The McMillan theorem for a class of asymptotically abelian C^*-algebras

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Abstract

An extension of the Shannon-McMillan-Breiman theorem to a class of non-commutative dynamical systems is given.

1 Introduction

In ergodic theory one of the main theorems on entropy is the McMillan theorem, also called the Shannon-McMillan-Breiman theorem. In one form it states that if (X, μ) is a probability space and T a measure preserving ergodic transformation, then for any finite measurable partition ξ and any $\varepsilon > 0$ there exists n_0 such that if $n \geq n_0$ then outside of a union of atoms of total measure $< \varepsilon$ every atom in $\vee_{i=0}^{n-1} T^{-i}\xi$ has measure in the interval $(e^{-n(H(T;\xi)+\varepsilon)}, e^{-n(H(T;\xi)-\varepsilon)})$.

In the present paper we shall give a non-commutative extension of the McMillan theorem. Our setting will be asymptotically abelian C*-dynamical systems with locality (A, τ, α) [NS], where we assume τ is an invariant ergodic trace. If N is a local subalgebra of A which is a mean generator [GS2], i.e. the C*-algebra $\vee_{i\in\mathbb{Z}}\alpha^i(N) = A$ and the dynamical entropy $H(\alpha)$ defined by [CS] and [CNT] satisfies $H(\alpha) = \lim \frac{1}{n} H(\vee_{i=0}^{n-1}\alpha^i(N))$, then we shall show that for any $\varepsilon > 0$ and any n sufficiently large their exists a central projection z_n in $N_n = \vee_{i=0}^{n-1}\alpha^i(N)$ such that $\tau(z_n) < \varepsilon$ and

$$e^{-n(H(\alpha)+\varepsilon)} < \tau(e) < e^{-n(H(\alpha)-\varepsilon)}$$

for any minimal projection e in $N_n(1 - z_n)$. The proof is based on ideas in [NS] and uses the classical McMillan theorem.

The above result forms the main contents of Section 2. Then in Section 3 we study sufficient conditions for a local algebra to be a mean generator. For example, we shall show that if N is a mean generator and M is a C*-algebra contained in $\vee_{-n}^{n}\alpha^i(N)$ then M is itself a mean generator for the C*-algebra $\vee_{i\in\mathbb{Z}}\alpha^i(M)$.

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2 The McMillan Theorem

Throughout the paper we consider C*-dynamical systems (A, τ, α) which are asymptotically abelian with locality [NS], i.e. we assume that there exists a dense α-invariant $*$-subalgebra $\mathcal{A}$ of A such that for all pairs $a, b \in \mathcal{A}$ the C*-algebra they generate is finite dimensional, and there is $p = p(a,b) \in \mathbb{N}$ such that $[\alpha^j(a), b] = 0$ for $|j| \geq p$. Recall that by a local algebra we mean a finite dimensional subalgebra of $\mathcal{A}$. We shall assume also that τ is an α-invariant trace.

Theorem 2.1 *Suppose the trace τ is ergodic, i.e. is an extremal α-invariant state. Suppose N is a local algebra which is a mean generator and the entropy $H(\alpha)$ is finite, so*

$$H(\alpha) = \lim_{n\to\infty} \frac{H(N_n)}{n} < \infty,$$

where $N_n = \vee_{i=0}^{n-1} \alpha^i(N)$. Then given $\varepsilon > 0$ there exist n_0 and central projections z_n in N_n such that $\tau(z_n) < \varepsilon$ for $n \geq n_0$ and

$$e^{-n(H(\alpha)+\varepsilon)} < \tau(e) < e^{-n(H(\alpha)-\varepsilon)}$$

for any minimal projection e in $N_n(1 - z_n)$.

Note that since τ is a trace, ergodicity of τ is equivalent to ergodicity of the automorphism α on the weak closure $\pi_\tau(A)''$ of the GNS-representation of A. Note also that we can consider $\pi_\tau(A)$ instead of A, and so assume that τ is faithful.

For all examples we know, the assumption on mean generator is always fulfilled. However, we were unable to prove that it is automatic. In the next section we shall discuss several sufficient conditions for a local algebra to be a mean generator.

To prove Theorem 2.1 we shall need the first part of the following proposition.

