POISSON BOUNDARIES, YETTER–DRINFELD ALGEBRAS, AND CLASSIFICATION OF NON-KAC COMPACT QUANTUM GROUPS OF $\mathrm{SU}(n)$ TYPE

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Abstract. We study tensor structures on $(\operatorname{Rep} G)$-module categories defined by actions of a compact quantum group G on unital C*-algebras. We show that having a tensor product which defines the module structure is equivalent to enriching the action of G to the structure of a braided-commutative Yetter–Drinfeld algebra. Applying this correspondence to noncommutative Poisson boundaries, we show that if G is coamenable, then any dimension-preserving unitary fiber functor on $\operatorname{Rep} G$ factors through $\operatorname{Rep} K$, where K is the maximal Kac quantum subgroup of G. In other words, any unitary dual 2-cocycle on G is cohomologous to a cocycle induced from K. As an application we obtain a complete list of non-Kac compact quantum groups with the same fusion rules and dimension function as $\mathrm{SU}(n)$.

Introduction

In his fundamental paper on deformations of $\mathrm{SU}(n)$ and the Tannaka–Krein duality for compact quantum groups, Woronowicz [Wor88] formulated the following problem: classify quantum groups having the same representation theory, meaning the same fusion rules and dimensions of irreducible representations, as $\mathrm{SU}(n)$. Since then, this and similar questions for other representation rings have been studied by a number of authors, see e.g. [WZ94, PM98, Ban96, Ohn99, Hai00, Bic03, Ohn05, Mro12, Mro13] and references therein.

Despite some success, a complete answer has only been obtained in a few low rank cases. At the current stage this is hardly surprising, and a complete explicit answer to Woronowicz's question is not to be expected. Indeed, to get such an answer requires, in particular, understanding of all unitary fiber functors on $\operatorname{Rep}\mathrm{SU}(n)$. This is equivalent to classifying full multiplicity ergodic actions of $\mathrm{SU}(n)$ on C*-algebras. As it was known already to Wassermann [Was88a], see also the discussion on pp. 1240–1241 in [NT11], this, in turn, includes classifying finite central type factor groups inside $\mathrm{SU}(n)$ up to conjugacy. For small n there are not many such subgroups, and for $n = 2, 3$ it is, indeed, possible to classify all full multiplicity ergodic actions [Was88b, Was88c]. However, the problem rapidly becomes unfeasible as n grows larger. Since fiber functors on $\operatorname{Rep}\mathrm{SU}(n)$ lead to Kac quantum groups, we therefore should not expect an explicit answer to Woronowicz's question for the class of compact quantum groups of Kac type. Note that nevertheless for genuine compact groups we have a complete answer: by a result of McMullen [McM84], see also [Han93, KLV10], a compact group with the same fusion rules as for $\mathrm{SU}(n)$ is itself isomorphic to $\mathrm{SU}(n)$.

We can also forget about compactness and try to describe all cosemisimple Hopf algebras with corepresentation theory of $\mathrm{SU}(n)$. For $n = 2$ this was done by Podleś and Müller [PM98], and by Bichon [Bic03] without the restriction on the dimension function (extending the result of Banica in the compact case [Ban96]). For $n = 3$, a complete classification of such Hopf algebras was obtained by Ohn [Ohn99, Ohn05]. Using a formidable amount of computations, he obtained a long

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list of various multiparametric deformations. Doing anything similar for $n \geq 4$, even with computer assistance, seems like an overwhelming task.

Our main result is that if we stay away from the Kac and noncompact cases, the question of Woronowicz has a very simple answer: the only quantum groups we have are $\mathrm{SU}_q(n)$ for $0 < q < 1$, the categorical twists $\mathrm{SU}_q^\tau(n)$ of $\mathrm{SU}_q(n)$ studied in our previous paper [NY13], and the deformations of such quantum groups by 2-cocycles on the dual of the maximal torus, see Theorems 5.4 and 5.5 for the precise statement. It is worth remembering that, on the purely algebraic level, compact quantum groups are simply Hopf $*$-algebras generated by matrix coefficients of their finite dimensional unitary corepresentations. Therefore our result describes all such Hopf $*$-algebras with corepresentation theory of $\mathrm{SU}(n)$ and noninvolutive antipode. For $n = 3$ this implies that a majority of Hopf algebras in the list of Ohn either do not admit a $*$-structure or have nonunitarizable corepresentations, which can also be checked by a careful inspection of his classification.

The technical core of the paper is a refinement of the categorical approach to actions of compact quantum groups on C*-algebras. The idea of this approach can be traced to works of Wassermann [Was88a] and Landstad [Lan92] in the 1980s. From the modern point of view, they proved that there is a one-to-one correspondence between full multiplicity ergodic actions of a compact group G and (dimension-preserving) unitary fiber functors $\operatorname{Rep} G \to \mathrm{Hilb}_f$. The quantum analogue of this result in the purely algebraic setting was proved by Ulbrich [Ulb89] and Schauenburg [Sch96], and the corresponding result in the C*-algebraic setting was proved by Bichon, De Rijdt and Vaes [BDRV06]. Thus, for a compact quantum group G, there is a correspondence between unitary fiber functors on $\operatorname{Rep} G$ and full quantum multiplicity ergodic actions of G. It is natural to ask then what corresponds to the unitary tensor functors from $\operatorname{Rep} G$ into arbitrary C*-tensor categories. We show that this is braided-commutative Yetter–Drinfeld algebras, which are C*-algebras equipped with actions of G and $\hat{G}$ satisfying certain compatibility conditions. Therefore such algebras play the same role for general tensor functors as Hopf–Galois objects for fiber functors.

This can also be interpreted as follows. As has recently been shown in [DCY12, Nes13], actions of G can be described in terms of $(\operatorname{Rep} G)$-module categories. Then our result says that such a module category structure is defined by a tensor functor if and only if we can also define an action of $\hat{G}$ to get a braided-commutative Yetter–Drinfeld algebra.

Another key idea of the paper is the construction of the Poisson boundary of a C*-tensor category. Noncommutative Poisson boundaries of discrete quantum groups were introduced by Izumi [Izu02]. As was shown by De Rijdt and Vander Vennet [DRVV10], if two compact quantum groups G_1 and G_2 are monoidally equivalent, then there is a simple way of passing from the Poisson boundary of $\hat{G}_1$ to that of $\hat{G}_2$. This suggests that there should be a purely categorical definition of the Poisson boundary. We show that this is indeed the case. Given an essentially small rigid C*-tensor category $\mathcal{C}$ with simple unit and a probability measure μ on the set of isomorphism classes of its simple objects, we define a new C*-tensor category $\mathcal{P}$ (in general with nonsimple unit) and a unitary tensor functor $\Pi\colon \mathcal{C} \to \mathcal{P}$. We call the pair $(\mathcal{P}, \Pi)$ the Poisson boundary of $(\mathcal{C}, \mu)$. In disguise, this notion has appeared before, in fact as early as in the paper of Longo and Roberts [LR97], which preceded the work of Izumi. A detailed study of this notion and a comparison of our definition with different constructions in the literature will be carried out in a separate paper [NY]. The only thing we need and prove here is that the Yetter–Drinfeld algebra corresponding to the Poisson boundary of $\operatorname{Rep} G$ is exactly the Poisson boundary of $\hat{G}$ as defined by Izumi.

The way we use Poisson boundaries can be explained as follows. Assume that we want to understand fiber functors on a C*-tensor category $\mathcal{C}$ with Poisson boundary $\mathcal{P}$. Such a functor $\mathcal{F}$ defines a compact quantum group G. If we happen to know that the Poisson boundary of $\hat{G}$ is G/K for some closed quantum subgroup K of G, then using our general results we may conclude that $\mathcal{P}$ and $\operatorname{Rep} K$ are monoidally equivalent and therefore $\mathcal{F}$ factors through $\mathcal{P}$. In other words, some fiber functors on $\mathcal{C}$ can be understood in terms of fiber functors on the potentially simpler category $\mathcal{P}$. The assumption on the Poisson boundary of $\hat{G}$ holds quite generally by the theory developed by

Tomatsu [Tom07]. This allows us to prove the following: if G is a coamenable compact quantum group, then any dimension-preserving unitary fiber functor on $\operatorname{Rep} G$ factors through $\operatorname{Rep} K$, where K is the maximal Kac quantum subgroup of G. This, together with results of Kazhdan and Wenzl [KW93] on monoidal categories with fusion rules of $\mathrm{SU}(n)$, eventually leads to our description of non-Kac compact quantum groups of $\mathrm{SU}(n)$-type.

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1. Preliminaries

In this section we briefly summarize the theory of compact quantum groups and their actions on operator algebras in the C*-algebraic formulation, as well as discuss an algebraic approach to Yetter–Drinfeld C*-algebras.

We mainly follow the conventions of [NT13]. When A and B are C*-algebras, $A \otimes B$ denotes their minimal tensor product. Unless said otherwise, we assume that C*-categories are closed under subobjects. On the other hand, for C*-tensor categories we do not assume that the unit object is simple. For objects U and V in a category $\mathcal{C}$ we denote by $\mathcal{C}(U, V)$ the set of morphisms $U \to V$.

1.1. **Compact quantum groups.** A *compact quantum* group G is represented by a unital C*-algebra $C(G)$ equipped with a unital $*$-homomorphism $\Delta\colon C(G) \to C(G) \otimes C(G)$ satisfying the coassociativity $(\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta$ and cancellation properties, meaning that $(C(G) \otimes 1)\Delta(C(G))$ and $(1 \otimes C(G))\Delta(C(G))$ are dense in $C(G) \otimes C(G)$. There is a unique state h satisfying $(h \otimes \iota)\Delta = h$ (and/or $(\iota \otimes h)\Delta = h$) called the Haar state. If h is faithful, G is called a reduced quantum group, and we are mainly interested in such cases.

A finite dimensional unitary representation of G is a unitary element $U \in B(H_U) \otimes C(G)$, where H_U is a finite dimensional Hilbert space, such that $(\iota \otimes \Delta)(U) = U_{12}U_{13}$. The dense $*$-subalgebra of $C(G)$ spanned by matrix coefficients of finite dimensional representations is denoted by $\mathbb{C}[G]$. The intertwiners between two representations U and V are the linear maps T from H_U to H_V satisfying $V(T \otimes 1) = (T \otimes 1)U$. The tensor product of two representations U and V is defined by $U_{13}V_{23}$ and denoted by $U \mathbin{⫪} V$. The category $\operatorname{Rep} G$ of finite dimensional unitary representations with intertwiners as morphisms and with tensor product $⫪$ becomes a semisimple C*-tensor category.

Using the monoidal structure on $\operatorname{Rep} G$, for any $W \in \operatorname{Rep} G$, we can define an endofunctor $\iota \otimes W$ on $\operatorname{Rep} G$ which maps an object U to $U \mathbin{⫪} W$ and a morphism T to $T \otimes \iota$. A natural transformation between such functors $\iota \otimes W$ and $\iota \otimes V$ is given by a collection of morphisms $\eta_U\colon U \mathbin{⫪} W \to U \mathbin{⫪} V$ for $U \in \operatorname{Rep} G$ that are natural in U.

Denote the Woronowicz character $f_1 \in \mathcal{U}(G) = \mathbb{C}[G]^*$ by ρ. The space $\mathcal{U}(G)$ has a structure of a $*$-algebra, defined by duality from the Hopf $*$-algebra $(\mathbb{C}[G], \Delta)$. Every finite dimensional unitary representation U of G defines a $*$-representation π_U of $\mathcal{U}(G)$ on H_U by $\pi_U(\omega) = (\iota \otimes \omega)(U)$. We will often omit π_U in expressions. Using the element ρ the conjugate unitary representation to U is defined by

$$\bar{U} = (j(\rho)^{1/2} \otimes 1)(j \otimes \iota)(U^*)(j(\rho)^{-1/2} \otimes 1) \in B(\bar{H}_U) \otimes \mathbb{C}[G],$$

where j denotes the canonical $*$-anti-isomorphism $B(H_U) \cong B(\bar{H}_U)$ defined by $j(T)\bar{\xi} = \overline{T^*\xi}$. We have morphisms $R_U\colon \mathbb{1} \to \bar{U} \mathbin{⫪} U$ and $\bar{R}_U\colon \mathbb{1} \to U \mathbin{⫪} \bar{U}$ defined by

$$R_U(1) = \sum_i \bar{\xi}_i \otimes \rho^{-1/2}\xi_i \quad \text{and} \quad \bar{R}_U(1) = \sum_i \rho^{1/2}\xi_i \otimes \bar{\xi}_i,$$

where $\{\xi_i\}_i$ is an orthonormal basis in H_U. They solve the conjugate equations for U and $\bar{U}$, meaning that

$$(R_U^* \otimes \iota)(\iota \otimes \bar{R}_U) = \iota_{\bar{U}} \quad \text{and} \quad (\bar{R}_U^* \otimes \iota)(\iota \otimes R_U) = \iota_U.$$

Therefore $\operatorname{Rep} G$ is a rigid C*-tensor category. Woronowicz's Tannaka–Krein duality theorem recovers the $*$-Hopf algebra $\mathbb{C}[G]$ from the rigid semisimple C*-tensor category $\operatorname{Rep} G$ and the forgetful fiber functor $U \mapsto H_U$.

1.2. **G-algebras and $(\operatorname{Rep} G)$-module categories.** Given a compact quantum group G, a unital G-C*-algebra is a unital C*-algebra B equipped with a continuous left action $\alpha\colon B \to C(G) \otimes B$ of G. This means that α is an injective unital $*$-homomorphism such that $(\Delta \otimes \iota)\alpha = (\iota \otimes \alpha)\alpha$ and such that the space $(C(G) \otimes 1)\alpha(B)$ is dense in $C(G) \otimes B$. The linear span of spectral subspaces,

$$\mathcal{B} = \{x \in B \mid \alpha(x) \in \mathbb{C}[G] \otimes_{\mathrm{alg}} B\},$$

which is a dense $*$-subalgebra of B, is called the *regular subalgebra* of B, and the elements of $\mathcal{B}$ are called *regular*. More concretely, the algebra $\mathcal{B}$ is spanned by elements of the form $(h\otimes\iota)((x\otimes 1)\alpha(a))$ for $x \in \mathbb{C}[G]$ and $a \in B$. This algebra is of central importance for the categorical reconstruction of B.

When $\mathcal{D}$ is a C*-category, the category $\operatorname{End}(\mathcal{D})$ of endofunctors of $\mathcal{D}$, with uniformly bounded natural transformations as morphisms, forms a C*-tensor category. A C*-category $\mathcal{D}$ endowed with a unitary tensor functor $\operatorname{Rep} G \to \operatorname{End}(\mathcal{D})$ is called a $(\operatorname{Rep} G)$-*module category.* For $U \in \operatorname{Rep} G$, we denote the induced functor on $\mathcal{D}$ by $X \mapsto X \times U$. An object X in a $(\operatorname{Rep} G)$-module category $\mathcal{D}$ is said to be *generating* if any other object $Y \in \mathcal{D}$ is isomorphic to a subobject of $X \times U$ for some $U \in \operatorname{Rep} G$.

Let us summarize the categorical duality theory of continuous actions of reduced compact quantum groups on unital C*-algebras developed in [DCY12] and [Nes13].

Theorem 1.1 ([DCY12, Theorem 6.4; Nes13, Theorem 3.3])**.** *Let G be a reduced compact quantum group. Then the following two categories are equivalent:*

(i) *The category of unital G-C*-algebras with unital G-equivariant $*$-homomorphisms as morphisms.*
(ii) *The category of pairs $(\mathcal{D}, M)$, where $\mathcal{D}$ is a right* $(\operatorname{Rep} G)$*-module C*-category and M is a generating object in $\mathcal{D}$, with equivalence classes of unitary* $(\operatorname{Rep} G)$*-module functors respecting the prescribed generating objects as morphisms.*

We omit the precise definition of the equivalence relation on functors between pairs $(\mathcal{D}, M)$, since it will not be important to us, see [DCY12, Theorem 7.1] for details. Note also that, as follows from the proof, under the above correspondence the fixed point algebra B^G is isomorphic to $\operatorname{End}_{\mathcal{D}}(M)$.

In the following subsections we overview the proof of the theorem.

1.3. **From algebras to module categories.** Given a G-C*-algebra (B, α), we consider the category $\mathcal{D}_B$ of G-equivariant finitely generated right Hilbert B-modules. In other words, objects of $\mathcal{D}_B$ are finitely generated right Hilbert B-modules X equipped with a linear map $\delta = \delta_X\colon X \to C(G)\otimes X$ which satisfies the comultiplicativity property $(\Delta\otimes\iota)\delta = (\iota\otimes\delta)\delta$, such that $(C(G)\otimes 1)\delta(X)$ is dense in $C(G) \otimes X$, and such that δ is compatible with the Hilbert B-module structure in the sense that

$$\delta(\xi a) = \delta(\xi)\alpha(a), \qquad \langle\delta(\xi), \delta(\zeta)\rangle = \alpha(\langle\xi, \zeta\rangle),$$

for $\xi, \zeta \in X$ and $a \in B$. Here, $C(G) \otimes X$ is considered as a right Hilbert $(C(G) \otimes B)$-module.

For $X \in \mathcal{D}_B$ and $U \in \operatorname{Rep} G$, we obtain a new object $X \times U$ in $\mathcal{D}_B$ given by the linear space $H_U \otimes X$, which is a right Hilbert B-module such that

$$(\xi \otimes x)a = \xi \otimes xa \quad\text{and}\quad \langle\xi \otimes x, \eta \otimes y\rangle_B = (\eta, \xi)\langle x, y\rangle_B \quad\text{for}\quad \xi, \eta \in H_U,\ x, y \in X,\ a \in B,$$

together with the compatible $C(G)$-coaction map

$$\delta = \delta_{H_U\otimes X}\colon H_U \otimes X \to C(G) \otimes H_U \otimes X, \quad \delta(\xi \otimes x) = U^*_{21}(\xi \otimes \delta_X(x))_{213}. \tag{1.1}$$

This construction is natural both in X and U, and satisfies $(X \times U) \times V \cong X \times (U \mathbin{\bar{\otimes}} V)$, with the obvious isomorphism mapping $\zeta \otimes \xi \otimes x \in H_V \otimes H_U \otimes X$ into $\xi \otimes \zeta \otimes x$. We keep the notation

$H_U \otimes X$ when we want to emphasize the realization of the object $X \times U$ as a Hilbert module. This way $\mathcal{D}_B$ becomes a right $(\operatorname{Rep} G)$-module category.

It is known that by the stabilization argument, any object in $\mathcal{D}_B$ is a direct summand of $B \times U$ for some $U \in \operatorname{Rep} G$. Thus we may, and often will, consider $\mathcal{D}_B$ as an idempotent completion of $\operatorname{Rep} G$ via the correspondence $U \mapsto B \times U$. To be precise, we start from a C*-category with the same objects as in $\operatorname{Rep} G$, but with the new enlarged morphism sets

$$\mathcal{C}_B(U, V) = \operatorname{Hom}_{G,B}(H_U \otimes B, H_V \otimes B),$$

and form new objects from projections in the C*-algebras $\mathcal{C}_B(U, U)$, thus getting a C*-category $\mathcal{C}_B$. Note that, more explicitly, the set $\mathcal{C}_B(U, V)$ consists of elements $T \in B(H_U, H_V) \otimes B$ such that

$$V_{12}^*(\iota \otimes \alpha)(T)U_{12} = T_{13}.$$

Note also that we automatically have $\mathcal{C}_B(U, V) \subset B(H_U, H_V) \otimes \mathcal{B}$.

For every $W \in \operatorname{Rep} G$, the functor $\iota \otimes W$ on $\operatorname{Rep} G$ extends to $\mathcal{C}_B$ in the obvious way: given a morphism $T \in \mathcal{C}_B(U, V) \subset B(H_U, H_V) \otimes \mathcal{B}$ the corresponding morphism $T \otimes \iota \in \mathcal{C}_B(U \mathbin{\bigcirc\!\!\!\!\!\top} W, V \mathbin{\bigcirc\!\!\!\!\!\top} W)$ is $T_{13} \in B(H_U, H_V) \otimes B(H_W) \otimes \mathcal{B}$. The right $(\operatorname{Rep} G)$-module C*-categories $\mathcal{C}_B$ and $\mathcal{D}_B$ are equivalent via a functor mapping U into $B \times U$.

Although the category $\mathcal{C}_B$ might appear somewhat ad hoc compared to $\mathcal{D}_B$, it is more convenient for computations and some of the constructions become simpler for $\mathcal{C}_B$. For example, suppose that $f\colon B_0 \to B_1$ is a morphism of G-C*-algebras. Then, $\iota \otimes f$ defines linear transformations $f_{\#,U,V}\colon \mathcal{C}_{B_0}(U, V) \to \mathcal{C}_{B_1}(U, V)$, which together define a functor $f_\#\colon \mathcal{C}_{B_0} \to \mathcal{C}_{B_1}$. The pair $(f_\#, \iota)_{U,V}$ gives a $(\operatorname{Rep} G)$-module homomorphism in the sense of [DCY12, Definition 3.17]. Under the above equivalence $\mathcal{D}_B \simeq \mathcal{C}_B$, this obvious construction corresponds to the scalar extension functor $\mathcal{D}_{B_0} \to \mathcal{D}_{B_1}$, mapping X into $X \otimes_{B_0} B_1$, discussed in [DCY12]. Note also that for the composition of G-equivariant maps we have the desired equality of functors $f_\# g_\# = (fg)_\#$ between the categories $\mathcal{C}_B$, rather than a natural isomorphism of functors, which we would have for the categories $\mathcal{D}_B$.

1.4. **From module categories to algebras.** We recall the construction of an action from a pair $(\mathcal{D}, M)$ following [Nes13]. Without loss of generality we may assume that the $(\operatorname{Rep} G)$-module category $\mathcal{D}$ is strict. Furthermore, in order to simplify the notation, by replacing $\mathcal{D}$ by an equivalent category we may assume that it is the idempotent completion of the category $\operatorname{Rep} G$ with larger morphism sets $\mathcal{D}(U, V)$ than in $\operatorname{Rep} G$, such that M is the unit object $\mathbb{1}$ in $\operatorname{Rep} G$ and the functor $\iota \times U$ on $\mathcal{D}$ is an extension of the functor $\iota \otimes U$ on $\operatorname{Rep} G$. Namely, simply define the new set of morphisms between U and V as $\mathcal{D}(M \times U, M \times V)$.

Choose representatives U_s of isomorphism classes of irreducible representations of G, and assume that $U_e = \mathbb{1}$ for some index e. We write H_s instead of H_{U_s}. Consider the linear space

$$\mathcal{B} = \bigoplus_s (\bar{H}_s \otimes \mathcal{D}(\mathbb{1}, U_s)). \tag{1.2}$$

We may assume that $\operatorname{Rep} G$ is small and consider also the much larger linear space

$$\tilde{\mathcal{B}} = \bigoplus_U (\bar{H}_U \otimes \mathcal{D}(\mathbb{1}, U)), \tag{1.3}$$

where the summation is over all objects in $\operatorname{Rep} G$. Define a linear map $\pi\colon \tilde{\mathcal{B}} \to \mathcal{B}$ as follows. Given a finite dimensional unitary representation U, choose isometries $w_i\colon H_{s_i} \to H_U$ defining a decomposition of U into irreducibles. Then, for $\bar{\xi} \otimes T \in \bar{H}_U \otimes \mathcal{D}(\mathbb{1}, U)$, put

$$\pi(\bar{\xi} \otimes T) = \sum_i \overline{w_i^* \xi} \otimes w_i^* T.$$

This map is independent of any choices. The space $\tilde{\mathcal{B}}$ is an associative algebra with product

$$(\bar{\xi} \otimes T) \cdot (\bar{\zeta} \otimes S) = \overline{(\xi \otimes \zeta)} \otimes (T \otimes \iota)S.$$

This product defines a product on $\mathcal{B}$ such that $\pi(x)\pi(y) = \pi(x \cdot y)$ for all $x, y \in \tilde{\mathcal{B}}$.

