The McMillan theorem for a class of asymptotically abelian C*-algebras

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Abstract

An extension of the Shannon-McMillan-Breiman theorem to a class of non-commutative dynamical systems is given.

1 Introduction

In ergodic theory one of the main theorems on entropy is the McMillan theorem, also called the Shannon-McMillan-Breiman theorem. In one form it states that if \((X, \mu)\) is a probability space and \(T\) a measure preserving ergodic transformation, then for any finite measurable partition \(\xi\) and any \(\varepsilon > 0\) there exists \(n_0\) such that if \(n \geq n_0\) then outside of a union of atoms of total measure < \(\varepsilon\) every atom in \(\bigvee_{i=0}^{n-1} T^{-i} \xi\) has measure in the interval \((e^{-n(H(T;\xi)+\varepsilon)}, e^{-n(H(T;\xi)-\varepsilon)})\).

In the present paper we shall give a non-commutative extension of the McMillan theorem. Our setting will be asymptotically abelian C*-dynamical systems with locality \((A, \tau, \alpha)\) [NS], where we assume \(\tau\) is an invariant ergodic trace. If \(N\) is a local subalgebra of \(A\) which is a mean generator [GS2], i.e. the C*-algebra \(\bigvee_{i\in\mathbb{Z}} \alpha^i(N) = A\) and the dynamical entropy \(H(\alpha)\) defined by [CS] and [CNT] satisfies \(H(\alpha) = \lim_{n \to \infty} \frac{1}{n} H(\bigvee_{i=0}^{n-1} \alpha^i(N))\), then we shall show that for any \(\varepsilon > 0\) and any \(n\) sufficiently large their exists a central projection \(z_n\) in \(N_n = \bigvee_{i=0}^{n-1} \alpha^i(N)\) such that \(\tau(z_n) < \varepsilon\) and

\[ e^{-n(H(\alpha)+\varepsilon)} < \tau(e) < e^{-n(H(\alpha)-\varepsilon)} \]

for any minimal projection \(e\) in \(N_n(1-z_n)\). The proof is based on ideas in [NS] and uses the classical McMillan theorem.

The above result forms the main contents of Section 2. Then in Section 3 we study sufficient conditions for a local algebra to be a mean generator. For example, we shall show that if \(N\) is a mean generator and \(M\) is a C*-algebra contained in \(\bigvee_{i\in\mathbb{Z}} \alpha^i(M)\) then \(M\) is itself a mean generator for the C*-algebra \(\bigvee_{i\in\mathbb{Z}} \alpha^i(M)\).

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2 The McMillan Theorem

Throughout the paper we consider $C^*$-dynamical systems $(A,\tau,\alpha)$ which are asymptotically abelian with locality [NS], i.e. we assume that there exists a dense $\alpha$-invariant $*$-subalgebra $A$ of $A$ such that for all pairs $a, b \in A$ the $C^*$-algebra they generate is finite dimensional, and there is $p = p(a, b) \in \mathbb{N}$ such that $|\alpha^j(a), b| = 0$ for $|j| \geq p$. Recall that by a local algebra we mean a finite dimensional subalgebra of $A$. We shall assume also that $\tau$ is an $\alpha$-invariant trace.

**Theorem 2.1** Suppose the trace $\tau$ is ergodic, i.e. is an extremal $\alpha$-invariant state. Suppose $N$ is a local algebra which is a mean generator and the entropy $H(\alpha)$ is finite, so

$$H(\alpha) = \lim_{n \to \infty} \frac{H(N_n)}{n} < \infty,$$

where $N_n = \vee_{i=0}^{n-1} \alpha^i(N)$. Then given $\varepsilon > 0$ there exist $n_0$ and central projections $z_n$ in $N_n$ such that $\tau(z_n) < \varepsilon$ for $n \geq n_0$ and

$$e^{-n(H(\alpha)+\varepsilon)} < \tau(e) < e^{-n(H(\alpha)-\varepsilon)}$$

for any minimal projection $e$ in $N_n(1-z_n)$.