Proposition 2.2 *Let N be a local algebra, $p \in \mathbb{N}$ such that $\alpha^j(N)$ commutes with N for $|j| \geq p$. Then*

(i) $H(\alpha|_{A(N)}) = \lim_{n\to\infty} \frac{H(N_{n-p}\,, \alpha^n)}{n}$, *where $A(N) = \vee_{i\in\mathbb{Z}} \alpha^i(N)$;*

(ii) $hcpa_\tau(\alpha|_{A(N)}) = \lim_{n\to\infty} \frac{hcpa_\tau(N_{n-p}\,, \alpha^n)}{n}$, *where $hcpa_\tau(N_{n-p}\,, \alpha^n)$ is Voiculescu's completely positive approximation entropy of α^n computed with respect to any finite set spanning N_{n-p}.*

Proof. (i) Since $H(N_{n-p}\,, \alpha^n) \leq nH(\alpha|_{A(N)})$, it is enough to prove that for any $m \in \mathbb{N}$

$$H(N_m\,, \alpha) \leq \liminf_{n\to\infty} \frac{H(N_{n-p}\,, \alpha^n)}{n}.$$

For $k \in \mathbb{N}$ set $I_k = [0, kn-1]$ and $I'_k = \cup_{l=0}^{k-1} [ln, (l+1)n - p - m]$. Since $\alpha^i(N_m) \subset \alpha^{ln}(N_{n-p})$ for $i \in [ln, (l+1)n - p - m]$, we have

$$\begin{aligned} H(\{\alpha^i(N_m)\}_{i\in I_k}) &\leq H(\{\alpha^i(N_m)\}_{i\in I'_k}) + |I_k \backslash I'_k| H(N_m) \\ &\leq H(\{\alpha^{ln}(N_{n-p})\}_{l=0}^{k-1}) + k(p + m - 1)H(N_m). \end{aligned}$$

Dividing by kn and letting $k \to \infty$ we get

$$H(N_m\,, \alpha) \leq \frac{H(N_{n-p}\,, \alpha^n)}{n} + \frac{p + m - 1}{n} H(N_m).$$

(ii) This part is proved analogously to [NS, Lemma 3.4]. We have to prove that for any $m \in \mathbb{N}$

$$hcpa_\tau(N_m\,, \alpha) \leq \liminf_{n\to\infty} \frac{hcpa_\tau(N_{n-p}\,, \alpha^n)}{n}.$$

For this fix m_0 and take $n \geq m + m_0 + p$ and $k \in \mathbb{N}$. Let $(B_0, \psi_0, \rho_0) \in \mathrm{CPA}(A, \tau)$ be such that $||(\psi_0 \circ \rho_0)(a) - a||_\tau \leq \varepsilon ||a||$ for any $a \in \cup_{j=0}^k \alpha^{jn}(N_{n-p})$. Consider the triple (B_1, ψ_1, ρ_1), where $B_1 = B_0^{m_0}$,

$$\psi_1(b_1, \ldots, b_{m_0}) = \frac{1}{m_0} \sum_{i=1}^{m_0} \alpha^{-i+1}(\psi_0(b_i)),$$

$$\rho_1(a) = (\rho_0(a), (\rho_0 \circ \alpha)(a), \ldots, (\rho_0 \circ \alpha^{m_0-1})(a)).$$

Then, for $i = 0, \ldots, kn-1$ and $a \in N_m$, we have as in the proof of [NS, Lemma 3.4]

$$||(\psi_1 \circ \rho_1)(\alpha^i(a)) - \alpha^i(a)||_\tau \leq \left(\varepsilon + \frac{2p}{m_0}\right) ||a||.$$

It follows that

$$hcpa_\tau(N_m, \alpha; \varepsilon + 2p/m_0) \leq \frac{hcpa_\tau(N_{n-p}, \alpha^n; \varepsilon)}{n} \leq \frac{hcpa_\tau(N_{n-p}, \alpha^n)}{n}.$$

Letting first $n \to \infty$ and then $\varepsilon \to 0$ and $m_0 \to \infty$, we obtain the desired inequality.