In order to define the $*$-structure on $\mathcal{B}$, first define an antilinear map $\bullet$ on $\tilde{\mathcal{B}}$ by

$$(\bar\xi\otimes T)^\bullet=\overline{\overline{\rho^{-1/2}\xi}}\otimes(T^*\otimes\iota)\bar R_U\quad\text{for}\quad \bar\xi\otimes T\in\bar H_U\otimes\mathcal{D}(\mathbb{1},U). \tag{1.4}$$

This map does not define an involution on $\tilde{\mathcal{B}}$, but on $\mathcal{B}$ we get an involution such that $\pi(x)^*=\pi(x^\bullet)$ for all $x\in\tilde{\mathcal{B}}$.

The $*$-algebra $\mathcal{B}$ has a natural left $\mathbb{C}[G]$-comodule structure defined by the map $\alpha\colon\mathcal{B}\to\mathbb{C}[G]\otimes\mathcal{B}$ such that if U is a finite dimensional unitary representation of G, $\{\xi_i\}_i$ is an orthonormal basis in H_U and u_{ij} are the matrix coefficients of U in this basis, then

$$\alpha(\pi(\bar\xi_i\otimes T))=\sum_j u_{ij}\otimes\pi(\bar\xi_j\otimes T). \tag{1.5}$$

It is shown then that the action α is algebraic in the sense of [DCY12, Definition 4.2], meaning that the fixed point algebra $A=\mathcal{B}^G\cong\mathrm{End}_{\mathcal{D}}(\mathbb{1})$ is a unital C*-algebra and the conditional expectation $(h\otimes\iota)\alpha\colon\mathcal{B}\to A$ is positive and faithful. It follows that there is a unique completion of $\mathcal{B}$ to a C*-algebra B such that α extends to an action of the reduced form of G on B. This finishes the construction of an action from a module category.

Example 1.2. Consider the action of G on itself by left translations, so we consider the coproduct map of $C(G)$ as an action of G on $C(G)$. It corresponds to the module category $\mathcal{D}=\mathrm{Hilb}_f$ of finite dimensional Hilbert spaces, with the distinguished object $\mathbb{C}$, considered as a $(\mathrm{Rep}\,G)$-module category using the forgetful tensor functor $U\mapsto H_U$. Explicitly, identifying $\mathcal{D}(\mathbb{1},H_U)$ with H_U, we get an isomorphism of the algebra $\tilde{\mathcal{B}}$ constructed from the pair $(\mathrm{Hilb}_f,\mathbb{C})$ onto $\widetilde{\mathbb{C}[G]}=\bigoplus_U\bar H_U\otimes H_U$, and then an isomorphism $\mathcal{B}\cong\mathbb{C}[G]$ such that $\pi\colon\tilde{\mathcal{B}}\to\mathcal{B}$ turns into the map $\pi_G\colon\widetilde{\mathbb{C}[G]}\to\mathbb{C}[G]$ that sends $\bar\xi\otimes\zeta\in\bar H_U\otimes H_U$ into the matrix coefficient $((\cdot\,\zeta,\xi)\otimes\iota)(U)$ of U. $\diamondsuit$

Returning to the general case, consider the action $\alpha\colon B\to C(G)\otimes B$ defined by a pair $(\mathcal{D},M)$ as described above. The equivalence between the $(\mathrm{Rep}\,G)$-module categories $\mathcal{D}$ and $\mathcal{D}_B$ can be very concretely described as follows. First of all, as we have discussed, by replacing $\mathcal{D}$ by an equivalent category we may assume that it is the idempotent completion of $\mathrm{Rep}\,G$ with new morphisms sets. Similarly, instead of $\mathcal{D}_B$ we consider the category $\mathcal{C}_B$. Then in order to define an equivalence we just have to describe the isomorphisms $\mathcal{D}(U,V)\cong\mathcal{C}_B(U,V)$. The equivalence between $\mathcal{D}$ and $\mathcal{C}_B$ constructed in the proof of [Nes13, Theorem 2.3] (see also Section 3 there) has the property that a morphism $T\in\mathcal{D}(\mathbb{1},V)$ is mapped into

$$\sum_j\zeta_j\otimes\pi(\bar\zeta_j\otimes T)\in\mathcal{C}_B(\mathbb{1},V)\subset B(\mathbb{C},H_V)\otimes B,$$

where $\{\zeta_j\}_j$ is an orthonormal basis in H_V and we identify $B(\mathbb{C},H_V)\otimes B$ with $H_V\otimes B$. Now assume that we have a morphism $T\in\mathcal{D}(U,V)$. We can write it as $(\iota\otimes R_U^*)(S\otimes\iota)$, with $S=(T\otimes\iota)\bar R_U\in\mathcal{D}(\mathbb{1},V\otimes\bar U)$. Choose an orthonormal basis $\{\xi_i\}_i$ in H_U. Then the morphism $S\otimes\iota_U$ defines the element

$$\sum_{i,j}\zeta_j\otimes\bar\xi_i\otimes1\otimes\pi\big(\overline{(\zeta_j\otimes\bar\xi_i)}\otimes S\big)\in\mathcal{C}_B(U,V\otimes\bar U\otimes U)\subset B(H_U,H_V\otimes\bar H_U\otimes H_U)\otimes B,$$

where we identify $B(H_U,H_V\otimes\bar H_U\otimes H_U)$ with $H_V\otimes\bar H_U\otimes B(H_U)$. It follows that $T=(\iota\otimes R_U^*)(S\otimes\iota)$ is mapped into

$$\sum_{ij}R_U^*(\xi_i\otimes\,\cdot\,)\zeta_j\otimes\pi\big(\overline{(\zeta_j\otimes\bar\xi_i)}\otimes S\big)\in\mathcal{C}_B(U,V)\subset B(H_U,H_V)\otimes B.$$

Since $R_U^*(\xi_i\otimes\xi)=(\rho^{-1/2}\xi,\xi_i)$, we conclude that the isomorphism $\mathcal{D}(U,V)\cong\mathcal{C}_B(U,V)$ is such that

$$\mathcal{D}(U,V)\ni T\mapsto\sum_{i,j}\theta_{\zeta_j,\xi_i}\pi_U(\rho^{-1/2})\otimes\pi\big(\overline{(\zeta_j\otimes\bar\xi_i)}\otimes(T\otimes\iota)\bar R_U\big)\in\mathcal{C}_B(U,V),$$

where $\theta_{\zeta_j,\xi_i} \in B(H_U, H_V)$ is the operator defined by $\theta_{\zeta_j,\xi_i}\xi = (\xi, \xi_i)\zeta_j$. This can also be written as

$$\mathcal{D}(U,V) \ni T \mapsto \sum_{i,j} \theta_{\zeta_j,\xi_i} \otimes \pi\big(\overline{(\zeta_j \otimes \overline{\rho^{-1/2}\xi_i})} \otimes (T \otimes \iota)\bar{R}_U\big) \in \mathcal{C}_B(U,V) \subset B(H_U, H_V) \otimes B. \quad (1.6)$$

1.5. **Yetter–Drinfeld algebras.** Assume we have a continuous left action of $\alpha\colon B \to C(G) \otimes B$ of a compact quantum group G on a unital C*-algebra B, as well as a continuous right action $\beta\colon B \to \mathcal{M}(B \otimes c_0(\hat{G}))$ of the dual discrete quantum group $\hat{G}$. The action β defines a left $\mathbb{C}[G]$-module algebra structure $\triangleright\colon \mathbb{C}[G] \otimes B \to B$ on B by

$$x \triangleright a = (\iota \otimes x)\beta(x) \quad \text{for} \quad x \in \mathbb{C}[G] \quad \text{and} \quad a \in B.$$

Here we view $c_0(\hat{G})$ as a subalgebra of $\mathcal{U}(G) = \mathbb{C}[G]^*$. This structure is compatible with involution, in the sense that

$$x \triangleright a^* = (S(x)^* \triangleright a)^*. \quad (1.7)$$

We say that B is a *Yetter–Drinfeld G-C*-algebra* if the following identity holds for all $x \in \mathbb{C}[G]$ and $a \in \mathcal{B}$:

$$\alpha(x \triangleright a) = x_{(1)}a_{(1)}S(x_{(3)}) \otimes (x_{(2)} \triangleright a_{(2)}), \quad (1.8)$$

where we use Sweedler's sumless notation, so we write $\Delta(x) = x_{(1)} \otimes x_{(2)}$ and $\alpha(a) = a_{(1)} \otimes a_{(2)}$. Note that the above identity implies that $\mathcal{B} \subset B$ is a submodule over $\mathbb{C}[G]$.

Yetter–Drinfeld C*-algebras in the more general setting of locally compact quantum groups have been studied by Nest and Voigt [NV10]. It is not difficult to see that our definition is equivalent to theirs,* but the case of compact quantum groups allows for the above familiar algebraic formulation, which is more convenient for our purposes. In the case of reduced compact quantum groups we can make it purely algebraic by getting rid of the right action β altogether.

Proposition 1.3. *Assume that G is a reduced compact quantum group and $\alpha\colon B \to C(G) \otimes B$ is a continuous action of G on a unital C*-algebra B. Let $\mathcal{B} \subset B$ be the subalgebra of regular elements. Suppose that $\mathcal{B}$ is also a left $\mathbb{C}[G]$-module algebra such that conditions* (1.7) *and* (1.8) *are satisfied for all $x \in \mathbb{C}[G]$ and $a \in \mathcal{B}$. Then there exists a unique continuous right action $\beta\colon B \to \mathcal{M}(B \otimes c_0(\hat{G}))$ such that $x \triangleright a = (\iota \otimes x)\beta(a)$ for all $x \in \mathbb{C}[G]$ and $a \in \mathcal{B}$.*

Proof. Let us show first that for any finite dimensional unitary representation $U = \sum_{i,j} m_{ij} \otimes u_{ij}$ of G, where m_{ij} are matrix units in $B(H_U)$, there exists a unital $*$-homomorphism $\beta_U\colon B \to B \otimes B(H_U)$ such that

$$\beta_U(a) = \sum_{i,j} (u_{ij} \triangleright a) \otimes m_{ij} \quad \text{for all} \quad a \in \mathcal{B}.$$

From the assumption that $\mathcal{B}$ is a $\mathbb{C}[G]$-module algebra we immediately get that $\beta_U\colon \mathcal{B} \to \mathcal{B} \otimes B(H_U)$ is a unital homomorphism. Condition (1.7) implies that this homomorphism is $*$-preserving. Thus, all we have to do is to show that β_U extends to a $*$-homomorphism $B \to B \otimes B(H_U)$. For this observe that the Yetter–Drinfeld condition (1.8) implies that

$$(\alpha \otimes \iota)\beta_U(a) = U_{31}(\iota \otimes \beta_U)\alpha(a)U_{31}^*.$$

It follows that if we let B_U to be the norm closure of $\beta_U(\mathcal{B})$ in $B \otimes B(H_U)$, then the restriction of the map

$$B \otimes B(H_U) \ni y \mapsto U_{31}^*(\alpha \otimes \iota)(y)U_{31} \in C(G) \otimes B \otimes B(H_U)$$

to B_U gives us a well-defined unital $*$-homomorphism $\gamma\colon B_U \to C(G) \otimes B_U$. Furthermore, since $\gamma(\beta_U(a)) = (\iota \otimes \beta_U)\alpha(a)$ for $a \in \mathcal{B}$, the map γ defines a continuous action of G on B_U. It follows that if we define a new C*-norm $\|\cdot\|'$ on $\mathcal{B}$ by

$$\|a\|' = \max\{\|a\|, \|\beta_U(a)\|\},$$

*It should also be taken into account that the definition of coproduct on $C_0(\hat{G})$ used in the theory of locally compact quantum groups is opposite to the one usually used for compact quantum groups.

then the action α of G on $\mathcal{B}$ extends to a continuous action on the completion of $\mathcal{B}$ in this norm. But according to [DCY12, Proposition 4.4] a C*-norm with such property is unique. Hence $\|a\|' = \|a\|$ for all $a \in \mathcal{B}$, and therefore the map β_U extends by continuity to B.

Since $c_0(\hat{G}) \cong c_0\text{-}\bigoplus_s B(H_s)$, the homomorphisms β_{U_s} define a unital $*$-homomorphism $\beta\colon B \to \mathcal{M}(B \otimes c_0(\hat{G})) = \ell^\infty\text{-}\bigoplus_s (B \otimes B(H_s))$ such that $(\iota \otimes x)\beta(a) = x \rhd a$ for all $x \in \mathbb{C}[G]$ and $a \in \mathcal{B}$. It is then straightforward to check that β is a continuous action. The uniqueness is also clear. $\square$

2. Yetter–Drinfeld algebras and tensor functors

In this section we prove one of our main result, a categorical description of a class of Yetter–Drinfeld C*-algebras.

2.1. **Two categories.** A Yetter–Drinfeld G-C*-algebra B is said to be *braided-commutative* if for all $a, b \in \mathcal{B}$ we have

$$ab = b_{(2)}(S^{-1}(b_{(1)}) \rhd a). \tag{2.1}$$

When b is in the fixed point algebra $A = \mathcal{B}^G$, the right hand side reduces to ba, and we see that A is contained in the center of $\mathcal{B}$.

Theorem 2.1. *Let G be a reduced compact quantum group. Then the following two categories are equivalent:*

(i) *The category $\mathcal{YD}_{\mathrm{brc}}(G)$ of unital braided-commutative Yetter–Drinfeld G-C*-algebras with unital G- and $\hat{G}$-equivariant $*$-homomorphisms as morphisms.*

(ii) *The category $\mathcal{Tens}(\mathrm{Rep}\, G)$ of pairs $(\mathcal{C}, \mathcal{E})$, where $\mathcal{C}$ is a C*-tensor category and $\mathcal{E}\colon \mathrm{Rep}\, G \to \mathcal{C}$ is a unitary tensor functor such that $\mathcal{C}$ is generated by the image of $\mathcal{E}$. The set of morphisms $(\mathcal{C}, \mathcal{E}) \to (\mathcal{C}', \mathcal{E}')$ in this category is the set of equivalence classes of pairs $(\mathcal{F}, \eta)$, where $\mathcal{F}$ is a unitary tensor functor $\mathcal{F}\colon \mathcal{C} \to \mathcal{C}'$ and η is a natural unitary monoidal isomorphism $\eta\colon \mathcal{F}\mathcal{E} \to \mathcal{E}'$.*

Moreover, given a morphism $[(\mathcal{F}, \eta)]\colon (\mathcal{C}, \mathcal{E}) \to (\mathcal{C}', \mathcal{E}')$, the corresponding homomorphism of Yetter–Drinfeld C*-algebras is injective if and only if $\mathcal{F}$ is faithful, and it is surjective if and only if $\mathcal{F}$ is full.

The condition that $\mathcal{C}$ is generated by the image of $\mathcal{E}$ means that any object in $\mathcal{C}$ is isomorphic to a subobject of $\mathcal{E}(U)$ for some $U \in \mathrm{Rep}\, G$.

We remind that we assume that C*-categories are closed under subobjects. We also stress that we do not assume that the unit in $\mathcal{C}$ is simple. In fact, as will be clear from the proof, the C*-algebra $\mathrm{End}_{\mathcal{C}}(\mathbb{1})$ is exactly the fixed point algebra B^G in the C*-algebra B corresponding to $(\mathcal{C}, \mathcal{E})$.

We have to explain how we define the equivalence relation on pairs $(\mathcal{F}, \eta)$. Assume $(\mathcal{F}, \eta)$ is a pair consisting of a unitary tensor functor $\mathcal{F}\colon \mathcal{C} \to \mathcal{C}'$ and a natural unitary monoidal isomorphism $\eta\colon \mathcal{F}\mathcal{E} \to \mathcal{E}'$. Then, for all objects U and V in $\mathrm{Rep}\, G$, we get linear maps

$$\mathcal{C}(\mathcal{E}(U), \mathcal{E}(V)) \to \mathcal{C}'(\mathcal{E}'(U), \mathcal{E}'(V)), \quad T \mapsto \eta_V \mathcal{F}(T) \eta_U^{-1}.$$

We say that two pairs $(\mathcal{F}, \eta)$ and $(\tilde{\mathcal{F}}, \tilde{\eta})$ are equivalent, if the corresponding maps $\mathcal{C}(\mathcal{E}(U), \mathcal{E}(V)) \to \mathcal{C}'(\mathcal{E}'(U), \mathcal{E}'(V))$ are equal for all U and V.

A somewhat more concrete way of thinking of the category $\mathcal{Tens}(\mathrm{Rep}\, G)$ of pairs $(\mathcal{C}, \mathcal{E})$ is as follows. Assume $(\mathcal{C}, \mathcal{E})$ is such a pair. First of all observe that the functor $\mathcal{E}$ is automatically faithful by semisimplicity and existence of conjugates in $\mathrm{Rep}\, G$. Then replacing the pair $(\mathcal{C}, \mathcal{E})$ by an isomorphic one, we may assume that $\mathcal{C}$ is a strict C*-tensor category containing $\mathrm{Rep}\, G$ and $\mathcal{E}$ is simply the embedding functor. Namely, similarly to our discussion in Section 1.4, define new sets of morphisms between objects U and V in $\mathrm{Rep}\, G$ as $\mathcal{C}(\mathcal{E}(U), \mathcal{E}(V))$ and then complete the new category we thus obtain with respect to subobjects.

Assume now that we have two strict C*-tensor categories $\mathcal{C}$ and $\mathcal{C}'$ containing $\operatorname{Rep} G$, and consider the embedding functors $\mathcal{E}\colon \operatorname{Rep} G \to \mathcal{C}$ and $\mathcal{E}'\colon \operatorname{Rep} G \to \mathcal{C}'$. Assume $[(\mathcal{F},\eta)]\colon (\mathcal{C},\mathcal{E}) \to (\mathcal{C}',\mathcal{E}')$ is a morphism. This means that the unitary isomorphisms $\eta_U\colon \mathcal{F}(U) \to U$ in $\mathcal{C}'$ are such that $\mathcal{F}(T) = \eta_V^{-1} T \eta_U$ for any morphism $T\colon U \to V$ in $\operatorname{Rep} G$, and the morphisms

$$\mathcal{F}_{2;U,V}\colon \mathcal{F}(U)\otimes\mathcal{F}(V) \to \mathcal{F}(U\otimes V)$$

defining the tensor structure of $\mathcal{F}$ are given by $\mathcal{F}_{2;U,V} = \eta^{-1}_{U\textcircled{T}V}(\eta_U\otimes\eta_V)$. We can then define a new unitary tensor functor $\tilde{\mathcal{F}}$ from the full subcategory of $\mathcal{C}$ formed by the objects in $\operatorname{Rep} G \subset \mathcal{C}$ into $\mathcal{C}'$ by letting $\tilde{\mathcal{F}}(U) = U$, $\tilde{\mathcal{F}}(T) = \eta_V \mathcal{F}(T)\eta_U^{-1}$ for $T \in \mathcal{C}(U,V)$, and $\tilde{\mathcal{F}}_{2;U,V} = \iota$. This functor can be extended to $\mathcal{C}$, by sending any subobject $X \subset U$ with corresponding projection $p_X \in \operatorname{End}_{\mathcal{C}}(U)$ to an object corresponding to the projection $\tilde{\mathcal{F}}(p_X) \in \operatorname{End}_{\mathcal{C}'}(U)$. Such an extension is unique up to a natural unitary monoidal isomorphism. Then by definition $[(\mathcal{F},\eta)] = [(\tilde{\mathcal{F}},\iota)]$.

Therefore morphisms $(\mathcal{C},\mathcal{E}) \to (\mathcal{C}',\mathcal{E}')$ are equivalence classes of unitary tensor functors $\mathcal{F}\colon \mathcal{C}\to\mathcal{C}'$ such that $\mathcal{F}$ is the identity functor on $\operatorname{Rep} G \subset \mathcal{C}$ and $\mathcal{F}_{2;U,V} = \iota$ for all objects U and V in $\operatorname{Rep} G$. Two such functors $\mathcal{F}$ and $\mathcal{G}$ are equivalent, or in other words they define the same morphism, if $\mathcal{F}(T) = \mathcal{G}(T)$ for all morphisms $T \in \mathcal{C}(U,V)$ and all objects U and V in $\operatorname{Rep} G$.

The rest of this section is devoted to the proof of Theorem 2.1.

2.2. **From Yetter–Drinfeld algebras to tensor categories.** In this subsection the assumption that G is reduced will not be important.

Assume that B is a braided-commutative Yetter–Drinfeld G-C*-algebra. Consider the category $\mathcal{D}_B$ of G-equivariant finitely generated right Hilbert B-modules discussed in Section 1.3. Then $\mathcal{D}_B$ can be turned into a C*-tensor category. A similar result in the purely algebraic setting was already shown in [CVOZ94]. The key observation is the following lemma. Let us say that a vector $\xi \in X$ is regular if $\delta_X(\xi)$ lies in the algebraic tensor product $\mathbb{C}[G]\otimes_{\mathrm{alg}} X$.

Lemma 2.2. *Assume that X is a G-equivariant finitely generated right Hilbert B-module, and let $\mathcal{X}$ be its subspace of regular vectors. Then there exists a unique unital $*$-homomorphism $\pi_X\colon B \to \operatorname{End}_B(X)$ such that $\pi_X(a)\xi = \xi_{(2)}(S^{-1}(\xi_{(1)}) \rhd a)$ for all $a \in \mathcal{B}$ and $\xi \in \mathcal{X}$. Furthermore, we have $\delta_X(\pi_X(a)\xi) = (\iota\otimes\pi_X)\alpha(a)\delta_X(\xi)$ for all $a \in B$ and $\xi \in X$.*

Proof. It suffices to consider the case $X = H_U \otimes B$ for an irreducible unitary representation $U = U_s$ of G, since any other module embeds into a finite direct sum of such modules as a direct summand. Then, using the action $\beta\colon B \to \mathcal{M}(B\otimes c_0(\hat{G}))$ and the projection $B\otimes c_0(\hat{G}) \to B\otimes B(H_s) \cong B(H_s)\otimes B$, we get a unital $*$-homomorphism $\pi_X\colon B \to \operatorname{End}_B(H_U\otimes B) = B(H_U)\otimes B$ such that

$$\pi_X(a) = \sum_{i,j} m_{ij}\otimes(u_{ij}\rhd a),$$

where $U = \sum_{i,j} m_{ij}\otimes u_{ij}$ and m_{ij} are the matrix units in $B(H_U)$ defined by an orthonormal basis $\{\xi_i\}_i$ in H_U. In order to see that this gives the correct definition of π_X, take $b \in \mathcal{B}$. Recalling definition (1.1) of $\delta_{H_U\otimes B}$, we get

$$(\xi_i\otimes b)_{(1)}\otimes(\xi_i\otimes b)_{(2)} = \sum_j u_{ij}^* b_{(1)}\otimes(\xi_j\otimes b_{(2)}).$$

Hence

$$(\xi_i\otimes b)_{(2)}(S^{-1}((\xi_i\otimes b)_{(1)})\rhd a) = \sum_j \xi_j\otimes b_{(2)}(S^{-1}(u_{ij}^* b_{(1)})\rhd a) = \sum_j \xi_j\otimes(S^{-1}(u_{ij}^*)\rhd a)b,$$

where the last equality follows by braided commutativity. Since $S^{-1}(u_{ij}^*) = u_{ji}$, we see that $\pi_X(a)$ acts as stated in the formulation of the lemma.