Note that since $\tau$ is a trace, ergodicity of $\tau$ is equivalent to ergodicity of the automorphism $\alpha$ on the weak closure $\pi_\tau(A)'$ of the GNS-representation of $A$. Note also that we can consider $\pi_\tau(A)$ instead of $A$, and so assume that $\tau$ is faithful.

For all examples we know, the assumption on mean generator is always fulfilled. However, we were unable to prove that it is automatic. In the next section we shall discuss several sufficient conditions for a local algebra to be a mean generator.

To prove Theorem 2.1 we shall need the first part of the following proposition.

**Proposition 2.2** Let $N$ be a local algebra, $p \in \mathbb{N}$ such that $\alpha^j(N)$ commutes with $N$ for $|j| \geq p$. Then

(i) $H(\alpha|_{A(N)}) = \lim_{n \to \infty} \frac{H(N_{n-p}, \alpha^n)}{n}$, where $A(N) = \vee_{i \in \mathbb{Z}} \alpha^i(N)$;

(ii) $\text{hcpa}_\tau(\alpha|_{A(N)}) = \lim_{n \to \infty} \frac{\text{hcpa}_\tau(N_{n-p}, \alpha^n)}{n}$, where $\text{hcpa}_\tau(N_{n-p}, \alpha^n)$ is Voiculescu’s completely positive approximation entropy of $\alpha^n$ computed with respect to any finite set spanning $N_{n-p}$.

**Proof.** (i) Since $H(N_{n-p}, \alpha^n) \leq nH(\alpha|_{A(N)})$, it is enough to prove that for any $m \in \mathbb{N}$

$$H(N_m, \alpha) \leq \liminf_{n \to \infty} \frac{H(N_{n-p}, \alpha^n)}{n}.$$ 

For $k \in \mathbb{N}$ set $I_k = [0, kn-1]$ and $I_k^l = \cup_{l=0}^{k-1} [ln, (l+1)n-p-m]$. Since $\alpha^i(N_m) \subset \alpha^\ln(N_{n-p})$ for $i \in [ln, (l+1)n-p-m]$, we have

$$H(\{\alpha^i(N_m)\}_{i \in I_k}) \leq H(\{\alpha^i(N_m)\}_{i \in I_k^l}) + |I_k \setminus I_k^l| H(N_m)$$

$$\leq H(\{\alpha^\ln(N_{n-p})\}_{l=0}^{k-1}) + k(p + m - 1) H(N_m).$$

Dividing by $kn$ and letting $k \to \infty$ we get

$$H(N_m, \alpha) \leq \frac{H(N_{n-p}, \alpha^n)}{n} + \frac{p + m - 1}{n} H(N_m).$$
(ii) This part is proved analogously to [NS, Lemma 3.4]. We have to prove that for any $m \in \mathbb{N}$
\[
hcpa_{\tau}(N_{m}, \alpha) \leq \liminf_{n \to \infty} \frac{hcpa_{\tau}(N_{n-p}, \alpha^{\alpha^{n}})}{n}.
\]
For this fix $m_0$ and take $n \geq m + m_0 + p$ and $k \in \mathbb{N}$. Let $(B_0, \psi_0, \rho_0) \in \text{CFA}(A, \tau)$ be such that $||(\psi_0 \circ \rho_0)(a) - a||_{\tau} \leq \varepsilon ||a||$ for any $a \in \bigcup_{j=0}^{k} \alpha^{j\alpha}(N_{n-p})$. Consider the triple $(B_1, \psi_1, \rho_1)$, where $B_1 = B_{0}^{m_0}$,
\[
\psi_1(b_1, \ldots, b_{m_0}) = \frac{1}{m_0} \sum_{i=1}^{m_0} \alpha^{-i}(\psi_0(b_i)),
\]
\[
\rho_1(a) = (\rho_0(a), (\rho_0 \circ \alpha)(a), \ldots, (\rho_0 \circ \alpha^{m_0-1})(a)).
\]
Then, for $i = 0, \ldots, kn - 1$ and $a \in N_m$, we have as in the proof of [NS, Lemma 3.4]
\[
||(\psi_1 \circ \rho_1)(\alpha^i(a)) - \alpha^i(a)||_{\tau} \leq \left( \varepsilon + \frac{2p}{m_0} \right) ||a||.
\]
It follows that
\[
hcpa_{\tau}(N_{m}, \alpha; \varepsilon + 2p/m_0) \leq \frac{hcpa_{\tau}(N_{n-p}, \alpha^{\alpha^{n}}; \varepsilon)}{n} \leq \frac{hcpa_{\tau}(N_{n-p}, \alpha^{\alpha^{n}})}{n}.
\]
Letting first $n \to \infty$ and then $\varepsilon \to 0$ and $m_0 \to \infty$, we obtain the desired inequality.