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Since $H_{\tau_1 \otimes \tau_2}(M_1 \otimes M_2) = H_{\tau_1}(M_1) + H_{\tau_2}(M_2)$ for finite dimensional C*-algebras M_1 and M_2 with normalized traces τ_1 and τ_2 respectively, we have

Corollary 2.3 *For systems which are asymptotically abelian with locality the tensor product formula for the entropy with respect to tracial states holds.*

Let $\{M_n\}_n$ be a sequence of finite dimensional C*-algebras, h a non-negative number, $\varepsilon > 0$. We shall say that the McMillan theorem holds for $\{M_n\}_n$, h and ε, if the conclusion of Theorem 2.1 holds for M_n instead of N_n, and h instead of $H(\alpha)$. So there exist n_0 and central projections z_n in M_n such that $\tau(z_n) < \varepsilon$ for $n \geq n_0$ and

$$e^{-n(h+\varepsilon)} < \tau(e) < e^{-n(h-\varepsilon)}$$

for any minimal projection e in $M_n(1 - z_n)$.

Lemma 2.4 *For any $\varepsilon > 0$ there exists $\varepsilon_1 > 0$ such that if $\{\xi_n\}_n$ and $\{\zeta_n\}_n$ are sequences of finite partitions of a Lebesgue space (X, μ), $\zeta_n \prec \xi_n$, $h = \lim \frac{1}{n} H(\xi_n)$, $h_1 = \lim \frac{1}{n} H(\zeta_n)$, $h < h_1 + \varepsilon_1$ and the McMillan theorem holds for $\{\zeta_n\}_n$, h_1 and ε_1, then it holds also for $\{\xi_n\}_n$, h and ε.*

Proof. Let $\{C_{ni}\}_{i \in X_n}$ be the atoms of ξ_n, $\{D_{nj}\}_{j \in Y_n}$ the atoms of ζ_n. Let $\tilde{Y}_n$ be the set of $j \in Y_n$ for which

$$e^{-n(h_1+\varepsilon_1)} < \mu(D_{nj}) < e^{-n(h_1-\varepsilon_1)}.$$

For n large enough

$$0 \leq \frac{1}{n}(H(\xi_n) - H(\zeta_n)) < \varepsilon_1,$$

or

$$0 \le \frac{1}{n} \sum_{\substack{i \in X_n \\ j \in Y_n}} \mu(C_{ni} \cap D_{nj}) \log \frac{\mu(D_{nj})}{\mu(C_{ni})} < \varepsilon_1. \tag{2.1}$$

For $i \in X_n$ let $j(i)$ be a unique index such that $C_{ni} \subset D_{nj(i)}$. Let

$$X'_n = \{i \in X_n \mid j(i) \in \tilde{Y}_n\}, \quad X''_n = \{i \in X_n \mid \mu(D_{nj(i)}) < e^{n\sqrt{\varepsilon_1}} \mu(C_{ni})\}, \quad \tilde{X}_n = X'_n \cap X''_n.$$

By virtue of (2.1)

$$\sum_{i \in X_n \setminus X''_n} \mu(C_{ni}) < \sqrt{\varepsilon_1}.$$

Hence, for n large enough,

$$\sum_{i \notin \tilde{X}_n} \mu(C_{ni}) \le \sum_{i \notin X'_n} \mu(C_{ni}) + \sum_{i \notin X''_n} \mu(C_{ni}) \le \sum_{j \notin \tilde{Y}_n} \mu(D_{nj}) + \sum_{i \notin X''_n} \mu(C_{ni}) < \varepsilon_1 + \sqrt{\varepsilon_1}.$$

For $i \in \tilde{X}_n$

$$\mu(C_{ni}) \le \mu(D_{nj(i)}) < e^{-n(h_1 - \varepsilon_1)} < e^{-n(h - 2\varepsilon_1)}$$

and

$$\mu(C_{ni}) \ge e^{-n\sqrt{\varepsilon_1}} \mu(D_{nj(i)}) > e^{-n(h_1 + \varepsilon_1 + \sqrt{\varepsilon_1})} \ge e^{-n(h + \varepsilon_1 + \sqrt{\varepsilon_1})}.$$

Thus we can take ε_1 such that $2\varepsilon_1 + \sqrt{\varepsilon_1} < \varepsilon$.