In order to show that $\delta_X(\pi_X(a)\xi) = (\iota\otimes\pi_X)\alpha(a)\delta_X(\xi)$ we take an arbitrary X. It suffices to consider $a\in\mathcal{B}$. Then for $\xi\in\mathcal{X}$ we have

$$\delta_X(\pi_X(a)\xi) = \delta_X(\xi_{(2)}(S^{-1}(\xi_{(1)})\rhd a)) = \xi_{(2)}(S^{-1}(\xi_{(1)})\rhd a)_{(1)}\otimes\xi_{(3)}(S^{-1}(\xi_{(1)})\rhd a)_{(2)}.$$

Applying the Yetter–Drinfeld condition (1.8) we see that the last expression equals

$$\xi_{(4)}S^{-1}(\xi_{(3)})a_{(1)}\xi_{(1)}\otimes\xi_{(5)}(S^{-1}(\xi_{(2)})\rhd a_{(2)}) = a_{(1)}\xi_{(1)}\otimes\xi_{(3)}(S^{-1}(\xi_{(2)})\rhd a_{(2)}),$$

and this is exactly $(\iota\otimes\pi_X)\alpha(a)\delta_X(\xi)$. □

If X is as in the lemma, we conclude that X has the structure of an equivariant B-B-correspondence. If f is a G-equivariant endomorphism of the right Hilbert B-module X, it is automatically a B-bimodule map because of the way the left action of B is defined. Therefore the category $\mathcal{D}_B$ can be considered as a full subcategory of the C*-category of G-equivariant B-B-correspondences. The latter category has a natural C*-tensor structure. In order to show that $\mathcal{D}_B$ forms a C*-tensor subcategory it suffices to show that, given objects X and Y in $\mathcal{D}_B$, we have:

(i) $X\otimes_B Y$ is a finitely generated right B-module;
(ii) the left B-module structure on $X\otimes_B Y$ induced by that on X coincides with the left B-module structure given by Lemma 2.2 using the action of G and the right B-module structure on $X\otimes_B Y$.

The second property is a routine computation similar to the one in the proof of the second part of Lemma 2.2, so we omit it. In order to check (i) it suffices to consider modules of the form $H_U\otimes B$. For such modules we have the following more precise result.

Lemma 2.3. *For any finite dimensional unitary representations U and V of G, the map*

$$T_{U,V}\colon (H_V\otimes B)\otimes_B(H_U\otimes B)\to H_{U\mathbin{⊤\!\!\!\!\bigcirc} V}\otimes B,\quad (\zeta\otimes b)\otimes(\xi\otimes a)\mapsto \xi_{(2)}\otimes\zeta\otimes(S^{-1}(\xi_{(1)})\rhd b)a,$$

is a G-equivariant unitary isomorphism of right Hilbert B-modules. Furthermore, the isomorphisms $T_{U,V}$ have the property $T_{U\mathbin{⊤\!\!\!\!\bigcirc} V,W}(\iota\otimes T_{U,V}) = T_{U,V\mathbin{⊤\!\!\!\!\bigcirc} W}(T_{V,W}\otimes\iota)$.

Recall that the $\mathbb{C}[G]$-comodule structure on H_U is given by $\xi\mapsto\xi_{(1)}\otimes\xi_{(2)} = U^*_{21}(1\otimes\xi)$.

Proof of Lemma 2.3. Since $\xi_{(2)}\otimes(S^{-1}(\xi_{(1)})\rhd b) = b(\xi\otimes 1)$, it is clear that the map $T_{U,V}$ defines a right $\mathcal{B}$-module isomorphism

$$(H_V\otimes\mathcal{B})\otimes_{\mathcal{B}}(H_U\otimes\mathcal{B})\cong H_{U\mathbin{⊤\!\!\!\!\bigcirc} V}\otimes\mathcal{B}.$$

It is also obvious that $T_{U,V}$ is isometric on the subspace spanned by vectors of the form $(\zeta\otimes1)\otimes(\xi\otimes1)$. Since such vectors generate $(H_V\otimes\mathcal{B})\otimes_{\mathcal{B}}(H_U\otimes\mathcal{B})$ as a right $\mathcal{B}$-module, and this module is dense in $(H_V\otimes B)\otimes_B(H_U\otimes B)$, it follows that $T_{U,V}$ extends by continuity to a unitary isomorphism of right Hilbert B-modules.

Next let us check the G-equivariance. The $\mathbb{C}[G]$-comodule structure on $(H_V\otimes\mathcal{B})\otimes_{\mathcal{B}}(H_U\otimes\mathcal{B})$ is given by

$$\begin{aligned}\delta((\zeta\otimes b)\otimes(\xi\otimes a)) &= (\zeta\otimes b)_{(1)}(\xi\otimes a)_{(1)}\otimes(\zeta\otimes b)_{(2)}\otimes(\xi\otimes a)_{(2)}\\ &= \zeta_{(1)}b_{(1)}\xi_{(1)}a_{(1)}\otimes(\zeta_{(2)}\otimes b_{(2)})\otimes(\xi_{(2)}\otimes a_{(2)}).\end{aligned}$$

Applying $\iota\otimes T_{U,V}$ we get

$$\zeta_{(1)}b_{(1)}\xi_{(1)}a_{(1)}\otimes\xi_{(3)}\otimes\zeta_{(2)}\otimes(S^{-1}(\xi_{(2)})\rhd b_{(2)})a_{(2)}.$$

On the other hand, using the same symbol δ for the comodule structure on $H_{U\mathbin{⊤\!\!\!\!\bigcirc} V}\otimes\mathcal{B}$, since $(U\mathbin{⊤\!\!\!\!\bigcirc} V)^* = V^*_{23}U^*_{13}$ we get

$$\begin{aligned}&(\iota\otimes\delta)T_{U,V}((\zeta\otimes b)\otimes(\xi\otimes a))\\ &\qquad= \delta(\xi_{(2)}\otimes\zeta\otimes(S^{-1}(\xi_{(1)})\rhd b)a)\\ &\qquad= \zeta_{(1)}\xi_{(2)}(S^{-1}(\xi_{(1)})\rhd b)_{(1)}a_{(1)}\otimes\xi_{(3)}\otimes\zeta_{(2)}\otimes(S^{-1}(\xi_{(1)})\rhd b)_{(2)}a_{(2)}.\end{aligned}$$

Applying (1.8) we see that the last expression equals

$$\begin{aligned}\zeta_{(1)}\xi_{(2)}S^{-1}(\xi_{(1)})_{(1)}b_{(1)}S(S^{-1}(\xi_{(1)})_{(3)})a_{(1)}\otimes\xi_{(3)}\otimes\zeta_{(2)}\otimes(S^{-1}(\xi_{(2)})\rhd b_{(2)})a_{(2)}&\\ =\zeta_{(1)}\xi_{(4)}S^{-1}(\xi_{(3)})b_{(1)}\xi_{(1)}a_{(1)}\otimes\xi_{(5)}\otimes\zeta_{(2)}\otimes(S^{-1}(\xi_{(2)})\rhd b_{(2)})a_{(2)}&\\ =\zeta_{(1)}b_{(1)}\xi_{(1)}a_{(1)}\otimes\xi_{(3)}\otimes\zeta_{(2)}\otimes(S^{-1}(\xi_{(2)})\rhd b_{(2)})a_{(2)}.&\end{aligned}$$

Therefore the map $T_{U,V}$ is indeed G-equivariant.

Finally, in order to prove the equality $T_{U\text{ⓣ}V,W}(\iota\otimes T_{U,V})=T_{U,V\text{ⓣ}W}(T_{V,W}\otimes\iota)$ it suffices to check it on tensor products of vectors of the form $\xi\otimes1$, since such tensor products generate a dense subspace of triple tensor products as right $\mathcal{B}$-modules. But for such vectors the statement is obvious. □

Therefore the category $\mathcal{D}_B$ can be considered as a full C*-tensor subcategory of the category of G-equivariant B-B-correspondences. In view of the previous lemma, it is convenient to replace the tensor product by the opposite one, so we put $X\times Y=Y\otimes_B X$. Furthermore, the functor $\mathcal{E}_B\colon \operatorname{Rep}G\to\mathcal{C}_B$ mapping U into the module $H_U\otimes B$, together with the unitary isomorphisms $T_{U,V}\colon\mathcal{E}_B(U)\otimes\mathcal{E}_B(V)\to\mathcal{E}_B(U\text{ⓣ}V)$ from Lemma 2.3, is a unitary tensor functor. We have thus proved the following result.

Theorem 2.4. *Let G be a compact quantum group and B be a unital braided-commutative Yetter–Drinfeld G-C*-algebra. Then the G-equivariant finitely generated right Hilbert B-modules form a C*-tensor category $\mathcal{D}_B$ with tensor product $X\times Y=Y\otimes_B X$. Furthermore, there is a unitary tensor functor $\mathcal{E}_B\colon\operatorname{Rep}G\to\mathcal{D}_B$ mapping U to the module $H_U\otimes B$.*

Up to an isomorphism, the pair $(\mathcal{D}_B,\mathcal{E}_B)$ can be more concretely described as follows. As we discussed in Section 1.3, the category $\mathcal{D}_B$ is equivalent to the category $\mathcal{C}_B$, which is the idempotent completion of the category with the same objects as in $\operatorname{Rep}G$, but with the new morphism sets

$$\mathcal{C}_B(U,V)\subset B(H_U,H_V)\otimes\mathcal{B}$$

consisting of elements T such that $V_{12}^*(\iota\otimes\alpha)(T)U_{12}=T_{13}$. We define the tensor product of objects in $\mathcal{C}_B$ as in $\operatorname{Rep}G$, and in order to completely describe the tensor structure it remains to write down a formula for the linear maps

$$\mathcal{C}_B(U,V)\otimes\mathcal{C}_B(W,Z)\to\mathcal{C}_B(U\text{ⓣ}W,V\text{ⓣ}Z).$$

This can be done using Lemma 2.3. First, note that by the proof of Lemma 2.2, if $U=\sum_{i,j}m_{ij}\otimes u_{ij}$ then for any $\xi\in H_U$ and $b\in\mathcal{B}$ we have

$$\xi_{(2)}\otimes(S^{-1}(\xi_{(1)})\rhd b)=\sum_{ij}(m_{ij}\otimes(u_{ij}\rhd b))(\xi\otimes1).$$

Therefore the map $T_{U,V}$ from Lemma 2.2 can be written as

$$(\zeta\otimes b)\otimes(\xi\otimes a)\mapsto\sum_{i,j}m_{ij}\xi\otimes\zeta\otimes(u_{ij}\rhd b)a.$$

It follows that given $T=\sum_l T_l\otimes b_l\in\mathcal{C}_B(W,Z)$, the morphism $\iota\otimes T\in\mathcal{C}_B(U\text{ⓣ}W,U\text{ⓣ}Z)$ considered as a map $H_{U\text{ⓣ}W}\otimes B\to H_{U\text{ⓣ}Z}\otimes B$ acts by

$$\xi\otimes\zeta\otimes1\mapsto T_{U,Z}\left(\sum_l(T_l\zeta\otimes b_l)\otimes(\xi\otimes1)\right)=\sum_{i,j,l}m_{ij}\xi\otimes T_l\zeta\otimes(u_{ij}\rhd b_l).$$

On the other hand, if $S=\sum_k S_k\otimes a_k\in\mathcal{C}_B(U,V)$, then the morphism $S\otimes\iota\in\mathcal{C}_B(U\text{ⓣ}Z,V\text{ⓣ}Z)$ considered as a map $H_{U\text{ⓣ}W}\otimes B\to H_{U\text{ⓣ}Z}\otimes B$ acts by

$$\xi\otimes\zeta\otimes1\mapsto T_{V,Z}\left(\sum_k(\zeta\otimes1)\otimes(S_k\xi\otimes a_k)\right)=\sum_k S_k\xi\otimes\zeta\otimes a_k.$$

To summarize, the tensor structure on $\mathcal{C}_B$ is described by the following rules:

$$\text{if } \ T=\sum_l T_l\otimes b_l\in\mathcal{C}_B(W,Z), \ \text{ then } \ \iota_U\otimes T=\sum_{i,j,l} m_{ij}\otimes T_l\otimes(u_{ij}\rhd b_l); \tag{2.2}$$

$$\text{if } \ S\in\mathcal{C}_B(U,V), \ \text{ then } \ S\otimes\iota_Z=S_{13}. \tag{2.3}$$

In this picture the functor $\mathcal{E}_B\colon \operatorname{Rep}G\to\mathcal{D}_B$ becomes the strict tensor functor $\mathcal{F}_B\colon \operatorname{Rep}G\to\mathcal{C}_B$ which is the identity map on objects, while on morphisms it is $T\mapsto T\otimes 1$.

2.3. **From tensor categories to Yetter–Drinfeld algebras.** Let us turn to the construction of a Yetter–Drinfeld algebra from a pair $(\mathcal{C},\mathcal{E})\in\mathcal{T}ens(\operatorname{Rep}(G))$. The category $\mathcal{C}$ can be considered as a right $(\operatorname{Rep}G)$-module category with the distinguished object $\mathbb{1}$. Therefore by Theorem 1.1 we can construct a C*-algebra $B=B_{\mathcal{C}}$ together with a left continuous action $\alpha\colon B\to C(G)\otimes B$. Our goal is to prove the following.

Theorem 2.5. *The G-C*-algebra B corresponding to the* $(\operatorname{Rep}G)$*-module category $\mathcal{C}$ with the distinguished object $\mathbb{1}$ has a natural structure of a braided-commutative Yetter–Drinfeld C*-algebra.*

The construction of the Yetter–Drinfeld structure can be described for any pair $(\mathcal{C},\mathcal{E})$, but in order to simplify the notation we assume that $\mathcal{C}$ is strict, $\operatorname{Rep}G$ is a C*-tensor subcategory of $\mathcal{C}$ and $\mathcal{E}$ is simply the embedding functor. This is enough by the discussion following the formulation of Theorem 2.1.

Recall from Section 1 that the subalgebra $\mathcal{B}\subset B$ of regular elements is given by (1.2). By Proposition 1.3, to prove the theorem we have to define a $\mathbb{C}[G]$-module algebra structure on $\mathcal{B}$ satisfying properties (1.7), (1.8), and (2.1).

In Section 1 we also defined a 'universal' algebra $\tilde{\mathcal{B}}=\bigoplus_U(\bar H_U\otimes\mathcal{C}(\mathbb{1},U))$, together with a homomorphism $\pi\colon\tilde{\mathcal{B}}\to\mathcal{B}$. Recall from Example 1.2 that we denote by $\widetilde{\mathbb{C}[G]}=\bigoplus_U(\bar H_U\otimes H_U)$ the algebra $\tilde{\mathcal{B}}$ corresponding to the forgetful fiber functor $\operatorname{Rep}G\to\mathrm{Hilb}_f$, and then the corresponding homomorphism $\pi_G\colon\widetilde{\mathbb{C}[G]}\to\mathbb{C}[G]$ maps $\bar\xi\otimes\zeta\in\bar H_U\otimes H_U$ into $((\cdot\,\zeta,\xi)\otimes\iota)(U)$.

Define a linear map

$$\tilde\rhd\colon\widetilde{\mathbb{C}[G]}\otimes\tilde{\mathcal{B}}\to\tilde{\mathcal{B}}$$

by letting, for $\bar\xi\otimes\zeta\in\bar H_U\otimes H_U$ and $\bar\eta\otimes T\in\bar H_V\otimes\mathcal{C}(\mathbb{1},V)$,

$$(\bar\xi\otimes\zeta)\,\tilde\rhd\,(\bar\eta\otimes T)=\overline{(\xi\otimes\eta\otimes\overline{\rho^{-1/2}\zeta})}\otimes(\iota\otimes T\otimes\iota)\bar R_U\in\bar H_{U\circledcirc V\circledcirc\bar U}\otimes\mathcal{C}(\mathbb{1},U\circledcirc V\circledcirc\bar U). \tag{2.4}$$

We remind that $\bar R_U\colon\mathbb{1}\to U\circledcirc\bar U$ is given by $\bar R_U(1)=\sum_i\rho^{1/2}\xi_i\otimes\bar\xi_i$ for an orthonormal basis $\{\xi_i\}$ in H_U. Identifying $\mathbb{C}[G]$ with the subspace $\bigoplus_s(\bar H_s\otimes H_s)\subset\widetilde{\mathbb{C}[G]}$, we define a linear map

$$\rhd\colon\mathbb{C}[G]\otimes\mathcal{B}\to\mathcal{B} \ \text{ by letting } \ x\rhd a=\pi(x\,\tilde\rhd\,a) \ \text{ for } \ x\in\mathbb{C}[G] \ \text{ and } \ a\in\mathcal{B}.$$

Lemma 2.6. *The map $\rhd$ defines a left $\mathbb{C}[G]$-module algebra structure on $\mathcal{B}$, and we have*

$$\pi_G(x)\rhd\pi(a)=\pi(x\,\tilde\rhd\,a) \ \text{ for all } \ x\in\widetilde{\mathbb{C}[G]} \ \text{ and } \ a\in\tilde{\mathcal{B}}.$$

Proof. We start with the second statement. We have to show that $\pi_G(x)\rhd\pi(a)=\pi(x\,\tilde\rhd\,a)$ for $x\in\widetilde{\mathbb{C}[G]}$ and $a\in\tilde{\mathcal{B}}$. Take $x=\bar\xi\otimes\zeta\in\bar H_U\otimes H_U$ and $a=\bar\eta\otimes T\in\bar H_V\otimes\mathcal{C}(\mathbb{1},V)$. Choose isometries $u_i\colon H_{s_i}\to H_U$ and $v_j\colon H_{s_j}\to H_V$ defining decompositions of U and V into irreducibles. Then

$$\begin{aligned}\pi_G(x)\rhd\pi(a)&=\pi\left(\sum_{i,j}(\overline{u_i^*\xi}\otimes u_i^*\xi)\,\tilde\rhd\,(\overline{v_j^*\eta}\otimes v_j^*T)\right)\\&=\pi\left(\sum_{i,j}\overline{(u_i^*\xi\otimes v_j^*\eta\otimes\overline{\rho^{-1/2}u_i^*\zeta})}\otimes(\iota\otimes v_j^*T\otimes\iota)\bar R_{s_i}\right),\end{aligned} \tag{2.5}$$

where $\bar R_{s_i}=\bar R_{U_{s_i}}$. On the other hand,

$$\pi(x\,\tilde\rhd\, a)=\pi\Big(\overline{(\xi\otimes\eta\otimes\overline{\rho^{-1/2}\zeta})}\otimes(\iota\otimes T\otimes\iota)\bar R_U\Big)$$
$$=\pi\left(\sum_{i,j,k}\overline{(u_i^*\xi\otimes v_j^*\eta\otimes\bar u_k^*\overline{\rho^{-1/2}\zeta})}\otimes(u_i^*\otimes v_j^*T\otimes\bar u_k^*)\bar R_U\right),\tag{2.6}$$

where the morphism $\bar u_k\colon H_{\bar U_{s_k}}=\bar H_{s_k}\to H_{\bar U}=\bar H_U$ is defined by $\bar u_k\bar\xi=\overline{u_k\xi}$. Since $u_k^*\pi_U(\rho)=\pi_{U_{s_k}}(\rho)u_k^*$, $\bar R_U=\sum_i(u_i\otimes\bar u_i)\bar R_{s_i}$ and the partial isometries u_i have mutually orthogonal images, we see that expressions (2.5) and (2.6) are equal.

In order to show that $\rhd$ defines a left $\mathbb{C}[G]$-module structure, take $x=\bar\xi\otimes\zeta\in\bar H_U\otimes H_U$, $y=\bar\mu\otimes\nu\in\bar H_W\otimes H_W$ and $a=\bar\eta\otimes T\in\bar H_V\otimes\mathcal{C}(\mathbb{1},V)$. Then

$$x\,\tilde\rhd\,(y\,\tilde\rhd\, a)=\overline{(\xi\otimes\mu\otimes\eta\otimes\overline{\rho^{-1/2}\nu}\otimes\overline{\rho^{-1/2}\zeta})}\otimes(\iota\otimes\iota\otimes T\otimes\iota\otimes\iota)(\iota\otimes\bar R_W\otimes\iota)\bar R_U$$

is an element in $\bar H_{U\mathbin{\circledR}W\mathbin{\circledR}V\mathbin{\circledR}\bar W\mathbin{\circledR}\bar U}\otimes\mathcal{C}(\mathbb{1},U\mathbin{\circledR}W\mathbin{\circledR}V\mathbin{\circledR}\bar W\mathbin{\circledR}\bar U)$, and

$$(x\cdot y)\,\tilde\rhd\, a=\overline{(\xi\otimes\mu\otimes\eta\otimes\overline{(\rho^{-1/2}\zeta\otimes\rho^{-1/2}\nu)})}\otimes(\iota\otimes T\otimes\iota)\bar R_{U\mathbin{\circledR}W}$$

is an element in $\bar H_{(U\mathbin{\circledR}W)\mathbin{\circledR}V\mathbin{\circledR}\overline{(U\mathbin{\circledR}W)}}\otimes\mathcal{C}(\mathbb{1},(U\mathbin{\circledR}W)\mathbin{\circledR}V\mathbin{\circledR}\overline{(U\mathbin{\circledR}W)})$. The only reason why these two elements are different is that the representations $\bar W\mathbin{\circledR}\bar U$ and $\overline{U\mathbin{\circledR}W}$ are equivalent, but not equal. The map $\sigma\colon\bar H_W\otimes\bar H_U\to\overline{H_U\otimes H_W}$, $\sigma(\bar\mu\otimes\bar\xi)=\overline{\xi\otimes\mu}$ defines such an equivalence, and we have $\bar R_{U\mathbin{\circledR}W}=(\iota\otimes\iota\otimes\sigma)(\iota\otimes\bar R_W\otimes\iota)\bar R_U$. It follows that upon projecting to $\mathcal{B}$ we get an honest equality

$$\pi(x\,\tilde\rhd\,(y\,\tilde\rhd\, a))=\pi((x\cdot y)\,\tilde\rhd\, a),$$

that is, $\pi_G(x)\rhd(\pi_G(y)\rhd\pi(a))=(\pi_G(x)\pi_G(y))\rhd\pi(a)$.

It remains to show that $\rhd$ respects the algebra structure on $\mathcal{B}$, that is, $x\rhd(ab)=(x_{(1)}\rhd a)(x_{(2)}\rhd b)$ for $x\in\mathbb{C}[G]$ and $a,b\in\mathcal{B}$.