Since $H_{\tau_1 \otimes \tau_2}(M_1 \otimes M_2) = H_{\tau_1}(M_1) + H_{\tau_2}(M_2)$ for finite dimensional $C^*$-algebras $M_1$ and $M_2$ with normalized traces $\tau_1$ and $\tau_2$ respectively, we have

**Corollary 2.3** For systems which are asymptotically abelian with locality the tensor product formula for the entropy with respect to tracial states holds.

Let $\{M_n\}_n$ be a sequence of finite dimensional $C^*$-algebras, $h$ a non-negative number, $\varepsilon > 0$. We shall say that the McMillan theorem holds for $\{M_n\}_n$, $h$ and $\varepsilon$, if the conclusion of Theorem 2.1 holds for $M_n$ instead of $N_n$, and $h$ instead of $H(\alpha)$. So there exist $n_0$ and central projections $z_n$ in $M_n$ such that $\tau(z_n) < \varepsilon$ for $n \geq n_0$ and
\[
e^{-n(h+\varepsilon)} < \tau(e) < e^{-n(h-\varepsilon)}
\]
for any minimal projection $e$ in $M_n(1 - z_n)$.

**Lemma 2.4** For any $\varepsilon > 0$ there exists $\varepsilon_1 > 0$ such that if $\{\xi_n\}_n$ and $\{\zeta_n\}_n$ are sequences of finite partitions of a Lebesgue space $(X, \mu)$, $\zeta_n < \xi_n$, $h = \lim \frac{1}{n} H(\xi_n)$, $h_1 = \lim \frac{1}{n} H(\zeta_n)$, $h < h_1 + \varepsilon_1$ and the McMillan theorem holds for $\{\xi_n\}_n$, $h_1$ and $\varepsilon_1$, then it holds also for $\{\zeta_n\}_n$, $h$ and $\varepsilon$.

**Proof.** Let $\{C_n\}_{i \in X_n}$ be the atoms of $\xi_n$, $\{D_{nj}\}_{j \in Y_n}$ the atoms of $\zeta_n$. Let $Y_n$ be the set of $j \in Y_n$ for which
\[
e^{-n(h_1+\varepsilon_1)} < \mu(D_{nj}) < e^{-n(h_1-\varepsilon_1)}.
\]
For $n$ large enough
\[
0 \leq \frac{1}{n} (H(\xi_n) - H(\zeta_n)) < \varepsilon_1,
\]
or

\[ \frac{1}{n} \sum_{i \in X_n, j \in Y_n} \mu(C_{ni} \cap D_{nj}) \log \frac{\mu(D_{nj})}{\mu(C_{ni})} < \varepsilon_1. \] (2.1)

For \( i \in X_n \) let \( j(i) \) be a unique index such that \( C_{ni} \subset D_{nj(i)} \). Let

\[ X'_n = \{ i \in X_n \mid j(i) \in \tilde{Y}_n \}, \quad X''_n = \{ i \in X_n \mid \mu(D_{nj(i)}) \leq e^{n\sqrt{\varepsilon_1}} \mu(C_{ni}) \}, \quad \tilde{X}_n = X'_n \cap X''_n. \]

By virtue of (2.1)

\[ \sum_{i \in X_n \setminus X''_n} \mu(C_{ni}) < \sqrt{\varepsilon_1}. \]

