

TWISTING THE q -DEFORMATIONS OF COMPACT SEMISIMPLE LIE GROUPS

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ABSTRACT. Given a compact semisimple Lie group G of rank r , and a parameter $q > 0$, we can define new associativity morphisms in $\text{Rep}(G_q)$ using a 3-cocycle Φ on the dual of the center of G , thus getting a new tensor category $\text{Rep}(G_q)^\Phi$. For a class of cocycles Φ we construct compact quantum groups G_q^τ with representation categories $\text{Rep}(G_q)^\Phi$. The construction depends on the choice of an r -tuple τ of elements in the center of G . In the simplest case of $G = \text{SU}(2)$ and $\tau = -1$, our construction produces Woronowicz's quantum group $\text{SU}_{-q}(2)$ out of $\text{SU}_q(2)$. More generally, for $G = \text{SU}(n)$, we get quantum group realizations of the Kazhdan–Wenzl categories.

INTRODUCTION

A known problem in the theory of quantum groups is classification of quantum groups with fusion rules of a given Lie group G , see e.g. [Wor88, WZ94, Ban96, Ohn99, Bic03, Ohn05, Mro13]. Although this problem has been completely solved in a few cases, most notably for $G = \text{SL}(2, \mathbb{C})$ [Ban96, Bic03], as the rank of G grows the situation quickly becomes complicated. Already for $G = \text{SL}(3, \mathbb{C})$, even when requiring the dimensions of the representations to remain classical, one gets a large list of quantum groups that is not easy to grasp [Ohn99, Ohn05]. A categorical version of the same problem turns out to be more manageable. Namely, the problem is to classify semisimple rigid monoidal \mathbb{C} -linear categories with fusions rules of G . As was shown by Kazhdan and Wenzl [KW93], for $G = \text{SL}(n, \mathbb{C})$ such categories \mathcal{C} are parametrized by pairs $(q_{\mathcal{C}}, \tau_{\mathcal{C}})$ of nonzero complex numbers, defined up to replacing $(q_{\mathcal{C}}, \tau_{\mathcal{C}})$ by $(q_{\mathcal{C}}^{-1}, \tau_{\mathcal{C}}^{-1})$, such that $q_{\mathcal{C}}^{n(n-1)/2} = \tau_{\mathcal{C}}^n$ and $q_{\mathcal{C}}$ is not a nontrivial root of unity.¹ Concretely, these are twisted representation categories $\mathcal{C} = \text{Rep}(\text{SL}_q(n))^\zeta$, where q is not a nontrivial root of unity and ζ is a root of unity of order n ; the corresponding parameters are $q_{\mathcal{C}} = q^2$ and $\tau_{\mathcal{C}} = \zeta^{-1}q^{n-1}$. The twists are defined by choosing a \mathbb{T} -valued 3-cocycle on the dual of the center of $\text{SL}(n, \mathbb{C})$ and by using this cocycle to define new associativity morphisms in $\text{Rep}(\text{SL}_q(n))$. The third cohomology group of the dual of the center is cyclic of order n , and this explains the parametrization of twists of $\text{Rep}(\text{SL}_q(n))$ by roots of unity. A partial extension of the result of Kazhdan and Wenzl to types BCD was obtained by Tuba and Wenzl [TW05].

Although two problems are clearly related, a solution of the latter does not immediately say much about the former. The present work is motivated by the natural question whether there exist quantum groups with representation categories $\text{Rep}(\text{SL}_q(n))^\zeta$ for all ζ such that $\zeta^n = 1$. Equivalently, do the categories $\text{Rep}(\text{SL}_q(n))^\zeta$ always admit fiber functors? For $n = 2$ there is essentially nothing to solve, since for $q \neq 1$ the category $\text{Rep}(\text{SL}_q(2))^{-1}$ is equivalent to $\text{Rep}(\text{SL}_{-q}(2))$. For $q = 1$ the answer is also known: the quantum group $\text{SU}_{-1}(2)$ defined by Woronowicz (which has nothing to do with the quantized universal enveloping algebra $\mathcal{U}_q(\mathfrak{sl}_2)$ at $q = -1$) has representation category $\text{Rep}(\text{SL}(2, \mathbb{C}))^{-1}$. For $n \geq 2$, quantum groups with fusion rules of $\text{SL}(n, \mathbb{C})$ have been studied by many authors, see e.g. [Hai00] and the references therein. Usually, one starts by finding a solution of

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¹This is not how the result is formulated in [KW93]. There is a known mistake in [KW93, Proposition 5.1], see [PR11, Section 7] for a discussion.

the quantum Yang–Baxter equation satisfying certain conditions, and from this derives a presentation of the algebra of functions on the quantum group [RTF89]. This approach cannot work in our case, since the category $\text{Rep}(\text{SL}_q(n))^\zeta$ does not have a braiding unless $\zeta^2 = 1$.

The approach we take works, to some extent, for any compact semisimple simply connected Lie group G . Assume that Φ is a \mathbb{T} -valued 3-cocycle on the dual of the center of G . To construct a fiber functor φ from the category $\text{Rep}(G_q)^\Phi$ with associativity morphisms defined by Φ , such that $\dim \varphi(U) = \dim U$, is the same as to find an invertible element F in a completion $\mathcal{U}(G_q \times G_q)$ of $\mathcal{U}_q(\mathfrak{g}) \otimes \mathcal{U}_q(\mathfrak{g})$ satisfying

$$\Phi = (\iota \otimes \hat{\Delta}_q)(F^{-1})(1 \otimes F^{-1})(F \otimes 1)(\hat{\Delta}_q \otimes \iota)(F).$$

Then, using the twist (or a pseudo-2-cocycle in the terminology of [EV96]) F , we can define a new comultiplication on $\mathcal{U}(G_q)$, thus getting a new quantum group with representation category $\text{Rep}(G_q)^\Phi$.

Our starting point is the simple remark that to solve the above cohomological equation we do not have to go all the way to G_q , it might suffice to pass from the center $Z(G)$ to a (quantum) subgroup of G_q , for example, to the maximal torus T . For simple G this is indeed enough: any 3-cocycle on $\widehat{Z(G)}$ becomes a coboundary when lifted to the dual $P = \hat{T}$ of T . The reason is that, for simple G , the center is contained in a torus of dimension at most 2. However, a 2-cochain f on P such that $\partial f = \Phi$ is unique only up to a 2-cocycle on P . Already for trivial Φ this leads to deformations of G_q by 2-cocycles on P that are not very well studied [AST91, LS91], with associated C^* -algebras of functions (for $q > 0$) that are typically not of type I.

Our next observation is that, for arbitrary G , if Φ lifts to a coboundary on P , then the cochain f can be chosen to be of a particular form. This leads to a very special class of quantum groups G_q^τ , whose construction depends on the choice of elements $\tau_1, \dots, \tau_r \in Z(G)$, where r is the rank of G . We show that the quantum groups G_q^τ are as close to G_q as one could hope. For example, they can be defined in terms of finite central extensions of $\mathcal{U}_q(\mathfrak{g})$.

Since we are, first of all, interested in compact quantum groups in the sense of Woronowicz, we will concentrate on the case $q > 0$, when the categories $\text{Rep}(G_q)^\Phi$ have a C^* -structure and, correspondingly, G_q^τ become compact quantum groups. We then show that the C^* -algebras $C(G_q^\tau)$ are KK-isomorphic to $C(G)$, they are of type I, and their primitive spectra are only slightly more complicated than that of $C(G_q)$. For $G = \text{SU}(n)$ we also find explicit generators and relations of the algebras $\mathbb{C}[\text{SU}_q^\tau(n)]$ of regular functions on $\text{SU}_q^\tau(n)$.

To summarize, our construction produces quantum groups with nice properties and with representation category $\text{Rep}(G_q)^\Phi$ for any 3-cocycle Φ on $\widehat{Z(G)}$ that lifts to a coboundary on \hat{T} . This covers the cases when G is simple, but in the general semisimple case there exist cocycles that do not have this property. For such cocycles the existence of fiber functors for $\text{Rep}(G_q)^\Phi$ remains an open problem.

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1. PRELIMINARIES

1.1. Compact quantum groups. A *compact quantum group* \mathbb{G} is given by a unital C^* -algebra $C(\mathbb{G})$ together with a coassociative unital $*$ -homomorphism $\Delta: C(\mathbb{G}) \rightarrow C(\mathbb{G}) \otimes C(\mathbb{G})$ satisfying the cancellation condition

$$[\Delta(C(\mathbb{G}))(C(\mathbb{G}) \otimes 1)] = C(\mathbb{G}) \otimes C(\mathbb{G}) = [\Delta(C(\mathbb{G}))(1 \otimes C(\mathbb{G}))],$$

where brackets denote the closed linear span. Here we only introduce the relevant terminology and summarize the essential results, see e.g. [NT13] for details.

A theorem of Woronowicz gives a distinguished state h , the Haar state, which is an analogue of the normalized Haar measure over compact groups. Denote by $C_r(\mathbb{G})$ the quotient of $C(\mathbb{G})$ by the kernel of the GNS-representation defined by h . We will be interested in the case where h is faithful, so that $C_r(\mathbb{G}) = C(\mathbb{G})$. This condition is automatically satisfied for coamenable compact quantum groups. The quantum groups studied in this paper will be coamenable thanks to Banica's theorem [Ban99, Proposition 6.1; NT13, Theorem 2.7.14].

A finite dimensional unitary representation of \mathbb{G} is given by a unitary element $U \in B(\mathcal{H}_U) \otimes C(\mathbb{G})$ satisfying the condition $U_{13}U_{23} = (\iota \otimes \Delta)(U)$. The tensor product of two representations is defined by $U \oplus V = U_{13}V_{23}$. The category $\text{Rep}(\mathbb{G})$ of finite dimensional unitary representations of \mathbb{G} has the structure of a rigid C^* -tensor category with a unitary fiber functor ('forgetful functor') $U \mapsto \mathcal{H}_U$ to the category Hilb_f of finite dimensional Hilbert spaces. Woronowicz's Tannaka–Krein duality theorem states that the reduced quantum group $(C_r(\mathbb{G}), \Delta)$ can be axiomatized in terms of $\text{Rep}(\mathbb{G})$ and the fiber functor.

We denote by $\mathbb{C}[\mathbb{G}] \subset C(\mathbb{G})$ the Hopf $*$ -algebra of matrix coefficients of finite dimensional representations of \mathbb{G} . Denote by $\mathcal{U}(\mathbb{G})$ the dual $*$ -algebra of $\mathbb{C}[\mathbb{G}]$, so $\mathcal{U}(\mathbb{G}) = \prod_{U \in \text{Irrep}(\mathbb{G})} B(\mathcal{H}_U)$. It can be considered from many different angles: as the algebra of functions on the dual discrete quantum group $\hat{\mathbb{G}}$, as the algebra of endomorphisms of the forgetful functor, as the multiplier algebra of the convolution algebra $\widehat{\mathbb{C}[\mathbb{G}]}$ of \mathbb{G} . We also write $\mathcal{U}(\mathbb{G}^n)$ for $n \geq 2$ to denote the 'tensor product' multipliers, such as

$$\mathcal{U}(\mathbb{G}^2) = \prod_{U, V \in \text{Irrep}(\mathbb{G})} B(\mathcal{H}_U) \otimes B(\mathcal{H}_V).$$

By duality, the multiplication map $m: \mathbb{C}[\mathbb{G}] \otimes \mathbb{C}[\mathbb{G}] \rightarrow \mathbb{C}[\mathbb{G}]$ defines a 'coproduct' $\hat{\Delta}: \mathcal{U}(\mathbb{G}) \rightarrow \mathcal{U}(\mathbb{G}^2)$.

1.2. Twisting of quantum groups. Let \mathbb{G} be a compact quantum group, and Φ be an invariant unitary 3-cocycle over the discrete dual of \mathbb{G} [NT13, Chapter 3]. Thus, Φ is a unitary element in $\mathcal{U}(\mathbb{G}^3)$ satisfying the cocycle condition

$$(1.1) \quad (1 \otimes \Phi)(\iota \otimes \hat{\Delta} \otimes \iota)(\Phi)(\Phi \otimes 1) = (\iota \otimes \iota \otimes \hat{\Delta})(\Phi)(\hat{\Delta} \otimes \iota \otimes \iota)(\Phi)$$

and the invariance condition $[\Phi, (\hat{\Delta} \otimes \iota)\hat{\Delta}(x)] = 0$ for $x \in \mathcal{U}(\mathbb{G})$.