Take elements $a=\bar\eta\otimes T\in\bar H_V\otimes\mathcal{C}(\mathbb{1},V)$ and $b=\bar\zeta\otimes S\in\bar H_W\otimes\mathcal{C}(\mathbb{1},W)$ in $\tilde{\mathcal{B}}$. Let U be a finite dimensional unitary representation of G. Choose an orthonormal bases $\{\xi_i\}_i$ in H_U and denote by u_{ij} the corresponding matrix coefficients of U. Since $u_{ij}=\pi_G(\bar\xi_i\otimes\xi_j)$ and $\Delta(u_{ij})=\sum_k u_{ik}\otimes u_{kj}$, we then have to show that

$$\pi\big((\bar\xi_i\otimes\xi_j)\tilde\rhd(a\cdot b)\big)=\sum_k\pi\big(((\bar\xi_i\otimes\xi_k)\tilde\rhd a)\cdot((\bar\xi_k\otimes\xi_j)\tilde\rhd b)\big).$$

We have

$$(\bar\xi_i\otimes\xi_j)\tilde\rhd(a\cdot b)=\overline{(\xi_i\otimes\eta\otimes\zeta\otimes\overline{\rho^{-1/2}\xi_j})}\otimes(\iota\otimes T\otimes S\otimes\iota)\bar R_U.\tag{2.7}$$

On the other hand,

$$\sum_k((\bar\xi_i\otimes\xi_k)\tilde\rhd a)\cdot((\bar\xi_k\otimes\xi_j)\tilde\rhd b)$$
$$=\sum_k\Big(\overline{(\xi_i\otimes\eta\otimes\overline{\rho^{-1/2}\xi_k})}\otimes(\iota\otimes T\otimes\iota)\bar R_U\Big)\cdot\Big(\overline{(\xi_k\otimes\zeta\otimes\overline{\rho^{-1/2}\xi_j})}\otimes(\iota\otimes S\otimes\iota)\bar R_U\Big)$$
$$=\sum_k\overline{(\xi_i\otimes\eta\otimes\overline{\rho^{-1/2}\xi_k}\otimes\xi_k\otimes\zeta\otimes\overline{\rho^{-1/2}\xi_j})}\otimes(\iota\otimes T\otimes\iota\otimes\iota\otimes S\otimes\iota)(\bar R_U\otimes\bar R_U)$$

Since $\sum_k\overline{\rho^{-1/2}\xi_k}\otimes\xi_k=R_U(1)$, and R_U is, up to a scalar factor, an isomeric embedding of $\mathbb{1}$ into $\bar U\mathbin{\circledR}U$, by applying π to the above expression we get

$$\pi\big(\overline{(\xi_i\otimes\eta\otimes\zeta\otimes\overline{\rho^{-1/2}\xi_j})}\otimes(\iota\otimes T\otimes R_U^*\otimes S\otimes\iota)(\bar R_U\otimes\bar R_U)\big).$$

Since $(R_U^*\otimes\iota)(\iota\otimes\bar R_U)=\iota$, this is exactly the expression we obtain by applying π to (2.7). $\square$

We next check compatibility (1.7) of $\rhd$ with the $*$-structure.

Lemma 2.7. *For all $x\in\mathbb{C}[G]$ and $a\in\mathcal{B}$ we have $x\rhd a^*=(S(x)^*\rhd a)^*$.*

Proof. Recall that the involution on $\mathcal{B}$ arises from the map $\bullet$ on $\tilde{\mathcal{B}}$ defined by (1.4), so for $a=\bar\eta\otimes T\in\bar H_V\otimes\mathcal{C}(\mathbb{1},V)$ we have

$$a^\bullet=\overline{\overline{\rho^{-1/2}\eta}}\otimes(T^*\otimes\iota)\bar R_V\in\bar H_{\bar V}\otimes\mathcal{C}(\mathbb{1},\bar V).$$

Let us also define an antilinear map $\dagger$ on $\widetilde{\mathbb{C}[G]}$ by letting, for $x=\bar\xi\otimes\zeta\in\bar H_U\otimes H_U$,

$$x^\dagger=\bar\zeta\otimes\xi.$$

We then have $\pi_G(x^\dagger)=S(\pi_G(x))^*$. Indeed, using that $(\iota\otimes S)(U)=U^*$, we compute:

$$S(\pi_G(x))^*=S(((\cdot\,\zeta,\xi)\otimes\iota)(U))^*=((\cdot\,\zeta,\xi)\otimes\iota)(U^*))^*=((\cdot\,\xi,\zeta)\otimes\iota)(U)=\pi_G(x^\dagger).$$

Turning now to the proof of the lemma, we have to show that

$$\pi(x\mathbin{\tilde\rhd}a^\bullet)=\pi((x^\dagger\mathbin{\tilde\rhd}a)^\bullet).$$

We compute:

$$\begin{aligned}x\mathbin{\tilde\rhd}a^\bullet&=(\bar\xi\otimes\zeta)\mathbin{\tilde\rhd}\Big(\overline{\overline{\rho^{-1/2}\eta}}\otimes(T^*\otimes\iota)\bar R_V\Big)\\&=\overline{(\xi\otimes\overline{\rho^{-1/2}\eta}\otimes\overline{\rho^{-1/2}\zeta})}\otimes(\iota\otimes T^*\otimes\iota\otimes\iota)(\iota\otimes\bar R_V\otimes\iota)\bar R_U\end{aligned}\tag{2.8}$$

and

$$\begin{aligned}(x^\dagger\mathbin{\tilde\rhd}a)^\bullet&=((\bar\zeta\otimes\xi)\mathbin{\tilde\rhd}a)^\bullet=\Big(\overline{(\zeta\otimes\eta\otimes\overline{\rho^{-1/2}\xi})}\otimes(\iota\otimes T\otimes\iota)\bar R_U\Big)^\bullet\\&=\overline{\overline{(\rho^{-1/2}\zeta\otimes\rho^{-1/2}\eta\otimes\bar\xi)}}\otimes\Big(\big((\iota\otimes T\otimes\iota)\bar R_U\big)^*\otimes\iota_{\overline{U\mathbin{\bar\oplus}V\mathbin{\bar\oplus}\bar U}}\Big)\bar R_{U\mathbin{\bar\oplus}V\mathbin{\bar\oplus}\bar U},\end{aligned}\tag{2.9}$$

where we used that $\pi_{\bar U}(\rho)=j(\pi_U(\rho))^{-1}$, that is, $\rho\bar\xi=\overline{\rho^{-1}\xi}$. Similarly to the proof of the previous lemma, the main reason why expressions (2.8) and (2.9) are not equal is that the representations $U\mathbin{\bar\oplus}\bar V\mathbin{\bar\oplus}\bar U$ and $\overline{U\mathbin{\bar\oplus}V\mathbin{\bar\oplus}\bar U}$ are equivalent, but not equal. The map $\sigma(\xi\otimes\bar\eta\otimes\bar\zeta)=\overline{\zeta\otimes\eta\otimes\bar\xi}$ defines such an equivalence, and then

$$\bar R_{U\mathbin{\bar\oplus}V\mathbin{\bar\oplus}\bar U}=(\iota\otimes\iota\otimes\iota\otimes\sigma)(\iota\otimes\iota\otimes R_U\otimes\iota\otimes\iota)(\iota\otimes\bar R_V\otimes\iota)\bar R_U.$$

Since $(\bar R_U^*\otimes\iota)(\iota\otimes R_U)=\iota$, we get

$$\Big(\big((\iota\otimes T\otimes\iota)\bar R_U\big)^*\otimes\iota\Big)\bar R_{U\mathbin{\bar\oplus}V\mathbin{\bar\oplus}\bar U}=\sigma(\iota\otimes T^*\otimes\iota\otimes\iota)(\iota\otimes\bar R_V\otimes\iota)\bar R_U.$$

From this we see that upon applying π expressions (2.8) and (2.9) indeed become equal. ☐

Our next goal is to check the Yetter–Drinfeld condition (1.8).

Lemma 2.8. *For all $x\in\mathbb{C}[G]$ and $a\in\mathcal{B}$ we have*

$$\alpha(x\rhd a)=x_{(1)}a_{(1)}S(x_{(3)})\otimes(x_{(2)}\rhd a_{(2)}).$$

Proof. Let U and V be finite dimensional unitary representations of G. Choose orthonormal bases $\{\xi_i\}_i$ in H_U and $\{\eta_k\}_k$ in H_V, and let u_{ij} and v_{kl} be the matrix coefficients of U and V, respectively. In order to simplify the computations assume that the vectors ξ_i are eigenvectors of ρ, so $\rho\xi_i=\rho_i\xi_i$ for some positive number ρ_i. Then the matrix coefficients of $\bar U$ in the basis $\{\bar\xi_i\}_i$ are given by

$$\bar u_{ij}=\rho_i^{1/2}\rho_j^{-1/2}u_{ij}^*=\rho_i^{1/2}\rho_j^{-1/2}S(u_{ji}).\tag{2.10}$$

Consider elements $x=u_{i_0j_0}\in\mathbb{C}[G]$ and $a=\pi(\eta_{k_0}\otimes T)\in\mathcal{B}$ for some $T\in\mathcal{C}(\mathbb{1},V)$. Recalling definition (1.5) of the action α, we have

$$\alpha(a)=\sum_k v_{k_0k}\otimes\pi(\bar\eta_k\otimes T).$$

It follows that

$$
\begin{aligned}
x_{(1)}a_{(1)}S(x_{(3)})\otimes(x_{(2)}\rhd a_{(2)}) &= \sum_{i,j,k} u_{i_0i}v_{k_0k}S(u_{jj_0})\otimes(u_{ij}\rhd\pi(\bar\eta_k\otimes T))\\
&= \sum_{i,j,k}\rho_j^{-1/2}u_{i_0i}v_{k_0k}S(u_{jj_0})\otimes\pi\Big(\overline{(\xi_i\otimes\eta_k\otimes\bar\xi_j)}\otimes(\iota\otimes T\otimes\iota)\bar R_U\Big). \qquad (2.11)
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\alpha(x\rhd a) &= \rho_{j_0}^{-1/2}\alpha\big(\pi\big(\overline{(\xi_{i_0}\otimes\eta_{k_0}\otimes\bar\xi_{j_0})}\otimes(\iota\otimes T\otimes\iota)\bar R_U\big)\big)\\
&= \rho_{j_0}^{-1/2}\sum_{i,j,k}u_{i_0i}v_{k_0k}\bar u_{j_0j}\otimes\pi\big(\overline{(\xi_i\otimes\eta_k\otimes\bar\xi_j)}\otimes(\iota\otimes T\otimes\iota)\bar R_U\big). \qquad (2.12)
\end{aligned}
$$

Since $\rho_j^{-1/2}S(u_{jj_0})=\rho_{j_0}^{-1/2}\bar u_{j_0j}$, we see that expressions (2.11) and (2.12) are equal. ☐

It remains to check the braided commutativity condition (2.1).

Lemma 2.9. *For all and $a,b\in\mathcal{B}$ we have $ab=b_{(2)}(S^{-1}(b_{(1)})\rhd a)$.*

Proof. Let U, V, $\{\xi_i\}_i$, u_{ij}, $\bar u_{ij}$ be as in the proof of the previous lemma. Note that by swapping the roles of U and $\bar U$ in (2.10) we get

$$S^{-1}(u_{ij})=\rho_i^{-1/2}\rho_j^{1/2}\bar u_{ji}.$$

(Recall again that $\rho\bar\xi_i=\overline{\rho^{-1}\xi_i}$.) Using this, take $P\in\mathcal{C}(\mathbb{1},U)$, $\bar\eta\otimes T\in\bar H_V\otimes\mathcal{C}(\mathbb{1},V)$ and for $a=\pi(\bar\eta\otimes T)$ and $b=\pi(\bar\xi_i\otimes P)$ compute:

$$
\begin{aligned}
b_{(2)}(S^{-1}(b_{(1)})\rhd a) &= \sum_j\pi(\bar\xi_j\otimes P)(S^{-1}(u_{ij})\rhd\pi(\bar\eta\otimes T))\\
&= \sum_j\rho_i^{-1/2}\rho_j^{1/2}\pi(\bar\xi_j\otimes P)\pi((\bar{\bar\xi}_j\otimes\bar\xi_i)\,\tilde\rhd\,(\bar\eta\otimes T))\\
&= \sum_j\rho_i^{-1/2}\rho_j^{1/2}\pi(\bar\xi_j\otimes P)\pi\Big(\overline{(\bar\xi_j\otimes\eta\otimes\overline{\rho^{-1/2}\bar\xi_i})}\otimes(\iota\otimes T\otimes\iota)\bar R_{\bar U}\Big)\\
&= \sum_j\rho_j^{1/2}\pi\Big(\overline{(\xi_j\otimes\bar\xi_j\otimes\eta\otimes\bar{\bar\xi}_i)}\otimes(P\otimes(\iota\otimes T\otimes\iota)\bar R_{\bar U})\Big).
\end{aligned}
$$

Denote by w the map $\xi\mapsto\bar{\bar\xi}$ defining an equivalence between U and $\bar{\bar U}$. Then $\bar R_{\bar U}=(\iota\otimes w)R_U$. Hence the above expression equals

$$
\begin{aligned}
&\sum_j\rho_j^{1/2}\pi\Big(\overline{(\xi_j\otimes\bar\xi_j\otimes\eta\otimes\xi_i)}\otimes(P\otimes(\iota\otimes T\otimes\iota)R_U)\Big)\\
&\qquad\qquad = \pi\Big(\overline{(\bar R_U(1)\otimes\eta\otimes\xi_i)}\otimes(\iota\otimes\iota\otimes T\otimes\iota)(\iota\otimes R_U)P\Big).
\end{aligned}
$$

Since $\bar R_U$ is, up to a scalar factor, an isometric embedding of $\mathbb{1}$ into $U\mathbin{\bar\otimes}\bar U$, the last expression equals

$$\pi\Big(\overline{(\eta\otimes\xi_i)}\otimes(\bar R_U^*\otimes T\otimes\iota)(\iota\otimes R_U)P\Big)=\pi(\overline{(\eta\otimes\xi_i)}\otimes(T\otimes P)).$$

But this is exactly ab. ☐

This finishes the proof of Theorem 2.5.

2.4. **Functoriality.** Consider the category $\mathcal{YD}_{\mathrm{brc}}(G)$ of unital braided-commutative Yetter–Drinfeld G-C*-algebras. For every object B we have constructed isomorphic pairs $(\mathcal{D}_B,\mathcal{E}_B)$ and $(\mathcal{C}_B,\mathcal{F}_B)$. Using the extension of scalars functor discussed at the end of Section 1.3, either of this constructions extends to a functor, giving us two naturally isomorphic functors $\mathcal{T}\colon \mathcal{YD}_{\mathrm{brc}}(G)\to \mathcal{T}ens(\operatorname{Rep} G)$ and $\tilde{\mathcal{T}}\colon \mathcal{YD}_{\mathrm{brc}}(G)\to \mathcal{T}ens(\operatorname{Rep} G)$. Namely, giving a morphism $f\colon B_0\to B_1$ we have a functor $f_\#\colon \mathcal{D}_{B_0}\to\mathcal{D}_{B_1}$ which maps a G-equivariant finitely generated right Hilbert B_0-module X into $X\otimes_{B_0}B_1$. We define a tensor structure on this functor by using the isomorphisms

$$(X\otimes_{B_0}B_1)\otimes_{B_1}(Y\otimes_{B_0}B_1)\cong (X\otimes_{B_0}Y)\otimes_{B_0}B_1$$

such that $(x\otimes a)\otimes(y\otimes b)\mapsto x\otimes y_{(2)}\otimes(S^{-1}(y_{(1)})\rhd a)b$. That these maps are indeed well-defined and that they give us a tensor structure on $f_\#$, is not difficult to check using arguments similar to those in the proof of Lemma 2.3. The tensor functor $f_\#$ together with the obvious isomorphisms $\eta_U\colon (H_U\otimes B_0)\otimes_{B_0}B_1\to H_U\otimes B_1$ define a morphism $(\mathcal{D}_{B_0},\mathcal{E}_{B_0})\to(\mathcal{D}_{B_1},\mathcal{E}_{B_1})$.

If we consider the map $B\mapsto(\mathcal{C}_B,\mathcal{F}_B)$ instead of $B\mapsto(\mathcal{D}_B,\mathcal{E}_B)$, then the situation is even better: in this case the functor $f_\#\colon\mathcal{C}_{B_0}\to\mathcal{C}_{B_1}$ defined by a morphism $f\colon B_0\to B_1$ is a strict tensor functor, meaning that $f_\#(T\otimes S)=f_\#(T)\otimes f_\#(S)$ on morphisms. This follows immediately from equations (2.2) and (2.3) describing the tensor structure on the categories $\mathcal{C}_B$.

Let us now construct a functor $\mathcal{S}$ in the opposite direction. It is possible to define this functor on the whole category $\mathcal{T}ens(\operatorname{Rep} G)$, but up to a natural isomorphism, such a functor is completely determined by its restriction to the full subcategory $\mathcal{T}ens_{\mathrm{si}}(\operatorname{Rep} G)$ consisting of pairs $(\mathcal{C},\mathcal{E})$ such that $\mathcal{C}$ is a strict C*-tensor category containing $\operatorname{Rep} G$, $\mathcal{C}$ is generated by $\operatorname{Rep} G$ and $\mathcal{E}$ is the embedding functor. So we will consider only the latter category. Given two objects $(\mathcal{C}_0,\mathcal{E}_0)$ and $(\mathcal{C}_1,\mathcal{E}_1)$ in $\mathcal{T}ens_{\mathrm{si}}(\operatorname{Rep} G)$, consider the corresponding Yetter–Drinfeld C*-algebras B_0 and B_1, and take a morphism $[(\mathcal{F},\eta)]\colon(\mathcal{C}_0,\mathcal{E}_0)\to(\mathcal{C}_1,\mathcal{E}_1)$. As we discussed after the formulation of Theorem 2.1, we may assume that the restriction of $\mathcal{F}$ to $\operatorname{Rep} G\subset\mathcal{C}_0$ is the identity tensor functor and $\eta_U=\iota$. In this case it is obvious from the construction of the algebras $\mathcal{B}_i$ that the maps $\bar H_U\otimes\mathcal{C}_0(\mathbb{1},U)\ni\bar\xi\otimes T\mapsto\bar\xi\otimes\mathcal{F}(T)\in\bar H_U\otimes\mathcal{C}_1(\mathbb{1},U)$ define a unital $*$-homomorphism $\mathcal{B}_0\to\mathcal{B}_1$ that respects the $\mathbb{C}[G]$-comodule and $\mathbb{C}[G]$-module structures. It extends to a homomorphism f of C*-algebras by [DCY12, Proposition 4.5]. It is also clear by our definition of morphisms in the category of pairs $(\mathcal{C},\mathcal{E})$ that f depends only on the equivalence class of $(\mathcal{F},\iota)$. We thus get a functor $\mathcal{S}\colon\mathcal{T}ens_{\mathrm{si}}(\operatorname{Rep} G)\to\mathcal{YD}_{\mathrm{brc}}(G)$.

Furthermore, it is clear from the construction that the morphism $f\colon B_0\to B_1$ defined by a morphism $[(\mathcal{F},\iota)]\colon(\mathcal{C}_0,\mathcal{E}_0)\to(\mathcal{C}_1,\mathcal{E}_1)$ is injective if and only if the maps $\mathcal{C}_0(\mathbb{1},U_s)\to\mathcal{C}_1(\mathbb{1},U_s)$, $T\mapsto\mathcal{F}(T)$, are injective for all s, and f is surjective if and only if these maps are surjective. Using Frobenius reciprocity it is easy to see that the maps $\mathcal{C}_0(\mathbb{1},U_s)\to\mathcal{C}_1(\mathbb{1},U_s)$ are injective, resp. surjective, for all s if and only if the maps $\mathcal{C}_0(U,U_V)\to\mathcal{C}_1(U,V)$ are injective, resp. surjective, for all objects U and V in $\operatorname{Rep} G\subset\mathcal{C}_0,\mathcal{C}_1$. Since the categories $\mathcal{C}_i$ are generated by $\operatorname{Rep} G$, it follows that f is injective if and only if $\mathcal{F}$ is faithful, and f is surjective if and only if $\mathcal{F}$ is full.

It is also worth noting that since a morphism $T\in\mathcal{C}(\mathbb{1},U)$ is zero if and only if $T^*T=0$ in $\operatorname{End}_{\mathcal{C}}(\mathbb{1})$, we have, given a morphism $[(\mathcal{F},\eta)]\colon(\mathcal{C}_0,\mathcal{E}_0)\to(\mathcal{C}_1,\mathcal{E}_1)$, that $\mathcal{F}$ is faithful if and only if the homomorphism $\operatorname{End}_{\mathcal{C}_0}(\mathbb{1})\to\operatorname{End}_{\mathcal{C}_1}(\mathbb{1})$ is injective. On the C*-algebra level this corresponds to the simple property that a morphism $B_0\to B_1$ of G-C*-algebras for a reduced compact quantum group G is injective if and only if its restriction to the fixed point algebra B_0^G is injective.

2.5. **Equivalence of categories.** To finish the proof of Theorem 2.1 it remains to show that the functors $\mathcal{T}\colon\mathcal{YD}_{\mathrm{brc}}(G)\to\mathcal{T}ens(\operatorname{Rep} G)$, or $\tilde{\mathcal{T}}\colon\mathcal{YD}_{\mathrm{brc}}(G)\to\mathcal{T}ens(\operatorname{Rep} G)$, and $\mathcal{S}\colon\mathcal{T}ens(\operatorname{Rep} G)\to\mathcal{YD}_{\mathrm{brc}}(G)$ are inverse to each other up to an isomorphism.

Let us start with a strict C*-tensor category $\mathcal{C}$ containing $\operatorname{Rep} G$ and construct a braided-commutative Yetter–Drinfeld C*-algebra B as described in Section 2.3. By Theorem 1.1 the $(\operatorname{Rep} G)$-module C*-categories $\mathcal{C}$ and $\mathcal{C}_B$ are equivalent. We will use the concrete form of this equivalence explained in Section 1.4. Recall that $\mathcal{C}_B$ is the idempotent completion of the category $\operatorname{Rep} G$ with morphisms

$\mathcal{C}_B(U,V)\subset B(H_U,H_V)\otimes\mathcal{B}$, and we have a unitary equivalence $\mathcal{F}\colon\mathcal{C}\to\mathcal{C}_B$ such that $\mathcal{F}(U)=U$ for $U\in\operatorname{Rep}G$, while the action of $\mathcal{F}$ on morphisms is given by (1.6), so

$$\mathcal{C}(U,V)\ni T\mapsto\sum_{i,j}\theta_{\zeta_j,\xi_i}\otimes\pi\big(\overline{(\zeta_j\otimes\overline{\rho^{-1/2}\xi_i})}\otimes(T\otimes\iota)\bar R_U\big),$$

where $\{\xi_i\}_i$ and $\{\zeta_j\}_j$ are orthonormal bases in H_U and H_V, respectively. We claim that $\mathcal{F}$ is a strict tensor functor on the full subcategory of $\mathcal{C}$ consisting of objects $U\in\operatorname{Rep}G$. This tensor functor extends then to a unitary tensor functor on the whole category $\mathcal{C}$. Thus, we have to show that $\mathcal{F}(S\otimes T)=\mathcal{F}(S)\otimes\mathcal{F}(T)$ on morphisms in $\mathcal{C}$. Since $\mathcal{F}$ is an equivalence of right $(\operatorname{Rep}G)$-module categories, we already know that this is true for morphisms S in $\mathcal{C}$ and morphisms T in $\operatorname{Rep}G$; this is also not difficult to check directly, since the formula for $\mathcal{F}(S)\otimes\iota$ does not involve the Yetter–Drinfeld structure, see (2.3). Therefore it remains to check that $\mathcal{F}(\iota\otimes T)=\iota\otimes\mathcal{F}(T)$ for morphisms T in $\mathcal{C}$.