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Lemma 2.5 *Let $\{\xi_n\}_n$ be an increasing sequence of finite partitions of a Lebesgue space (X, μ), ζ a finite partition such that $\zeta \prec \vee_n \xi_n$. Suppose the McMillan theorem holds for $\{\xi_n \vee \zeta\}_n$, h and ε. Then it holds also for $\{\xi_n\}_n$, h and 2ε.*

Proof. Let $\{C_{ni}\}_{i \in X_n}$ be the atoms of ξ_n, $\{D_j\}_{j \in Y}$ the atoms of ζ. Let

$$X'_n = \{i \in X_n \mid \exists\, j(i) \in Y \ : \ \mu(C_{ni} \cap D_{j(i)}) > (1 - \varepsilon/2)\mu(C_{ni})\}.$$

We assert that

$$\sum_{i \notin X'_n} \mu(C_{ni}) \to 0 \quad \text{as} \quad n \to \infty.$$

Indeed, otherwise there exist $j \in Y$ and $c > 0$ such that

$$\sum_{i \notin X'_n} \mu(C_{ni} \cap D_j) \ge c$$

for infinitely many n's. In other words, the set $E_n = \cup_{i \notin X'_n} (C_{ni} \cap D_j)$ has measure $\mu(E_n) \ge c$, and for the conditional expectation $E(D_j|\xi_n)$ of the characteristic function of the set D_j with respect to the partition ξ_n we have

$$E(D_j|\xi_n)(x) \le 1 - \frac{\varepsilon}{2} \quad \text{on} \quad E_n,$$

what contradicts the almost everywhere convergence of $\{E(D_j|\xi_n)\}_n$ to the characteristic function of D_j.

Now let Z_n be the set of pairs (i,j) such that

$$e^{-n(h+\varepsilon)} < \mu(C_{ni} \cap D_j) < e^{-n(h-\varepsilon)},$$

and

$$\tilde{X}_n = \{i \in X'_n \mid (i, j(i)) \in Z_n\}.$$

For $i \in \tilde{X}_n$

$$\mu(C_{ni}) \geq \mu(C_{ni} \cap D_{j(i)}) > e^{-n(h+\varepsilon)},$$

and for n large enough

$$\mu(C_{ni}) < \frac{1}{1-\varepsilon/2}\mu(C_{ni} \cap D_{j(i)}) < e^{n\varepsilon}\mu(C_{ni} \cap D_{j(i)}) < e^{-n(h-2\varepsilon)}.$$

Since for large n

$$\sum_{(i,j)\in Z_n} \mu(C_{ni} \cap D_j) > 1 - \varepsilon$$

and

$$\sum_{i\in X'_n} \mu(C_{ni} \cap D_{j(i)}) > (1-\varepsilon/2) \sum_{i\in X'_n} \mu(C_{ni}) > 1 - \varepsilon,$$

we have also

$$\sum_{i\in \tilde{X}_n} \mu(C_{ni}) \geq \sum_{i\in \tilde{X}_n} \mu(C_{ni} \cap D_{j(i)}) > 1 - 2\varepsilon.$$

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Proof of Theorem 2.1. Let C_n be a masa in N_n, i.e. a maximal abelian $*$-subalgebra. We may suppose that $C_n \subset C_{n+1}$. Then we have to prove that the McMillan theorem holds for $\{C_n\}_n$, $H(\alpha)$ and any $\varepsilon > 0$ (note that if a projection z'_n is chosen in C_n as in the statement of the McMillan theorem then we can replace $1-z'_n$ by its central support p_n in N_n and let $z_n = 1-p_n$).

Using Proposition 2.2(i) choose $m \in \mathbb{N}$ such that

$$\left| H(\alpha) - \frac{1}{m}H(N_{m-p}\,, \alpha^m) \right| < \varepsilon_1^2,$$

where ε_1 is as in Lemma 2.4. Let A_m be the von Neumann subalgebra of $\pi_\tau(A)''$ generated by $\alpha^{jm}(N_{m-p})$, $j \in \mathbb{Z}$, D a masa in N_{m-p}. Then $D_n = \bigvee_{j=0}^{[\frac{n}{m}]-1} \alpha^{jm}(D)$ is a masa in $\bigvee_{j=0}^{[\frac{n}{m}]-1} \alpha^{jm}(N_{m-p})$. If α^m was ergodic, we could apply the classical McMillan theorem to $\{D_n\}_n$ and then make use of Lemma 2.4 to conclude that it holds also for $\{C_n\}_n$. Since α^m can be non-ergodic, consider the fixed point algebra $Z \subset \pi_\tau(A)''$ with respect to α^m. Since α is asymptotically abelian and ergodic on $\pi_\tau(A)''$, Z is a finite dimensional subalgebra of the center of $\pi_\tau(A)''$. Let $\{z_i\}_{i\in X}$ be the atoms of Z. The automorphism α acts transitively on the set of atoms, so the systems $(Az_i, \tau_i, \alpha^m|_{Az_i})$ are pairwise conjugate, where $\tau_i = |X|\tau|_{Az_i}$ and $|X|$ denotes the cardinality of X. Since $\frac{1}{|X|}\sum_i H(\alpha^m|_{Az_i}) = H(\alpha^m)$, we conclude that $H(\alpha^m|_{Az_i}) = mH(\alpha)$, so for $H_i = \frac{1}{m}H(\alpha^m|_{A_m z_i})$ we have