Then, the representation category $\text{Rep}(\mathbb{G})$ can be twisted into a new C^* -tensor category $\text{Rep}(\mathbb{G})^\Phi$, by using the action by Φ on $\mathcal{H}_U \otimes \mathcal{H}_V \otimes \mathcal{H}_W$ as the new associativity morphism $(U \oplus V) \oplus W \rightarrow U \oplus (V \oplus W)$ for $U, V, W \in \text{Rep}(\mathbb{G})$. The category $\text{Rep}(\mathbb{G})^\Phi$ can be considered as the module category of the discrete quasi-bialgebra $(\widehat{\mathbb{C}[\mathbb{G}]}, \hat{\Delta}, \Phi)$ [Dri89].

Suppose the category $\text{Rep}(\mathbb{G})^\Phi$ is rigid. This is equivalent to the condition that the central element

$$\Phi_1 \hat{S}(\Phi_2) \Phi_3 = m(m \otimes \iota)(\iota \otimes \hat{S} \otimes \iota)(\Phi)$$

in $\mathcal{U}(\mathbb{G})$ is invertible. Suppose also that there exists a unitary $F \in \mathcal{U}(\mathbb{G}^2)$ such that

$$(1.2) \quad \Phi = (\iota \otimes \hat{\Delta})(F^*)(1 \otimes F^*)(F \otimes 1)(\hat{\Delta} \otimes \iota)(F).$$

Then the discrete quantum group $\mathcal{U}(\mathbb{G})$ can be deformed into another one, with the new coproduct $\hat{\Delta}_F(x) = F\hat{\Delta}(x)F^*$. By duality, the function algebra $\mathbb{C}[\mathbb{G}]$ can be endowed with the new product

$$x \cdot_F y = m(F^* \triangleright (x \otimes y) \triangleleft F).$$

Here, \triangleright and \triangleleft are the natural actions of $\mathcal{U}(\mathbb{G})$ on $\mathbb{C}[\mathbb{G}]$ given by $X \triangleright a = \langle X, a_{[2]} \rangle a_{[1]}$ and $a \triangleleft X = \langle X, a_{[1]} \rangle a_{[2]}$. We denote the corresponding compact quantum group by \mathbb{G}_F . Note that in general the involution on $\mathbb{C}[\mathbb{G}_F]$ differs from the original one, see [NT13, Example 2.3.9].

We have a unitary monoidal equivalence of the C^* -tensor categories $\text{Rep}(\mathbb{G})^\Phi$ and $\text{Rep}(\mathbb{G}_F)$. The tensor functor $\varphi: \text{Rep}(\mathbb{G})^\Phi \rightarrow \text{Rep}(\mathbb{G}_F)$ is given by the identity map on objects and morphisms, but with the nontrivial tensor transformation $\varphi(U) \oplus \varphi(V) \rightarrow \varphi(U \oplus V)$ defined by

$$\mathcal{H}_U \otimes \mathcal{H}_V \rightarrow \mathcal{H}_U \otimes \mathcal{H}_V, \quad \xi \otimes \eta \mapsto F^*(\xi \otimes \eta).$$

In terms of fiber functors, F gives a tensor functor $\text{Rep}(\mathbb{G})^\Phi \rightarrow \text{Hilb}_f$ which is the same as that of $\text{Rep}(\mathbb{G})$ on objects and morphisms, but with the modified tensor transformation $\mathcal{H}_U \otimes \mathcal{H}_V \rightarrow \mathcal{H}_{U \otimes V}$ given by $\xi \otimes \eta \mapsto F^*(\xi \otimes \eta)$.

Examples of invariant 3-cocycles can be obtained as follows. Assume \mathbb{H} is a closed central subgroup of \mathbb{G} , so \mathbb{H} is a compact abelian group and we are given a surjective homomorphism $\pi: \mathbb{C}[\mathbb{G}] \rightarrow \mathbb{C}[\mathbb{H}]$ of Hopf $*$ -algebras such that the image of $\mathcal{U}(\mathbb{H})$ under the dual homomorphism $\mathcal{U}(\mathbb{H}) \rightarrow \mathcal{U}(\mathbb{G})$ is a central subalgebra of $\mathcal{U}(\mathbb{G})$, or equivalently, for any irreducible unitary representation U of \mathbb{G} the element $(\iota \otimes \pi)(U)$ has the form $1 \otimes \chi_U$ for a character χ_U of \mathbb{H} . Unitary 3-cocycles in $\mathcal{U}(\mathbb{H}^3)$ are nothing else than \mathbb{T} -valued 3-cocycles on the Pontryagin dual $\hat{\mathbb{H}}$. Any such cocycle defines an invariant cocycle Φ in $\mathcal{U}(\mathbb{G}^3)$; when \mathbb{G} is itself compact abelian, this is just the usual pullback homomorphism $Z^3(\hat{\mathbb{H}}; \mathbb{T}) \rightarrow Z^3(\hat{\mathbb{G}}; \mathbb{T})$. Explicitly, the action of Φ on $\mathcal{H}_U \otimes \mathcal{H}_V \otimes \mathcal{H}_W$ is by multiplication by $\Phi(\chi_U, \chi_V, \chi_W)$. For such cocycles Φ the C^* -tensor category $\text{Rep}(\mathbb{G})^\Phi$ is always rigid.

1.3. Quantized universal enveloping algebra. Throughout the whole paper G denotes a semi-simple simply connected compact Lie group, and \mathfrak{g} denotes its complexified Lie algebra. We fix a maximal torus T in G , and denote the corresponding Cartan subalgebra by \mathfrak{h} . The root lattice is denoted by Q , and the weight lattice by P . We fix a choice of positive roots, and denote the corresponding positive simple roots by $\{\alpha_1, \dots, \alpha_r\}$. We also fix an ad-invariant symmetric form on \mathfrak{g} such that it is negative definite on the real Lie algebra of G . If G is simple, we assume that this form is standardly normalized, meaning that $(\alpha, \alpha) = 2$ for every short root α . The Cartan matrix is denoted by $(a_{ij})_{1 \leq i, j \leq r}$, and the Weyl group is denoted by W . The center $Z(G)$ of G is contained in T and can be identified with the dual of P/Q .

In what follows the variable q ranges over the strictly positive real numbers, although many results remain true for all $q \neq 0$ such that the numbers $q_i = q^{(\alpha_i, \alpha_i)/2}$ are not nontrivial roots of unity. For $q \neq 1$, the *quantized universal enveloping algebra* $\mathcal{U}_q(\mathfrak{g})$ is the universal algebra over \mathbb{C} generated by the elements E_i, F_i , and $K_i^{\pm 1}$ for $1 \leq i \leq r$ satisfying the relations

$$\begin{aligned} [K_i, K_j] &= 0, & K_i E_j K_i^{-1} &= q_i^{a_{ij}} E_j, & K_i F_j K_i^{-1} &= q_i^{-a_{ij}} F_j, \\ [E_i, F_j] &= \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}, \\ \sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1 - a_{ij} \\ k \end{bmatrix}_{q_i} E_i^k E_j E_i^{1-a_{ij}-k} &= 0, \\ \sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1 - a_{ij} \\ k \end{bmatrix}_{q_i} F_i^k F_j F_i^{1-a_{ij}-k} &= 0. \end{aligned}$$

It has the structure of a Hopf $*$ -algebra defined by the operations

$$\begin{aligned} \hat{\Delta}_q(E_i) &= E_i \otimes 1 + K_i \otimes E_i, & \hat{\Delta}_q(F_i) &= F_i \otimes K_i^{-1} + 1 \otimes F_i, & \hat{\Delta}_q(K_i) &= K_i \otimes K_i, \\ \hat{S}_q(E_i) &= -K_i^{-1} E_i, & \hat{S}_q(F_i) &= -F_i K_i^{-1}, & \hat{S}_q(K_i) &= K_i^{-1}, \\ \hat{\epsilon}_q(E_i) &= \hat{\epsilon}_q(F_i) = 0, & \hat{\epsilon}_q(K_i) &= 1, \\ E_i^* &= F_i K_i, & F_i^* &= K_i^{-1} E_i, & K_i^* &= K_i. \end{aligned}$$

A representation (π, V) of $\mathcal{U}_q(\mathfrak{g})$ is said to be *admissible* when V admits a decomposition $\bigoplus_{\chi \in P} V_\chi$ such that $\pi(K_i)|_{V_\chi}$ is equal to the scalar $q^{(\alpha_i, \chi)}$. The category of finite dimensional admissible $*$ -representations of $\mathcal{U}_q(\mathfrak{g})$ is a C^* -tensor category with the forgetful functor. We denote the associated compact quantum group by G_q . There is a natural inclusion of T into $\mathcal{U}(G_q)$. Then the set $Z(G_q)$ of group-like central elements in $\mathcal{U}(G_q)$ coincides with $Z(G)$. The class of representations of G_q on which $Z(G)$ acts trivially corresponds to a quotient quantum group denoted by $G_q/Z(G)$.

2. TWISTED q -DEFORMATIONS

2.1. Extension of the QUE-algebra. For $q > 0$, we let $\tilde{\mathcal{U}}_q(\mathfrak{g})$ denote the universal $*$ -algebra generated by $\mathcal{U}_q(\mathfrak{g})$ and unitary central elements C_1, \dots, C_r . It is not difficult to check that for $q \neq 1$ the following formulas define a Hopf $*$ -algebra structure on $\tilde{\mathcal{U}}_q(\mathfrak{g})$:

$$\hat{\Delta}(E_i) = E_i \otimes C_i + K_i \otimes E_i, \quad \hat{\Delta}(K_i) = K_i \otimes K_i, \quad \hat{\Delta}(C_i) = C_i \otimes C_i.$$

Similarly, for $q = 1$, we define

$$\hat{\Delta}(E_i) = E_i \otimes C_i + 1 \otimes E_i, \quad \hat{\Delta}(H_i) = H_i \otimes 1 + 1 \otimes H_i, \quad \hat{\Delta}(C_i) = C_i \otimes C_i.$$

There is a Hopf $*$ -algebra homomorphism from $\tilde{\mathcal{U}}_q(\mathfrak{g})$ onto $\mathcal{U}_q(\mathfrak{g})$, defined by $C_i \mapsto 1$ and by the identity map on the copy of $\mathcal{U}_q(\mathfrak{g})$. There is also a Hopf $*$ -algebra homomorphism onto $\mathbb{C}[(C_i)_{i=1}^r]$, given by $E_i \mapsto 0$, $F_i \mapsto 0$, $K_i \mapsto 1$, and by the identity map on the C_i 's. We regard representations of $\mathcal{U}_q(\mathfrak{g})$ and of $\mathbb{C}[(C_i)_{i=1}^r]$ as the ones of $\tilde{\mathcal{U}}_q(\mathfrak{g})$ via these homomorphisms.

Remark 2.1. The Hopf algebra $\tilde{\mathcal{U}}_q(\mathfrak{g})$ is closely related to the Drinfeld double $\mathcal{D}(\mathcal{U}_q(\mathfrak{b}_+))$ of $\mathcal{U}_q(\mathfrak{b}_+) = \langle E_i, K_i \mid 1 \leq i \leq r \rangle$. Namely, put

$$X_i^+ = E_i C_i^{-1}, \quad K_i^+ = K_i C_i^{-1}, \quad X_i^- = F_i, \quad K_i^- = K_i C_i.$$

Then we see that the elements X_i^+ and K_i^+ generate a copy of $\mathcal{U}_q(\mathfrak{b}_+)$, while the X_i^- and K_i^- generate a copy of $\mathcal{U}_q(\mathfrak{b}_-)$, and taking together these subalgebras give a copy of $\mathcal{D}(\mathcal{U}_q(\mathfrak{b}_+))$ in $\tilde{\mathcal{U}}_q(\mathfrak{g})$. The homomorphism $\tilde{\mathcal{U}}_q(\mathfrak{g}) \rightarrow \mathcal{U}_q(\mathfrak{g})$ is an extension of the standard projection $\mathcal{D}(\mathcal{U}_q(\mathfrak{b}_+)) \rightarrow \mathcal{U}_q(\mathfrak{g})$. If we add square roots of K_i^\pm to $\mathcal{D}(\mathcal{U}_q(\mathfrak{b}_+))$, thus getting a Hopf algebra $\mathcal{D}(\mathcal{U}_q(\mathfrak{b}_+))$, we can recover $\tilde{\mathcal{U}}_q(\mathfrak{g})$ by letting $C_i = (K_i^-)^{1/2} (K_i^+)^{-1/2}$. Therefore we have inclusions of Hopf algebras $\mathcal{D}(\mathcal{U}_q(\mathfrak{b}_+)) \subset \tilde{\mathcal{U}}_q(\mathfrak{g}) \subset \mathcal{D}(\mathcal{U}_q(\mathfrak{b}_+))$.

