ is a subalgebra of Mw_l of full rank, by [NS06, Theorem 2.2.2(vii)] we have

$$S(\psi|_{Mw_l}) \leq \sum_k S(\psi|_{M_{kl}}) = \sum_k \psi(z_k w_l) S(\psi_{kl}) + \sum_k \eta(\psi(z_k w_l)).$$

Therefore

$$S(\psi) \leq \sum_{k,l} \psi(z_k w_l) S(\psi_{kl}) + \sum_{k,l} \eta(\psi(z_k w_l)). \quad (\text{A.2})$$

Turning to $S(\psi|_N)$, by [NS06, Theorem 2.2.2(ii)] we have

$$S(\psi|_N) = \sum_k S(\psi|_{Nz_k}) \leq \sum_{k,l} S(\psi(\cdot w_l)|_{Nz_k}) = \sum_{k,l} \psi(z_k w_l) S(\psi_{kl}|_{N_{kl}}) + \sum_{k,l} \eta(\psi(z_k w_l)). \quad (\text{A.3})$$

On the other hand, since the von Neumann entropy is concave by [NS06, Theorem 2.2.2(ii)], we have

$$\begin{aligned} S(\psi|_{Nz_k}) &= \psi(z_k) S\left(\sum_l \frac{\psi(z_k w_l)}{\psi(z_k)} \psi(z_k w_l)^{-1} \psi(\cdot w_l)|_{Nz_k}\right) + \eta(\psi(z_k)) \\ &\geq \sum_l \psi(z_k w_l) S(\psi_{kl}|_{N_{kl}}) + \eta(\psi(z_k)). \end{aligned}$$

Therefore

$$S(\psi|_N) \geq \sum_{k,l} \psi(z_k w_l) S(\psi_{kl}|_{N_{kl}}) + \sum_k \eta(\psi(z_k)). \quad (\text{A.4})$$

Now, by [NS06, Theorem 2.2.2(vi)] we have

$$|S(\psi_{kl}) - S(\psi_{kl}|_{N_{kl}})| \leq \log a_{kl}.$$

This, together with (A.2) and (A.4), gives

$$S(\psi) - S(\psi|_N) \leq \sum_{k,l} \psi(z_k w_l) \log a_{kl} + \sum_{k,l} \eta(\psi(z_k w_l)) - \sum_k \eta(\psi(z_k)),$$

which is what we need as

$$\eta(\psi(z_k)) = - \sum_l \psi(z_k w_l) \log \psi(z_k).$$

The lower bound for $S(\psi) - S(\psi|_N)$ follows similarly from (A.1) and (A.3), if we in addition use that

$$S(\psi_{kl}) - S(\psi_{kl}|_{N_{kl}}) \geq -S(\psi_{kl}|_{N_{kl}}) \geq -\log n_k,$$

so that

$$S(\psi_{kl}) - S(\psi_{kl}|_{N_{kl}}) \geq -\log \min\{a_{kl}, n_k\}.$$

□

Proof of Proposition A.1. Given a finite decomposition $\varphi = \sum_i \varphi_i$ we want to obtain an upper bound on

$$S(\varphi) - S(\varphi|_N) + \sum_i (S(\varphi_i|_N) - S(\varphi_i)).$$

By Lemma A.2 we have

$$S(\varphi) - S(\varphi|_N) \leq \sum_{k,l} \varphi(z_k w_l) \log \frac{\varphi(z_k) a_{kl}}{\varphi(z_k w_l)}$$

and

$$\sum_i (S(\varphi_i|_N) - S(\varphi_i)) \leq \sum_{i,k,l} \varphi_i(z_k w_l) \log \frac{\varphi_i(w_l) \min\{a_{kl}, n_k\}}{\varphi_i(z_k w_l)}.$$

Since $\sum_i \varphi_i(z_k w_l) = \varphi(z_k w_l)$, using the concavity of \log we get

$$\begin{aligned} \sum_i \varphi_i(z_k w_l) \log \frac{\varphi_i(w_l) \min\{a_{kl}, n_k\}}{\varphi_i(z_k w_l)} &\leq \varphi(z_k w_l) \log \left(\sum_i \frac{\varphi_i(z_k w_l) \varphi_i(w_l) \min\{a_{kl}, n_k\}}{\varphi_i(z_k w_l)} \right) \\ &= \varphi(z_k w_l) \log \frac{\varphi(w_l) \min\{a_{kl}, n_k\}}{\varphi(z_k w_l)}. \end{aligned}$$

Putting all this together we get the required upper bound on $H_\varphi(M|N)$. That this bound is exactly the value of $H_\varphi(M|N)$ for tracial φ is proved in [PP86, Section 6], see also [NS06, Theorem 10.1.4]. \square

The following result generalizes another estimate of Pimsner and Popa for tracial states, given in [PP91, Theorem 2.6].

