CONTRACT ADJUSTMENT UNDER UNCERTAINTY

HELGE HOLDEN, LARS HOLDEN, AND STEINAR HOLDEN

Abstract. Consider trade in continuous time between two players. The gains from trade are divided according to a contract, and at each point in time, either player may unilaterally induce a costly adjustment of the contract. Players' payoffs from trade under the contract, as well as from trade under an adjusted contract, are exogenous and stochastic. We consider players' choice of whether and when to adjust the contract payment. We show that there exists a Nash equilibrium in thresholds, where each player adjusts the contract whenever the contract payment relative to the outcome of an adjustment passes the threshold. There is strategic substitutability in the choice of thresholds, so that if one player becomes more active by choosing a threshold closer to unity, the other player becomes more passive.

1. Introduction

In most economies, a large part of the transactions take place within long-term relationships. Employment relationships, marriages, business partnerships are obvious examples. In general, there is a surplus from the relationship that is shared between the parties. The sharing of the surplus may depend on an explicit or implicit contract, or it may depend on some rules or habits. In any case, there is usually some rigidity in the contract or sharing rule, in the sense that it may be constant over a long time, until one player demands that it is changed.

An important question in this setting is under which circumstances a player will demand an adjustment of the contract. Furthermore, if one player is active, being eager to improve his payoff, how will this affect the behavior of the opponent? For concreteness, we consider a specific setting, where one player undertakes an exogenous service for the other, and where the remuneration for the service is given in a contract; an employment relationship is a good example. Trade takes place in continuous time.

A demand for adjustment of the contract may be caused by a change in the payoffs from trade that is to one player's disadvantage, or because the outside alternatives change. In our setting, these effects are captured by assuming that the contract payment is set in nominal terms, so that the real value of the contract payment depends on the stochastic aggregate price level. Second, we assume that outside alternatives may change according to an exogenous stochastic process, known to both parties at the time when an adjustment is demanded.

Formally, we consider a two-player differential game. There is a contract, according to which player B (the buyer) makes a fixed nominal payment V per unit of time to player A (the seller), as a remuneration for some exogenous and unspecified service. The real value of the fixed payment, R, depends on the aggregate price level Q, where $R = V/Q$. At each point in time, either player may unilaterally induce an adjustment of the contract payment to a new real value Z, which we shall refer to as the real adjustment outcome (implying a new nominal value $V = ZQ$). Adjusting the contract payment carries an exogenous fee to both players. The aggregate price level Q and the real adjustment outcome Z are exogenous stochastic processes. We consider players' decision of

---

Date: July 24, 2009.

2000 *Mathematics Subject Classification.* Primary: 91A10, 91A15; Secondary: 91B40.

Key words and phrases. Contract adjustment, adjustment costs, renegotiation, differential games, (S,s) strategy.

JEL Classification: C72, C73, C78, E31.

We dedicate this paper to our parents.

We are grateful to three referees, Karl Ove Moene, Kjetil Storesletten and participants at seminars at the University of Oslo and the Institute for International Economics, Stockholm, for valuable comments to an earlier draft, and to Geir Asheim, Fred Espen Benth, Ethan Kaplan and Bernt Øksendal for helpful discussions.

when to adjust the payment given in the contract, allowing for an unlimited number of adjustments over an infinite horizon.

One interpretation of our assumption that one player may unilaterally induce a renegotiation of the contract, is that there is an explicit reopening clause in the existing contract. An alternative interpretation is that a player may unilaterally disrupt trade, or credibly threaten to do so, and thus enforce the opponent to enter a renegotiation process so that trade can be resumed. This may be possible if it is not verifiable for the court which party violates the contract (MacLeod and Malcomson [31] and Hart and Moore [24]), or if the courts will not enforce any penalty provisions (Grout [21]).

The model we consider is simple: There are only two players, and their only choice variable is at each point in time whether to demand an adjustment of the contract. The result of a possible adjustment is exogenous and known in advance to both players. Yet the decision problem facing the players is very complex. When deciding whether to require an adjustment of the contract, a player must weigh the gain from a possible improvement in contract terms against the costs of adjustment. However, the player must also take into consideration that an adjustment now, making the contract terms more favorable to himself, will make the contract less favorable to the opponent. This may cause the opponent to require an adjustment at an earlier point in time than he otherwise would have done, involving both adjustment costs and less favorable contract terms for the first player.

In principle, strategies may depend on anything that has happened in the history of the game, and thus be immensely complicated. To keep the analysis tractable, we follow the tradition of the differential games literature (see Isaacs [29] and Dockner, Jørgensen, Van Long, and Sorger [16]) of restricting attention to Markov strategies, i.e., strategies where actions are allowed to depend on past history through the current value of the state variables only.

We show that if the stochastic processes are continuous, or, if they include jumps, the size of the jumps is from a continuous decreasing distribution, then there exists a Nash equilibrium in thresholds for the ratio of the real contract payment, R, to the real adjustment outcome, Z. In equilibrium, player A will demand an adjustment whenever R/Z is below player A's threshold, irrespective of whether this is caused by high inflation eroding the real value of the contract payment, or by an increase in the real adjustment outcome. Conversely, player B will require an adjustment whenever R/Z is above player B's threshold. An adjustment will ensure that R is set equal to Z, implying that the ratio $R/Z = 1$. We then proceed by exploring how the thresholds depend on the various features of the model, as well as analyzing the strategic interdependence between the thresholds.