Let $\tau = (\tau_1, \dots, \tau_r)$ be an r -tuple of elements in $Z(G)$. We say that a representation (π, V) of $\tilde{\mathcal{U}}_q(\mathfrak{g})$ is τ -admissible if its restriction to $\mathcal{U}_q(\mathfrak{g})$ is admissible and the elements C_i act on the weight spaces V_χ as scalars $\langle \tau_i, \chi \rangle$. The category of τ -admissible representations is a rigid C^* -tensor category with forgetful functor. Moreover, the $G_q/Z(G)$ -representations are naturally included in the τ -admissible representations as a C^* -tensor subcategory.

Definition 2.2. We let G_q^τ denote the compact quantum group realizing the category of finite dimensional τ -admissible $*$ -representations of $\tilde{\mathcal{U}}_q(\mathfrak{g})$ together with its canonical fiber functor.

In other words, $\mathbb{C}[G_q^\tau] \subset \tilde{\mathcal{U}}_q(\mathfrak{g})^*$ is spanned by matrix coefficients of finite dimensional τ -admissible representations, and the Hopf $*$ -algebra structure on $\mathbb{C}[G_q^\tau]$ is defined by duality using that of $\tilde{\mathcal{U}}_q(\mathfrak{g})$.

Since every admissible representation of $\mathcal{U}_q(\mathfrak{g})$ extends uniquely to a τ -admissible representation of $\tilde{\mathcal{U}}_q(\mathfrak{g})$, and every τ -admissible representation is obtained this way, we can identify the $*$ -algebra $\mathcal{U}(G_q^\tau)$ with $\mathcal{U}(G_q)$. The image $\mathcal{U}_q^\tau(\mathfrak{g})$ of $\tilde{\mathcal{U}}_q(\mathfrak{g})$ in $\mathcal{U}(G_q^\tau) = \mathcal{U}(G_q)$ plays the role of a quantized universal enveloping algebra for G_q^τ . As an algebra it is generated by $E_i, F_i, K_i^{\pm 1}$ and τ_i (which is the image of C_i), but is endowed with a modified coproduct

$$(2.1) \quad \hat{\Delta}(E_i) = E_i \otimes \tau_i + K_i \otimes E_i, \quad \hat{\Delta}(K_i) = K_i \otimes K_i, \quad \hat{\Delta}(\tau_i) = \tau_i \otimes \tau_i.$$

To put it differently, as a $*$ -algebra, $\mathcal{U}_q^\tau(\mathfrak{g})$ is the tensor product of $\mathcal{U}_q(\mathfrak{g})$ and the group algebra of the group $T_\tau \subset Z(G)$ generated by τ_1, \dots, τ_r , while the coproduct is defined by (2.1). As a quotient of $\tilde{\mathcal{U}}_q(\mathfrak{g})$, the Hopf $*$ -algebra $\mathcal{U}_q^\tau(\mathfrak{g})$ is obtained by requiring that the unitaries C_1, \dots, C_r satisfy the same relations as $\tau_1, \dots, \tau_r \in Z(G)$.

2.2. Twisting and associator. Given $\tau = (\tau_1, \dots, \tau_r) \in Z(G)^r$, we obtain a 3-cocycle on $\widehat{Z(G)} = P/Q$ as follows.

First, let $f(\lambda, \mu)$ be a \mathbb{T} -valued function on $P \times P$ satisfying

$$(2.2) \quad f(\lambda, \mu + Q) = f(\lambda, \mu), \quad f(\lambda + \alpha_i, \mu) = \langle \tau_i, \mu \rangle f(\lambda, \mu).$$

These conditions imply that f can be determined by its restriction to the image of a set-theoretic section $(P/Q)^2 \rightarrow P^2$. For example, if $\lambda_1, \dots, \lambda_n$ is a system of representatives of P/Q , then we can put

$$f\left(\lambda_i + \sum_{j=1}^r m_j \alpha_j, \mu\right) = \prod_{j=1}^r \langle \tau_j, \mu \rangle^{m_j}$$

for all $1 \leq i \leq n$ and $(m_1, \dots, m_r) \in \mathbb{Z}^r$.

Using (2.2), the coboundary of f ,

$$(\partial f)(\lambda, \mu, \nu) = f(\mu, \nu) f(\lambda + \mu, \nu)^{-1} f(\lambda, \mu + \nu) f(\lambda, \mu)^{-1},$$

is seen to be invariant under the translation by Q in each variable. Thus, ∂f can be considered as a 3-cochain on P/Q with values in \mathbb{T} . By construction, it is a cocycle. If f' satisfies the same condition as f above, the difference $f' f^{-1}$ is Q^2 -invariant, that is, it defines a function on $(P/Q)^2$. Thus, the cohomology class of ∂f in $H^3(P/Q; \mathbb{T})$ depends only on τ . It also follows that the twisted coproduct $\hat{\Delta}_f(x) = f \hat{\Delta}_q(x) f^*$ does not depend on the choice of f .

Since $(\partial f)^*$ belongs to $\mathcal{U}(Z(G)^3)$, as we discussed in Section 1.2, it can be regarded as an invariant 3-cocycle in $\mathcal{U}(G_q^3)$ which is denoted by Φ^τ . Similarly, f can be considered as a unitary in $\mathcal{U}(G_q^2)$, and we have

$$\Phi^\tau = (\iota \otimes \hat{\Delta}_q)(f^*)(1 \otimes f^*)(f \otimes 1)(\hat{\Delta}_q \otimes \iota)(f).$$

Proposition 2.3. *The coproduct $\hat{\Delta}_f$ on $\mathcal{U}(G_q)$ coincides with the coproduct $\hat{\Delta}$ defined by (2.1).*

Proof. Since f is contained in $\mathcal{U}(T^2) \subset \mathcal{U}(G_q^2)$, $\hat{\Delta}_f = \hat{\Delta}_q$ on the elements K_i . For E_i , since the action of E_i on an admissible module increases the weight of a vector by α_i , identities (2.2) imply that $f(K_i \otimes E_i) f^* = K_i \otimes E_i$ and $f(E_i \otimes 1) f^* = E_i \otimes \tau_i$. Comparing these identities with (2.1), we obtain the assertion. \square

Corollary 2.4. *The representation category of G_q^τ is unitarily monoidally equivalent to $\text{Rep}(G_q)^{\Phi^\tau}$, the representation category of G_q with associativity morphisms defined by Φ^τ .*

This result can also be interpreted as follows. Let $\Phi_{\text{KZ},q} \in \mathcal{U}(G^3)$ be the Drinfeld associator coming from the Knizhnik–Zamolodchikov equations associated with the parameter $\hbar = \log(q)/\pi i$. The representation category of G_q is equivalent to that of G with associativity morphisms defined by $\Phi_{\text{KZ},q}$. The equivalence is given by a unitary Drinfeld twist $F_D \in \mathcal{U}(G^2)$ satisfying (1.2) for $\Phi_{\text{KZ},q}$ [NT13, Chapter 4]. It follows that $\text{Rep}(G_q^\tau)$ is unitarily monoidally equivalent to the category $\text{Rep}(G)$ with associativity morphisms defined by

$$\Phi_{\text{KZ},q}^\tau = (\iota \otimes \hat{\Delta})(F_D^*)(1 \otimes F_D^*) \Phi^\tau (F_D \otimes 1) (\hat{\Delta} \otimes \iota)(F_D) = \Phi^\tau \Phi_{\text{KZ},q},$$

where we now consider Φ^τ as an element of $\mathcal{U}(G^3)$. Correspondingly, the unitary $F_D^\tau = f F_D \in \mathcal{U}(G^2)$ plays the role of a Drinfeld twist for G_q^τ .

Remark 2.5. The construction of [NT10] can be carried out for G_q^τ to obtain a spectral triple over $\mathbb{C}[G_q^\tau]$ as an isospectral deformation of the spin Dirac operator on G . Indeed, it is enough to verify the boundedness of $[1 \otimes (\iota \otimes \gamma)(t), (\pi \otimes \iota \otimes \widetilde{\text{ad}})(\Phi_{\text{KZ},q}^\tau)]$ for any irreducible representation π , where t is the standard symmetric tensor $\sum_i x_i \otimes x_i$ [NT10, Corollary 3.2]. Since $(\pi \otimes \iota \otimes \widetilde{\text{ad}})(\Phi^\tau) \in \mathbb{C} \otimes \mathcal{U}(Z(G)) \otimes \mathbb{C}$ commutes with $1 \otimes (\iota \otimes \gamma)(t)$, we can reduce the proof to the case of trivial τ .

A natural question is how large the class of cocycles of the form Φ^τ is. These cocycles are analyzed in detail in Appendix. Using that analysis we point out the following.

Proposition 2.6. *A \mathbb{T} -valued 3-cocycle Φ on P/Q is cohomologous to Φ^τ for some $\tau_1, \dots, \tau_r \in Z(G)$ if and only if Φ lifts to a coboundary on P . This is always the case if P/Q can be generated by not more than two elements. For example, this is the case if G is simple.*

Proof. The first statement is proved in Corollary A.4. It is also shown there that another equivalent condition on Φ is that it vanishes on $\bigwedge^3(P/Q) \subset H_3(P/Q; \mathbb{Z})$. This condition is obviously satisfied if P/Q can be generated by two elements. Finally, if G is simple, then it is known that P/Q is cyclic in all cases except for $G = \text{Spin}(4n)$, in which case $P/Q \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. \square

Therefore for simple G the quantum groups G_q^τ realize all possible associativity morphisms on $\text{Rep}(G_q)$ defined by 3-cocycles on the dual of the center. In the semisimple case this is not true as soon as the center becomes slightly more complicated, namely, as soon as $\bigwedge^3(P/Q) \neq 0$. We conjecture that in this case, if we take a cocycle Φ on P/Q that does not lift to a coboundary on P , then there are no unitary fiber functors on $\text{Rep}(G)^\Phi$, that is, there are no compact quantum groups with this representation category. Note that by Corollary A.5 any such cocycle Φ is cohomologous to product of a cocycle Φ^τ and a 3-character on P/Q that is nontrivial on $\bigwedge^3(P/Q) \subset (P/Q)^{\otimes 3}$.

2.3. Isomorphisms of twisted quantum groups. Denote the cohomology class of the cocycle Φ^τ in $H^3(P/Q; \mathbb{T})$ by $\Theta(\tau)$. This way we obtain a homomorphism

$$\Theta: Z(G)^r \rightarrow H^3(P/Q; \mathbb{T}).$$

Assume $\tau \in \ker \Theta$. Let f be a function satisfying (2.2). Then there exists a 2-cochain $g: (P/Q)^2 \rightarrow \mathbb{T}$ such that $\partial f = \partial g$, so that fg^{-1} is a 2-cocycle on P . Another choice of f and g would give us a cocycle that differs from fg^{-1} by a 2-cocycle on P/Q . Therefore taking the cohomology class of fg^{-1} we get a well-defined homomorphism

$$\Upsilon: \ker \Theta \rightarrow H^2(P; \mathbb{T})/H^2(P/Q; \mathbb{T}).$$

Proposition 2.7. *Assume $\tau', \tau \in Z(G)^r$ are such that*

$$\tau'\tau^{-1} \in \ker \Theta \quad \text{and} \quad \tau'\tau^{-1} \in \ker \Upsilon.$$

Then the quantum groups $G_q^{\tau'}$ and G_q^τ are isomorphic.

Proof. Denote by $\hat{\Delta}'$ and $\hat{\Delta}$ the coproducts on $\mathcal{U}(G_q)$ defined by τ' and τ , see (2.1). Let f' and f be functions satisfying (2.2) for τ' and τ , respectively, so that $\hat{\Delta}' = \hat{\Delta}_{f'}$ and $\hat{\Delta} = \hat{\Delta}_f$. The assumptions of the proposition mean that there exists a 2-cochain g on P/Q such that $f'f^{-1}g$ is a coboundary on P . In other words, there exists a unitary $u \in \mathcal{U}(T^2) \subset \mathcal{U}(G_q^2)$ such that

$$f'g = (u \otimes u)f\hat{\Delta}_q(u)^*.$$

Then Ad_u is an isomorphism of $(\mathcal{U}(G_q), \hat{\Delta})$ onto $(\mathcal{U}(G_q), \hat{\Delta}')$, hence $G_q^\tau \cong G_q^{\tau'}$. \square

Apart from the isomorphisms given by this proposition, we have $G_q^\tau \cong G_{q^{-1}}^{\tau^{-1}}$. There also are isomorphisms induced by symmetries of the based root datum of G . Finally, for $q = 1$ there can be additional isomorphisms defined by conjugation by elements in $\mathcal{U}(G)$ that lie in the normalizer of the maximal torus.