Hence, for \( n \) large enough,

\[ \sum_{i \in X_n} \mu(C_{ni}) \leq \sum_{i \notin X'_n} \mu(C_{ni}) + \sum_{i \in X'_n} \mu(C_{ni}) \leq \sum_{j \notin Y_n} \mu(D_{nj}) + \sum_{i \notin X''_n} \mu(C_{ni}) < \varepsilon_1 + \sqrt{\varepsilon_1}. \]

For \( i \in \tilde{X}_n \)

\[ \mu(C_{ni}) \leq \mu(D_{nj(i)}) < e^{-n(h_1 - \varepsilon_1)} < e^{-n(h - 2\varepsilon_1)} \]

and

\[ \mu(C_{ni}) \geq e^{-n\sqrt{\varepsilon_1}} \mu(D_{nj(i)}) > e^{-n(h_1 + \varepsilon_1 + \sqrt{\varepsilon_1})} \geq e^{-n(h + \varepsilon_1 + \sqrt{\varepsilon_1})}. \]

Thus we can take \( \varepsilon_1 \) such that \( 2\varepsilon_1 + \sqrt{\varepsilon_1} < \varepsilon \).

\[ \square \]

**Lemma 2.5** Let \( \{\xi_n\}_n \) be an increasing sequence of finite partitions of a Lebesgue space \( (X, \mu) \), \( \zeta \) a finite partition such that \( \zeta \prec \vee_n \xi_n \). Suppose the McMillan theorem holds for \( \{\xi_n \vee \zeta\}_n \), \( h \) and \( \varepsilon \). Then it holds also for \( \{\xi_n\}_n \), \( h \) and \( 2\varepsilon \).

**Proof.** Let \( \{C_{ni}\}_{i \in X_n} \) be the atoms of \( \xi_n \), \( \{D_j\}_{j \in Y} \) the atoms of \( \zeta \). Let

\[ X'_n = \{ i \in X_n \mid \exists j(i) \in Y : \mu(C_{ni} \cap D_{j(i)}) > (1 - \varepsilon/2)\mu(C_{ni}) \}. \]

We assert that

\[ \sum_{i \notin X'_n} \mu(C_{ni}) \to 0 \quad \text{as} \quad n \to \infty. \]

Indeed, otherwise there exist \( j \in Y \) and \( c > 0 \) such that

\[ \sum_{i \notin X'_n} \mu(C_{ni} \cap D_j) \geq c \]

for infinitely many \( n \)’s. In other words, the set \( E_n = \cup_{i \notin X'_n} (C_{ni} \cap D_j) \) has measure \( \mu(E_n) \geq c \), and for the conditional expectation \( E(D_j | \xi_n) \) of the characteristic function of the set \( D_j \) with respect to the partition \( \xi_n \) we have

\[ E(D_j | \xi_n)(x) \leq 1 - \frac{\varepsilon}{2} \quad \text{on} \quad E_n, \]

what contradicts the almost everywhere convergence of \( \{E(D_j | \xi_n)\}_n \) to the characteristic function of \( D_j \).
Now let $Z_n$ be the set of pairs $(i, j)$ such that
\[ e^{-n(h+\varepsilon)} < \mu(C_{ni} \cap D_j) < e^{-n(h-\varepsilon)}, \]
and
\[ \tilde{X}_n = \{i \in X_n' \mid (i, j(i)) \in Z_n\}. \]
For $i \in \tilde{X}_n$
\[ \mu(C_{ni}) \geq \mu(C_{ni} \cap D_{j(i)}) > e^{-n(h+\varepsilon)}, \]
and for $n$ large enough
\[ \mu(C_{ni}) < \frac{1}{1-\varepsilon/2} \mu(C_{ni} \cap D_{j(i)}) < e^{n\varepsilon} \mu(C_{ni} \cap D_{j(i)}) < e^{-n(h-2\varepsilon)}. \]
Since for large $n$
\[ \sum_{(i, j) \in Z_n} \mu(C_{ni} \cap D_j) > 1 - \varepsilon \]
and
\[ \sum_{i \in X_n} \mu(C_{ni} \cap D_{j(i)}) > (1-\varepsilon/2) \sum_{i \in X_n} \mu(C_{ni}) > 1 - \varepsilon, \]
we have also
\[ \sum_{i \in \tilde{X}_n} \mu(C_{ni}) \geq \sum_{i \in \tilde{X}_n} \mu(C_{ni} \cap D_{j(i)}) > 1 - 2\varepsilon. \]