$$H_i \leq \frac{1}{m}H(\alpha^m|_{Az_i}) = H(\alpha).$$

On the other hand,

$$\frac{1}{|X|}\sum_i H_i = \frac{1}{m}H(\alpha^m|_{A_m\vee Z}) = \frac{1}{m}H(N_{m-p}\vee Z,\alpha^m) = \frac{1}{m}H(N_{m-p}\,,\alpha^m) > H(\alpha)-\varepsilon_1^2.$$

Thus if $\tilde{X}=\{i\in X \mid H_i > H(\alpha)-\varepsilon_1\}$ then

$$H(\alpha)-\varepsilon_1^2 < \frac{|\tilde{X}|}{|X|}H(\alpha)+\frac{|X\backslash\tilde{X}|}{|X|}(H(\alpha)-\varepsilon_1),$$

whence

$$\sum_{i\notin\tilde{X}}\tau(z_i) = \frac{|X\backslash\tilde{X}|}{|X|} < \varepsilon_1 < \varepsilon. \tag{2.2}$$

We have also

$$H(\alpha)=\frac{1}{m}H(\alpha^m|_{Az_i}) \le \liminf_{n\to\infty}\frac{H_{\tau_i}(C_nz_i)}{n}$$

and

$$\lim_{n\to\infty}\frac{1}{|X|}\sum_i\frac{H_{\tau_i}(C_nz_i)}{n} = \lim_{n\to\infty}\frac{H(C_n\vee Z)}{n} = \lim_{n\to\infty}\frac{H(C_n)}{n} = H(\alpha),$$

so

$$\lim_{n\to\infty}\frac{H_{\tau_i}(C_nz_i)}{n} = H(\alpha)\quad\forall\, i\in X.$$

For any $i\in X$ the automorphism α^m is ergodic on Az_i, so by the classical result the McMillan theorem holds for $\{D_nz_i\}_n$, H_i and ε_1. Then by Lemma 2.4, for $i\in\tilde{X}$, it holds for $\{C_nz_i\}_n$, $H(\alpha)$ and ε (with respect to the trace τ_i). By virtue of (2.2) it holds also for $\{C_n\vee Z\}_n$, $H(\alpha)$ and 2ε. Finally, by Lemma 2.5 the McMillan theorem holds for $\{C_n\}_n$, $H(\alpha)$ and 4ε.

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The method used in the proof can be applied to prove the following weak form of the McMillan theorem under more general assumptions.

Theorem 2.6 *Let (A,τ,α) be an asymptotically abelian system with locality, and τ an ergodic trace. Then the entropy $H(\alpha)$ of Connes and Størmer of the system coincides with Voiculescu's completely positive approximation entropy $hcpa_\tau(\alpha)$.*

Proof. Let N be a local subalgebra of A. It suffices to prove that $hcpa_\tau(N,\alpha)\le H(\alpha)$. Then by Proposition 2.2(ii) it is enough to prove that $hcpa_\tau(N_{m-p}\,,\alpha^m)\le mH(\alpha)$. Keep the notations of the proof of Theorem 2.1. By the classical McMillan theorem we have $hcpa_{\tau_i}(N_{m-p}z_i\,,\alpha^m) = H(N_{m-p}z_i\,,\alpha^m)$ (see the proof of [V, Proposition 1.7]). Since $H(N_{m-p}z_i\,,\alpha^m)\le H(\alpha^m|_{Az_i}) = mH(\alpha)$ and

$$hcpa_\tau(N_{m-p}\,,\alpha^m)\le hcpa_\tau(N_{m-p}\vee Z,\alpha^m)\le \max_i hcpa_{\tau_i}(N_{m-p}z_i\,,\alpha^m),$$

we obtain the desired inequality.