3. FUNCTION ALGEBRAS OF TWISTED QUANTUM GROUPS

3.1. Crossed product description. As before, assume $\tau = (\tau_1, \dots, \tau_r) \in Z(G)^r$. Recall that we denote by T_τ the subgroup of $Z(G)$ generated by the elements τ_1, \dots, τ_r . There is a homomorphism

$$\psi: \hat{T}_\tau \rightarrow T/Z(G)$$

defined as follows. Given $\chi \in \hat{T}_\tau$, we define a character on the root lattice Q by $\alpha_i \mapsto \chi(\tau_i)$. It can be extended to P , and we obtain an element $\tilde{\psi}(\chi) \in \hat{P} = T$. The ambiguity of this extension is in $Q^\perp \cap T = Z(G)$. Thus, the image $\psi(\chi)$ of $\tilde{\psi}(\chi)$ in $T/Z(G)$ is well-defined.

The homomorphism ψ allows us to define an action of \hat{T}_τ by conjugation on G_q , that is, we have an action $\text{Ad}\psi$ of \hat{T}_τ on $C(G_q)$ defined by

$$(\text{Ad}\psi(\chi))(a) = \left\langle \tilde{\psi}(\chi^{-1}), a_{[1]} \right\rangle \left\langle \tilde{\psi}(\chi), a_{[3]} \right\rangle a_{[2]};$$

recall that the elements of T define characters of $C(G_q)$, that is, they are group-like unitary elements in $\mathcal{U}(G_q)$.

Theorem 3.1. *There is a canonical isomorphism*

$$C(G_q^\tau) \cong (C(G_q) \rtimes_{\text{Ad}\psi} \hat{T}_\tau)^{T_\tau},$$

where the group T_τ acts on $C(G_q) \rtimes_{\text{Ad}\psi} \hat{T}_\tau$ by right translations ρ on $C(G_q)$ and by the dual action on $C^*(\hat{T}_\tau)$.

Proof. Let us first identify the compact quantum group \tilde{G}_q^τ defined by the category of finite dimensional representations of $\mathcal{U}_q^\tau(\mathfrak{g})$ such that their restrictions to $\mathcal{U}_q(\mathfrak{g})$ are admissible. Any such irreducible representation is tensor product of an irreducible admissible representation of $\mathcal{U}_q(\mathfrak{g})$ and a character of T_τ ; recall that these can be regarded as representations of $\mathcal{U}_q^\tau(\mathfrak{g})$. It follows that the Hopf $*$ -algebra $\mathbb{C}[\tilde{G}_q^\tau]$ contains copies of $\mathbb{C}[G_q]$ and $C^*(\hat{T}_\tau)$, and as a space it is tensor product of these Hopf $*$ -subalgebras. It remains to find relations between elements of $\mathbb{C}[G_q]$ and $C^*(\hat{T}_\tau)$ inside $\mathbb{C}[\tilde{G}_q^\tau]$.

Let (π, V) be a finite dimensional admissible representation of $\mathcal{U}_q(\mathfrak{g})$, and χ be a character of T_τ . Then, on the one hand, $\pi \otimes \chi$ is a representation on V with E_i acting by $\chi(\tau_i)\pi(E_i)$. On the other hand, $\chi \otimes \pi$ is also a representation on the same space V with E_i acting by $\pi(E_i)$. From this we see that the operator $\pi(\tilde{\psi}(\chi))$, where we consider the standard extension of π to $\mathcal{U}(G_q)$, intertwines $\chi \otimes \pi$ with $\pi \otimes \chi$. In other words, if $U_\pi \in B(V) \otimes \mathbb{C}[G_q]$ is the representation of G_q defined by π , then in $B(V) \otimes \mathbb{C}[\tilde{G}_q^\tau]$ we have

$$(\pi(\tilde{\psi}(\chi)) \otimes u_\chi)U_\pi = U_\pi(\pi(\tilde{\psi}(\chi)) \otimes u_\chi).$$

Since

$$(\pi(\tilde{\psi}(\chi)^{-1}) \otimes 1)U_\pi(\pi(\tilde{\psi}(\chi)) \otimes 1) = (\iota \otimes \text{Ad}\psi(\chi))(U_\pi),$$

this exactly means that if $a \in \mathbb{C}[G_q]$ is a matrix coefficient of π , then $u_\chi a = (\text{Ad}\psi(\chi))(a)u_\chi$. Therefore $\mathbb{C}[\tilde{G}_q^\tau] = \mathbb{C}[G_q] \rtimes_{\text{Ad}\psi} \hat{T}_\tau$.

Now, the quantum group G_q^τ is the quotient of \tilde{G}_q^τ defined by the category of τ -admissible representations. By definition, a representation $\pi \otimes \chi$ of $\mathcal{U}_q^\tau(\mathfrak{g})$ is τ -admissible if $\pi(\tau_i) = \chi(\tau_i)$. Therefore $\mathbb{C}[G_q^\tau] \subset \mathbb{C}[\tilde{G}_q^\tau] = \mathbb{C}[G_q] \rtimes_{\text{Ad}\psi} \hat{T}_\tau$ is spanned by elements of the form au_χ , where a is a matrix coefficient of an admissible representation π such that $\pi(\tau_i) = \chi(\tau_i)$. If π is irreducible, then $\pi(\tau_i)$ is scalar, and we have $\rho(\tau_i)(a) = \pi(\tau_i)a$. Hence $\mathbb{C}[G_q^\tau] = (\mathbb{C}[G_q] \rtimes_{\text{Ad}\psi} \hat{T}_\tau)^{T_\tau}$. \square

Corollary 3.2. *The C^* -algebra $C(G_q^\tau)$ is of type I.*

Proof. Since $C(G_q^\tau) \subset C(G_q) \rtimes_{\text{Ad}\psi} \hat{T}_\tau$, this follows from the known fact that the C^* -algebra $C(G_q)$ is of type I. \square

Recall that the family $(C(G_q))_{0 < q < \infty}$ has canonical structure of a continuous field of C^* -algebras [NT11].

Corollary 3.3. *The C^* -algebras $(C(G_q^\tau))_{0 < q < \infty}$ form a continuous field of C^* -algebras.*

3.2. Primitive spectrum. Let us turn to a description of the primitive spectrum of $C(G_q^\tau)$. We will concentrate on the case $q \neq 1$, the case $q = 1$ can be treated similarly. First of all observe that the action of T_τ on $C(G_q) \rtimes_{\text{Ad}\psi} \hat{T}_\tau$ is saturated, since every spectral subspace contains a unitary. We thus obtain a strong Morita equivalence

$$(3.1) \quad C(G_q^\tau) \sim_M C(G_q) \rtimes_{\text{Ad}\psi} \hat{T}_\tau \rtimes_{\rho, \widehat{\text{Ad}\psi}} T_\tau \cong C(G_q) \rtimes_\rho T_\tau \rtimes_{\text{Ad}\psi, \hat{\rho}} \hat{T}_\tau.$$

Recall how to describe primitive spectra of crossed products, see e.g. [Wil07]. Let Γ be a finite group acting on a separable C^* -algebra A . Then any primitive ideal J of $A \rtimes \Gamma$ is determined by the Γ -orbit of an ideal $I \in \text{Prim}(A)$ and an ideal $J_0 \in \text{Prim}(A \rtimes \text{Stab}_\Gamma(I))$ by the condition $J_0 \cap A = I$ and $J = \text{Ind } J_0$.

If A is of type I, the ideals J_0 can, in turn, be described as follows. Put $\Gamma_0 = \text{Stab}_\Gamma(I)$. We want to describe irreducible representations of $A \rtimes \Gamma_0$ whose restrictions to A have kernel I . Let H be the space of an irreducible representation π of A with kernel I . Then the action of Γ_0 on A/I is implemented by a projective unitary representation $\gamma \mapsto u_\gamma$ of Γ_0 on H . Let ω be the corresponding 2-cocycle. Consider the regular $\bar{\omega}$ -representation $\gamma \mapsto \lambda_\gamma^{\bar{\omega}}$ of Γ_0 on $\ell^2(\Gamma_0)$. Then $A \rtimes \Gamma_0$ has a representation on $H \otimes \ell^2(\Gamma_0)$ defined by $a \mapsto \pi(a) \otimes 1$, $\gamma \mapsto u_\gamma \otimes \lambda_\gamma^{\bar{\omega}}$. Any irreducible representation of $A \rtimes \Gamma_0$ whose restriction to A has kernel I is a subrepresentation of this representation. So it remains to decompose the representation of $A \rtimes \Gamma_0$ on $H \otimes \ell^2(\Gamma_0)$ into irreducible subrepresentations. The von Neumann algebra generated by the image of $A \rtimes \Gamma_0$ is $B(H) \otimes C^*(\Gamma_0; \bar{\omega})$. Therefore the representations we are interested in are in a one-to-one correspondence with irreducible representations of $C^*(\Gamma_0; \bar{\omega})$.

To summarize, if A is a separable C^* -algebra of type I and Γ is a finite group acting on A , then the primitive spectrum $\text{Prim}(A \rtimes \Gamma)$ can be identified with the set of pairs $([I], J)$, where $[I]$ is the Γ -orbit of an ideal $I \in \text{Prim}(A)$, $J \in \text{Prim}(C^*(\Gamma_I; \bar{\omega}_I))$, and ω_I is the 2-cocycle on $\Gamma_I = \text{Stab}_\Gamma(I)$ defined by a projective representation of Γ_I implementing the action of Γ_I on the image of A under an irreducible representation with kernel I .

Returning to $C(G_q^\tau)$, for an element $w \in W$ of the Weyl group and a character $\chi \in \hat{T}_\tau$, put $\theta_w(\chi) = w^{-1}(\tilde{\psi}(\chi))\tilde{\psi}(\chi)^{-1}$. This defines a homomorphism from \hat{T}_τ to T .

Proposition 3.4. *For $q > 0$, $q \neq 1$, the primitive spectrum of $C(G_q^\tau)$ can be identified with*

$$\coprod_{w \in W} (\theta_w(\hat{T}_\tau) \backslash T/T_\tau) \times \widehat{\theta_w^{-1}(T_\tau)}.$$

Proof. In view of the strong Morita equivalence (3.1) it suffices to describe the primitive spectrum of

$$C(G_q) \rtimes_\rho T_\tau \rtimes_{\text{Ad}\psi, \hat{\rho}} \hat{T}_\tau.$$

Recall that the spectrum of $C(G_q)$ is $W \times T$. The right translation action of T_τ on $C(G_q)$ defines an action on $W \times T$ that is simply the action by translations on T . Therefore $\text{Prim}(C(G_q) \rtimes_\rho T_\tau)$ can be identified with $W \times T/T_\tau$, and every irreducible representation of $C(G_q) \rtimes_\rho T_\tau$ is induced from an irreducible representation of $C(G_q)$.

Next, we have to understand the action of \hat{T}_τ on $\text{Prim}(C(G_q) \rtimes_\rho T_\tau)$. Since the dual action preserves the equivalence class of any induced representation, we just have to look at the action $\text{Ad}\psi$. Given a representation $\pi_w \otimes \pi_t$ of $C(G_q)$ corresponding to $(w, t) \in W \times T$, we have

$$(\pi_w \otimes \pi_t)(\text{Ad}\psi(\chi^{-1})) \sim \pi_w \otimes \pi_{\theta_w(\chi)t}$$

by [NT12, Lemma 3.4; Yam13, Lemma 8]. It follows that the action of \hat{T}_τ on $\text{Prim}(C(G_q) \rtimes_\rho T_\tau) = W \times T/T_\tau$ is by translations on T/T_τ via the homomorphisms $\theta_w: \hat{T}_\tau \rightarrow T$. Hence the space of \hat{T}_τ -orbits is $\coprod_{w \in W} \theta_w(\hat{T}_\tau) \backslash T/T_\tau$, and the stabilizer of a point (w, tT_τ) is $\theta_w^{-1}(T_\tau) \subset \hat{T}_\tau$.