Proposition A.3. *For any state φ on M we have*

$$H_\varphi(M|N) \leq 2 \log \|A\|.$$

Proof. Consider the sets $\Omega = \{(k, l) \in K \times L \mid a_{kl} \neq 0\}$ and

$$\Delta = \{\xi = (\xi_{kl})_{(k,l) \in \Omega} \mid \xi_{kl} \geq 0, \sum_{k,l} \xi_{kl} = 1\} \subset \mathbb{R}_+^\Omega.$$

Define a function f on \mathbb{R}_+^Ω by

$$f(\xi) = \sum_{k,l} \xi_{kl} \log \frac{\xi_k^{(1)} \xi_l^{(2)} a_{kl}^2}{\xi_{kl}^2},$$

where $\xi_k^{(1)} = \sum_l \xi_{kl}$ and $\xi_l^{(2)} = \sum_k \xi_{kl}$. By Proposition A.1 we have $H_\varphi(M|N) \leq f(\xi)$ for $\xi \in \Delta$ defined by $\xi_{kl} = \varphi(z_k w_l)$. Therefore it suffices to show that $f(\xi) \leq 2 \log \|A\|$ for all $\xi \in \Delta$. We will prove that this is the case for any nonzero matrix A with nonnegative real coefficients.

Let $\zeta \in \Delta$ be a maximum point of the function $f|_\Delta$. We may assume that $\zeta_{kl} > 0$ for all $(k, l) \in \Omega$, since otherwise we can simply modify the matrix A by letting $a_{kl} = 0$ for $(k, l) \in \Omega$ such that $\zeta_{kl} = 0$, which can only decrease the norm of A , since by the Perron–Frobenius theory the norm of A^*A is the maximum of the numbers $\mu \geq 0$ such that $A^*Aw \geq \mu w$ for some nonzero vector $w \in \mathbb{R}_+^L$. By removing zero rows and columns of A we may also assume that the projection maps $\Omega \rightarrow K$ and $\Omega \rightarrow L$ are surjective, so the numbers $\zeta_k^{(1)}$ and $\zeta_l^{(2)}$ are well-defined and strictly positive for all $k \in K$ and $l \in L$.

Using that

$$\frac{\partial}{\partial \xi_{kl}} \log \xi_i^{(1)} = \frac{\delta_{ik}}{\xi_k^{(1)}} \quad \text{and} \quad \frac{\partial}{\partial \xi_{kl}} \log \xi_j^{(2)} = \frac{\delta_{jl}}{\xi_l^{(2)}},$$

we get

$$\frac{\partial f}{\partial \xi_{kl}}(\zeta) = \log \frac{\zeta_k^{(1)} \zeta_l^{(2)} a_{kl}^2}{\zeta_{kl}^2}.$$

Since ζ is a maximum point of $f|_{\Delta}$, the gradient of f at this point is orthogonal to Δ , so

$$\log \frac{\zeta_k^{(1)} \zeta_l^{(2)} a_{kl}^2}{\zeta_{kl}^2} = \lambda \quad \text{for all } (k, l) \in \Omega$$

for some $\lambda \in \mathbb{R}$. Then $f(\zeta) = \lambda$, and it remains to show that $\lambda \leq 2 \log \|A\|$.

Put

$$v_k = (\zeta_k^{(1)})^{1/2} \quad \text{and} \quad w_l = (\zeta_l^{(2)})^{1/2}.$$

Then, using that $\zeta_{kl} = e^{-\lambda/2} a_{kl} v_k w_l$, we get

$$\sum_l a_{kl} w_l = e^{\lambda/2} \sum_l \frac{\zeta_{kl}}{v_k} = e^{\lambda/2} \frac{\zeta_k^{(1)}}{v_k} = e^{\lambda/2} v_k,$$

so that $Aw = e^{\lambda/2} v$. Similarly we get $A^*v = e^{\lambda/2} w$. We thus see that w is an eigenvector of A^*A with eigenvalue e^λ . Hence $e^\lambda \leq \|A\|^2$. \square

Note that the maximum of the function $f|_{\Delta}$ from the above proof is exactly $2 \log \|A\|$. Indeed, let $w \in \mathbb{R}_+^L$ be an eigenvector of A^*A with eigenvalue $\|A\|^2$ normalized so that $\|w\|_2 = 1$, which exists by the Perron–Frobenius theorem. Then letting $\xi_{kl} = \|A\|^{-2} a_{kl} (Aw)_k w_l$ we get $f(\xi) = 2 \log \|A\|$. This of course does not imply that the supremum of $H_\varphi(M|N)$ over all states φ equals $2 \log \|A\|$, even if $n_k \geq a_{kl}$, since we only know that our upper bound on $H_\varphi(M|N)$ is sharp for tracial states, and for tracial states the numbers $\varphi(z_k w_l)$ cannot be arbitrary.