■

Remarks 2.7 (i) The assumption that N is a mean generator in Theorem 2.1 is very close to being necessary. Indeed, suppose N satisfies the conclusion of Theorem 2.1. Then

$$\log\operatorname{rank} N_n(1-z_n) < n(H(\alpha)+\varepsilon).$$

Thus

$$H(N_n(1-z_n)\oplus\mathbb{C}z_n)\le\log\operatorname{rank}(N_n(1-z_n)\oplus\mathbb{C}z_n)\le\log(e^{n(H(\alpha)+\varepsilon)}+1),$$

so

$$\limsup_n\frac{1}{n}H(N_n(1-z_n)\oplus\mathbb{C}z_n)\le H(\alpha)+\varepsilon\le\liminf_n\frac{1}{n}H(N_n)+\varepsilon.$$

In particular, if the N_n's don't grow too fast, e.g. if $\operatorname{rank}N_n\le e^{Cn}$ for some $C>0$, then

$$\limsup_n\frac{1}{n}H(N_n)\le\limsup_n\left(C\tau(z_n)+\frac{1}{n}H(N_n(1-z_n)\oplus\mathbb{C}z_n)\right),$$

and hence

$$H(\alpha)=\lim_n\frac{H(N_n)}{n}.$$

(ii) It is not clear what the optimal assumptions are for Theorem 2.1 to be true. The conclusion holds in several cases when the dynamical system is not asymptotically abelian. Such an example is that of a binary shift, see [PP]. Then we are given a subset X of $\mathbb{N}$ and the algebra $A(X)$ generated by symmetries $\{s_n\}_{n\in\mathbb{Z}}$ satisfying the commutation relations

$$s_is_j=\begin{cases} s_js_i, & \text{if } |i-j|\notin X,\\ -s_js_i, & \text{if } |i-j|\in X.\end{cases}$$

The binary shift is the automorphism α defined by $\alpha(s_n)=s_{n+1}$. Let $A_n=C^*(s_0,\dots,s_{n-1})$. Then $A_n\cong\operatorname{Mat}_{2^{d_n}}(\mathbb{C})\otimes Z_n$, where Z_n is the diagonal in $\operatorname{Mat}_{2^{c_n}}(\mathbb{C})$, $n=2d_n+c_n$, and thus any minimal projection in A_n has trace $2^{-d_n-c_n}$. It follows that $H(A_n)=\log\operatorname{rank}A_n=(c_n+d_n)\log 2$. The sequence $\{c_n\}_n$ varies with different X. If we assume $-X\cup\{0\}\cup X$ is nonperiodic then $\{c_n\}_n$ satisfies $|c_n-c_{n+1}|=1$ and takes the value zero an infinite number of times. Thus $A(X)$ is the CAR-algebra with tracial state τ. By [GS1] the C*-dynamical system $(A(X),\tau,\alpha)$ is asymptotically abelian with locality if and only if X is finite, in which case the sequence $\{c_n\}_n$ is bounded. We show that the following three conditions are equivalent. By [GS1] they are satisfied not only for asymptotically abelian systems.

(i) The conclusion of Theorem 2.1 holds for each algebra A_n (instead of N).

(ii) $H(\alpha)=\frac{1}{2}\log 2$ and $\lim_n\frac{c_n}{n}=0$.

(iii) $H(\alpha)=\lim_n\frac{1}{n}H(A_n)$.

Indeed, the implication (i)⇒(iii) follows from the previous remark.

Since $H(\alpha)\le\liminf_n\frac{1}{n}H(A_n)=\frac{1}{2}\log 2$, see e.g. [GS2], (iii) implies $\lim_n\frac{1}{n}H(A_n)=\frac{1}{2}\log 2$, hence by [GS2, Lemma 4.7], $\lim_n\frac{c_n}{n}=0$. Thus (iii)⇒(ii).

If (ii) holds then $\frac{1}{n}(d_n+c_n)\to\frac{1}{2}$, and so (ii) implies (i).

3 Mean Generators

In this section we discuss several sufficient conditions for a local algebra to be a mean generator.

Our first result shows that it is enough to prove that at least one algebra is a mean generator.