The motivation for our paper is twofold. First, we argue that the problem itself is of great interest. Many economic transactions take place within bilateral relationships, and it seems important to analyze players' decisions of when to require an adjustment of the terms of the transactions. Obvious economic examples are labor contracts, tenancy contracts or delivery contracts, but essentially the same type of problem may be relevant for e.g., trade agreements between two countries, business relationships between two partners, or even for marriages. In many of these settings we observe rigidity in the terms of trade, yet occasionally one player may invoke an adjustment.

There is a fairly large literature on the corresponding problem in a unilateral setting, in particular where the focus is on the optimal choice of nominal prices (or wages) under a stochastic evolution of money or aggregate prices (so-called state-dependent pricing, e.g., Sheshinski and Weiss [34], Danziger [11], [13], [15], Caplin and Spulber [10], and Caplin and Leahy [9]). In contrast, the literature on bilateral adjustment is very small, with Andersen and Christensen [3, 4] as notable exceptions.

Compared to the literature on unilateral adjustment, we simplify by taking the adjustment outcome as exogenous, focusing solely on the timing decision. On the other hand, by considering a two-player game, we introduce a strategic dimension that makes the analysis much more complex. Thus, when deciding whether to adjust the contract payment now, a player must take into consideration that this may cause a subsequent adjustment by the opponent, at an earlier stage than he otherwise would have done, inflicting additional costs on both players. In view of this it is of interest to note that the optimal behavior in both types of models in most cases is characterized by threshold strategies, often termed (S, s) strategies, where the prevailing price is changed if it is sufficiently far from the optimal new price, so that the gain from adjustment covers the adjustment costs. However, we also find circumstances under which a player may use a mixed strategy in equilibrium.

A second motivation for our analysis is that we believe it to be of considerable relevance for related parts of economic literature. In the sticky-price macro literature, the standard approach is the Calvo specification, where the timing of adjustment follows an exogenous stochastic process. However, there is a number of recent advances with state-dependent pricing, including Dotsey, King, and Wolman [17], Bakhshi, Kahn, and Rudolf [5] and Gertler and Leahy [20]. Yet leading contributions studying wage setting, like Erceg, Henderson, and Levin [18], still use a Calvo formulation where households or workers are assumed to be able to set wages unilaterally. Thus it seems worthwhile to explore the alternative assumptions, that the timing of the contract renegotiation is endogenous and state-dependent, and that both players may affect the time when adjustment takes place.

Our paper also contributes to the literature on contract length, e.g., Danziger [14] on labour contracts and Bandiera [6] on tenancy agreements. Most of this literature is concerned with the duration of fixed-length (or time-dependent) contracts, empirically, or as seen from the point of optimality (e.g. in relation to investment incentives). In contrast, we let the length of the contract follow from optimal adjustment of the players.

Real world contracts of bilateral trade often specify an expiration date, seemingly making the decision of when to require an adjustment less relevant. However, as noted above, contracts may be renegotiated before any expiration date. Contracts may also be extended beyond the expiration date. In most European countries, the parties to a permanent employment contract are legally bound by the terms of the contract, unless the parties have agreed on a new contract, or one party has terminated the relationship, see Malcomson [32] and Holden [26]. Thus, even if there is an expiration date on the wage terms of the employment contract, the old wage prevails also after the expiration date, unless the parties have agreed to a change. In the sample of Israeli labor contracts studied by Danziger [12], 86 percent of all new contracts were signed after the expiration of the previous contract, with average delay of 213 days. Thus, the decision of when to adjust the contract is key.

The remainder of the paper is organized as follows. The basic model is described in Section 2. We assume that the stochastic processes follow the exponential of a Lévy processes. This is more general than most previous contributions, and it includes geometric Brownian motion as well as many other stochastic processes. We show that there exists no strategy that is better than a threshold strategy, specifying an adjustment of the contract whenever the ratio of the real contract payment to the real adjustment outcome passes a specific threshold. Section 3 derive formulaes for the expected objective functions, in the case when both players use threshold strategies. In Section 4, we prove the existence of a Nash equilibrium under the additional assumption that if there are jumps in the stochastic processes, the size of the jump is from a continuous, decreasing distribution. In Section 5, we show that if we also allow for jumps where the size of the distribution has a mass point for a fixed size, then Nash equilibrium still exists, but in this case it may require that agents randomize between two thresholds. In Section 6, we extend the basic model by allowing for a stage prior to the basic model, where players may invest in reducing the adjustment cost, and we consider the efficiency of this investment decision. Section 7 concludes. Approximate formulas for the equilibrium are given in Appendix C. Proofs are provided in Appendix D.[1]

2. The model

Formally, we consider a two-player differential game. There is a contract, according to which player B makes a fixed nominal payment $V(t_i)$ per unit of time to player A, as a remuneration for some exogenous and unspecified service. The time when the payment is set, is denoted t_i. At

[1] In a working paper we also consider the case where only one player is allowed to adjust the contract, cf. Holden et al. [25].

each point in time, either of the players may unilaterally adjust the nominal payment, inducing an adjustment cost on both players.