To finish the proof of the proposition it remains to show that the action $(\text{Ad}\psi, \hat{\rho})$ of $\theta_w^{-1}(T_\tau)$ on $C(G_q) \rtimes_\rho T_\tau$ can be implemented in the space of the induced representation $\text{Ind}(\pi_w \otimes \pi_t)$ by a unitary

representation of $\theta_w^{-1}(T_\tau)$. For this, in turn, it suffices to show that the equivalences

$$(\pi_w \otimes \pi_{t'})(\text{Ad}t^{-1}) \sim \pi_w \otimes \pi_{w^{-1}(t)t^{-1}t'}$$

from [NT12, Lemma 3.4; Yam13, Lemma 8] can be implemented by a unitary representation $t \mapsto v_t$ of $T/Z(G)$ on the space of representation π_w . But this is easy to see. Specifically, using the notation of [NT12, Yam13], if $w = s_i$ is the reflection corresponding to a simple root α_i , then the required representation $t \mapsto v_t$ on $\ell^2(\mathbb{Z}_+)$ can be defined by $v_t e_n = \langle t, \alpha_i \rangle^n e_n$. For arbitrary w we just have to take tensor products of such representations. \square

Remark 3.5. A description of the topology on $\text{Prim}(C(G_q))$ is given in [NT12]. The above argument is, however, not quite enough to understand the topology on $\text{Prim}(C(G_q^\tau))$.

3.3. K-theory. The maximal torus T is embedded in $\mathcal{U}(G_q^\tau)$, so it can be considered as a subgroup of G_q^τ . Let us consider the right translation action ρ of T on $C(G_q^\tau)$. The crossed product $C(G_q^\tau) \rtimes_\rho T$ is a \hat{T} - C^* -algebra with respect to the dual action.

Proposition 3.6. *The dual action of \hat{T} on $C(G_q^\tau) \rtimes_\rho T$ is equivariantly strongly Morita equivalent to an action on $C(G_q) \rtimes_\rho T$ that is homotopic to the dual action.*

Proof. If we identify $C(G_q^\tau)$ with $(C(G_q) \rtimes_{\text{Ad}\psi} \hat{T}_\tau)^{T_\tau}$, then the action of T by right translations on $C(G_q^\tau)$ extends to an action on $C(G_q) \rtimes_{\text{Ad}\psi} \hat{T}_\tau$ that is trivial on $C^*(T_\tau)$ and coincides with the action by right translations on $C(G_q)$. This action of T on $C(G_q) \rtimes_{\text{Ad}\psi} \hat{T}_\tau$ commutes with the action of T_τ . Hence the strong Morita equivalence (3.1) is T -equivariant, and taking crossed products we get a \hat{T} -equivariant strong Morita equivalence

$$(3.2) \quad C(G_q^\tau) \rtimes_\rho T \sim_M C(G_q) \rtimes_{\text{Ad}\psi} \hat{T}_\tau \rtimes_{\rho, \widehat{\text{Ad}\psi}} T_\tau \rtimes_\rho T.$$

Denote the C^* -algebra on the right hand side by A . We claim that A is isomorphic to

$$B = C(G_q) \rtimes_{\text{Ad}\psi} \hat{T}_\tau \rtimes_{\widehat{\text{Ad}\psi}} T_\tau \rtimes_\rho T.$$

Indeed, the map $au_\chi u_t u_{t'} \mapsto au_\chi u_t u_{tt'}$ for $a \in C(G_q)$, $\chi \in \hat{T}_\tau$, $t \in T_\tau$ and $t' \in T$ is the required isomorphism. The dual action of \hat{T} on A corresponds to an action β on B which is given by the dual action on the copy of $C^*(T)$ and by the dual action on the copy of $C^*(T_\tau)$ via the canonical homomorphism $r: \hat{T} \rightarrow \hat{T}_\tau$.

The map $\hat{T} \ni \chi \mapsto u_{r(\chi)} \in C^*(\hat{T}_\tau) \subset M(B)$ is a 1-cocycle for the action β . Therefore β is strongly Morita equivalent to the action γ defined by $\gamma_\chi = (\text{Ad}u_{r(\chi)})\beta_\chi$. This action is already trivial on $C^*(T_\tau)$, while on $C(G_q)$ it is given by $\text{Ad}\psi(r(\chi))$, and on $C^*(T)$ it coincides with the dual action.

Denote by δ the restriction of γ to $C(G_q) \rtimes_\rho T \subset M(B)$. Then, similarly to (3.2), the actions δ and γ are strongly Morita equivalent.

Combining the Morita equivalences that we have obtained, we conclude that the dual action of \hat{T} on $C(G_q^\tau) \rtimes_\rho T$ is strongly Morita equivalent to the action $\delta = (\text{Ad}\psi(r(\cdot)), \hat{\rho})$ on $C(G_q) \rtimes_\rho T$. Choosing a basis in $\hat{T} = P$ and paths from $\hat{\psi}(r(\chi))$ to the neutral element in T for every basis element χ , we see that δ is homotopic to the dual action on $C(G_q) \rtimes_\rho T$. \square

Theorem 3.7. *The C^* -algebra $C(G_q^\tau)$ is KK-isomorphic to $C(G_q)$, hence to $C(G)$.*

Proof. Since the torsion-free commutative group \hat{T} satisfies the strong Baum–Connes conjecture, the functor $A \mapsto A \rtimes \hat{T}$ maps homotopic actions into KK-isomorphisms of the corresponding crossed products. By the previous proposition, this, together with the Takesaki–Takai duality, implies that $C(G_q^\tau)$ and $C(G_q)$ are KK-isomorphic. By [NT12] we also know that $C(G_q)$ is KK-isomorphic to $C(G)$. \square

Remark 3.8.

(i) The above proof shows that the continuous field of Corollary 3.3 is a KK-fibration in the sense of [ENOO09]. The argument of [NT11] applies to the Dirac operator D given by Remark 2.5, and we obtain that the K-homology class of D is independent of q . The bi-equivariance of D and the construction in the proof of Proposition 3.6 imply that the K-homology class of D is also independent of τ up to the isomorphism of Theorem 3.7.

(ii) For the group \hat{T} the strong Baum–Connes conjecture is a consequence of the Pimsner–Voiculescu sequence in KK-theory. Therefore the proof of Theorem 3.7 can be written such that it relies only on this sequence, see e.g. [San11, Section 5.1] for a related argument.

4. TWISTED $SU_q(n)$

4.1. Special unitary group. Let us review the structure of $SU(n)$, see e.g. [FH91, Chapter 15]. For the sake of presentation, it is convenient to consider also the unitary group $U(n)$. We take the subgroup of the diagonal matrices \tilde{T} as a maximal torus of $U(n)$, and take $T = \tilde{T} \cap SU(n)$ as a maximal torus of $SU(n)$. We will often identify \tilde{T} with \mathbb{T}^n . We write the corresponding Cartan subalgebras as $\tilde{\mathfrak{h}} \subset \mathfrak{gl}_n$ and $\mathfrak{h} \subset \mathfrak{sl}_n$.

Let $\{e_{ij}\}_{i,j=1}^n$ be the matrix units in $M_n(\mathbb{C}) = \mathfrak{gl}_n$, and $\{\tilde{L}_i\}_{i=1}^n$ be the basis in $\tilde{\mathfrak{h}}^*$ dual to the basis $\{e_{ii}\}_{i=1}^n$ in $\tilde{\mathfrak{h}}$. Denote by L_i the image of \tilde{L}_i in \mathfrak{h}^* . Therefore any $n-1$ elements among L_1, \dots, L_n form a basis in \mathfrak{h}^* , and we have $\sum_i L_i = 0$.

The weight lattice $P \subset \mathfrak{h}^*$ is generated by the elements L_i . The pairing between T and P is given by $\langle t, L_i \rangle = t_i$ for $t \in T \subset \mathbb{T}^n$. As simple roots we take

$$\alpha_i = L_i - L_{i+1}, \quad 1 \leq i \leq n-1.$$

The fundamental weights are then given by

$$\varpi_i = L_1 + \dots + L_i, \quad 1 \leq i \leq n-1.$$

Consider the homomorphism $|\cdot|: P \rightarrow \mathbb{Z}$ such that $L_1 \mapsto n-1$ and $L_i \mapsto -1$ for $1 < i \leq n$. In other words,

$$|a_1 \varpi_1 + \dots + a_{n-1} \varpi_{n-1}| = \lambda_1 + \dots + \lambda_{n-1},$$

where λ_{n-i} is given by $a_1 + \dots + a_i$. The image of Q under $|\cdot|$ is $n\mathbb{Z}$, and therefore we can use this homomorphism to identify P/Q with $\mathbb{Z}/n\mathbb{Z}$.

4.2. Twisted quantum special unitary groups. By Proposition A.3, the cohomology group $H^3(\mathbb{Z}/n\mathbb{Z}; \mathbb{T})$ is isomorphic to $\mathbb{Z}/n\mathbb{Z}$, and a cocycle generating this group can be defined by

$$\phi(a, b, c) = \zeta_n^{\omega_n(a,b)c}, \quad \text{where } \zeta_n = e^{2\pi i/n} \quad \text{and} \quad \omega_n(a, b) = \left\lfloor \frac{a+b}{n} \right\rfloor - \left\lfloor \frac{a}{n} \right\rfloor - \left\lfloor \frac{b}{n} \right\rfloor.$$

Using this generator we identify $H^3(\mathbb{Z}/n\mathbb{Z}; \mathbb{T})$ with the group $\mu_n \subset \mathbb{T}$ of units of order n . Therefore, given $\zeta \in \mu_n$, we have a category $\text{Rep}(SU_q(n))^\zeta$ with associativity morphisms defined by multiplication by $\zeta^{\omega_n(|\lambda|, |\eta|) |\nu|}$ on the tensor product $V_\lambda \otimes V_\eta \otimes V_\nu$ of irreducible $\mathcal{U}_q(\mathfrak{g})$ -modules with highest weights λ, η, ν . This agrees with the conventions of Kazhdan and Wenzl [KW93].

It is also convenient to identify $Z(SU(n))$ with the group μ_n . Thus, for $\tau = (\tau_1, \dots, \tau_{n-1}) \in \mu_n^{n-1}$, we can define a twisting $SU_q^\tau(n)$ of $SU_q(n)$. Its representation category is one of $\text{Rep}(SU_q(n))^\zeta$, and to find ζ we have to compute the homomorphism $\Theta: Z(SU(n))^{n-1} \rightarrow H^3(P/Q; \mathbb{T})$ introduced in Section 2.3. Under our identifications this becomes a homomorphism $\mu_n^{n-1} \rightarrow \mu_n$.

Proposition 4.1. *We have $\Theta(\tau) = \prod_{i=1}^{n-1} \tau_i^{-i}$.*

Proof. Recall the construction of Θ . We choose a function $f: P \times P \rightarrow \mathbb{T}$ such that it factors through $P \times (P/Q)$ and $f(\lambda + \alpha_i, \mu) = \langle \tau_i, \mu \rangle f(\lambda, \mu)$. Then $\Theta(\tau)$ is the cohomology class of ∂f in $H^3(P/Q; \mathbb{T})$.

Note that $\langle \tau_i, \mu \rangle = \tau_i^{-|\mu|}$, which is immediate for $\mu = L_j$, and define a character χ of $Q \otimes (P/Q) = Q \otimes (\mathbb{Z}/n\mathbb{Z})$ by

$$\chi(\alpha_i \otimes k) = \tau_i^k \quad \text{for } 1 \leq i \leq n-1 \text{ and } k \in \mathbb{Z}/n\mathbb{Z},$$

so that $f(\lambda + \alpha, \mu) = \chi(\alpha \otimes |\mu|)f(\lambda, \mu)$ for all $\alpha \in Q$. By Proposition A.6, the cohomology class of ∂f depends only on the restriction of χ to

$$\ker(Q \otimes (\mathbb{Z}/n\mathbb{Z}) \rightarrow P \otimes (\mathbb{Z}/n\mathbb{Z})) \cong \text{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z},$$

and by varying τ we get this way an isomorphism $\text{Hom}(\text{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}), \mathbb{T}) \cong H^3(\mathbb{Z}/n\mathbb{Z}; \mathbb{T})$. In order to compute this isomorphism we can use the resolution $n\mathbb{Z} \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}/n\mathbb{Z}$ instead of $Q \rightarrow P \xrightarrow{|\cdot|} \mathbb{Z}/n\mathbb{Z}$. Define a morphism between these resolutions by $\mathbb{Z} \rightarrow P$, $1 \mapsto \varpi_{n-1} = -L_n$. By pulling back χ under this morphism, we get a character $\tilde{\chi}$ of $(n\mathbb{Z}) \otimes (\mathbb{Z}/n\mathbb{Z})$ such that

$$\tilde{\chi}(n \otimes k) = \chi(n\varpi_{n-1} \otimes k).$$

We have $n\varpi_{n-1} = \sum_{i=1}^{n-1} i\alpha_i$. Therefore

$$\tilde{\chi}(n \otimes k) = \zeta^k, \quad \text{where } \zeta = \prod_{i=1}^{n-1} \tau_i^i.$$

Then the function $\tilde{f}: \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{T}$ defined by

$$\tilde{f}(a, b) = \zeta^{\lfloor \frac{a}{n} \rfloor b},$$

factors through $\mathbb{Z} \times (\mathbb{Z}/n\mathbb{Z})$, $\tilde{f}(a+n, b) = \tilde{\chi}(n \otimes b)\tilde{f}(a, b)$ and $(\partial \tilde{f})(a, b, c) = \zeta^{-\omega_n(a,b)c}$. Therefore the class of $\partial \tilde{f}$ in $H^3(\mathbb{Z}/n\mathbb{Z}; \mathbb{T}) = \mu_n$ is ζ^{-1} . \square

In Section 2.3 we also introduced a homomorphism Υ . In the present case we have $H^2(P/Q; \mathbb{T}) = 0$, so Υ is a homomorphism $\ker \Theta \rightarrow H^2(P; \mathbb{T})$.