APPENDIX B. CANONICAL SHIFT ON THE TOWER OF RELATIVE COMMUTANTS

Let $N \subset M$ be a finite index inclusion of II_1 factors. Put $M_{-1} = N$ and $M_0 = M$. Iterating the basic extension with respect to the trace-preserving conditional expectations we get the Jones tower

$$M_{-1} \subset M_0 \subset M_1 \subset M_2 \subset \dots$$

We also choose a tunnel

$$\dots \subset M_{-3} \subset M_{-2} \subset M_{-1} \subset M_0.$$

Let $e_j \in M_{j+1}$, $j \in \mathbb{Z}$, be the corresponding Jones projections. Denote by τ the unique tracial state on $\bigcup_n M_n$ and by E_j the τ -preserving conditional expectation $\bigcup_n M_n \rightarrow M_j$. Thus, $M_{j+1} = M_j e_j M_j$, meaning that M_{j+1} is spanned by the elements $x e_j y$ for $x, y \in M_j$, $e_j x e_j = E_{j-1}(x) e_j$ for $x \in M_j$, and $E_j(e_j) = \lambda 1$, where $\lambda = [M : N]^{-1}$.

Consider the canonical shift γ on $\bigcup_{j < k} (M'_j \cap M_k)$. This is an automorphism such that $\gamma(e_j) = e_{j+2}$ and $\gamma(M'_j \cap M_k) = M'_{j+2} \cap M_{k+2}$. It can be defined as follows, see, e.g., [NS06, Section 10.4]. The representation of M_{j+1} on $L^2(M_j)$ given by the definition of the basic extension extends uniquely to a representation of $\bigcup_n M_n$ such that

$$e_{j+n} \Lambda_j(x) = J_j e_{j-n} J_j \Lambda_j(x) = \Lambda_j(x e_{j-n}) \quad \text{for all } n \geq 1,$$

where $\Lambda_j: M_j \rightarrow L^2(M_j)$ is the GNS-map and J_j is the modular conjugation. In this representation we have $M_{j+n} = J_j M'_{j-n} J_j$. Define a $*$ -anti-automorphism γ_j of $\bigcup_{i < k} (M'_i \cap M_k)$ by

$$\gamma_j(x) = J_j x^* J_j \quad \text{on } L^2(M_j).$$

The canonical shift is defined by $\gamma = \gamma_{j+1} \gamma_j$. This definition is independent of $j \in \mathbb{Z}$. The automorphism γ is completely characterized by the properties that it maps $M'_j \cap M_k$ into $M'_{j+2} \cap M_{k+2}$ and satisfies

$$\gamma(x) e_{j+1} = \lambda^{j-k} e_{j+1} \dots e_k x e_{k+1} e_k \dots e_{j+1} \quad \text{for } x \in M'_j \cap M_k, \quad j < k. \quad (\text{B.1})$$

The following proposition, which is at the origin of the classification theory of subfactors, is a well-known result of Ocneanu.

Proposition B.1. *There is an isomorphism of the inductive system*

$$\mathrm{End}_{N-N}(L^2(M)) \xrightarrow{\iota_{L^2(M)}^{\otimes \cdot}} \mathrm{End}_{N-N}(L^2(M)^{\otimes N^2}) \xrightarrow{\iota_{L^2(M)}^{\otimes \cdot}} \mathrm{End}_{N-N}(L^2(M)^{\otimes N^3}) \rightarrow \dots$$

onto the system

$$M'_{-1} \cap M_1 \rightarrow M'_{-3} \cap M_1 \rightarrow M'_{-5} \cap M_1 \rightarrow \dots,$$

where all the arrows are the inclusion maps, such that the shift endomorphism $T \mapsto T \otimes \iota_{L^2(M)}$ of $\bigcup_{n \geq 1} \mathrm{End}_{N-N}(L^2(M)^{\otimes N^n})$ corresponds to the endomorphism γ^{-1} of $\bigcup_{n \geq 1} (M'_{-2n+1} \cap M_1)$.

Despite being well-known, this is usually formulated in a weaker form, see, e.g., [Bis97, Section 4], and it seems to be difficult to find a clear complete proof of the proposition as it is stated above in the literature. We will therefore sketch a possible proof for the reader's convenience.