**Proof of Theorem 2.1.** Let $C_n$ be a masa in $N_n$, i.e. a maximal abelian $*$-subalgebra. We may suppose that $C_n \subset C_{n+1}$. Then we have to prove that the McMillan theorem holds for $\{C_n\}_n$, $H(\alpha)$ and any $\varepsilon > 0$ (note that if a projection $z'_n$ is chosen in $C_n$ as in the statement of the McMillan theorem then we can replace $1-z'_n$ by its central support $p_n$ in $N_n$ and let $z_n = 1-p_n$).

Using Proposition 2.2(i) choose $m \in \mathbb{N}$ such that
\[ \left| H(\alpha) - \frac{1}{m} H(N_{m-p}, \alpha^m) \right| < \varepsilon_1^2, \]
where $\varepsilon_1$ is as in Lemma 2.4. Let $A_m$ be the von Neumann subalgebra of $\pi_\tau(A)^m$ generated by $[\frac{\pi}{m}]^{-1}$, $j \in \mathbb{Z}$, $D$ a masa in $N_{m-p}$. Then $D_n = \bigvee_{j=0}^{m-1} \alpha^j(D)$ is a masa in $\bigvee_{j=0}^{m-1} \alpha^j(N_{m-p})$.

If $\alpha^m$ was ergodic, we could apply the classical McMillan theorem to $\{D_n\}_n$ and then make use of Lemma 2.4 to conclude that it holds also for $\{C_n\}_n$. Since $\alpha^m$ can be non-ergodic, consider the fixed point algebra $Z \subset \pi_\tau(A)^m$ with respect to $\alpha^m$. Since $\alpha$ is asymptotically abelian and ergodic on $\pi_\tau(A)^m$, $Z$ is a finite dimensional subalgebra of the center of $\pi_\tau(A)^m$. Let $\{z_i\}_{i \in X}$ be the atoms of $Z$. The automorphism $\alpha$ acts transitively on the set of atoms, so the systems $(A_{z_i}, \tau_i, \alpha^m|_{A_{z_i}})$ are pairwise conjugate, where $\tau_i = |X||_{A_{z_i}}$ and $|X|$ denotes the cardinality of $X$. Since $\frac{1}{|X|} \sum_{i \in X} H(\alpha^m|_{A_{z_i}}) = H(\alpha^m)$, we conclude that $H(\alpha^m|_{A_{z_i}}) = mH(\alpha)$, so for $H_i = \frac{1}{m} H(\alpha^m|_{A_{z_i}})$ we have
\[ H_i \leq \frac{1}{m} H(\alpha^m|_{A_{z_i}}) = H(\alpha). \]
On the other hand,
\[ \frac{1}{|X|} \sum_{i} H_i = \frac{1}{m} H(\alpha^m|_{A_m \vee Z}) = \frac{1}{m} H(N_{m-p} \vee Z, \alpha^m) = \frac{1}{m} H(N_{m-p}, \alpha^m) > H(\alpha) - \varepsilon_1^2. \]
Thus if \( \tilde{X} = \{i \in X \mid H_i > H(\alpha) - \varepsilon_1\} \) then
\[ H(\alpha) - \varepsilon_1^2 < \frac{|\tilde{X}|}{|X|} H(\alpha) + \frac{|X \setminus \tilde{X}|}{|X|} (H(\alpha) - \varepsilon_1), \]
whence
\[ \sum_{i \notin \tilde{X}} \tau(z_i) = \frac{|X \setminus \tilde{X}|}{|X|} < \varepsilon_1 < \varepsilon. \tag{2.2} \]