Lemma 4.2. *The homomorphism $\Upsilon: \ker \Theta \rightarrow H^2(P; \mathbb{T})$ is injective.*

Proof. Assume $\tau \in \ker \Theta$, so $\prod_{i=1}^{n-1} \tau_i^i = 1$. In this case the character χ of $Q \otimes (P/Q)$ from the proof of the previous proposition extends to $P \otimes (P/Q)$ by

$$\chi(L_i \otimes \mu) = (\tau_1 \dots \tau_{i-1})^{-|\mu|} \quad \text{for } 1 \leq i \leq n \text{ and } \mu \in P.$$

Therefore if we consider χ as a function on $P \times P$, we can take it as a function f in that proof. Then f is a 2-cocycle, and by definition, the image of τ under Υ is the cohomology class of \tilde{f} . It is well-known, and also follows from Proposition A.1, that f is a coboundary if and only if f is symmetric. For $1 < i < j \leq n$ we have

$$f(L_i, L_j) \overline{f(L_j, L_i)} = (\tau_i \dots \tau_{j-1})^{-1}.$$

So if f is symmetric, then $\tau_2 = \dots = \tau_{n-1} = 1$, but then also $\tau_1 = 1$. \square

Therefore Proposition 2.7 does not give us any nontrivial isomorphisms between the quantum groups $\text{SU}_q^\tau(n)$. On the other hand, the flip map on the Dynkin diagram induces an automorphism of $\mathcal{U}(\text{SU}_q(n))$ such that $K_i \mapsto K_{n-i}$ and $E_i \mapsto E_{n-i}$ for $1 \leq i \leq n-1$. On $Z(\text{SU}(n)) \subset \mathcal{U}(\text{SU}_q(n))$ this automorphism is $t \mapsto t^{-1}$. It follows that it induces isomorphisms

$$\text{SU}_q^{(\tau_1, \dots, \tau_{n-1})}(n) \cong \text{SU}_q^{(\tau_{n-1}^{-1}, \dots, \tau_1^{-1})}(n).$$

For $0 < q < 1$, these seem to be the only obvious isomorphisms between the quantum groups $\text{SU}_q^\tau(n)$.

4.3. Generators and relations. The C^* -algebra $C(SU_q(n))$ is generated by the matrix coefficients $(u_{ij})_{1 \leq i, j \leq n}$ of the natural representation of $SU_q(n)$ on \mathbb{C}^n , the fundamental representation with highest weight ϖ_1 . They satisfy the relations [Dri87, Wor88]

$$(4.1) \quad u_{ij}u_{il} = qu_{il}u_{ij} \quad (j < l), \quad u_{ij}u_{kj} = qu_{kj}u_{ij} \quad (i < k),$$

$$(4.2) \quad u_{ij}u_{kl} = u_{kl}u_{ij} \quad (i > k, j < l), \quad u_{ij}u_{kl} - u_{kl}u_{ij} = (q - q^{-1})u_{il}u_{kj} \quad (i < k, j < l),$$

$$(4.3) \quad \text{qdet}((u_{ij})_{i,j}) = \sum_{\sigma \in S_n} (-q)^{|\sigma|} u_{1\sigma(1)} \cdots u_{n\sigma(n)} = 1.$$

Here, $|\sigma|$ is the inversion number of the permutation σ . The involution is defined by

$$u_{ij}^* = (-q)^{j-i} \text{qdet}(U_j^i),$$

where U_j^i is the matrix obtained from $U = (u_{kl})_{k,l}$ by deleting the i -th row and j -th column.

In order to find generators and relations of $\mathbb{C}[SU_q^\tau(n)]$, we will use the embedding of the algebra $\mathbb{C}[SU_q^\tau(n)]$ into $\mathbb{C}[SU_q(n)] \rtimes_{\text{Ad}\psi} \hat{T}_\tau$ described in Theorem 3.1. Recall that $\psi: \hat{T}_\tau \rightarrow T/Z(SU(n)) = T/\mu_n$ is the homomorphism such that $\langle \tilde{\psi}(\chi), \alpha_i \rangle = \chi(\tau_i)$, where $\tilde{\psi}(\chi)$ is a lift of $\psi(\chi)$ to T . Hence

$$\tilde{\psi}(\chi) = (z, z\chi(\tau_1)^{-1}, \dots, z\chi(\tau_1 \dots \tau_{n-1})^{-1}) \in T \subset \mathbb{T}^n,$$

where $z \in \mathbb{T}$ is a number such that $z^n = \prod_{i=1}^{n-1} \chi(\tau_i)^{-i}$. It follows that

$$(4.4) \quad (\text{Ad}\psi(\chi))(u_{ij}) = \left(\prod_{1 \leq p < i} \chi(\tau_p) \right) \left(\prod_{1 \leq p < j} \chi(\tau_p)^{-1} \right) u_{ij}.$$

Now, the algebra $\mathbb{C}[SU_q^\tau(n)]$ is generated by matrix coefficients of the fundamental representation of $SU_q^\tau(n)$ with highest weight ϖ_1 . Under the embedding $\mathbb{C}[SU_q^\tau(n)] \hookrightarrow \mathbb{C}[SU_q(n)] \rtimes_{\text{Ad}\psi} \hat{T}_\tau$, these matrix coefficients correspond to $v_{ij} = u_{ij}u_{\chi_{\text{nat}}}$, where $\chi_{\text{nat}} \in \hat{T}_\tau$ is the character determined by the natural representation of $SU_q(n)$ on \mathbb{C}^n , so $\chi_{\text{nat}}(\tau_i) = \tau_i$. From (4.1)-(4.3) we then get the following relations:

$$(4.5) \quad v_{ij}v_{il} = \left(\prod_{j \leq p < l} \tau_p^{-1} \right) qv_{il}v_{ij} \quad (j < l), \quad v_{ij}v_{kj} = \left(\prod_{i \leq p < k} \tau_p \right) qv_{kj}v_{ij} \quad (i < k),$$

$$(4.6) \quad v_{ij}v_{kl} = \left(\prod_{k \leq p < i} \tau_p^{-1} \right) \left(\prod_{j \leq p < l} \tau_p^{-1} \right) v_{kl}v_{ij} \quad (i > k, j < l),$$

$$(4.7) \quad \left(\prod_{j \leq p < l} \tau_p \right) v_{ij}v_{kl} - \left(\prod_{i \leq p < k} \tau_p \right) v_{kl}v_{ij} = (q - q^{-1})v_{il}v_{kj} \quad (i < k, j < l),$$

$$(4.8) \quad \sum_{\sigma \in S_n} \tau^{m(\sigma)} (-q)^{|\sigma|} v_{1\sigma(1)} \cdots v_{n\sigma(n)} = 1,$$

where $m(\sigma) = (m(\sigma)_1, \dots, m(\sigma)_{n-1})$ is the multi-index given by $m(\sigma)_i = \sum_{k=2}^n (k-1)m_i^{(k, \sigma(k))}$, and

$$m_i^{(k,j)} = \begin{cases} 1, & \text{if } k \leq i < j, \\ -1, & \text{if } j \leq i < k, \\ 0, & \text{otherwise.} \end{cases}$$

Proposition 4.3. *For any $\tau \in \mu_n^{n-1}$, the algebra $\mathbb{C}[SU_q^\tau(n)]$ is a universal algebra generated by elements v_{ij} satisfying relations (4.5)-(4.8).*

Proof. We already know that relations (4.5)-(4.8) are satisfied in $\mathbb{C}[SU_q^\tau(n)]$, so we just have to show that there are no other relations. Let \mathcal{A} be a universal algebra generated by elements w_{ij} satisfying relations (4.5)-(4.8). We can define an action of \hat{T}_τ on \mathcal{A} by (4.4). Then in $\mathcal{A} \rtimes \hat{T}_\tau$ the elements $w_{ij}u_{\chi_{\text{nat}}}^{-1}$ satisfy the defining relations of $\mathbb{C}[SU_q(n)]$, so we have a homomorphism $\mathbb{C}[SU_q(n)] \rightarrow \mathcal{A} \rtimes \hat{T}_\tau$

mapping u_{ij} into $w_{ij}u_{\chi_{\text{nat}}}^{-1}$. It extends to a homomorphism $\mathbb{C}[\text{SU}_q(n)] \rtimes \hat{T}_\tau \rightarrow \mathcal{A} \rtimes \hat{T}_\tau$ that is identity on the group algebra of \hat{T}_τ . Restricting to $\mathbb{C}[\text{SU}_q^\tau(n)] \subset \mathbb{C}[\text{SU}_q(n)] \rtimes \hat{T}_\tau$, we get a homomorphism $\mathbb{C}[\text{SU}_q^\tau(n)] \rightarrow \mathcal{A}$ mapping v_{ij} into w_{ij} . \square

The involution on $\mathbb{C}[\text{SU}_q^\tau(n)]$ is determined by requiring the invertible matrix $(v_{ij})_{i,j}$ to be unitary. An explicit formula can be easily found using that for $\mathbb{C}[\text{SU}_q(n)]$.

Remark 4.4. The relations in $\mathbb{C}[\text{SU}_q^\tau(n)]$ cannot be obtained using the FRT-approach, since the categories $\text{Rep}(\text{SU}_q(n))^\zeta$ are typically not braided. More precisely, $\text{Rep}(\text{SU}_q(n))^\zeta$ has a braiding if and only if either $\zeta = 1$ or n is even and $\zeta = -1$. This statement is already implicit in [KW93], and it can be proved as follows. If $\zeta = 1$ or n is even and $\zeta = -1$, then a braiding indeed exists, see e.g. [Pin07]. Conversely, suppose we have a braiding. In other words, there exists an R -matrix \mathcal{R} for $(\mathcal{U}(\text{SU}_q(n)), \hat{\Delta}_q, \Phi)$, where $\Phi = \zeta^{\omega_n(|\lambda|, |\eta|)|\nu|}$. Recall that this means that \mathcal{R} is an invertible element in $\mathcal{U}(\text{SU}_q(n) \times \text{SU}_q(n))$ such that $\hat{\Delta}_q^{\text{op}} = \mathcal{R}\hat{\Delta}_q(\mathcal{R})^{-1}$ and

$$(\hat{\Delta}_q \otimes \iota)(\mathcal{R}) = \Phi_{312}\mathcal{R}_{13}\Phi_{132}^{-1}\mathcal{R}_{23}\Phi, \quad (\iota \otimes \hat{\Delta}_q)(\mathcal{R}) = \Phi_{231}^{-1}\mathcal{R}_{13}\Phi_{213}\mathcal{R}_{12}\Phi^{-1}.$$

Since Φ is central and symmetric in the first two variables, the last two identities can be written as

$$(\hat{\Delta}_q^{\text{op}} \otimes \iota)(\mathcal{R}) = \mathcal{R}_{23}\mathcal{R}_{13}\Phi, \quad (\iota \otimes \hat{\Delta}_q)(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{12}\Phi_{321}^{-1}.$$

On the other hand, we know that $\text{Rep}(\text{SU}_q(n))$ is braided, so there exists an element \mathcal{R}_q satisfying the above properties with Φ replaced by 1. Consider the element $F = \mathcal{R}_q^{-1}\mathcal{R}$. Then F is invariant, meaning that it commutes with the image of $\hat{\Delta}_q$. Furthermore, we have

$$\begin{aligned} (F \otimes 1)(\hat{\Delta}_q \otimes \iota)(F) &= (\mathcal{R}_q^{-1} \otimes 1)(\hat{\Delta}_q^{\text{op}} \otimes \iota)(\mathcal{R}_q^{-1})(\hat{\Delta}_q^{\text{op}} \otimes \iota)(\mathcal{R})(\mathcal{R} \otimes 1) \\ &= ((\mathcal{R}_q)_{23}(\mathcal{R}_q)_{13}(\mathcal{R}_q)_{12})^{-1}\mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}\Phi, \end{aligned}$$

and similarly

$$\begin{aligned} (1 \otimes F)(\iota \otimes \hat{\Delta}_q)(F) &= (\iota \otimes \hat{\Delta}_q)(\mathcal{R}_q^{-1})(1 \otimes \mathcal{R}_q^{-1})(1 \otimes \mathcal{R})(\iota \otimes \hat{\Delta}_q)(\mathcal{R}) \\ &= ((\mathcal{R}_q)_{23}(\mathcal{R}_q)_{13}(\mathcal{R}_q)_{12})^{-1}\mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}\Phi_{321}^{-1}. \end{aligned}$$

Therefore

$$(\iota \otimes \hat{\Delta}_q)(F^{-1})(1 \otimes F^{-1})(F \otimes 1)(\hat{\Delta}_q \otimes \iota)(F) = \Phi_{321}\Phi.$$

This implies that $\text{Rep}(\text{SU}_q(n))$ is monoidally equivalent to $\text{Rep}(\text{SU}_q(n))^{\Phi_{321}\Phi}$. Since the cocycle $\Phi_{321}\Phi$ on the dual of the center is cohomologous to the cocycle $\zeta^{2\omega_n(|\lambda|, |\eta|)|\nu|}$, this means that $\text{Rep}(\text{SU}_q(n))$ is monoidally equivalent to $\text{Rep}(\text{SU}_q(n))^{\zeta^2}$. By the Kazhdan–Wenzl classification this is the case only if $\zeta^2 = 1$.