Take $T\in\mathcal{C}(V,W)$. Let $\{\eta_k\}_k$ be an orthonormal basis in H_W. We then have

$$\mathcal{F}(\iota_U\otimes T)=\sum_{i,j,k,l}\theta_{\xi_i\otimes\eta_k,\xi_j\otimes\zeta_l}\otimes\pi\Big(\overline{(\xi_i\otimes\eta_k\otimes\overline{(\rho^{-1/2}\xi_j\otimes\rho^{-1/2}\zeta_l)})}\otimes((\iota\otimes T)\otimes\iota_{\overline{U\mathbin{\bar\otimes}W}})\bar R_{U\mathbin{\bar\otimes}W}\Big).$$

Similarly to the proof of Lemma 2.6, using that $\overline{U\mathbin{\bar\otimes}W}$ is equivalent to $\bar W\mathbin{\bar\otimes}\bar U$ and that modulo this equivalence $\bar R_{U\mathbin{\bar\otimes}W}$ coincides with $(\iota\otimes\bar R_W\otimes\iota)\bar R_U$, we see that the above expression equals

$$\sum_{i,j,k,l}\theta_{\xi_i\otimes\eta_k,\xi_j\otimes\zeta_l}\otimes\pi\Big(\overline{(\xi_i\otimes\eta_k\otimes\overline{\rho^{-1/2}\zeta_l}\otimes\overline{\rho^{-1/2}\xi_j})}\otimes(\iota\otimes(T\otimes\iota)\bar R_W\otimes\iota)\bar R_U\Big).$$

The operators θ_{ξ_i,ξ_j} are the matrix units m_{ij} in $B(H_U)$. Recalling the definition of $\triangleright$ we can therefore write the above expression as

$$\sum_{i,j,k,l}m_{ij}\otimes\theta_{\eta_k,\zeta_l}\otimes\Big(u_{ij}\triangleright\pi\Big(\overline{(\eta_k\otimes\overline{\rho^{-1/2}\zeta_l})}\otimes(T\otimes\iota)\bar R_W\Big)\Big),$$

where u_{ij} are the matrix units of U. According to (2.2) this is exactly the formula for $\iota_U\otimes\mathcal{F}(T)$.

Conversely, consider a unital braided-commutative Yetter–Drinfeld C*-algebra B and the corresponding pair $(\mathcal{C}_B,\mathcal{F}_B)$. Let $B_{\mathcal{C}}$ be the Yetter–Drinfeld C*-algebra constructed from this pair. By Theorem 1.1 we know that there exists an isomorphism $\lambda\colon B_{\mathcal{C}}\to B$ intertwining the actions of G. So all we have to do is to check that λ is also a $\mathbb{C}[G]$-module map. The isomorphism λ is defined by

$$\lambda(\pi(\bar\zeta\otimes T))=(\bar\zeta\otimes\iota)(T)\tag{2.13}$$

for $\zeta\in H_V$ and $T\in\mathcal{C}_B(\mathbb{1},V)\subset B(\mathbb{C},H_V)\otimes\mathcal{B}=H_V\otimes\mathcal{B}$, see the proof of [Nes13, Theorem 2.3]. As above, fix finite dimensional unitary representations U and V of G and orthonormal bases $\{\xi_i\}_i$ and $\{\zeta_k\}_k$ in H_U and H_V, and let u_{ij} be the matrix coefficients of U. Take

$$T=\sum_k\zeta_k\otimes b_k\in\mathcal{C}_B(\mathbb{1},V)\subset H_V\otimes\mathcal{B}.$$

Then $\lambda(\pi(\bar\zeta_{k_0}\otimes T))=b_{k_0}$, and we want to check that

$$\lambda(u_{i_0j_0}\triangleright\pi(\bar\zeta_{k_0}\otimes T))=u_{i_0j_0}\triangleright b_{k_0}.$$

By definition we have

$$u_{i_0j_0}\triangleright\pi(\bar\zeta_{k_0}\otimes T)=\pi((\bar\xi_{i_0}\otimes\xi_{j_0})\mathbin{\tilde\triangleright}(\bar\zeta_{k_0}\otimes T))=\pi\Big(\overline{(\xi_{i_0}\otimes\zeta_{k_0}\otimes\overline{\rho^{-1/2}\xi_{j_0}})}\otimes(\iota\otimes T\otimes\iota)\bar R_U\Big).$$

In order to compute the image of this element under λ, we need an explicit formula for $(\iota\otimes T\otimes\iota)\bar R_U\in\mathcal{C}_B(\mathbb{1},U\otimes V\otimes\bar U)\subset H_{U\mathbin{\bar\otimes}V\mathbin{\bar\otimes}\bar U}\otimes\mathcal{B}$. By (2.2) and (2.3), the element

$$\iota_U\otimes T\otimes\iota_U\in\mathcal{C}_B(U\mathbin{\bar\otimes}\bar U,U\mathbin{\bar\otimes}V\mathbin{\bar\otimes}\bar U)\subset B(H_U)\otimes H_V\otimes B(\bar H_U)\otimes\mathcal{B}$$

equals $\sum_{i,j,k} m_{ij} \otimes \zeta_k \otimes 1 \otimes (u_{ij} \rhd b_k)$. It follows that

$$(\iota \otimes T \otimes \iota)\bar{R}_U = \sum_{i,j,k} (\xi_i \otimes \zeta_k \otimes \overline{\rho^{1/2}\xi_j}) \otimes (u_{ij} \rhd b_k).$$

Therefore

$$u_{i_0j_0} \rhd \pi(\bar{\zeta}_{k_0} \otimes T) = \pi \left(\overline{(\xi_{i_0} \otimes \zeta_{k_0} \otimes \overline{\rho^{-1/2}\xi_{j_0}})} \otimes \left(\sum_{i,j,k} (\xi_i \otimes \zeta_k \otimes \overline{\rho^{1/2}\xi_j}) \otimes (u_{ij} \rhd b_k) \right) \right).$$

Applying λ we get the required equality $\lambda(u_{i_0j_0} \rhd \pi(\bar{\zeta}_{k_0} \otimes T)) = u_{i_0j_0} \rhd b_{k_0}$. Since the algebra $\mathcal{B}$ is spanned by such elements b_{k_0} for different V, it follows that λ is a $\mathbb{C}[G]$-module map. This completes the proof of Theorem 2.1.

3. Examples and applications

Let us present some known examples of Yetter–Drinfeld algebras from the point of view of the equivalence of categories established in Theorem 2.1.

3.1. **Quotient type coideals.** By a closed quantum subgroup of G we mean a compact quantum group H together with a surjective homomorphism $\pi\colon \mathbb{C}[G] \to \mathbb{C}[H]$ of Hopf $*$-algebras. This is consistent with the definition used in the theory of locally compact quantum groups, but is weaker than e.g. the definition used in [Tom07]. Assuming that both G and H are reduced, the homomorphism π does not always extend to a homomorphism $C(G) \to C(H)$. Nevertheless the algebra $C(G/H)$ of continuous functions on the quantum homogeneous space G/H is always well-defined: it is the norm closure of

$$\mathbb{C}[G/H] = \{x \in \mathbb{C}[G] \mid (\iota \otimes \pi)\Delta(x) = x \otimes 1\}.$$

The algebra $C(G/H)$ is a braided-commutative Yetter–Drinfeld G-C*-algebra, with the left action of G defined by the restriction of Δ to $C(G/H)$, and the action of $\hat{G}$ defined by the restriction of the right adjoint action on $C(G)$ to $C(G/H)$. In other words, the $\mathbb{C}[G]$-module structure on $\mathbb{C}[G/H]$ is defined by

$$x \rhd a = x_{(1)} a S(x_{(2)}).$$

It is known and is easy to see that the G-C*-algebra $C(G/H)$ corresponds to the category $\operatorname{Rep} H$ with the distinguished object $\mathbb{1}$, viewed as a $(\operatorname{Rep} G)$-module category via the forgetful tensor functor $\operatorname{Rep} G \to \operatorname{Rep} H$. Namely, in the notation of Section 2.3, by identifying $\operatorname{Hom}_H(\mathbb{C}, H_U)$ with a subspace of H_U, we can view the algebra $\tilde{\mathcal{B}}$ corresponding to the functor $\operatorname{Rep} G \to \operatorname{Rep} H$ as a subalgebra of $\widetilde{\mathbb{C}[G]} = \bigoplus_U (\bar{H}_U \otimes H_U)$. Then the map $\pi_G\colon \widetilde{\mathbb{C}[G]} \to \mathbb{C}[G]$ induces a G-equivariant isomorphism $\mathcal{B} \cong \mathbb{C}[G/H]$.

We claim that the $\mathbb{C}[G]$-module structure on $\mathbb{C}[G/H]$ defined by the tensor functor $\operatorname{Rep} G \to \operatorname{Rep} H$ is exactly the adjoint action. In order to show this it is enough to consider the case of trivial H, since it corresponds to the inclusion $\operatorname{Rep} G \hookrightarrow \mathrm{Hilb}_f$, while the general case corresponds to the intermediate inclusion $\operatorname{Rep} G \hookrightarrow \operatorname{Rep} H$. As in the proof of Lemma 2.8, fix unitary representations U and V and orthonormal bases $\{\xi_i\}_i$ in H_U and $\{\zeta_k\}_k$ in H_V such that $\rho\xi_i = \rho_i\xi_i$. Denote matrix coefficients of U, V and $\bar{U}$ by u_{ij}, v_{kl}, $\bar{u}_{ij}$. Recall that by (2.10) we have $\bar{u}_{ij} = \rho_i^{1/2}\rho_j^{-1/2}S(u_{ji})$. Then

$$(\bar{\xi}_i \otimes \xi_j) \mathbin{\tilde{\rhd}} (\bar{\zeta}_k \otimes \zeta_l) = \sum_m \overline{(\xi_i \otimes \zeta_k \otimes \rho_j^{-1/2}\bar{\xi}_j)} \otimes (\rho_m^{1/2}\xi_m \otimes \zeta_l \otimes \bar{\xi}_m)$$

It follows that

$$u_{ij} \rhd v_{kl} = \sum_m \rho_j^{-1/2}\rho_m^{1/2} u_{im} v_{kl} \bar{u}_{jm} = \sum_m u_{im} v_{kl} S(u_{mj}),$$

which is exactly the formula for the adjoint action.

As a simple application of Theorem 2.1 we now get the following result, which under slightly stronger assumptions has been already established in [Tom07] and [Sal11].

Theorem 3.1. *Let G be a reduced compact quantum group. Then any unital left G- and right $\hat{G}$-invariant C*-subalgebra of $C(G)$ has the form $C(G/H)$ for a closed quantum subgroup H of G.*

Proof. Let $B \subset C(G)$ be a unital left G- and right $\hat{G}$-invariant C*-algebra. Consider the corresponding pair $(\mathcal{D}_B, \mathcal{E}_B) = \mathcal{T}(B) \in \mathcal{Tens}(\operatorname{Rep} G)$. By the ergodicity of the G-action on B, the unit object in $\mathcal{D}_B$ is simple. Since $\mathcal{D}_B$ is generated by the image of $\operatorname{Rep} G$ and the category $\operatorname{Rep} G$ is rigid, the C*-tensor category $\mathcal{D}_B$ is rigid as well. The inclusion $B \hookrightarrow C(G)$ defines a morphism

$$(\mathcal{D}_B, \mathcal{E}_B) \to (\mathcal{D}_{C(G)}, \mathcal{E}_{C(G)}) \cong (\mathrm{Hilb}_f, \mathcal{F}),$$

where $\mathcal{F}\colon \operatorname{Rep} G \to \mathrm{Hilb}_f$ is the forgetful fiber functor. This means that $\mathcal{D}_B$ has a unitary fiber functor $\mathcal{E}\colon \mathcal{D}_B \to \mathrm{Hilb}_f$ such that $\mathcal{F} = \mathcal{E}\mathcal{E}_B$. By Woronowicz's Tannaka–Krein duality theorem, the pair $(\mathcal{D}_B, \mathcal{E})$ defines a compact quantum group H. Then the functor $\mathcal{E}_B$ defines a functor $\operatorname{Rep} G \to \operatorname{Rep} H$ such that the forgetful fiber functor $\mathcal{F}$ on $\operatorname{Rep} G$ factors through that on $\operatorname{Rep} H$. It follows that H can be regarded as a quantum subgroup of G.

Since by the discussion preceding the theorem the factorization of the fiber functor $\mathcal{F}\colon \operatorname{Rep} G \to \mathrm{Hilb}_f$ through $\operatorname{Rep} G \to \operatorname{Rep} H$ corresponds to the inclusion $C(G/H) \hookrightarrow C(G)$, we have therefore shown that there exists a closed quantum subgroup $H \subset G$ and an isomorphism $(\mathcal{D}_B, \mathcal{E}_B) \cong (\mathcal{D}_{C(G/H)}, \mathcal{E}_{C(G/H)})$ such that the morphism $(\mathcal{D}_B, \mathcal{E}_B) \to (\mathcal{D}_{C(G)}, \mathcal{E}_{C(G)})$ under this isomorphism becomes the morphism $(\mathcal{D}_{C(G/H)}, \mathcal{E}_{C(G/H)}) \to (\mathcal{D}_{C(G)}, \mathcal{E}_{C(G)})$ defined by the inclusion $C(G/H) \hookrightarrow C(G)$. Since $\mathcal{T}$ is an equivalence of categories, this implies that $B = C(G/H)$. ☐

3.2. **Linking algebras.** The considerations of the previous subsection can be generalized to the linking algebras defined by monoidal equivalences. Let $\mathcal{F}$ be the forgetful functor $\operatorname{Rep} G \to \mathrm{Hilb}_f$, and $\mathcal{F}'\colon \operatorname{Rep} G \to \mathrm{Hilb}_f, U \mapsto H'_U$, be another unitary fiber functor. We denote the compact quantum group corresponding to $\mathcal{F}'$ by G'. Then it is not difficult to check that the linking algebra between G and G', introduced in the C*-algebraic setting in [BDRV06] and in the purely algebraic setting earlier in [Sch96], is exactly the C*-algebra $B(\mathcal{F}, \mathcal{F}')$ corresponding to the pair $(\mathrm{Hilb}_f, \mathcal{F}')$ by our construction. In addition to the left action of G it carries also a commuting right action of G', which is easy to see using that the regular subalgebra of $B(\mathcal{F}, \mathcal{F}')$ is $\mathcal{B}(\mathcal{F}, \mathcal{F}') = \bigoplus_s (\bar{H}_s \otimes H'_s)$.

The G-C*-algebras B of the form $B(\mathcal{F}, \mathcal{F}')$ can be abstractly characterized by saying that the regular subalgebra $\mathcal{B} \subset B$ is a Hopf–Galois extension of $\mathbb{C}$ over $\mathbb{C}[G]$, which is a well-studied notion in the algebraic approach to quantum groups, see [Bic10]. By definition, this means that the Galois map

$$\Gamma\colon \mathcal{B} \otimes \mathcal{B} \to \mathbb{C}[G] \otimes \mathcal{B}, \quad x \otimes y \mapsto x_{(1)} \otimes x_{(2)} y,$$

is bijective. Analogously to the case of $\mathbb{C}[G]$ (which is the linking algebra $\mathcal{B}(\mathcal{F}, \mathcal{F})$), there is a standard structure of a braided-commutative Yetter–Drinfeld algebra over G on $\mathcal{B}$. Namely, the action of $\mathbb{C}[G]$ on $\mathcal{B}$ is the so called Miyashita–Ulbrich action, defined by

$$x \triangleright a = \Gamma^{-1}(x \otimes 1)_1 a \Gamma^{-1}(x \otimes 1)_2.$$

We claim that this action is the same as the one induced by the pair $(\mathrm{Hilb}_f, \mathcal{F}')$ by our construction. In order to show this, replace $(\mathrm{Hilb}_f, \mathcal{F}')$ by an isomorphic pair consisting of a strict C*-tensor category $\mathcal{C}$ containing $\operatorname{Rep} G$ and the embedding functor $\operatorname{Rep} G \to \mathcal{C}$, as explained in Section 2.1. What is now special about $\mathcal{C}$, is that the unit object is simple and the maps $\mathcal{C}(\mathbb{1}, U) \otimes \mathcal{C}(\mathbb{1}, V) \to \mathcal{C}(\mathbb{1}, U \oplus V)$ are bijective. As in the previous subsection, fix unitary representations U and V of G and an orthonormal basis $\{\xi_i\}_i$ in H_U such that $\rho\xi_i = \rho_i \xi_i$. We can find elements $T_l \in \mathcal{C}(\mathbb{1}, U)$ and $S_l \in \mathcal{C}(\mathbb{1}, \bar{U})$ such that

$$\bar{R}_U = \sum_l T_l \otimes S_l \ \text{ in } \ \mathcal{C}.$$

Then, for any $P \in \mathcal{C}(\mathbb{1}, V)$ and $\zeta \in H_V$, we have

$$(\bar{\xi}_i \otimes \xi_j)\tilde{\triangleright}(\bar{\zeta} \otimes P) = \rho_j^{-1/2}\overline{(\xi_i \otimes \zeta \otimes \bar{\xi}_j)} \otimes (\iota \otimes P \otimes \iota)\bar{R}_U = \rho_j^{-1/2}\sum_l (\bar{\xi}_i \otimes T_l) \cdot (\bar{\zeta} \otimes P) \cdot (\bar{\bar{\xi}}_j \otimes S_l).$$

Therefore in order to prove the claim it suffices to check that

$$\sum_l \Gamma(\pi(\bar{\xi}_i \otimes T_l) \otimes \pi(\bar{\bar{\xi}}_j \otimes S_l)) = \rho_j^{1/2} u_{ij} \otimes 1.$$

But this is true by the following simple computation:

$$\begin{aligned}\sum_l \Gamma(\pi(\bar{\xi}_i \otimes T_l) \otimes \pi(\bar{\bar{\xi}}_j \otimes S_l)) &= \sum_{k,l} u_{ik} \otimes \pi(\bar{\xi}_k \otimes T_l)\pi(\bar{\bar{\xi}}_j \otimes S_l)\\ &= \sum_k u_{ik} \otimes \pi(\overline{(\xi_k \otimes \bar{\xi}_j)} \otimes \bar{R}_U) = \sum_k u_{ik} \otimes \overline{\bar{R}_U^*(\xi_k \otimes \bar{\xi}_j)} = \rho_j^{1/2} u_{ij} \otimes 1.\end{aligned}$$

If H' is a closed quantum subgroup of G', then, similarly to the C*-algebras $C(G/H) \subset C(G)$, we may define C*-algebras $B(\mathcal{F}, \mathcal{F}')^{H'} \subset B(\mathcal{F}, \mathcal{F}')$. Then by a completely analogous argument to that in the proof of Theorem 3.1 we obtain the following result.

Theorem 3.2. *Let G be a reduced compact quantum group and $B = B(\mathcal{F}, \mathcal{F}')$ be the linking C*-algebra defined by the forgetful fiber functor $\mathcal{F}\colon \operatorname{Rep} G \to \mathrm{Hilb}_f$ and a unitary fiber functor $\mathcal{F}'\colon \operatorname{Rep} G \to \mathrm{Hilb}_f$. Then any unital left G- and right $\hat{G}$-invariant C*-subalgebra of $B(\mathcal{F}, \mathcal{F}')$ has the form $B(\mathcal{F}, \mathcal{F}')^{H'}$ for a closed quantum subgroup $H' \subset G'$.*

Let us finally say a few words about the differences between our approach to reconstructing the tensor functor $\mathcal{F}'$ from $B(\mathcal{F}, \mathcal{F}')$ and that in [BDRV06]. Assume B is a unital G-C*-algebra such that $B^G = \mathbb{C}1$. We can define a weak unitary tensor functor $\mathcal{E}\colon \operatorname{Rep} G \to \mathrm{Hilb}_f$, called the spectral functor, by letting

$$\mathcal{E}(U) = \mathcal{D}_B(B, B \times U) \ \text{ and } \ \mathcal{E}_{2;U,V}\colon \mathcal{E}(U) \otimes \mathcal{E}(V) \to \mathcal{E}(U \mathbin{\text{⫪}} V), \quad T \otimes S \mapsto (T \otimes \iota)S.$$

The scalar product on $\mathcal{E}(U)$ is defined by $S^*T = (T, S)1$, which makes sense by the ergodicity assumption. In general the maps $\mathcal{E}_{2;U,V}$ are not unitary but only isometric. When they are unitary, so that $(\mathcal{E}, \mathcal{E}_2)$ becomes a unitary tensor functor, then the ergodic action of G on B is said to be of full quantum multiplicity. In this case, if G is reduced, then $B \cong B(\mathcal{F}, \mathcal{E})$ as G-C*-algebras [BDRV06] (see also [Nes13], where a more general result is proved in the notation consistent with the present work). In particular, another way of formulating the Hopf–Galois condition is to say that the action of G is of full quantum multiplicity.

The spectral functor is constructed in a simple way using only the action of G, while in order to construct a tensor functor in our approach we also have to use the Miyashita–Ulbrich action. The reason why the two constructions give isomorphic functors is basically the following observation. Given a unitary fiber functor $\mathcal{F}'\colon \operatorname{Rep} G \to \mathrm{Hilb}_f$, we can define a new unitary fiber functor $\mathcal{E}\colon \operatorname{Rep} G \to \mathrm{Hilb}_f$ by letting

$$\mathcal{E}(U) = \operatorname{Hom}(\mathbb{C}, \mathcal{F}'(U)) \ \text{ and } \ \mathcal{E}_{2;U,V}\colon \mathcal{E}(U) \otimes \mathcal{E}(V) \to \mathcal{E}(U \mathbin{\text{⫪}} V), \quad T \otimes S \mapsto \mathcal{F}'_{2;U,V}(T \otimes \iota)S.$$

But it is clear that under the identification of $\operatorname{Hom}(\mathbb{C}, H)$ with H, the tensor functor $\mathcal{E}$ becomes identical to $\mathcal{F}'$.