We have also
\[ H(\alpha) = \frac{1}{m} H(\alpha^m |_{A_{z_i}}) \leq \liminf_{n \to \infty} \frac{H_{\tau_i}(C_n z_i)}{n}, \]
and
\[ \lim_{n \to \infty} \frac{1}{|X|} \sum_{i} \frac{H_{\tau_i}(C_n z_i)}{n} = \lim_{n \to \infty} \frac{H(C_n \vee Z)}{n} = \lim_{n \to \infty} \frac{H(C_n)}{n} = H(\alpha), \]
so
\[ \lim_{n \to \infty} \frac{H_{\tau_i}(C_n z_i)}{n} = H(\alpha) \quad \forall \ i \in X. \]

For any \( i \in X \) the automorphism \( \alpha^m \) is ergodic on \( A_{z_i} \), so by the classical result the McMillan theorem holds for \( \{D_n z_i\}_n \), \( H_i \) and \( \varepsilon_1 \). Then by Lemma 2.4, for \( i \in \tilde{X} \), it holds for \( \{C_n z_i\}_n \), \( H(\alpha) \) and \( \varepsilon \) (with respect to the trace \( \tau_i \)). By virtue of (2.2) it holds also for \( \{C_n \vee Z\}_n \), \( H(\alpha) \) and \( 2\varepsilon \). Finally, by Lemma 2.5 the McMillan theorem holds for \( \{C_n\}_n \), \( H(\alpha) \) and \( 4\varepsilon \).  

The method used in the proof can be applied to prove the following weak form of the McMillan theorem under more general assumptions.

**Theorem 2.6** Let \((A, \tau, \alpha)\) be an asymptotically abelian system with locality, and \( \tau \) an ergodic trace. Then the entropy \( H(\alpha) \) of Connes and Størmer of the system coincides with Voiculescu’s completely positive approximation entropy \( \text{hcpa}_\tau(\alpha) \).

**Proof.** Let \( N \) be a local subalgebra of \( A \). It suffices to prove that \( \text{hcpa}_\tau(N, \alpha) \leq H(\alpha) \). Then by Proposition 2.2(ii) it is enough to prove that \( \text{hcpa}_\tau(N_{m-p}, \alpha^m) \leq mH(\alpha) \). Keep the notations of the proof of Theorem 2.1. By the classical McMillan theorem we have \( \text{hcpa}_\tau(N_{m-p z_i}, \alpha^m) = H(N_{m-p z_i}, \alpha^m) \) (see the proof of [V, Proposition 1.7]). Since \( H(N_{m-p z_i}, \alpha^m) \leq H(\alpha^m |_{A_{z_i}}) = mH(\alpha) \) and
\[ \text{hcpa}_\tau(N_{m-p}, \alpha^m) \leq \text{hcpa}_\tau(N_{m-p} \vee Z, \alpha^m) \leq \max_i \text{hcpa}_\tau(N_{m-p z_i}, \alpha^m), \]
we obtain the desired inequality.  