APPENDIX A. COCYCLES ON ABELIAN GROUPS

Let Γ be a discrete abelian group. As is common in operator algebra, we denote the generators of the group algebra $\mathbb{Z}[\Gamma]$ by λ_γ ($\gamma \in \Gamma$). Let $(C_*(\Gamma), d)$ be the nonnormalized bar-resolution of the $\mathbb{Z}[\Gamma]$ -module \mathbb{Z} , so $C_n(\Gamma)$ ($n \geq 0$) is the free $\mathbb{Z}[\Gamma]$ -module with basis consisting of n -tuples of elements in Γ , written as $[\gamma_1 | \dots | \gamma_n]$, and the differential $d: C_n(\Gamma) \rightarrow C_{n-1}(\Gamma)$ is defined by

$$d[\gamma_1 | \dots | \gamma_n] = \lambda_{\gamma_1}[\gamma_2 | \dots | \gamma_n] + \sum_{i=1}^{n-1} (-1)^i [\gamma_1 | \dots | \gamma_i + \gamma_{i+1} | \dots | \gamma_n] + (-1)^n [\gamma_1 | \dots | \gamma_{n-1}].$$

Let M be a commutative group endowed with the trivial Γ -module structure. The group cohomology $H^*(\Gamma; M)$ can be computed from the standard complex induced by the bar-resolution. Concretely, we have a cochain complex

$$C^*(\Gamma; M) = \text{Hom}_{\mathbb{Z}[\Gamma]}(C_*(\Gamma), M) = \text{Map}(\Gamma^*, M),$$

endowed with the boundary map $\partial: C^n(\Gamma; M) \rightarrow C^{n+1}(\Gamma; M)$ defined by

$$\begin{aligned} (\partial\phi)(\gamma_1, \dots, \gamma_{n+1}) &= \phi(\gamma_2, \dots, \gamma_{n+1}) - \phi(\gamma_1 + \gamma_2, \gamma_3, \dots, \gamma_{n+1}) + \dots \\ &\quad + (-1)^n \phi(\gamma_1, \dots, \gamma_{n-1}, \gamma_n + \gamma_{n+1}) + (-1)^{n+1} \phi(\gamma_1, \dots, \gamma_n). \end{aligned}$$

By M -valued cocycles on Γ we mean cocycles in $(C^*(\Gamma; M), \partial)$. We will consider only \mathbb{T} -valued cocycles, but with minor modifications everything what we say remains true for cocycles with values in any divisible group M .

For the sake of computation, it is also convenient to introduce the integer homology $H_*(\Gamma) = H_*(\Gamma; \mathbb{Z})$, which is given as the homology of the complex $C_*(\Gamma; \mathbb{Z}) = \mathbb{Z} \otimes_{\mathbb{Z}[\Gamma]} C_*(\Gamma)$. Since the action of Γ on \mathbb{T} is trivial, we have $C^*(\Gamma; \mathbb{T}) = \text{Hom}_{\mathbb{Z}[\Gamma]}(C_*(\Gamma), \mathbb{T}) = \text{Hom}(C_*(\Gamma; \mathbb{Z}), \mathbb{T})$. Moreover, the injectivity of \mathbb{T} as a \mathbb{Z} -module implies that any character of $H_n(\Gamma; \mathbb{Z})$ can be lifted to a character of $C_n(\Gamma; \mathbb{Z})$. It follows that the groups $H^n(\Gamma; \mathbb{T})$ and $H_n(\Gamma)$ are Pontryagin dual to each other. This is a particular case of the Universal Coefficient Theorem.

A map $\phi: \Gamma^n \rightarrow \mathbb{T}$ ($n \geq 1$) is called an n -character on Γ if it is a character in every variable, so it is defined by a character on $\Gamma^{\otimes n}$ (unless specified otherwise, all tensor products in this appendix are over \mathbb{Z}). It is easy to see that every n -character is a \mathbb{T} -valued cocycle. An n -character ϕ is called alternating if $\phi(\gamma_1, \dots, \gamma_n) = 1$ as long as $\gamma_i = \gamma_{i+1}$ for some i ; then $\phi(\gamma_{\sigma(1)}, \dots, \gamma_{\sigma(n)}) = \phi(\gamma_1, \dots, \gamma_n)^{\text{sgn}(\sigma)}$ for any $\sigma \in S_n$. In other words, a n -character is alternating if it factors through the exterior power $\bigwedge^n \Gamma$, which is the quotient of $\Gamma^{\otimes n}$ by the subgroup generated by elements $\gamma_1 \otimes \dots \otimes \gamma_n$ such that $\gamma_i = \gamma_{i+1}$ for some i . It will sometimes be convenient to view $\bigwedge^n \Gamma$ as a subgroup of $\Gamma^{\otimes n}$ via the embedding

$$\gamma_1 \wedge \dots \wedge \gamma_n \mapsto \sum_{\sigma \in S_n} \text{sgn}(\sigma) \gamma_{\sigma(1)} \otimes \dots \otimes \gamma_{\sigma(n)}.$$

We will also consider $\bigwedge^n \Gamma$ as a subgroup of $H_n(\Gamma)$. The embedding $\bigwedge^n \Gamma \hookrightarrow H_n(\Gamma)$ is constructed using the canonical isomorphism $\Gamma \cong H_1(\Gamma)$ and the Pontryagin product on $H_*(\Gamma)$, see [Bro94, Theorem V.6.4]. On the chain level the latter product can be defined using the shuffle product, so that $\gamma_1 \wedge \dots \wedge \gamma_n$ is identified with the homology class of the cycle

$$\sum_{\sigma \in S_n} \text{sgn}(\sigma) (1 \otimes [\gamma_{\sigma(1)} | \dots | \gamma_{\sigma(n)}]) \in C_n(\Gamma; \mathbb{Z}).$$

For free abelian groups we have $\bigwedge^n \Gamma = H_n(\Gamma)$. By duality we get the following description of cocycles.

Proposition A.1. *If Γ is free abelian, then for every $n \geq 1$ we have:*

- (i) *any \mathbb{T} -valued n -cocycle on Γ is cohomologous to an alternating n -character;*
- (ii) *an n -character is a coboundary if and only if it vanishes on $\bigwedge^n \Gamma \subset \Gamma^{\otimes n}$; in particular, an alternating n -character is a coboundary if and only if its order divides $n!$.*

Proof. The value of an n -cocycle ϕ on $\gamma_1 \wedge \dots \wedge \gamma_n \in H_n(\Gamma)$ is

$$\langle \phi, \gamma_1 \wedge \dots \wedge \gamma_n \rangle = \prod_{\sigma \in S_n} \phi(\gamma_{\sigma(1)}, \dots, \gamma_{\sigma(n)})^{\text{sgn}(\sigma)}.$$

This immediately implies (ii), since if ϕ is an n -character, then the above product is exactly the value of ϕ on $\gamma_1 \wedge \dots \wedge \gamma_n$ considered as an element of $\Gamma^{\otimes n}$.

Turning to (i), assume ψ is an n -cocycle. It defines a character χ of $H_n(\Gamma) = \bigwedge^n \Gamma$. Let ϕ be a character of $\bigwedge^n \Gamma$ such that $\phi^{n!} = \chi$. Then ϕ is an alternating n -character, and ϕ is cohomologous to ψ , since both cocycles ϕ and ψ define the same character χ of $H_n(\Gamma) = \bigwedge^n \Gamma$. \square

We now turn to the more complicated case of finite abelian groups and concentrate on 3-cocycles. In this case $\bigwedge^3 \Gamma$ is a proper subgroup of $H_3(\Gamma)$: as follows from Proposition A.3 below, the quotient $H_3(\Gamma)/\bigwedge^3 \Gamma$ is (noncanonically) isomorphic to $\Gamma \oplus (\Gamma \wedge \Gamma)$. Correspondingly, not every third

cohomology class can be represented by a 3-character. Additional 3-cocycles can be obtained by the following construction.

Lemma A.2. *Assume $\Gamma = \Gamma_1/\Gamma_0$ for some abelian groups Γ_1 and Γ_0 . Suppose $f: \Gamma_1 \times \Gamma_1 \rightarrow \mathbb{T}$ is a function such that*

$$f(\alpha, \beta + \gamma) = f(\alpha, \beta) \quad \text{and} \quad f(\alpha + \gamma, \beta) = \chi(\gamma \otimes \beta)f(\alpha, \beta)$$

for all $\alpha, \beta \in \Gamma_1$ and $\gamma \in \Gamma_0$, where χ is a character of $\Gamma_0 \otimes \Gamma$. Then the function

$$(\partial f)(\alpha, \beta, \gamma) = f(\beta, \gamma)f(\alpha + \beta, \gamma)^{-1}f(\alpha, \beta + \gamma)f(\alpha, \beta)^{-1}$$

on Γ_1^3 is Γ_0^3 -invariant, hence it defines a \mathbb{T} -valued 3-cocycle on Γ .

Proof. This is a straightforward computation. □

In order to describe explicitly generators of $H^3(\Gamma; \mathbb{T})$, let us introduce some notation. For natural numbers n_1, \dots, n_k , denote by (n_1, \dots, n_k) their greatest common divisor. For $n \in \mathbb{N}$, denote by χ_n the character of $\mathbb{Z}/n\mathbb{Z}$ defined by $\chi_n(1) = e^{2\pi i/n}$. Finally, for integers a and b and a natural number n , put

$$\omega_n(a, b) = \left\lfloor \frac{a+b}{n} \right\rfloor - \left\lfloor \frac{a}{n} \right\rfloor - \left\lfloor \frac{b}{n} \right\rfloor.$$

Note that ω_n is a well-defined function on $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ with values 0 or 1.