We start by considering the N -bimodule maps

$$u_{m,n}: L^2(M_m) \otimes_N L^2(M_n) \rightarrow L^2(M_{m+n+1}) \quad (m, n \geq 0),$$

$$\begin{aligned} u_{m,n}(\Lambda_m(x) \otimes \Lambda_n(y)) &= \lambda^{-(m+1)(n+1)/2} \Lambda_{m+n+1}(x e_m \dots e_0 e_{m+1} \dots e_1 \dots e_{m+n} \dots e_n y) \\ &= \lambda^{-(m+1)(n+1)/2} \Lambda_{m+n+1}(x e_m \dots e_{m+n} e_{m-1} \dots e_{m+n-1} \dots e_0 \dots e_n y). \end{aligned}$$

Lemma B.2. *The maps $u_{m,n}$ are unitary, and the following identities hold:*

$$u_{k+m+1,n}(u_{k,m} \otimes \iota_{L^2(M_n)}) = u_{k,m+n+1}(\iota_{L^2(M_k)} \otimes u_{m,n}). \quad (\text{B.2})$$

Proof. Using the identities

$$E_{m+n-k}(e_{m+n-k} \dots e_{m-k} E_{m-k}(x^* x) e_{m-k} \dots e_{m+n-k}) = \lambda^{n+1} E_{m-k-1}(x^* x)$$

for $k = 0, \dots, m$, it is easy to check that the maps $u_{m,n}$ are isometric. Identity (B.2) is also straightforward. To prove surjectivity of $u_{m,n}$, observe first that

$$M_m e_m \dots e_{m+n} M_{m+n} = M_{m+n+1}.$$

This can be seen by induction on n , using that

$$M_m e_m \dots e_{m+n} M_{m+n} = M_m e_m \dots e_{m+n-1} M_{m+n-1} e_{m+n} M_{m+n}$$

and $M_{m+n} e_{m+n} M_{m+n} = M_{m+n+1}$. From this, in turn, by induction on m we get

$$M_m e_m \dots e_{m+n} e_{m-1} \dots e_{m+n-1} \dots e_0 \dots e_n M_n = M_{m+n+1},$$

since the left hand side can be written as

$$M_m e_m \dots e_{m+n} M_{m-1} e_{m-1} \dots e_{m+n-1} \dots e_0 \dots e_n M_n.$$

This proves surjectivity of $u_{m,n}$. □

By distributing parentheses in $L^2(M)^{\otimes N^n}$, e.g., as

$$L^2(M) \otimes_N (L^2(M) \otimes_N (\dots (L^2(M) \otimes_N L^2(M)) \dots)),$$

and using the isomorphisms $u_{k,m}$, we get an isomorphism $v_n: L^2(M)^{\otimes N^n} \rightarrow L^2(M_{n-1})$, and hence an isomorphism

$$\psi_n: \mathrm{End}_{N-N}(L^2(M)^{\otimes N^n}) \rightarrow N' \cap M_{2n-1}.$$

By (B.2) the isomorphism v_n , and hence ψ_n , is independent of the way we distribute parentheses in $L^2(M)^{\otimes N^n}$.

Lemma B.3. *For any $T \in \mathrm{End}_{N-N}(L^2(M)^{\otimes N^n})$ we have*

$$\psi_{n+1}(T \otimes \iota_{L^2(M)}) = \psi_n(T) \quad \text{and} \quad \psi_{n+1}(\iota_{L^2(M)} \otimes T) = \gamma(\psi_n(T)).$$

Proof. As $v_{n+1} = u_{n-1,0}(v_n \otimes \iota)$, for the first equality it suffices to show that

$$u_{n-1,0}: L^2(M_{n-1}) \otimes_N L^2(M) \rightarrow L^2(M_n)$$

is a left M_{2n-1} -module map. It is clear that $u_{n-1,0}$ is an M_{n-1} -module map. Therefore it is enough to show that $u_{n-1,0}e_k = e_k u_{n-1,0}$ for $k = n-1, \dots, 2n-2$. Consider three cases.

(i) $k = n-1$. We have, for $x \in M_{n-1}$ and $y \in M$, that

$$u_{n-1,0}(e_{n-1}\Lambda_{n-1}(x) \otimes \Lambda(y)) = \lambda^{-n/2}\Lambda_n(E_{n-2}(x)e_{n-1} \dots e_0y) = \lambda^{-n/2}\Lambda_n(e_{n-1}xe_{n-1} \dots e_0y),$$

which is what we need.