**Remarks 2.7** (i) The assumption that \( N \) is a mean generator in Theorem 2.1 is very close to being necessary. Indeed, suppose \( N \) satisfies the conclusion of Theorem 2.1. Then
\[ \log \text{rank} N_n (1-z_n) < n(H(\alpha) + \varepsilon). \]
Thus
\[ H(N_n(1 - z_n) \oplus \mathbb{C}z_n) \leq \log \text{rank } (N_n(1 - z_n) \oplus \mathbb{C}z_n) \leq \log(e^{n(H(\alpha)+\varepsilon)} + 1), \]
so
\[ \limsup_n \frac{1}{n} H(N_n(1 - z_n) \oplus \mathbb{C}z_n) \leq H(\alpha) + \varepsilon \leq \liminf_n \frac{1}{n} H(N_n) + \varepsilon. \]
In particular, if the \( N_n \)'s don't grow too fast, e.g. if \( \text{rank } N_n \leq e^{Cn} \) for some \( C > 0 \), then
\[ \limsup_n \frac{1}{n} H(N_n) \leq \limsup_n \left( C\tau(z_n) + \frac{1}{n} H(N_n(1 - z_n) \oplus \mathbb{C}z_n) \right), \]
and hence
\[ H(\alpha) = \lim_n \frac{H(N_n)}{n}. \]
(ii) It is not clear what the optimal assumptions are for Theorem 2.1 to be true. The conclusion holds in several cases when the dynamical system is not asymptotically abelian. Such an example is that of a binary shift, see [PP]. Then we are given a subset \( X \) of \( \mathbb{N} \) and the algebra \( A(X) \) generated by symmetries \( \{s_n\}_{n \in \mathbb{Z}} \) satisfying the commutation relations
\[ s_is_j = \begin{cases} s_js_i, & \text{if } |i - j| \notin X, \\ -s_js_i, & \text{if } |i - j| \in X. \end{cases} \]
The binary shift is the automorphism \( \alpha \) defined by \( \alpha(s_n) = s_{n+1} \). Let \( A_n = C^*(s_0, \ldots, s_{n-1}) \). Then \( A_n \cong \text{Mat}_{2d_n}(\mathbb{C}) \otimes \mathbb{Z}_n \), where \( \mathbb{Z}_n \) is the diagonal in \( \text{Mat}_{2^{\infty}}(\mathbb{C}) \), \( n = 2d_n + c_n \), and thus any minimal projection in \( A_n \) has trace \( 2^{-d_n-c_n} \). It follows that \( H(A_n) = \log \text{rank } A_n = (c_n + d_n) \log 2 \). The sequence \( \{c_n\}_n \) varies with different \( X \). If we assume \( -X \cup \{0\} \cup X \) is nonperiodic then \( \{c_n\}_n \) satisfies \( |c_n - c_{n+1}| = 1 \) and takes the value zero an infinite number of times. Thus \( A(X) \) is the CAR-algebra with tracial state \( \tau \). By [GS1] the \( C^* \)-dynamical system \( (A(X), \tau, \alpha) \) is asymptotically abelian with locality if and only if \( X \) is finite, in which case the sequence \( \{c_n\}_n \) is bounded. We show that the following three conditions are equivalent. By [GS1] they are satisfied not only for asymptotically abelian systems.

(i) The conclusion of Theorem 2.1 holds for each algebra \( A_n \) (instead of \( N \)).
(ii) \( H(\alpha) = \frac{1}{2} \log 2 \) and \( \lim_{n \to \infty} \frac{\alpha_n}{n} = 0 \).
(iii) \( H(\alpha) = \lim_n \frac{1}{n} H(A_n). \)

Indeed, the implication (i) \( \Rightarrow \) (iii) follows from the previous remark.

Since \( H(\alpha) \leq \liminf_n \frac{1}{n} H(A_n) = \frac{1}{2} \log 2 \), see e.g. [GS2], (iii) implies \( \lim_n \frac{1}{n} H(A_n) = \frac{1}{2} \log 2 \), hence by [GS2, Lemma 4.7], \( \lim_n \frac{\alpha_n}{n} = 0 \). Thus (iii) \( \Rightarrow \) (ii).

If (ii) holds then \( \frac{1}{n}(d_n + c_n) \to \frac{1}{2} \), and so (ii) implies (i).

3 Mean Generators

In this section we discuss several sufficient conditions for a local algebra to be a mean generator.

Our first result shows that it is enough to prove that at least one algebra is a mean generator.