Proposition A.3. *Assume $\Gamma = \bigoplus_{i=1}^m \mathbb{Z}/n_i\mathbb{Z}$ for some $n_i \geq 1$. Then*

$$H^3(\Gamma; \mathbb{T}) \cong \bigoplus_i \mathbb{Z}/n_i\mathbb{Z} \oplus \bigoplus_{i<j} \mathbb{Z}/(n_i, n_j)\mathbb{Z} \oplus \bigoplus_{i<j<k} \mathbb{Z}/(n_i, n_j, n_k)\mathbb{Z}.$$

Explicitly, generators ϕ_i of $\mathbb{Z}/n_i\mathbb{Z}$, ϕ_{ij} of $\mathbb{Z}/(n_i, n_j)\mathbb{Z}$ and ϕ_{ijk} of $\mathbb{Z}/(n_i, n_j, n_k)\mathbb{Z}$ can be defined by

$$\phi_i(a, b, c) = \chi_{n_i}(\omega_{n_i}(a_i, b_i)c_i), \quad \phi_{ij}(a, b, c) = \chi_{n_j}(\omega_{n_i}(a_i, b_i)c_j), \quad \phi_{ijk}(a, b, c) = \chi_{(n_i, n_j, n_k)}(a_i b_j c_k).$$

Proof. Recall first how to compute the homology of finite cyclic groups. Consider the group $\mathbb{Z}/n\mathbb{Z}$. Then there is a free resolution (P_*, d) of the $\mathbb{Z}[\mathbb{Z}/n\mathbb{Z}]$ -module \mathbb{Z} such that P_k is generated by one basis element e_k , and

$$de_{2k+1} = \lambda_1 e_{2k} - e_{2k} \quad \text{and} \quad de_{2k+2} = \sum_{a \in \mathbb{Z}/n\mathbb{Z}} \lambda_a e_{2k+1} \quad \text{for } k \geq 0.$$

The morphism $P_0 \rightarrow \mathbb{Z}$ is given by $e_0 \mapsto 1$. Using this resolution we get

$$H_{2k+1}(\mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z} \quad \text{and} \quad H_{2k+2}(\mathbb{Z}/n\mathbb{Z}) = 0 \quad \text{for } k \geq 0.$$

Turning to the proof of the proposition, the first statement is equivalent to

$$H_3(\Gamma) \cong \bigoplus_i \mathbb{Z}/n_i\mathbb{Z} \oplus \bigoplus_{i<j} \mathbb{Z}/(n_i, n_j)\mathbb{Z} \oplus \bigoplus_{i<j<k} \mathbb{Z}/(n_i, n_j, n_k)\mathbb{Z}.$$

This, in turn, is proved by induction on m using the isomorphisms

$$H_1(\Gamma) \cong \Gamma, \quad H_2(\Gamma) \cong \Gamma \bigwedge \Gamma,$$

which are valid for any abelian group Γ , and the Künneth formula, which gives that $H_3(\Gamma \oplus \mathbb{Z}/n\mathbb{Z})$ is isomorphic to

$$H_3(\Gamma) \oplus (H_2(\Gamma) \otimes H_1(\mathbb{Z}/n\mathbb{Z})) \oplus H_3(\mathbb{Z}/n\mathbb{Z}) \oplus \text{Tor}_1^{\mathbb{Z}}(H_1(\Gamma), H_1(\mathbb{Z}/n\mathbb{Z})).$$

Note only that

$$\text{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/k\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/(k, n)\mathbb{Z} \cong \mathbb{Z}/k\mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z}.$$

Let us check next that the functions ϕ_i , ϕ_{ij} and ϕ_{ijk} are indeed 3-cocycles. For ϕ_{ijk} this is clear, since it is a 3-character. Concerning ϕ_i , consider the function

$$f_i(a, b) = \chi_{n_i} \left(- \left\lfloor \frac{a_i}{n_i} \right\rfloor b_i \right)$$

on $\mathbb{Z}^m \times \mathbb{Z}^m$. It is of the type described in Lemma A.2 for $\Gamma_1 = \mathbb{Z}^m$ and $\Gamma_0 = \bigoplus_{i=1}^m n_i \mathbb{Z}$, so $\phi_i(a, b, c) = (\partial f_i)(a, b, c)$ is a 3-cocycle on Γ . Similarly, consider the function

$$f_{ij}(a, b) = \chi_{n_j} \left(- \left\lfloor \frac{a_i}{n_i} \right\rfloor b_j \right).$$

It is again of the type described in Lemma A.2, so $\phi_{ij} = \partial f_{ij}$ is a 3-cocycle.

Our next goal is to construct a ‘dual basis’ in $H_3(\Gamma)$. Let u_i be the generator $1 \in \mathbb{Z}/n_i \mathbb{Z} \subset \Gamma$. Denote by θ_{ijk} the cycle representing $u_i \wedge u_j \wedge u_k \in \bigwedge^3 \Gamma \subset H_3(\Gamma)$ obtained by the shuffle product, so

$$\theta_{ijk} = \sum_{\sigma \in S_3} \text{sgn}(\sigma) (1 \otimes [u_{\sigma(i)} | u_{\sigma(j)} | u_{\sigma(k)}]),$$

where we consider S_3 as the group of permutations of $\{i, j, k\}$.

Consider the $\mathbb{Z}[\mathbb{Z}/n_i \mathbb{Z}]$ -resolution (P_*^i, d) of \mathbb{Z} described at the beginning of the proof. Let e_n^i be the basis element of P_n^i . We have a chain map $P_*^i \rightarrow C_*(\mathbb{Z}/n_i \mathbb{Z})$ of resolutions of \mathbb{Z} defined by

$$(A.1) \quad e_0^i \mapsto [\emptyset], \quad e_1^i \mapsto [1], \quad e_2^i \mapsto \sum_{a \in \mathbb{Z}/n_i \mathbb{Z}} [a|1], \quad e_3^i \mapsto \sum_{a \in \mathbb{Z}/n_i \mathbb{Z}} [1|a|1], \dots$$

It follows that we have a 3-cycle $\theta_i \in C_3(\Gamma; \mathbb{Z})$ defined by

$$\theta_i = \sum_{a=0}^{n_i-1} 1 \otimes [u_i | a u_i | u_i].$$

Finally, consider the $\mathbb{Z}[\mathbb{Z}/n_i \mathbb{Z} \oplus \mathbb{Z}/n_j \mathbb{Z}]$ -resolution $P_*^i \otimes P_*^j$ of \mathbb{Z} . Using this resolution we get a third homology class represented by

$$\frac{n_j}{(n_i, n_j)} 1 \otimes e_2^i \otimes e_1^j + \frac{n_i}{(n_i, n_j)} 1 \otimes e_1^i \otimes e_2^j.$$

A chain map between the resolutions $P_*^i \otimes P_*^j$ and $C_*(\mathbb{Z}/n_i \mathbb{Z} \oplus \mathbb{Z}/n_j \mathbb{Z})$ can be defined by the tensor product of the chain maps (A.1) and the shuffle product. This gives us a 3-cycle $\theta_{ij} \in C_3(\Gamma; \mathbb{Z})$. Explicitly,

$$\begin{aligned} \theta_{ij} &= \frac{n_j}{(n_i, n_j)} \sum_{a=0}^{n_i-1} 1 \otimes ([a u_i | u_i | u_j] - [a u_i | u_j | u_i] + [u_j | a u_i | u_i]) \\ &\quad + \frac{n_i}{(n_i, n_j)} \sum_{b=0}^{n_j-1} 1 \otimes ([u_i | b u_j | u_j] - [b u_j | u_i | u_j] + [b u_j | u_j | u_i]). \end{aligned}$$

The only nontrivial pairings between the cocycles ϕ_i , ϕ_{ij} , ϕ_{ijk} and the cycles θ_i , θ_{ij} , θ_{ijk} are

$$\langle \phi_i, \theta_i \rangle = \zeta_{n_i}, \quad \langle \phi_{ij}, \theta_{ij} \rangle = \zeta_{n_j}^{n_j / (n_i, n_j)} = \zeta_{(n_i, n_j)}, \quad \langle \phi_{ijk}, \theta_{ijk} \rangle = \zeta_{(n_i, n_j, n_k)},$$

where $\zeta_n = e^{2\pi i/n}$. This implies that these cocycles and cycles are the required generators of the Pontryagin dual groups $H^3(\Gamma; \mathbb{T})$ and $H_3(\Gamma)$. \square

Corollary A.4. *Assume Γ is a finite abelian group. Write Γ as Γ_1/Γ_0 for a finite rank free abelian group Γ_1 . Then for any \mathbb{T} -valued 3-cocycle ϕ on Γ the following conditions are equivalent:*

- (i) ϕ vanishes on $\bigwedge^3 \Gamma \subset H_3(\Gamma)$;
- (ii) ϕ lifts to a coboundary on Γ_1 ;

(iii) $\phi = \partial f$ for a function $f: \Gamma_1 \times \Gamma_1 \rightarrow \mathbb{T}$ as in Lemma A.2.

Proof. The equivalence of (i) and (ii) is clear, since a cocycle on Γ_1 is a coboundary if and only if it vanishes on $H_3(\Gamma_1) = \bigwedge^3 \Gamma_1$. Also, obviously (iii) implies (ii). Therefore the only nontrivial statement is that (i), or (ii), implies (iii). Assume ϕ is a cocycle that vanishes on $\bigwedge^3 \Gamma \subset H_3(\Gamma)$. We can identify Γ_1 with \mathbb{Z}^m in such a way that $\Gamma_0 = \bigoplus_{i=1}^m n_i \mathbb{Z}$ for some $n_i \geq 1$. Then in the notation of the proof of the above proposition the assumption on ϕ means that ϕ vanishes on the cycles θ_{ijk} , whose homology classes are exactly $u_i \wedge u_j \wedge u_k \in \bigwedge^3 \Gamma \subset H_3(\Gamma)$. It follows that ϕ is cohomologous to product of powers of cocycles ϕ_i and ϕ_{ij} . But the cocycles ϕ_i and ϕ_{ij} are of the form ∂f with $f: \Gamma_1 \times \Gamma_1 \rightarrow \mathbb{T}$ as in Lemma A.2. Therefore ϕ is cohomologous to a cocycle of the form ∂f , hence ϕ itself is of the same form. \square

Since every character of $\bigwedge^3 \Gamma \subset \Gamma^{\otimes 3}$ extends to a 3-character on Γ , this corollary can also be formulated as follows.

Corollary A.5. *With $\Gamma = \Gamma_1/\Gamma_0$ as in the previous corollary, any \mathbb{T} -valued 3-cocycle ϕ on Γ can be written as product of a 3-character χ on Γ and a cocycle ∂f with $f: \Gamma_1 \times \Gamma_1 \rightarrow \mathbb{T}$ as in Lemma A.2. Such a cocycle ϕ lifts to a coboundary on Γ_1 if and only if χ vanishes on $\bigwedge^3 \Gamma \subset \Gamma^{\otimes 3}$, and in this case $\phi = \partial g$ with $g: \Gamma_1 \times \Gamma_1 \rightarrow \mathbb{T}$ as in Lemma A.2.*

Let us now look more carefully at the construction of cocycles described in Lemma A.2. As Corollary A.4 shows, the class of 3-cocycles obtained by this construction does not depend on the presentation of Γ as quotient of a finite rank free abelian group. It is also clear that there is a lot of redundancy in this construction, since the group $H_3(\Gamma)$ can be much smaller than $\Gamma_0 \otimes \Gamma$. The following proposition makes these observations a bit more precise.

Proposition A.6. *Assume Γ is a finite abelian group, and write Γ as Γ_1/Γ_0 for a finite rank free abelian group Γ_1 . Let $f: \Gamma_1 \times \Gamma_1 \rightarrow \mathbb{T}$ be a function as in Lemma A.2, and χ be the associated character of $\Gamma_0 \otimes \Gamma$. Then the cohomology class of ∂f in $H^3(\Gamma; \mathbb{T})$ depends only on the restriction of χ to*

$$\ker(\Gamma_0 \otimes \Gamma \rightarrow \Gamma_1 \otimes \Gamma) \cong \mathrm{Tor}_1^{\mathbb{Z}}(\Gamma, \Gamma) \cong \Gamma \otimes \Gamma.$$

Therefore by varying χ we get a natural in Γ homomorphism

$$\mathrm{Hom}(\mathrm{Tor}_1^{\mathbb{Z}}(\Gamma, \Gamma), \mathbb{T}) \rightarrow H^3(\Gamma; \mathbb{T}),$$

whose image is the annihilator of $\bigwedge^3 \Gamma \subset H_3(\Gamma)$.

Proof. It is easy to see that the cohomology class of ∂f depends only on χ , so we have a homomorphism $\mathrm{Hom}(\Gamma_0 \otimes \Gamma, \mathbb{T}) \rightarrow H^3(\Gamma; \mathbb{T})$. We have to check that if a character χ of $\Gamma_0 \otimes \Gamma$ vanishes on $\ker(\Gamma_0 \otimes \Gamma \rightarrow \Gamma_1 \otimes \Gamma)$, then the image of χ in $H^3(\Gamma; \mathbb{T})$ is zero. But this is clear, since we can extend χ to a character f of $\Gamma_1 \otimes \Gamma$, and then f , considered as a function on $\Gamma_1 \times \Gamma_1$, is of the type described in Lemma A.2, with associated character χ , and f is a 2-character, so $\partial f = 0$.

Naturality of the homomorphism $\mathrm{Hom}(\mathrm{Tor}_1^{\mathbb{Z}}(\Gamma, \Gamma), \mathbb{T}) \rightarrow H^3(\Gamma; \mathbb{T})$ in Γ is straightforward to check. The statement that its image coincides with the annihilator of $\bigwedge^3 \Gamma \subset H_3(\Gamma)$ follows from Corollary A.4. \square

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