3.3. **Discrete dual.** Consider the algebra $\ell^\infty(\hat{G}) \subset \mathcal{U}(G) = \mathbb{C}[G]^*$ of bounded functions on $\hat{G}$. We have a left adjoint action α of G on $\ell^\infty(G) \cong \ell^\infty\text{-}\bigoplus_s B(H_s)$ defined by

$$B(H_s) \ni T \mapsto (U_s)^*_{21}(1 \otimes T)(U_s)_{21}. \tag{3.1}$$

This action is continuous only in the von Neumann algebraic sense, so in order to stay within the class of G-C*-algebras, instead of $\ell^\infty(\hat{G})$ we should rather consider the norm closure $B(\hat{G})$ of the regular subalgebra $\ell^\infty_{\mathrm{alg}}(\hat{G}) \subset \ell^\infty(\hat{G})$. Then the right action $\hat{\Delta}$ of $\hat{G}$ on $\ell^\infty(\hat{G})$ makes this algebra into a unital braided-commutative Yetter–Drinfeld C*-algebra. In other words, the left $\mathbb{C}[G]$-module structure on $\ell^\infty_{\mathrm{alg}}(\hat{G})$ is defined by

$$x \rhd a = (\iota \otimes x)\hat{\Delta}(a). \tag{3.2}$$

In the subsequent computations we will use the notation $\hat{\Delta}(a) = a^{(1)} \otimes a^{(2)}$. Literally this does not make sense, but the expressions like $x(a^{(2)})a^{(1)}$ for $x \in \mathbb{C}[G]$ are still meaningful.

We want to describe the corresponding C*-tensor category $\mathcal{C} = \mathcal{C}_{B(\hat{G})}$ and the unitary tensor functor $\mathcal{F} = \mathcal{F}_{B(\hat{G})}\colon \operatorname{Rep} G \to \mathcal{C}$. By definition, the category $\mathcal{C}$ is the idempotent completion of the category with the same objects as in $\operatorname{Rep} G$, but with the morphism sets $\mathcal{C}(U, V) \subset B(H_U, H_V) \otimes \ell^\infty_{\mathrm{alg}}(\hat{G})$. In fact, for the reasons that will become apparent in a moment, it is more convenient to consider $\mathcal{C}(U, V)$ as a subset of $\ell^\infty_{\mathrm{alg}}(\hat{G}) \otimes B(H_U, H_V)$. Thus, we define $\mathcal{C}(U, V)$ as the set of elements $T \in \ell^\infty_{\mathrm{alg}}(\hat{G}) \otimes B(H_U, H_V)$ such that

$$V^*_{31}(\alpha \otimes \iota)(T)U_{31} = 1 \otimes T.$$

From the definition of the adjoint action α we see that an element $T \in \ell^\infty_{\mathrm{alg}}(\hat{G}) \otimes B(H_U, H_V)$ lies in $\mathcal{C}(U, V)$ if and only if it defines a G-equivariant map $H_s \otimes H_U \to H_s \otimes H_V$ for all s. It follows that $\mathcal{C}(U, V)$ can be identified with the space $\mathrm{Nat}_b(\iota \otimes U, \iota \otimes V)$ of bounded natural transformations between the functors $\iota \otimes U$ and $\iota \otimes V$ on $\operatorname{Rep} G$.

Using this picture we get a natural tensor structure on $\mathcal{C}$: the tensor product of objects is defined as in $\operatorname{Rep} G$, while the tensor product of natural transformations $\nu\colon \iota \otimes U \to \iota \otimes V$ and $\eta\colon \iota \otimes W \to \iota \otimes Z$ is defined by

$$\nu \otimes \eta = (\nu \otimes \iota)\eta.$$

Here we consider η as a natural transformation $\iota \otimes U \mathbin{⦶} W \to \iota \otimes U \mathbin{⦶} Z$ and $\nu \otimes \iota$ as a natural transformation $\iota \otimes U \mathbin{⦶} Z \to \iota \otimes V \mathbin{⦶} Z$. Note that by naturality of η we have $(\nu_X \otimes \iota)\eta_{X ⦶ U} = \eta_{X ⦶ V}(\nu_X \otimes \iota)$, which by slightly abusing notation can be written as $(\nu \otimes \iota)\eta = \eta(\nu \otimes \iota)$, so that we can also write

$$\nu \otimes \eta = \eta(\nu \otimes \iota).$$

Explicitly, if $\nu = \sum_i a_i \otimes T_i \in \ell^\infty_{\mathrm{alg}}(\hat{G}) \otimes B(H_U, H_V)$ and $\eta = \sum_j b_j \otimes S_j \in \ell^\infty_{\mathrm{alg}}(\hat{G}) \otimes B(H_W, H_Z)$, then

$$\nu \otimes \eta = \sum_{i,j} a_i b_j^{(1)} \otimes (T_i \pi_U(b_j^{(2)}) \otimes S_j) \in \ell^\infty_{\mathrm{alg}}(\hat{G}) \otimes B(H_U \otimes H_W, H_V \otimes H_Z). \tag{3.3}$$

The functor $\mathcal{F}\colon \operatorname{Rep} G \to \mathcal{C}$ is now the strict tensor functor such that $\mathcal{F}(U) = U$ on objects and $\mathcal{F}(T) = 1 \otimes T$ on morphisms.

It remains to show that the tensor structure on $\mathcal{C}$ defines the same $\mathbb{C}[G]$-module structure on $\ell^\infty_{\mathrm{alg}}(\hat{G})$ as (3.2). Consider an element $\bar{\xi} \otimes \eta \in \bar{H}_V \otimes \mathcal{C}(\mathbb{1}, V)$. Identifying $B(\mathbb{C}, H_V)$ with H_V we can write $\eta = \sum_k a_k \otimes \zeta_k$ for some $a_k \in \ell^\infty_{\mathrm{alg}}(\hat{G})$ and $\zeta_k \in H_V$. Then, identifying the algebra $\mathcal{B}$ constructed from the pair $(\mathcal{C}, \mathcal{F})$ with $\ell^\infty_{\mathrm{alg}}(\hat{G})$, the element $a = \pi(\bar{\xi} \otimes \eta) \in \mathcal{B} = \ell^\infty_{\mathrm{alg}}(\hat{G})$ equals $\sum_k (\zeta_k, \zeta) a_k$, see equation (2.13). Choose a unitary representations U and an orthonormal basis $\{\xi_i\}_i$ in H_U consisting of eigenvectors of ρ, so $\rho\xi_i = \rho_i \xi_i$. Then by (3.3) and definition (2.4) of the map $\tilde{\rhd}$ we get

$$(\bar{\xi}_i \otimes \xi_j) \mathbin{\tilde{\rhd}} \left(\bar{\zeta} \otimes \left(\sum_k a_k \otimes \zeta_k\right)\right) = \overline{(\xi_i \otimes \zeta \otimes \rho_j^{-1/2} \bar{\xi}_j)} \otimes \left(\sum_{k,l} a_k^{(1)} \otimes \left(\rho_l^{1/2} a_k^{(2)} \xi_l \otimes \zeta_k \otimes \bar{\xi}_l\right)\right),$$

whence

$$u_{ij} \rhd a = \sum_k (a_k^{(2)} \xi_j, \xi_i)(\zeta_k, \zeta) a_k^{(1)} = (a^{(2)} \xi_j, \xi_i) a^{(1)}.$$

But this is exactly how the action (3.2) is defined.

4. Poisson boundaries

In this section we prove our main application of the correspondence between braided-commutative Yetter–Drinfeld algebras and tensor categories by applying it to a particular class of algebras.

4.1. **Poisson boundaries of discrete quantum groups.** Let us briefly overview the theory of noncommutative Poisson boundaries developed by Izumi [Izu02]. We follow the conventions of [INT06].

In the following, for a representation U of G, we use the notation $d(U)$ for the quantum dimension of U instead of the more standard notation $\dim_q U$. Consider the state ϕ_U on $B(H_U)$ defined by

$$\phi_U(T) = \frac{\mathrm{Tr}(T\pi_U(\rho)^{-1})}{d(U)} \quad \text{for } \ T \in B(H). \tag{4.1}$$

If U is irreducible, it can be characterized as the unique state satisfying

$$(\iota \otimes \phi_U)(U_{21}^*(1 \otimes T)U_{21}) = \phi_U(T).$$

For our fixed representatives of irreducible representations $\{U_s\}_s$ of G, we write ϕ_s instead of ϕ_{U_s}.

When ϕ is a normal state on $\ell^\infty(\hat{G})$, we define a completely positive map P_ϕ on $\ell^\infty(\hat{G})$

$$P_\phi(a) = (\phi \otimes \iota)\hat{\Delta}(a).$$

If μ is a probability measure on the set $\mathrm{Irr}(G)$ of isomorphism classes of irreducible representations of G, we define a normal unital completely positive map P_μ on $\ell^\infty(\hat{G})$ by $P_\mu = \sum_s \mu(s)P_{\phi_s}$. The space

$$H^\infty(\hat{G}, \mu) = \{x \in \ell^\infty(\hat{G}) \mid x = P_\mu(x)\}$$

of P_μ-harmonic elements is called the *noncommutative Poisson boundary* of $\hat{G}$ with respect to μ. This is an operator subspace of $\ell^\infty(\hat{G})$ closed under the left adjoint action α of G defined by (3.1) and the right action $\hat{\Delta}$ of $\hat{G}$ on itself by translations. It has a new product structure

$$x \cdot y = \lim_{n\to\infty} P_\mu^k(xy),$$

where the limit is taken in the strong* operator topology. With this product $H^\infty(\hat{G}, \mu)$ becomes a von Neumann algebra (with the original operator space structure), and the actions of G and $\hat{G}$ on $\ell^\infty(\hat{G})$ define continuous, in the von Neumann algebraic sense, actions on $H^\infty(\hat{G}, \mu)$.

Consider the regular subalgebra $H^\infty_{\mathrm{alg}}(\hat{G}, \mu) = H^\infty(\hat{G}, \mu) \cap \ell^\infty_{\mathrm{alg}}(\hat{G})$ of $H^\infty(\hat{G}, \mu)$ and denote by $B(\hat{G}, \mu)$ its norm closure. In other words, in the notation of Section 3.3, $B(\hat{G}, \mu) = B(\hat{G}) \cap H^\infty(\hat{G}, \mu)$. We will show in Theorem 4.5 that the action of $\hat{G}$ on $H^\infty(\hat{G}, \mu)$ restricts to a continuous action on $B(\hat{G}, \mu)$ and that $B(\hat{G}, \mu)$ becomes a braided-commutative Yetter–Drinfeld G-C*-algebra.

Since the G-action on $\ell^\infty(\hat{G})$ is implemented by the conjugation by the multiplicative unitary, the G-fixed point subalgebra $H^\infty(\hat{G}, \mu)^G$ coincides with the intersection $H^\infty(\hat{G}, \mu) \cap Z(\ell^\infty(\hat{G}))$. The Markov operator P_μ leaves the algebra $Z(\ell^\infty(\hat{G}))$ invariant. If we think of elements of $Z(\ell^\infty(\hat{G}))$ as bounded functions on $\mathrm{Irr}(G)$, then the action of the Markov operator on $Z(\ell^\infty(\hat{G})) = \ell^\infty(\mathrm{Irr}(\hat{G}))$ can be expressed as

$$(P_\mu f)(s) = \sum_{t,r} \mu(t) m_{t,s}^r \frac{d(r)}{d(t)d(s)} f(r),$$

where $m_{t,s}^r$ is the multiplicity of U_r in $U_t \mathbin{\bigcirc\!\!\!\!\top} U_s$. Therefore $H^\infty(\hat{G}, \mu)^G$ is the algebra of bounded measurable functions on the Poisson boundary, in the usual probabilistic sense, of the random walk on $\mathrm{Irr}(G)$ with transition probabilities

$$p_\mu(s, r) = \sum_t \mu(t) m_{t,s}^r \frac{d(r)}{d(t)d(s)}.$$

The composition of Markov operators P_μ can be expressed as $P_\mu P_\nu = P_{\nu*\mu}$, where the convolution of probability measures ν and μ on $\mathrm{Irr}(G)$ is defined by

$$(\nu * \mu)(t) = \sum_{s,r} m^t_{s,r} \frac{d(t)}{d(s)d(r)} \nu(s)\mu(r).$$

The following conditions on probability measures will be of interest to us. A probability measure μ is said to be:

(i) *symmetric*, if $\mu(\bar{s}) = \mu(s)$ for any $s \in \mathrm{Irr}(G)$, where $\bar{s}$ is such that $\bar{U}_s \cong U_{\bar{s}}$;
(ii) *generating*, if $\cup_{k=1}^\infty \operatorname{supp} \mu^{*k} = \mathrm{Irr}(G)$;
(iii) *ergodic*, if the only P_μ-harmonic functions on $\mathrm{Irr}(G)$ are the constant functions, that is, $H^\infty(\hat{G}, \mu)^G$ reduces to $\mathbb{C}1$.

We remark that a symmetric ergodic measure μ, or even an ergodic measure with symmetric support, is automatically generating. Indeed, the symmetry assumptions implies that we have a well-defined equivalence relation on $\mathrm{Irr}(G)$ such that $s \sim t$ iff t can be reached from s with nonzero probability in a finite nonzero number of steps. Then any bounded function that is constant on equivalence classes is P_μ-harmonic. Hence μ is generating by the ergodicity assumption.

A compact quantum group G is said to be *coamenable* if the counit of $\mathbb{C}[G]$ is continuous with respect to the reduced norm. By the main result of [Tom06], coamenability is equivalent to existence of a two-sided invariant mean, which is a (singular) state m on $\ell^\infty(\hat{G})$ satisfying

$$m = (m \otimes \iota)\hat{\Delta} = (\iota \otimes m)\hat{\Delta}.$$

The following proposition is a simple extension of a result of Kaimanovich and Vershik on random walks on groups [KV83], it is essentially contained in [HY00].

Proposition 4.1. *Let G be a coamenable compact quantum group. Then there is a symmetric ergodic probability measure μ on* $\mathrm{Irr}(G)$*.*

Proof. The restriction of an invariant mean on $\ell^\infty(\hat{G})$ to $\ell^\infty(\mathrm{Irr}(G)) = Z(\ell^\infty(\hat{G}))$ gives an invariant mean on the representation ring $R(G)$ of G equipped with the quantum dimension function d. Thus, the quantum dimension is weakly amenable in the sense of [HI98], and by [HY00, Theorem 2.5] there is a symmetric ergodic probability measure on $\mathrm{Irr}(G)$. □

Our use of Poisson boundaries will rely on a result of Tomatsu [Tom07]. In order to formulate it, recall that any compact quantum group G contains a unique maximal quantum subgroup K among its closed Kac quantum subgroups. If the homomorphism $\mathbb{C}[G] \to \mathbb{C}[K]$ extends to a homomorphism $C(G) \to C(K)$, which is the case if G is coamenable, then $C(K)$ is called the canonical Kac quotient of $C(G)$, see [Soł05, Appendix].

Theorem 4.2 (cf. [Tom07, Theorem 4.8])**.** *Let G be a coamenable compact quantum group, and μ be a symmetric ergodic probability measure on* $\mathrm{Irr}(G)$*. Then the Poisson boundary $H^\infty(\hat{G}, \mu)$ is G- and $\hat{G}$-equivariantly isomorphic to $L^\infty(G/K)$, where K is the maximal Kac quantum subgroup of G.*

Proof. In [Tom07, Theorem 4.8] instead of ergodicity Tomatsu assumes that μ is generating and the representation ring $R(G)$ is commutative. But in the general case the proof is the same. Using the G-action α and an invariant mean m, one constructs a map $\Lambda\colon H^\infty(\hat{G}, \mu) \to L^\infty(G)$ by setting $\Lambda(a) = (\iota \otimes m)\alpha(a)$. This gives the desired bijection of $H^\infty(\hat{G}, \mu)$ onto $L^\infty(G/K)$. The commutativity of the fusion rules is used in [Tom07] to ensure ergodicity of μ, which, in turn, implies that $h \circ \Lambda(a) = \hat{\varepsilon}(a)$, where $\hat{\varepsilon}$ is the counit on $\ell^\infty(\hat{G})$. Since we do start from an ergodic measure, we obtain the same formula, and the rest of the argument is completely identical. □

4.2. **Poisson boundaries of monoidal categories.** By a result of De Rijdt and Vander Vennet [DRVV10], noncommutative Poisson boundaries are compatible with monoidal equivalences of compact quantum groups. More precisely, given a compact quantum group G with the forgetful fiber functor $\mathcal{F}\colon \operatorname{Rep} G \to \mathrm{Hilb}_f$, another unitary fiber functor $\mathcal{F}'\colon \operatorname{Rep} G \to \mathrm{Hilb}_f$ defining a compact quantum group G', and a probability measure μ on $\operatorname{Irr}(G) = \operatorname{Irr}(G')$, there is a G-equivariant isomorphism

$$H^\infty_{\mathrm{alg}}(\hat G, \mu) \cong \{a \in \mathcal{B}(\mathcal{F}, \mathcal{F}') \otimes_{\mathrm{alg}} H^\infty_{\mathrm{alg}}(\hat G', \mu) \mid (\beta' \otimes \iota)(a) = (\iota \otimes \alpha')(a)\},$$

where α' is the left adjoint action of G' on $H^\infty_{\mathrm{alg}}(\hat G', \mu)$ and β' is the right action of G' on the linking algebra $\mathcal{B}(\mathcal{F}, \mathcal{F}')$. This suggests that one may define the associated $(\operatorname{Rep} G)$-module category purely in terms of the category $\operatorname{Rep} G$. We will show that this is indeed the case.

Throughout this subsection we assume that $\mathcal{C}$ is an essentially small rigid C*-tensor category with simple unit, although the only case we will be later interested in, is $\mathcal{C} = \operatorname{Rep} G$. Given an object U in $\mathcal{C}$, we can define the so called normalized partial traces

$$\operatorname{tr}_U \otimes \iota\colon \mathcal{C}(U \otimes V, U \otimes W) \to \mathcal{C}(V, W)$$

by letting

$$(\operatorname{tr}_U \otimes \iota)(T) = d(U)^{-1}(R_U^* \otimes \iota_W)(\iota_{\bar U} \otimes T)(R_U \otimes \iota_V),$$

where $d(U)$ is the intrinsic dimension of U and R_U is part of a standard solution $(R_U, \bar R_U)$ of the conjugate equations for U, see e.g. [NT13, Section 2.5]. For $\mathcal{C} = \operatorname{Rep} G$ the morphisms R_U and $\bar R_U$ from Section 1.1 do form a standard solution, and we see that in this case we get

$$(\operatorname{tr}_U \otimes \iota)(T) = (\phi_U \otimes \iota)(T) \quad \text{for} \quad T \in \mathcal{C}(U \mathbin{\bar\otimes} V, U \mathbin{\bar\otimes} W) \subset B(H_U) \otimes B(H_V, H_W), \tag{4.2}$$

where ϕ_U is the state defined by (4.1).

Using the partial trace $\operatorname{tr}_U \otimes \iota$ we define an operator P_U on the space of natural transformations $\operatorname{Nat}(\iota \otimes V, \iota \otimes W)$ by

$$P_U(\eta)_X = (\operatorname{tr}_U \otimes \iota)(\eta_{U \otimes X}).$$

It is easy to see that this operation preserves the subspace $\operatorname{Nat}_b(\iota \otimes V, \iota \otimes W)$ of uniformly bounded natural transformations.

Let $\operatorname{Irr}(\mathcal{C})$ be the set of isomorphism classes of simple objects in $\mathcal{C}$. For every $s \in \operatorname{Irr}(\mathcal{C})$ fix a representative U_s. Then, given a probability measure μ on $\operatorname{Irr}(\mathcal{C})$, we define an operator P_μ acting on $\operatorname{Nat}_b(\iota \otimes U, \iota \otimes V)$ by $P_\mu = \sum_s \mu(s) P_{U_s}$.

We say that a bounded natural transformation $\eta\colon \iota \otimes U \to \iota \otimes V$ is *P_μ-harmonic* if $P_\mu(\eta) = \eta$. Any morphism $T\colon U \to V$ defines a bounded natural transformation $\iota \otimes T$, which is obviously P_μ-harmonic for every μ.

Proposition 4.3. *Given bounded P_μ-harmonic natural transformations $\eta\colon \iota \otimes U \to \iota \otimes V$ and $\nu\colon \iota \otimes V \to \iota \otimes W$, the limit*

$$(\nu \cdot \eta)_X = \lim_{n \to \infty} P_\mu^n(\nu\eta)_X$$

exists for all objects X and defines a bounded P_μ-harmonic natural transformation $\iota \otimes U \to \iota \otimes W$. Furthermore, the composition $\cdot$ is associative.

Note that since the spaces $\mathcal{C}(X \otimes U, X \otimes W)$ are finite dimensional by our assumptions on $\mathcal{C}$, the notion of a limit is unambiguous.

Proof of Proposition 4.3. This is an immediate consequence of results of Izumi [Izu12]. Namely, replacing U, V and W by their direct sum we may assume that $U = V = W$. Then $\operatorname{End}_b(\iota \otimes U) = \operatorname{Nat}_b(\iota \otimes U, \iota \otimes U) \cong \ell^\infty\text{-}\bigoplus_s \operatorname{End}_{\mathcal{C}}(U_s \otimes U)$ is a von Neumann algebra and P_μ is a normal unital completely positive map on it. By [Izu12, Corollary 5.2] the subspace of P_μ-invariant elements is itself a von Neumann algebra with product $\cdot$ such that $x \cdot y$ is the strong* operator limit of the sequence $\{P_\mu^n(xy)\}_n$. $\square$

We can now give the main definition.

Definition 4.4. The *categorical Poisson boundary* $(\mathcal{P}, \Pi)$ of $(\mathcal{C}, \mu)$ consists of the C*-tensor category $\mathcal{P}$ and the strict unitary tensor functor $\Pi\colon \mathcal{C} \to \mathcal{P}$ defined as follows. The category $\mathcal{P}$ is the subobject completion of $\mathcal{C}$ with the new morphism sets

$$\mathcal{P}(U, V) = \{\eta \in \mathrm{Nat}_b(\iota \otimes U, \iota \otimes V) \mid P_\mu \eta = \eta\},$$

endowed with the composition law

$$(\eta \cdot \eta')_X = \lim_{n\to\infty} P_\mu^n(\eta\eta')_X.$$

On objects in $\mathcal{C}$ the tensor product in $\mathcal{P}$ is the same as in $\mathcal{C}$, while on morphisms it is given by

$$\eta \otimes \eta' = (\eta \otimes \iota) \cdot \eta'.$$

The functor $\Pi\colon \mathcal{C} \to \mathcal{P}$ is defined by letting $\Pi(U) = U$ on objects and $\Pi(T) = \iota \otimes T$ on morphisms.

In more detail, the tensor product of morphisms in $\mathcal{P}$ is described as follows. When $X \in \mathcal{C}$ and $\eta \in \mathcal{P}(U, V)$, $\eta \otimes \iota_X$ is simply given by the family $(\eta_Y \otimes \iota_X)_{Y\in\mathcal{C}}$. On the other hand, $\iota_X \otimes \eta$ is given by the family $(\eta_{X\otimes Y})_{Y\in\mathcal{C}}$. If $\eta \in \mathcal{P}(U, V)$ and $\eta' \in \mathcal{P}(U', V')$, their tensor product $\eta \otimes \eta'$ is defined as the composition $(\eta \otimes \iota_{V'}) \cdot (\iota_U \otimes \eta')$ (which is also equal to $(\iota_V \otimes \eta') \cdot (\eta \otimes \iota_{U'})$ by naturality, see the similar discussion in Section 3.3).

We remark that from the proof of Proposition 4.3 it is clear that $\mathcal{P}$ is in fact a W*-tensor category.