**Proposition 3.1** If \( N \) is a mean generator for \( A(N) \), \( H(\alpha|_{A(N)}) < \infty \) and \( M \) is a subalgebra of \( \bigvee_{-n}^n \alpha^k(N) \) for some \( n \), then \( M \) is a mean generator for \( A(M) \).
Proof. Without loss of generality we may suppose that $M \subset N$. As above, let $p$ be such that $\alpha^j(N)$ commutes with $N$ for $|j| \geq p$. Let $D_n$ be a masa in $M_{n-p}$, $C_n$ a masa in $N_{n-p}$ containing $D_n$. We have

$$0 \leq H(D_n) - H(D_n, \alpha^n) = H(C_n) - (H(D_n, \alpha^n) + H(C_n|D_n)) \leq H(C_n) - H(C_n, \alpha^n).$$

Since $\frac{1}{n}(H(C_n) - H(C_n, \alpha^n)) \to 0$ by assumption and Proposition 2.2, we conclude that

$$\frac{1}{n}(H(D_n) - H(D_n, \alpha^n)) \to 0,$$

hence $M$ is a mean generator.

We have used in the proof that if $H(\alpha|A(N)) < \infty$ then $N$ is a mean generator if and only if

$$\lim_{n \to \infty} \frac{1}{n}(H(N_{n-p}) - H(N_{n-p}, \alpha^n)) = 0.$$

Using the following lemma we shall show that it suffices to check this condition using any subalgebra of $N_n$ containing the center $Z(N_n)$ instead of $N_n$.

Lemma 3.2 Let $M_1, \ldots, M_n$ be commuting finite dimensional algebras, $Z(M_i) \subset A_i \subset M_i$. Then

$$H(\lor_i M_i) - H(\lor_i A_i) = \sum_i (H(M_i) - H(A_i)).$$

Proof. It suffices to consider the case $n = 2$. Let $\{e_i\}_i$ be the atoms of a masa in $A_1$, $\{f_j\}_j$ the atoms of a masa in $A_2$. Let $(M_1)_{e_i}$ be a factor of type $I_{m_i}$, $(M_2)_{f_j}$ a factor of type $I_{n_j}$. Then

$$H(M_1 \lor M_2) - H(A_1 \lor A_2) = -\sum_{i,j} \tau(e_i f_j) \log \frac{\tau(e_i f_j)}{m_i n_j} + \sum_{i,j} \tau(e_i f_j) \log \tau(e_i f_j)
\quad = \sum_i \tau(e_i) \log m_i + \sum_j \tau(f_j) \log n_j
\quad = (H(M_1) - H(A_1)) + (H(M_2) - H(A_2)).$$

Proposition 3.3 Let $N$ be a local algebra such that $H(\alpha|A(N)) < \infty$. Let $\{M_n\}_n$ be a sequence of algebras such that $Z(N_{n-p}) \subset M_n \subset N_{n-p}$. Then $N$ is a mean generator for $A(N)$ if and only if

$$\lim_{n \to \infty} \frac{1}{n}(H(M_n) - H(M_n, \alpha^n)) = 0.$$

In particular, $N$ is a mean generator if $\lim_n \frac{1}{n} H(Z(N_n)) = 0$.

Proof. For any $m \in \mathbb{N}$, by Lemma 3.2

$$H(\lor_j \alpha^{-j}(N_{n-p})) - H(\lor_j \alpha^{-j}(M_n)) = m(H(N_{n-p}) - H(M_n)),$$

hence

$$H(N_{n-p}) - H(N_{n-p}, \alpha^n) = H(M_n) - H(M_n, \alpha^n),$$

what gives the result.

Finally, recall the following simple condition (see e.g. [Ch]).
Proposition 3.4  Let $N$ be a local algebra. Suppose there exists $p \in \mathbb{N}$ such that

$$\tau(ab) = \tau(a)\tau(b)$$

for any $a \in \vee_{-\infty}^{0} \alpha^{1}(N)$ and $b \in \vee_{p}^{\infty} \alpha^{1}(N)$. Then $N$ is a mean generator for $A(N)$.

The last condition shows that Theorem 2.1 can be applied to asymptotically abelian binary shifts and canonical shifts on towers of relative commutants. On the other hand, Proposition 3.1 allows to apply Theorem 2.1 to systems arising from topological dynamics, which were considered in [NS, Section 5]. Indeed, if a local algebra has a masa lying in the diagonal then it is a mean generator since all computations are reduced to the abelian case. Then Proposition 3.1 shows that any local algebra is a mean generator.

References