Proposition 3.1 *If N is a mean generator for $A(N)$, $H(\alpha|_{A(N)})<\infty$ and M is a subalgebra of $\vee_{-n}^{n}\alpha^i(N)$ for some n, then M is a mean generator for $A(M)$.*

Proof. Without loss of generality we may suppose that $M \subset N$. As above, let p be such that $\alpha^j(N)$ commutes with N for $|j| \geq p$. Let D_n be a masa in M_{n-p}, C_n a masa in N_{n-p} containing D_n. We have

$$0 \leq H(D_n) - H(D_n, \alpha^n) = H(C_n) - (H(D_n, \alpha^n) + H(C_n|D_n)) \leq H(C_n) - H(C_n, \alpha^n).$$

Since $\frac{1}{n}(H(C_n) - H(C_n, \alpha^n)) \to 0$ by assumption and Proposition 2.2, we conclude that

$$\frac{1}{n}(H(D_n) - H(D_n, \alpha^n)) \to 0,$$

hence M is a mean generator.

■

We have used in the proof that if $H(\alpha|_{A(N)}) < \infty$ then N is a mean generator if and only if

$$\lim_{n\to\infty} \frac{1}{n}(H(N_{n-p}) - H(N_{n-p}, \alpha^n)) = 0.$$

Using the following lemma we shall show that it suffices to check this condition using any subalgebra of N_n containing the center $Z(N_n)$ instead of N_n.

Lemma 3.2 *Let $M_1, \dots, M_n$ be commuting finite dimensional algebras, $Z(M_i) \subset A_i \subset M_i$. Then*

$$H(\vee_i M_i) - H(\vee_i A_i) = \sum_i (H(M_i) - H(A_i)).$$

Proof. It suffices to consider the case $n = 2$. Let $\{e_i\}_i$ be the atoms of a masa in A_1, $\{f_j\}_j$ the atoms of a masa in A_2. Let $(M_1)_{e_i}$ be a factor of type I_{m_i}, $(M_2)_{f_j}$ a factor of type I_{n_j}. Then

$$\begin{aligned} H(M_1 \vee M_2) - H(A_1 \vee A_2) &= -\sum_{i,j} \tau(e_i f_j) \log \frac{\tau(e_i f_j)}{m_i n_j} + \sum_{i,j} \tau(e_i f_j) \log \tau(e_i f_j) \\ &= \sum_i \tau(e_i) \log m_i + \sum_j \tau(f_j) \log n_j \\ &= (H(M_1) - H(A_1)) + (H(M_2) - H(A_2)). \end{aligned}$$

■

Proposition 3.3 *Let N be a local algebra such that $H(\alpha|_{A(N)}) < \infty$. Let $\{M_n\}_n$ be a sequence of algebras such that $Z(N_{n-p}) \subset M_n \subset N_{n-p}$. Then N is a mean generator for $A(N)$ if and only if*

$$\lim_{n\to\infty} \frac{1}{n}(H(M_n) - H(M_n, \alpha^n)) = 0.$$

In particular, N is a mean generator if $\lim_n \frac{1}{n} H(Z(N_n)) = 0$.

Proof. For any $m \in \mathbb{N}$, by Lemma 3.2

$$H(\vee_{j=0}^{m-1} \alpha^{jn}(N_{n-p})) - H(\vee_{j=0}^{m-1} \alpha^{jn}(M_n)) = m(H(N_{n-p}) - H(M_n)),$$

hence

$$H(N_{n-p}) - H(N_{n-p}, \alpha^n) = H(M_n) - H(M_n, \alpha^n),$$

what gives the result.

■

Finally, recall the following simple condition (see e.g. [Ch]).

Proposition 3.4 *Let N be a local algebra. Suppose there exists $p \in \mathbb{N}$ such that*

$$\tau(ab) = \tau(a)\tau(b)$$

for any $a \in \vee_{-\infty}^{0}\alpha^i(N)$ and $b \in \vee_{p}^{\infty}\alpha^i(N)$. Then N is a mean generator for $A(N)$.

The last condition shows that Theorem 2.1 can be applied to asymptotically abelian binary shifts and canonical shifts on towers of relative commutants. On the other hand, Proposition 3.1 allows to apply Theorem 2.1 to systems arising from topological dynamics, which were considered in [NS, Section 5]. Indeed, if a local algebra has a masa lying in the diagonal then it is a mean generator since all computations are reduced to the abelian case. Then Proposition 3.1 shows that any local algebra is a mean generator.

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