In general the unit object is no longer simple in $\mathcal{P}$. If we identify endomorphisms of the identity functor on $\mathcal{C}$ with functions on $\mathrm{Irr}(\mathcal{C})$, then the operator P_μ defines a Markov operator on $\ell^\infty(\mathrm{Irr}(\mathcal{C}))$, and $\mathrm{End}_{\mathcal{P}}(\mathbb{1})$ becomes the algebra of harmonic functions with respect to this operator. Therefore in the language of the previous subsection, the unit object in $\mathcal{P}$ is simple if and only if μ is ergodic.

A detailed treatment of categorical Poisson boundaries will be given in a separate paper [NY]. Let us just mention that the notion is interesting only for infinite categories: if $\mathrm{Irr}(\mathcal{C})$ is finite, then $\Pi\colon \mathcal{C} \to \mathcal{P}$ is a monoidal equivalence for any generating measure μ.

We now specialize to the categories $\mathrm{Rep}\, G$.

Theorem 4.5. *Let G be a compact quantum group and μ be a probability measure on* $\mathrm{Irr}(G)$*. Then the dense C*-subalgebra $B(\hat{G}, \mu) \subset H^\infty(\hat{G}, \mu)$ is a unital braided-commutative Yetter–Drinfeld G-C*-algebra and the pair $(\mathcal{D}_{B(\hat{G},\mu)}, \mathcal{E}_{B(\hat{G},\mu)})$, consisting of the C*-tensor category $\mathcal{D}_{B(\hat{G},\mu)}$ of G-equivariant finitely generated Hilbert $B(\hat{G}, \mu)$-modules and the unitary tensor functor $\mathcal{E}_{B(\hat{G},\mu)}\colon$ $\mathrm{Rep}\, G \to \mathcal{D}_{B(\hat{G},\mu)}$, is isomorphic to the categorical Poisson boundary of* $(\mathrm{Rep}\, G, \mu)$.

Proof. For $\mu = \delta_e$, when $H^\infty(\hat{G}, \mu) = \ell^\infty(\hat{G})$, this theorem is the contents of Section 3.3. The general case easily follows from this. Indeed, denote by $\tilde{\mathcal{B}}_\mu$ and $\mathcal{B}_\mu$ the algebras constructed from the Poisson boundary $(\mathcal{P}, \Pi)$ of $(\mathrm{Rep}\, G, \mu)$ as described in Section 2.3. If $\mu = \delta_e$, we simply write $\tilde{\mathcal{B}}$ and $\mathcal{B}$. Thus,

$$\tilde{\mathcal{B}} = \bigoplus_U (\bar{H}_U \otimes \mathrm{Nat}_b(\iota, \iota \otimes U)).$$

As we showed in Section 3.3, the Yetter–Drinfeld algebra $\mathcal{B}$ can be identified with $\ell^\infty_{\mathrm{alg}}(\hat{G})$, and then the homomorphism $\pi\colon \tilde{\mathcal{B}} \to \mathcal{B} = \ell^\infty_{\mathrm{alg}}(\hat{G})$ is given by

$$\pi\left(\bar{\xi} \otimes \left(\sum_k a_k \otimes \zeta_k\right)\right) = \sum_k (\zeta_k, \xi) a_k,$$

if we view $\mathrm{Nat}_b(\iota, \iota \otimes U)$ as a subspace of $\ell^\infty\text{-}\bigoplus_s (B(H_s) \otimes H_U)$.

The Markov operators P_μ on $\mathrm{Nat}_b(\iota, \iota\otimes U)$ define an operator $\iota \otimes P_\mu$ on $\tilde{\mathcal{B}}$. Then by definition, the algebra $\tilde{\mathcal{B}}_\mu$ is the subspace of $(\iota\otimes P_\mu)$-invariant elements in $\tilde{\mathcal{B}}$. Furthermore, as follows from (4.2), we have $\pi(\iota\otimes P_\mu) = P_\mu \pi$, where on the right hand side by P_μ we mean the operator on $\ell^\infty(\hat{G})$ defined in Section 4.1. This already implies that the restriction of π to $\tilde{\mathcal{B}}_\mu$ defines a surjective homomorphism

$\tilde{\mathcal{B}}_\mu \to H^\infty_{\mathrm{alg}}(\hat{G},\mu)$. Recalling how $\mathcal{B}_\mu$ is obtained from $\tilde{\mathcal{B}}_\mu$, we then conclude that this restriction factors through $\mathcal{B}_\mu$ and defines a G-equivariant $*$-isomorphism $\mathcal{B}_\mu \cong H^\infty_{\mathrm{alg}}(\hat{G},\mu)$.

It remains to compare the $\mathbb{C}[G]$-module structures. For this part the computation is in fact exactly the same as for $\mu=\delta_e$. The point is that, in the formula (2.4) for the $\mathbb{C}[G]$-action, one only needs to compute the compositions of the form $(\iota\otimes T\otimes\iota)\bar{R}_U$ for $U,V\in\operatorname{Rep}G$ and $T\in\mathcal{P}(\mathbb{1},V)$. In general, if $\eta\in\mathcal{P}(U,V)$ and $S\in W\to U$ is a morphism in $\operatorname{Rep}G$, the composition $\eta\cdot S$ (or more precisely $\eta\cdot(\iota\otimes S)$) is represented by the family $(\eta_X(\iota_X\otimes S))_X$, which is independent of μ. Thus, the $\mathbb{C}[G]$-module structure on $H^\infty_{\mathrm{alg}}(\hat{G},\mu)$ induced by the tensor category structure of $\mathcal{P}$ via the isomorphism $\mathcal{B}_\mu\cong H^\infty_{\mathrm{alg}}(\hat{G},\mu)$, is the restriction of that on $\ell^\infty_{\mathrm{alg}}(\hat{G})$. But this is exactly how the original $\mathbb{C}[G]$-module structure was defined on $H^\infty_{\mathrm{alg}}(\hat{G},\mu)$. □

Corollary 4.6. *Let G be a compact quantum group and μ be an ergodic probability measure on* $\operatorname{Irr}(G)$*. Assume that the Poisson boundary $H^\infty(\hat{G},\mu)$ is G- and $\hat{G}$-equivariantly isomorphic to $L^\infty(G/H)$ for a closed quantum subgroup H of G. Then the categorical Poisson boundary of* $(\operatorname{Rep}G,\mu)$ *is isomorphic to* $(\operatorname{Rep}H,\mathcal{F})$*, where $\mathcal{F}\colon \operatorname{Rep}G\to\operatorname{Rep}H$ is the forgetful tensor functor.*

Proof. This follows immediately from Theorem 4.5 and the results of Section 3.1, where we showed that we have an isomorphism $(\mathcal{D}_{C(G/H)},\mathcal{E}_{C(G/H)})\cong(\operatorname{Rep}H,\mathcal{F})$. □

4.3. **Factorization of fiber functors.** Let G be a compact quantum group. We say that a unitary fiber functor $\mathcal{E}\colon \operatorname{Rep}G\to\mathrm{Hilb}_f$ is *dimension-preserving* if $\dim\mathcal{E}(U)=\dim H_U$ for all $U\in\operatorname{Rep}G$.

As an application of the theory developed in the previous sections we can now prove the following result.

Theorem 4.7. *Let G be a coamenable compact quantum group and K be the maximal Kac quantum subgroup of G. Then any dimension-preserving unitary fiber functor on* $\operatorname{Rep}G$ *factors through* $\operatorname{Rep}K$*.*

Proof. Let $\mathcal{E}\colon \operatorname{Rep}G\to\mathrm{Hilb}_f$ be a dimension-preserving unitary fiber functor. It defines a new compact quantum group G' and a unitary monoidal equivalence $\mathcal{G}\colon\operatorname{Rep}G\to\operatorname{Rep}G'$ such that if $\mathcal{F}'_G\colon\operatorname{Rep}G'\to\mathrm{Hilb}_f$ is forgetful fiber functor, then $\mathcal{E}\cong\mathcal{F}'_G\mathcal{G}$. Since the quantum group G' has the same classical dimension function as G, it is also coamenable, see e.g. [NT13, Theorem 2.7.10].

Let $K'\subset G'$ be the maximal Kac quantum subgroup, and $\mathcal{E}_K\colon\operatorname{Rep}G\to\operatorname{Rep}K$, $\mathcal{E}'_K\colon\operatorname{Rep}G'\to\operatorname{Rep}K'$ and $\mathcal{F}'_K\colon\operatorname{Rep}K'\to\mathrm{Hilb}_f$ be the forgetful tensor functors. By Proposition 4.1 there exists a symmetric ergodic probability measure μ on $\operatorname{Irr}(G)$. Then by Theorem 4.2 and Corollary 4.6, the categorical Poisson boundary of $(\operatorname{Rep}G,\mu)$ is isomorphic to $(\operatorname{Rep}K,\mathcal{E}_K)$, while that of $(\operatorname{Rep}G',\mu)$ is isomorphic to $(\operatorname{Rep}K',\mathcal{E}'_K)$. Therefore

$$(\operatorname{Rep}K,\mathcal{E}_K)\cong(\operatorname{Rep}K',\mathcal{E}'_K\mathcal{G}).$$

This means that there exists a unitary monoidal equivalence $\mathcal{G}_K\colon\operatorname{Rep}K\to\operatorname{Rep}K'$ such that $\mathcal{G}_K\mathcal{E}_K\cong\mathcal{E}'_K\mathcal{G}$, whence $\mathcal{F}'_K\mathcal{G}_K\mathcal{E}_K\cong\mathcal{F}'_K\mathcal{E}'_K\mathcal{G}=\mathcal{F}'_G\mathcal{G}\cong\mathcal{E}$. □

We remark that any unitary fiber functor on $\operatorname{Rep}K$ is dimension-preserving. Indeed, clearly K is coamenable, since the restriction of an invariant mean on $\ell^\infty(\hat{G})$ to $\ell^\infty(\hat{K})$ defines an invariant mean. Since K is of Kac type, it follows that $\operatorname{Rep}K$ is amenable, so the claim follows from [NT13, Corollary 2.7.9]. Therefore the above theorem can be formulated by saying that a unitary fiber functor on $\operatorname{Rep}G$ is dimension-preserving if and only if it factors through $\operatorname{Rep}K$.

It is well-known that dimension-preserving unitary fiber functors can be described in terms of 2-cocycles on $\hat{G}$, see e.g. [NT13, Chapter 3]. Briefly, recall that a unitary 2-cocycle Ω on $\hat{G}$, or a unitary dual 2-cocycle on G, is a unitary element in $\ell^\infty(\hat{G})\mathbin{\bar{\otimes}}\ell^\infty(\hat{G})\simeq\ell^\infty\text{-}\bigoplus_{s,t}B(H_s)\otimes B(H_t)$ satisfying

$$(\Omega\otimes1)(\hat{\Delta}\otimes\iota)(\Omega)=(1\otimes\Omega)(\iota\otimes\hat{\Delta})(\Omega).$$

Such a cocycle defines a fiber functor $\mathcal{F}_\Omega\colon \operatorname{Rep} G \to \mathrm{Hilb}_f$, which is the same as the forgetful fiber functor on objects and morphisms, but has the new tensor structure

$$H_U \otimes H_V \to H_{U \mathbin{\bar\otimes} V}, \quad \xi\otimes\eta \mapsto \Omega^*(\xi\otimes\eta).$$

We denote by G_Ω the new compact quantum group defined by this functor. More concretely, $\mathbb{C}[G_\Omega]=\mathbb{C}[G]$ as coalgebras, while the new $*$-algebra structure is defined by duality from $(\mathcal{U}(G),\hat\Delta_\Omega)$, where $\hat\Delta_\Omega=\Omega\hat\Delta(\cdot)\Omega^*$.

Two cocycles Ω_1 and Ω_2 are called cohomologous if there exists a unitary $u\in\ell^\infty(\hat G)$ such that

$$\Omega_2=(u\otimes u)\Omega_1\hat\Delta(u)^*.$$

The cohomology classes of unitary 2-cocycles form a set $H^2(\hat G;\mathbb{T})$. The map $\Omega\mapsto\mathcal{F}_\Omega$ defines a bijection between $H^2(\hat G;\mathbb{T})$ and the set of natural unitary monoidal isomorphism classes of dimension-preserving unitary fiber functors on $\operatorname{Rep} G$.

If H is a closed quantum subgroup of G and Ω is a dual 2-cocycle on H, the natural inclusion $\ell^\infty(\hat H)\subset\ell^\infty(\hat G)$ allows us to consider Ω as a dual 2-cocycle on G. We call such cocycles *induced* from H. A dual 2-cocycle Ω is cohomologous a cocycle induced from H if and only if the fiber functor $\mathcal{F}_\Omega\colon \operatorname{Rep} G\to\mathrm{Hilb}_f$ factors through $\operatorname{Rep} H$ and the corresponding functor on $\operatorname{Rep} H$ is dimension-preserving.

This discussion leads to the following reformulation of Theorem 4.7.

Theorem 4.8. *Let G be a coamenable compact quantum group and K be the maximal Kac quantum subgroup of G. Then any unitary dual 2-cocycle on G is cohomologous to a cocycle induced from K.*

We finish this section with the following simple observation.

Proposition 4.9. *Let G be a compact quantum group, K be its maximal Kac quantum subgroup and Ω be a unitary dual 2-cocycle on K. Then K_Ω is the maximal Kac quantum subgroup of G_Ω.*

Proof. The quantum group K_Ω is again of Kac type, e.g. because such quantum groups can be abstractly characterized as compact quantum groups for which the classical and quantum dimension functions coincide. Hence K_Ω is contained in the maximal Kac quantum subgroup of G_Ω. Since Ω^* is a unitary dual 2-cocycle on G_Ω and $(G_\Omega)_{\Omega^*}=G$, by swapping the roles of G and G_Ω we conclude that K_Ω must be maximal. □

5. Compact quantum groups of Lie type

In this section we classify dimension-preserving fiber functors for a class of quantum groups by applying the results we have obtained so far.

5.1. **Twisted q-deformations of compact Lie groups.** Let G be a compact simply connected semisimple Lie group, and T be its maximal torus. Suppose that c is a $\mathbb{T}$-valued 2-cochain on the dual group $\hat T$ such that its coboundary ∂c defines a 3-cocycle Φ^c on $\widehat{Z(G)}$. Then Φ^c can be considered as an associator Φ^c on $\operatorname{Rep} G$: for irreducible representations U, V and W, the new associativity morphisms

$$\Phi^c(U,V,W)\colon (U \mathbin{\bar\otimes} V)\mathbin{\bar\otimes} W\to U\mathbin{\bar\otimes}(V\mathbin{\bar\otimes} W),$$

are given by multiplication by $\Phi^c(\omega_U,\omega_V,\omega_W)$, where $\omega_U\in\widehat{Z(G)}$ is the central character of U. This gives us a new rigid C*-tensor category $(\operatorname{Rep} G,\Phi^c)$, which has the same data as $\operatorname{Rep} G$ except for the new associativity morphisms Φ^c. Since Φ^c is the coboundary of c over $\hat T$, we have a unitary fiber functor $\mathcal{F}_c\colon (\operatorname{Rep} G,\Phi^c)\to\mathrm{Hilb}_f$, which is identical to the forgetful fiber functor on $\operatorname{Rep} G$, except that the tensor structure is given by

$$H_U\otimes H_V\to H_{U\mathbin{\bar\otimes} V},\quad \xi\otimes\eta\mapsto c^*(\xi\otimes\eta).$$

This functor defines a new compact quantum group G^c. Explicitly, similarly to the discussion in the previous section, $\mathbb{C}[G^c]=\mathbb{C}[G]$ as coalgebras, while the new $*$-algebra structure is defined by duality from $(\mathcal{U}(G),c\hat\Delta(\cdot)c^*)$.

Similarly, for any $0<q<\infty$, the q-deformation G_q contains T as a closed subgroup, and by the same construction we obtain a new quantum group G_q^c such that $\mathcal{U}(G_q^c)=(\mathcal{U}(G_q),c\hat\Delta_q(\cdot)c^*)$. Because c is defined on $\hat T$, the coproduct of any element $a\in\mathcal{U}(T)$ computed in $\mathcal{U}(G_q^c)$ is the same as $\hat\Delta_q(a)=\hat\Delta(a)$. In particular, T is a closed subgroup of G_q^c. We studied particular examples of this construction in [NY13].

As was shown by Tomatsu [Tom07, Lemma 4.10], for $q\ne1$ the maximal Kac quantum subgroup of G_q is T. We have the following generalization of this result to the quantum groups G_q^c.

Theorem 5.1. *Let G be a compact simply connected semisimple Lie group, T be its maximal torus, and c be a $\mathbb{T}$-valued 2-cochain on $\hat T$ such that ∂c is defined on $\widehat{Z(G)}$. Then for any $q>0$, $q\ne1$, the maximal Kac quantum subgroup of G_q^c is the maximal torus T.*

For the proof we will need the following simple lemma.

Lemma 5.2. *Under the identification $\mathcal{U}(G_q^c)\simeq\mathcal{U}(G_q)$ as $*$-algebras, the Woronowicz character f_1 of G_q^c is given by the same positive element as for G_q.*

Proof. Since G_q^c does not change when we multiply c by a 2-cochain living on $\widehat{Z(G)}$, without loss of generality we may assume that the cocycle Φ^c is normalized. Put $\lambda_s=\Phi^c(\omega_{\bar U_s},\omega_{U_s},\omega_{\bar U_s})$. Let us also write R_s and $\bar R_s$ for the solutions R_{U_s} and $\bar R_{U_s}$ of the conjugate equations for U_s in $\operatorname{Rep}G$ defined in Section 1.1. Then in the category $(\operatorname{Rep}G_q,\Phi^c)$, the pair $(\lambda_sR_s,\bar R_s)$ solves the conjugate equations for U_s. Under the identification of $\mathcal{U}(G_q^c)$ with $\mathcal{U}(G_q)$, in the category $\operatorname{Rep}(G_q^c)$ this solution becomes equal to $(\lambda_s cR_s,c\bar R_s)$.

Now, it is well-known that the Woronowicz character ρ of G_q lies in $\mathcal{U}(T)\subset\mathcal{U}(G_q)$, see e.g. [NT13, Proposition 2.4.10]. Namely, it equals $q^{-2\rho^*}$, where $\rho^*\in\mathfrak{h}$ is the unique vector satisfying $\alpha(\rho^*)=d_\alpha=(\alpha,\alpha)/2$ for each simple root α. In other words, under the isomorphism $\mathfrak{h}\cong\mathfrak{h}^*$ the element ρ^* corresponds to half the sum of positive roots. Since we also have $c\in\mathcal{U}(T\times T)$, it follows that if we choose an orthonormal basis $\{\xi_i\}_i$ in H_s consisting of weight vectors, then the vector $cR_s(1)\in\bar H_s\otimes H_s$ has the form $\sum_i\bar\xi_i\otimes\beta_i\rho^{-1/2}\xi_i$ for some $\beta_i\in\mathbb{T}$. It follows that the Woronowicz character $f_{-1/2}$ for G_q^c has the form $\rho^{-1/2}v$ for a unitary v commuting with $\mathcal{U}(T)$. Hence $f_1=\rho$ and $v=1$. ☐

Observe now that by Proposition 4.9, if Theorem 5.1 is true for a cochain c then it is true for any other cochain that differs from c by a $\mathbb{T}$-valued 2-cocycle on $\hat T$. Since G_q^c also does not change when we multiply c by a 2-cochain living on $\widehat{Z(G)}$, it follows that in order to prove the theorem it suffices to consider cochains representing every cohomology class in $H^3(\widehat{Z(G)};\mathbb{T})$ of the form $[\Phi^c]$. Such representatives were constructed in [NY13], see Proposition 2.6 there. We will not need an explicit form of these cochains, but let is briefly review the structure of the corresponding quantum groups G_q^c, which we denoted by G_q^τ in [NY13].

Given $\tau=(\tau_i)_{i=1}^{\operatorname{rk}G}\in Z(G)^{\operatorname{rk}G}$, consider the finite subgroup T_τ of T generated by the components τ_i. There is a group homomorphism $\psi\colon\hat T_\tau\to T/Z(G)$ characterized by $\langle\psi(\chi),\alpha_i\rangle=\tau_i$ for any $\chi\in\hat T_\tau$ and any positive simple root α_i. Composing this homomorphism with the conjugation action of $T/Z(G)$ on $C(G_q)$, we obtain an action of $\hat T_\tau$ on $\mathbb{C}[G_q]$, denoted by $\operatorname{Ad}\psi$. The crossed product $C(G_q)\rtimes_{\operatorname{Ad}\psi}\hat T_\tau$ has an action of T_τ which is given by right translations on the copy of $C(G_q)$ and by the dual action on the copy of $\hat T_\tau$. The C*-algebra $(C(G_q)\rtimes_{\operatorname{Ad}\psi}\hat T_\tau)^{T_\tau}$ has the structure of a compact quantum group, induced by those of G_q and $\hat T_\tau$. This is our quantum group G_q^τ.

Using the crossed product presentation of $C(G_q^\tau)$ and results of Soibelman on the representation theory of $C(G_q)$ for $q \neq 1$, it is not difficult to obtain information on representations of the C*-algebra $C(G_q^\tau)$, see [NY13, Proposition 3.4]:

$$\operatorname{Prim}(C(G_q^\tau)) = \coprod_{w \in W} (\theta_w(\hat{T}_\tau)\backslash T/T_\tau) \times \widehat{\theta_w^{-1}(T_\tau)}, \tag{5.1}$$

where W is the Weyl group and θ_w is a certain homomorphism of $\hat{T}_\tau$ into T expressed in terms of ψ and $w \in W$.

Proof of Theorem 5.1. By the preceding discussion it suffices to prove the theorem for quantum groups G_q^τ. By Lemma 5.2 and its proof, the scaling groups of G_q^τ and G_q are both given by the conjugation action by $q^{-2it\rho^*} \in T$, $t \in \mathbb{R}$. We denote this common scaling group by $(\tau_t)_t$. We claim that the only irreducible representations of $C(G_q^\tau)$ that are fixed, up to an isomorphism, under $(\tau_t)_t$ are the evaluations at the points of T.

Since $\operatorname{Ad}\psi$ commutes with τ_t, we obtain a natural extension of τ_t to $\mathbb{C}[G_q] \rtimes_{\operatorname{Ad}\psi} \hat{T}_\tau$ by letting it act trivially on the copy of $\hat{T}_\tau$. The embedding of $\mathbb{C}[G_q^\tau]$ into the crossed product is compatible with this extension of τ_t. The action of T_τ on $\mathbb{C}[G_q] \rtimes_{\operatorname{Ad}\psi} \hat{T}_\tau$, being implemented by the right torus translation action rt and the dual action, also commutes with the above extension of τ_t. Thus, up to a strong Morita equivalence, the action of the scaling group $(\tau_t)_t$ on $C(G_q^\tau)$ can be identified with the action of $\mathbb{R}$ on $C(G_q) \rtimes_{\mathrm{rt}} T_\tau \rtimes_{\operatorname{Ad}\psi,\widehat{\mathrm{rt}}} \hat{T}_\tau$ such that $\mathbb{R}$ acts by $(\tau_t)_t$ on $C(G_q)$ and trivially on the rest.