(ii) $k = n$. In this case, using that $e_n\Lambda_{n-1}(x) = \Lambda_{n-1}(xe_{n-2})$, we get

$$u_{n-1,0}(e_n\Lambda_{n-1}(x) \otimes \Lambda(y)) = \lambda^{-n/2}\Lambda_n(xe_{n-2}e_{n-1} \dots e_0y) = \lambda^{-n/2+1}\Lambda_n(xe_{n-2} \dots e_0y).$$

On the other hand,

$$e_n u_{n-1,0}(\Lambda_{n-1}(x) \otimes \Lambda(y)) = \lambda^{-n/2}\Lambda_n(E_{n-1}(xe_{n-1} \dots e_0y)) = \lambda^{-n/2+1}\Lambda_n(xe_{n-2} \dots e_0y),$$

so again $u_{n-1,0}e_n = e_n u_{n-1,0}$.

(iii) $n < k \leq 2n-2$. Using again that $e_k = J_{n-1}e_{2n-k-2}J_{n-1}$ on $L^2(M_{n-1})$ and $e_k = J_n e_{2n-k} J_n$ on $L^2(M_n)$, we get

$$u_{n-1,0}(e_k\Lambda_{n-1}(x) \otimes \Lambda(y)) = \lambda^{-n/2}\Lambda_n(xe_{2n-k-2}e_{n-1} \dots e_0y)$$

and

$$e_k u_{n-1,0}(\Lambda_{n-1}(x) \otimes \Lambda(y)) = \lambda^{-n/2}\Lambda_n(xe_{n-1} \dots e_0y e_{2n-k}).$$

As e_{2n-k} commutes with $y \in M$ and

$$e_{2n-k-2}e_{n-1} \dots e_0 = \lambda e_{n-1} \dots e_{2n-k} e_{2n-k-2} \dots e_0 = e_{n-1} \dots e_0 e_{2n-k},$$

this gives $u_{n-1,0}e_k = e_k u_{n-1,0}$.

Turning to the second identity, as $v_{n+1} = u_{0,n-1}(\iota \otimes v_n)$, it suffices to show that

$$u_{0,n-1}(\iota \otimes x) = \gamma(x)u_{0,n-1} \quad \text{for } x \in N' \cap M_{2n-1}.$$

Since $\gamma(x) \in M'_1 \cap M_{2n+1}$ commutes with M , recalling the definition of $u_{0,n-1}$ we see that this boils down to showing that $w_{n-1}x = \gamma(x)w_{n-1}$, where $w_{n-1}: L^2(M_{n-1}) \rightarrow L^2(M_n)$ is defined by

$$w_{n-1}\Lambda_{n-1}(y) = \Lambda_n(e_0 \dots e_{n-1}y).$$

It is convenient to prove a stronger statement. Define a map $\pi: M_{2n-1} \rightarrow M_{2n+1}$ by

$$\pi(x) = \lambda^{-2n}e_0 \dots e_{2n-1}x e_{2n}e_{2n-1} \dots e_0.$$

It is easy to see that π is a $*$ -homomorphism. By (B.1) we also have $\pi(x) = \gamma(x)e_0$ for $x \in N' \cap M_{2n-1}$. It follows that in order to prove the second part of the lemma it suffices to show that

$$w_{n-1}x = \pi(x)w_{n-1} \quad \text{for all } x \in M_{2n-1}.$$

Since $\pi(x) = xe_0$ for $x \in N$, this identity holds for $x \in N$. Hence to finish the proof it is enough to check this identity for $x = e_k$, $k = -1, \dots, 2n-2$. That is, we have to show that $w_{n-1}e_{-1} = \lambda^{-1}e_0e_{-1}e_1e_0w_{n-1}$ and $w_{n-1}e_k = e_{k+2}e_0w_{n-1}$ for $k = 0, \dots, 2n-2$. This is done similarly to the first part of the proof of the lemma by considering different cases: two cases for $w_{n-1}e_{-1} = \lambda^{-1}e_0e_{-1}e_1e_0w_{n-1}$ corresponding to $n = 1$ and $n > 1$, and four cases for $w_{n-1}e_k = e_{k+2}e_0w_{n-1}$ corresponding to $0 \leq k < n-2$, $k = n-2$ ($n \geq 2$), $k = n-1$ ($n \geq 1$) and $n \leq k \leq 2n-2$. \square

Proof of Proposition B.1. It follows from Lemma B.3 that the isomorphisms

$$\gamma^{-(n-1)}\psi_n: \text{End}_{N-N}(L^2(M)^{\otimes N^n}) \rightarrow M'_{-2n+1} \cap M_1$$

define the required isomorphism of the inductive systems. \square

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