The description (5.1) of the primitive spectrum of $C(G_q)$ is obtained from the Morita equivalence $C(G_q^\tau) \sim C(G_q) \rtimes_{\mathrm{rt}} T_\tau \rtimes_{\operatorname{Ad}\psi,\widehat{\mathrm{rt}}} \hat{T}_\tau$ and the identification of $\operatorname{Prim}(C(G_q))$ with $\coprod_{w\in W} T$. It follows from the above considerations that on the part of the spectrum of $C(G_q^\tau)$ labeled by $w \in W$ in (5.1), the scaling group induces the translation by $q^{-2it(w\rho^*-\rho^*)}$, $t \in \mathbb{R}$, on $\theta_w(\hat{T}_\tau)\backslash T/T_\tau$ (see [NT12, Lemma 3.4; Yam13, Lemma 8] for the action of T on $\operatorname{Prim}(C(G_q))$ induced by the translations on $C(G_q)$). Since $w\rho^* \neq \rho^*$ unless $w = e$, we obtain the claim.

The rest of the argument is identical to the proof of [Tom07, Lemma 4.10]. Namely, let K be the maximal Kac quantum subgroup of G_q^τ, and π be an irreducible representation of $C(K)$. Composing π with the restriction map $C(G_q^\tau) \to C(K)$, we obtain an irreducible representation of $C(G_q^\tau)$, again denoted by π. Since the restriction map $C(G_q^\tau) \to C(K)$ intertwines the scaling groups, and the scaling group of K is trivial, π has to be the evaluation at some point of T. This proves that K is contained in T, hence $K = T$. ☐

Combining this theorem with Theorems 4.7 and 4.8 we get the following result.

Corollary 5.3. *Let G_q^c ($q > 0$, $q \neq 1$) be as in Theorem 5.1. Then any dimension-preserving unitary fiber functor $\operatorname{Rep} G_q^c \to \mathrm{Hilb}_f$ factors through $\operatorname{Rep} T$. Equivalently, any unitary dual 2-cocycle on G_q^c is cohomologous to a cocycle induced from T.*

In particular, for $q > 0$, $q \neq 1$, the class of compact quantum groups G_q^c is closed under cocycle deformations.

5.2. **Non-Kac compact quantum groups of $\mathrm{SU}(n)$-type.** For $G = \mathrm{SU}(n)$ the results of the previous subsection can be further strengthened thanks to a classification theorem of Kazhdan and Wenzl [KW93]. It states that any semisimple rigid monoidal category with fusion rules of $\mathrm{SL}(n)$ must be equivalent to one of the categories $(\operatorname{Rep}(\mathrm{SL}_q(n)), \Phi^c)$, where q is not a nontrivial root of unity and c is a 2-cochain as in the previous subsection. The corresponding result for C*-tensor categories states that any rigid C*-tensor category with fusion rules of $\mathrm{SU}(n)$ is unitarily monoidally equivalent to $(\operatorname{Rep}(\mathrm{SU}_q(n)), \Phi^c)$ for some $q > 0$ and c, and the pair $(q, [\Phi^c])$ is uniquely determined up to the transformation $(q, [\Phi^c]) \mapsto (q^{-1}, [\Phi^c]^{-1})$, see [Jor] for details, as well as [Pin07; PR11, Section 7] for related slightly weaker results. In particular, any such C*-tensor category is unitarily monoidally equivalent to $\mathrm{Rep}(\mathrm{SU}_q^c(n))$.

Let us say that a compact quantum group G is of $\mathrm{SU}(n)$-type if there is a bijection between $\mathrm{Irr}(G)$ and $\mathrm{Irr}(\mathrm{SU}(n))$ inducing a dimension-preserving isomorphism of the representation rings $R(G) \cong R(\mathrm{SU}(n))$. Since as we have shown, for $q \neq 1$ the class of quantum groups $\mathrm{SU}_q^c(n)$ is closed under cocycle deformations, from the result of Kazhdan and Wenzl we obtain an answer to the question of Woronowicz raised at the end of [Wor88], in the non-Kac case.

Theorem 5.4. *Let G be a non-Kac compact quantum group of $\mathrm{SU}(n)$-type for some $n \geq 2$. Then $G \cong \mathrm{SU}_q^c(n)$ for some $q \in (0,1)$ and a $\mathbb{T}$-valued 2-cochain c on $\hat{T} \cong \mathbb{Z}^{n-1}$ such that ∂c defines a 3-cocycle Φ^c on $Z(\widehat{\mathrm{SU}(n)})$. The number $q \in (0,1)$ and the cohomology class $[\Phi^c] \in H^3(Z(\widehat{\mathrm{SU}(n)}); \mathbb{T})$ of the cocycle Φ^c are uniquely defined by G.*

The cochain c is by no means unique. Since the isomorphism class of $\mathrm{SU}_q^c(n)$ does not change if we multiply c by a coboundary, we can get any non-Kac compact quantum group G of $\mathrm{SU}(n)$-type, up to an isomorphism, by the following procedure. Choose cochains $c_1, \dots, c_n$ such that $[\Phi^{c_i}]$ exhaust the group $H^3(Z(\widehat{\mathrm{SU}(n)}); \mathbb{T}) \cong \mathbb{Z}/n\mathbb{Z}$. Choose also 2-cocycles ω on $\hat{T}$ representing elements of $H^2(\hat{T}; \mathbb{T})$. Then $G \cong \mathrm{SU}_q^{c_k\omega}(n)$ for some $q \in (0,1)$, $1 \leq k \leq n$ and ω. The numbers q and k are uniquely determined, but the cocycle ω is still not unique: the flip map on the Dynkin diagram defines nontrivial isomorphisms between the quantum groups $\mathrm{SU}_q^{c_k\omega}(n)$. It is plausible, however, that these are now the only isomorphisms.

As we already mentioned, explicit examples of cochains were constructed in [NY13]. Namely, we showed that every $(n-1)$-tuple $\tau = (\tau_1, \dots, \tau_{n-1})$ of roots of unity of order n defines a cochain c_τ, and by [NY13, Proposition 4.1] we have $[\Phi^{c_\tau}] = [\Phi^{c_\nu}]$ if and only if

$$\prod_{i=1}^{n-1} \tau_i^i = \prod_{i=1}^{n-1} \nu_i^i.$$

Therefore the required representatives c_k can be obtained by taking, for example,

$$\tau = (e^{2\pi(k-1)i/n}, 1, \dots, 1).$$

The corresponding quantum groups $\mathrm{SU}_q^{c_\tau}(n)$ were denoted by $\mathrm{SU}_q^\tau(n)$ in [NY13]. We will not reproduce the explicit form of c_τ here and only use the relations in $\mathbb{C}[\mathrm{SU}_q^\tau(n)]$ obtained using the crossed product decomposition.

We next have to deform $\mathrm{SU}_q^{c_\tau}(n)$ by a 2-cocycle ω on $\hat{T}$ to get a quantum group $\mathrm{SU}_q^{c_\tau\omega}(n)$, which we will also denote by $\mathrm{SU}_q^{\tau,\omega}(n)$. To get explicit generators and relations in $\mathbb{C}[\mathrm{SU}_q^{\tau,\omega}(n)]$, it is convenient to enlarge the maximal torus $T \simeq \mathbb{T}^{n-1}$ of $\mathrm{SU}(n)$ to that of $\mathrm{U}(n)$, that is, to the group $\tilde{T} \cong \mathbb{T}^n$ of diagonal unitary matrices. Lifting 2-cocycles on $\hat{T}$ to $\hat{\tilde{T}} \cong \mathbb{Z}^n$, we conclude that up to coboundaries such cocycles are represented by alternating bicharacters $\omega\colon \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{T}$ which satisfy $\omega(e_1 + \dots + e_n, x) = 1$ for any $x \in \mathbb{Z}^n$. Putting $\omega_{ij} = \omega(e_i, e_j) \in \mathbb{T}$, the matrix $(\omega_{ij})_{i,j=1}^n$ satisfies $\omega_{ii} = 1$, $\omega_{ji} = \bar{\omega}_{ij}$ and $\prod_i \omega_{ij} = 1$ for any j. Two such matrices $\omega = (\omega_{ij})_{i,j}$ and $\tilde{\omega} = (\tilde{\omega}_{ij})_{i,j}$ represent the same element of $H^2(\hat{T}; \mathbb{T})$ if and only if $\omega_{ij}^2 = \tilde{\omega}_{ij}^2$ for all i, j.

The algebra $\mathbb{C}[\mathrm{SU}_q^\tau(n)]$ is generated by the matrix coefficients v_{ij}, $1 \leq i, j \leq n$, of the canonical n-dimensional representation. Then $\mathbb{C}[\mathrm{SU}_q^{\tau,\omega}(n)] = \mathbb{C}[\mathrm{SU}_q^\tau(n)]$ as coalgebras, while the new product $\cdot_\omega$ is determined by the following rule: if $x, y \in \mathbb{C}[\mathrm{SU}_q^\tau(n)]$ are such that

$$(\pi \otimes \iota \otimes \pi)\Delta^{(2)}(x) = z_i \otimes x \otimes z_j \quad \text{and} \quad (\pi \otimes \iota \otimes \pi)\Delta^{(2)}(y) = z_k \otimes x \otimes z_l,$$

where $\pi\colon \mathbb{C}[\mathrm{SU}_q^\tau(n)] \to \mathbb{C}[T]$ is the restriction map and we write $t = \mathrm{diag}(z_1(t), \dots, z_n(t))$ for elements $t \in T$, so that $\pi(v_{ij}) = \delta_{ij} z_i$, then

$$x \cdot_\omega y = \omega_{ik} \bar{\omega}_{jl} xy.$$

From the relations in $\mathbb{C}[\mathrm{SU}^\tau_q(n)]$ given in [NY13, Section 4.3] we conclude that $\mathbb{C}[\mathrm{SU}^{\tau,\omega}_q]$ can be described as the universal algebra with generators v_{ij}, $1\le i,j\le n$, and relations

$$v_{ij}v_{il}=\Big(\prod_{j\le p<l}\tau_p^{-1}\Big)q\bar\omega_{jl}^2v_{il}v_{ij}\quad (j<l),\quad v_{ij}v_{kj}=\Big(\prod_{i\le p<k}\tau_p\Big)q\omega_{ik}^2v_{kj}v_{ij}\quad (i<k),$$

$$v_{ij}v_{kl}=\Big(\prod_{i<p\le k}\tau_p^{-1}\Big)\Big(\prod_{j\le p<l}\tau_p^{-1}\Big)\omega_{ik}^2\bar\omega_{jl}^2v_{kl}v_{ij}\quad (i>k,j<l),$$

$$\Big(\prod_{j\le p<l}\tau_p\Big)\omega_{jl}^2v_{ij}v_{kl}-\Big(\prod_{i\le p<k}\tau_p\Big)\bar\omega_{ki}^2v_{kl}v_{ij}=(q-q^{-1})v_{il}v_{kj}\quad (i<k,j<l),$$

and

$$\sum_{\sigma\in S_n}\tau^{m(\sigma)}(-q)^{|\sigma|}\bar\omega(1,\dots,n)\omega(\sigma(1),\dots,\sigma(n))v_{1\sigma(1)}\cdots v_{n\sigma(n)}=1,$$

where $m(\sigma)=(m(\sigma)_1,\dots,m(\sigma)_{n-1})$ is the multi-index given by $m(\sigma)_i=\sum_{k=2}^n(k-1)m_i^{(k,\sigma(k))}$, with

$$m_i^{(k,j)}=\begin{cases}1, & \text{if } k\le i<j,\\ -1, & \text{if } j\le i<k,\\ 0, & \text{otherwise},\end{cases}$$

and the function $\omega(i_1,\dots,i_n)$ is defined by $\prod_{k<l}\omega_{i_k,i_l}$. The $*$-structure is uniquely determined by requiring the invertible matrix $(v_{ij})_{i,j}$ to be unitary.

To summarize, Theorem 5.4 can be reformulated as follows.

Theorem 5.5. *Let G be a non-Kac compact quantum group of $\mathrm{SU}(n)$-type for some $n\ge 2$. Then $G\cong \mathrm{SU}^{\tau,\omega}_q(n)$ for some $q\in(0,1)$, τ and ω as above. The numbers q and $\prod_{i=1}^{n-1}\tau_i^i$ are uniquely defined by G.*

References

[Ban96] T. Banica, *Théorie des représentations du groupe quantique compact libre* O(n), C. R. Acad. Sci. Paris Sér. I Math. **322** (1996), no. 3, 241–244, available at `arXiv:math/9806063v4 [math.QA]`. MR1378260 (97A:46108)

[Bic03] J. Bichon, *The representation category of the quantum group of a non-degenerate bilinear form*, Comm. Algebra **31** (2003), no. 10, 4831–4851, DOI 10.1081/AGB-120023135. MR1998031 (2004g:20068)

[Bic10] J. Bichon, *Hopf-Galois objects and cogroupoids.* preprint (2010), available at `arXiv:1006.3014 [math.QA]`.

[BDRV06] J. Bichon, A. De Rijdt, and S. Vaes, *Ergodic coactions with large multiplicity and monoidal equivalence of quantum groups*, Comm. Math. Phys. **262** (2006), no. 3, 703–728, DOI 10.1007/s00220-005-1442-2. MR2202309 (2007a:46072)

[CVOZ94] S. Caenepeel, F. Van Oystaeyen, and Y. H. Zhang, *Quantum Yang-Baxter module algebras*, Proceedings of Conference on Algebraic Geometry and Ring Theory in honor of Michael Artin, Part III (Antwerp, 1992), 1994, pp. 231–255, DOI 10.1007/BF00960863. MR1291020 (95e:16031)

[DCY12] K. De Commer and M. Yamashita, *Tannaka–Kreĭn duality for compact quantum homogeneous spaces. I. General theory.* preprint (2012), available at `arXiv:1211.6552 [math.OA]`, to appear in Theory Appl. Categ.

[DRVV10] A. De Rijdt and N. Vander Vennet, *Actions of monoidally equivalent compact quantum groups and applications to probabilistic boundaries*, Ann. Inst. Fourier (Grenoble) **60** (2010), no. 1, 169–216. MR2664313 (2011g:46128)

[Hai00] P. H. Hai, *On matrix quantum groups of type A_n*, Internat. J. Math. **11** (2000), no. 9, 1115–1146, DOI 10.1142/S0129167X00000581, available at `arXiv:q-alg/9708007 [math.QA]`. MR1809304 (2001m:16064)

[Han93] D. Handelman, *Representation rings as invariants for compact groups and limit ratio theorems for them*, Internat. J. Math. **4** (1993), no. 1, 59–88, DOI 10.1142/S0129167X93000054. MR1209960 (94c:22005)

[HY00] T. Hayashi and S. Yamagami, *Amenable tensor categories and their realizations as AFD bimodules*, J. Funct. Anal. **172** (2000), no. 1, 19–75, DOI 10.1006/jfan.1999.3521. MR1749868 (2001d:46092)

[HI98] F. Hiai and M. Izumi, *Amenability and strong amenability for fusion algebras with applications to subfactor theory*, Internat. J. Math. **9** (1998), no. 6, 669–722, DOI 10.1142/S0129167X98000300. MR1644299 (99h:46116)

[Izu02] M. Izumi, *Non-commutative Poisson boundaries and compact quantum group actions*, Adv. Math. **169** (2002), no. 1, 1–57, DOI 10.1006/aima.2001.2053. MR1916370 (2003j:46105)

[Izu12] M. Izumi, *E_0-semigroups: around and beyond Arveson's work*, J. Operator Theory **68** (2012), no. 2, 335–363. MR2995726

[INT06] M. Izumi, S. Neshveyev, and L. Tuset, *Poisson boundary of the dual of* $\mathrm{SU}_q(n)$, Comm. Math. Phys. **262** (2006), no. 2, 505–531, DOI 10.1007/s00220-005-1439-x. MR2200270 (2007f:58012)

[Jor] B. Jordans. in preparation.

[KLV10] D. Kazhdan, M. Larsen, and Y. Varshavsky, *The Tannakian Formalism and the Langlands Conjectures*. preprint (2010), available at `arXiv:1006.3864 [math.NT]`.

[KW93] D. Kazhdan and H. Wenzl, *Reconstructing monoidal categories*, I. M. Gel′fand Seminar, 1993, pp. 111–136. MR1237835 (95e:18007)

[KV83] V. A. Kaĭmanovich and A. M. Vershik, *Random walks on discrete groups: boundary and entropy*, Ann. Probab. **11** (1983), no. 3, 457–490. MR704539 (85d:60024)

[Lan92] M. B. Landstad, *Ergodic actions of nonabelian compact groups*, Ideas and methods in mathematical analysis, stochastics, and applications (Oslo, 1988), Cambridge Univ. Press, Cambridge, 1992, pp. 365–388. MR1190512 (93j:46072)

[LR97] R. Longo and J. E. Roberts, *A theory of dimension*, K-Theory **11** (1997), no. 2, 103–159, DOI 10.1023/A:1007714415067. MR1444286 (98i:46065)

[McM84] J. R. McMullen, *On the dual object of a compact connected group*, Math. Z. **185** (1984), no. 4, 539–552, DOI 10.1007/BF01236263. MR733774 (85e:22010)

[Mro12] C. Mrozinski, *Quantum groups of* $\mathrm{GL}(2)$ *representation type*. preprint (2012), available at `arXiv:1201.3494 [math.QA]`.

[Mro13] C. Mrozinski, *Quantum automorphism groups and* $\mathrm{SO}(3)$*-deformations*. preprint (2013), available at `arXiv:1303.7091 [math.QA]`.

[Nes13] S. Neshveyev, *Duality theory for nonergodic actions*, Münster J. Math. (2013), available at `arXiv:1303.6207 [math.OA]`. to appear in print.

[NT11] S. Neshveyev and L. Tuset, *On second cohomology of duals of compact groups*, Internat. J. Math. **22** (2011), no. 9, 1231–1260, DOI 10.1142/S0129167X11007239. MR2844801 (2012k:22010)

[NT12] S. Neshveyev and L. Tuset, *Quantized algebras of functions on homogeneous spaces with Poisson stabilizers*, Comm. Math. Phys. **312** (2012), no. 1, 223–250, DOI 10.1007/s00220-012-1455-6, available at `arXiv:1103.4346 [math.OA]`. MR2914062

[NT13] S. Neshveyev and L. Tuset, *Compact quantum groups and their representation categories*. manuscript (2013), to be published in Cours Spécialsés de la SMF; available at the first author's website.

[NY13] S. Neshveyev and M. Yamashita, *Twisting the q-deformations of compact semisimple Lie groups*. preprint (2013), available at `arXiv:1305.6949 [math.OA]`.

[NY] S. Neshveyev and M. Yamashita, *Poisson boundaries of monoidal categories*. in preparation.

[NV10] R. Nest and C. Voigt, *Equivariant Poincaré duality for quantum group actions*, J. Funct. Anal. **258** (2010), no. 5, 1466–1503, DOI 10.1016/j.jfa.2009.10.015, available at `arXiv:0902.3987 [math.KT]`. MR2566309

[Ohn99] C. Ohn, *Quantum* $\mathrm{SL}(3,\mathbf{C})$*'s with classical representation theory*, J. Algebra **213** (1999), no. 2, 721–756, DOI 10.1006/jabr.1998.7658, available at `arXiv:q-alg/9711005 [math.QA]`. MR1673475 (2000c:17028)

[Ohn05] C. Ohn, *Quantum* $\mathrm{SL}(3,\mathbb{C})$*'s: the missing case*, Hopf algebras in noncommutative geometry and physics, 2005, pp. 245–255, available at `arXiv:math/0210244 [math.QA]`. MR2106933 (2005h:20110)

[Pin07] C. Pinzari, *The representation category of the Woronowicz quantum group* $\mathrm{S}_\mu\mathrm{U}(d)$ *as a braided tensor* C^**-category*, Internat. J. Math. **18** (2007), no. 2, 113–136, DOI 10.1142/S0129167X07003972. MR2307417 (2008k:46212)

[PR11] C. Pinzari and J. E. Roberts, *A rigidity result for extensions of braided tensor* C^**-categories derived from compact matrix quantum groups*, Comm. Math. Phys. **306** (2011), no. 3, 647–662, DOI 10.1007/s00220-011-1260-7, available at `arXiv:1007.4480 [mtah.OA]`. MR2825504 (2012h:46125)

[PM98] P. Podleś and E. Müller, *Introduction to quantum groups*, Rev. Math. Phys. **10** (1998), no. 4, 511–551, DOI 10.1142/S0129055X98000173. MR1629723 (99g:46106)

[Sal11] P. Salmi, *Compact quantum subgroups and left invariant* C^**-subalgebras of locally compact quantum groups*, J. Funct. Anal. **261** (2011), no. 1, 1–24, DOI 10.1016/j.jfa.2011.03.003. MR2785890 (2012f:46159)

[Sch96] P. Schauenburg, *Hopf bi-Galois extensions*, Comm. Algebra **24** (1996), no. 12, 3797–3825, DOI 10.1080/00927879608825788. MR1408508 (97f:16064)

[Soł05] P. M. Sołtan, *Quantum Bohr compactification*, Illinois J. Math. **49** (2005), no. 4, 1245–1270. MR2210362 (2007a:46079)

[Tom06] R. Tomatsu, *Amenable discrete quantum groups*, J. Math. Soc. Japan **58** (2006), no. 4, 949–964. MR2276175 (2008g:46122)

[Tom07] R. Tomatsu, *A characterization of right coideals of quotient type and its application to classification of Poisson boundaries*, Comm. Math. Phys. **275** (2007), no. 1, 271–296, DOI 10.1007/s00220-007-0267-6. MR2335776 (2008j:46058)

[Ulb89] K.-H. Ulbrich, *Fibre functors of finite-dimensional comodules*, Manuscripta Math. **65** (1989), no. 1, 39–46, DOI 10.1007/BF01168365. MR1006625 (90e:16014)

[Was88a] A. Wassermann, *Ergodic actions of compact groups on operator algebras. II. Classification of full multiplicity ergodic actions*, Canad. J. Math. **40** (1988), no. 6, 1482–1527, DOI 10.4153/CJM-1988-068-4. MR990110 (92d:46168)

[Was88b] A. Wassermann, *Ergodic actions of compact groups on operator algebras. III. Classification for* SU(2), Invent. Math. **93** (1988), no. 2, 309–354, DOI 10.1007/BF01394336. MR948104 (91e:46093)

[Was88c] A. Wassermann, *Coactions and Yang-Baxter equations for ergodic actions and subfactors*, Operator algebras and applications, Vol. 2, London Math. Soc. Lecture Note Ser., vol. 136, Cambridge Univ. Press, Cambridge, 1988, pp. 203–236. MR996457 (92d:46167)

[Wor88] S. L. Woronowicz, *Tannaka-Kreĭn duality for compact matrix pseudogroups. Twisted* $\mathrm{SU}(N)$ *groups*, Invent. Math. **93** (1988), no. 1, 35–76, DOI 10.1007/BF01393687. MR943923 (90e:22033)

[WZ94] S. L. Woronowicz and S. Zakrzewski, *Quantum deformations of the Lorentz group. The Hopf* **-algebra level*, Compositio Math. **90** (1994), no. 2, 211–243. MR1266253 (95b:17021)

[Yam13] M. Yamashita, *Equivariant comparison of quantum homogeneous spaces*, Comm. Math. Phys. **317** (2013), no. 3, 593–614, DOI 10.1007/s00220-012-1594-9, available at `arXiv:1109.2991 [math.OA]`. MR3009718

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