The real value of the contract payment at time t, $R(t,\omega)$ is found by deflating the nominal contract payment by the aggregate price level $Q(t,\omega)$ at time t, i.e., $R(t,\omega) = V(t_i)/Q(t,\omega)$.[2] The parameter ω denotes that $Q(t,\omega)$ and thus also $R(t,\omega)$ are stochastic. Players' flow payoffs are constant elasticity functions of the real contract payment, so that R^{η_A} (i.e., R raised to the power η_A) and R^{η_B} are the flow payoffs of player A and B, respectively, where $\eta_A > 0$ and $\eta_B < 0$ (implying that player A gains and player B loses from an increase in R).[3] This implies that both players have constant relative risk aversion, which is important to ensure an equilibrium with constant thresholds. The degree of risk aversion does not affect the qualitative results, but it has some impact on the numerical results, cf. simulations below.[4]

If a player demands an adjustment of the contract at time t, the real value of the new contract is set equal to the real adjustment outcome $Z(t,\omega)$, which is also an exogenous stochastic process, see section 2.1. The new nominal contract is thus $V(t,\omega) = Z(t,\omega)Q(t,\omega)$. Adjustment of the contract involves a fee that is proportional to the real adjustment outcome; specifically, the adjustment fee is $\tau_\nu Z^{\eta_\nu}$, where τ_ν ($\nu = A$ or B) is assumed to be strictly positive, deterministic and for simplicity independent of which player initiates the adjustment.[5]

Note that we do not consider the possibility that players care explicitly about the actions and intentions of the other player. Thus, we neglect that a player may care about an adjustment per se, viewing it as unfair or unwarranted; it is only the real contract payment and the adjustment costs that enter the payoff functions.

The overall objective function of the players is the discounted sum of flow payoffs, less the costs associated with adjusting the contract

$$U_\nu(t_1,\dots,\omega) = \int_0^\infty R^{\eta_\nu}(s,\omega)\exp(-\beta s)ds - \tau_\nu \sum_{j=0}^{\infty} Z^{\eta_\nu}(t_{j+1},\omega)\exp(-\beta t_{j+1}) \tag{1}$$

where the discount rate β is positive and t_j denote the times of contract adjustment. To avoid unimportant additional constants, we normalize by setting $R(0,\omega) = Z(0,\omega) = 1$, and $t_0 = 0$. As noted above, the players choose when to adjust the contract in order to maximize their objective function. At each time t, the contemporaneous values $Z(t,\omega)$ and $Q(t,\omega)$ are known to the players, but the future values $Z(s,\omega)$ and $Q(s,\omega)$ for $s > t$ are unknown.

Most previous studies in the (S,s) literature assume that the stochastic processes are continuous, e.g., according to a geometric Brownian motion. However, in real life situations, payoff functions may often be discontinues when important new events occur, or when new information is revealed. For instance, if player B signs a contract with a third party for delivery of the output produced

[2] If players agree on a contract in real terms, i.e. where V is continuously indexed to Q, this can be captured by setting $Q = 1$ at all times, and the analysis below is unaffected.

[3] A simple example of these payoff functions can be derived in a worker-firm framework. Let $R = V/Q$ be the real wage, and workers' flow payoff an increasing function of the real wage, R^{η_A}. The firm has a constant returns to scale production function $Y = L$, where Y is output and L is employment. Furthermore, the product demand facing the firm is $Y = (P/Q)^{-E}$, where P is the product price and $E > 1$ the elasticity of demand. The flow payoff of the firm is the real profit level, which is $(PY - VL)/Q$. The profit maximising price then satisfies the first order condition $P = (E/(E-1))V$. Substituting out for the first order condition in the profit function, we obtain the flow payoff of the firm as a decreasing isolelastic function of the real wage $\pi = CR^{1-E}$, where $C = E^{-E}(E-1)^{E+1}$ is a positive constant. Subject to an unimportant constant C, the flow payoff is then at the assumed form, where $\eta_B = 1 - E < 0$.

[4] Player A is risk averse or risk loving, depending on whether η_A is smaller than unity (payoff concave in R) or greater than unity (payoff convex in R). The payoff of player B is convex in R, indicating a preference for risk. Note however that as a profit function in general is convex in prices, player B may well be a risk averse owner of a firm, with utility being a concave function of profits, of the form π^σ, where $0 < \sigma < 1$. With profits being defined as in Footnote 3, the flow utility or payoff is then $(R^{1-E})^\sigma = R^{\eta_B}$, where $\eta_B = (1-E)\sigma < 0$. Note also that if $\eta_A = -\eta_B$, and the stochastic processes are symmetric geometrically, in the sense that Z and Z^{-1} have the same properties, and so do Q and Q^{-1}, then the model is also symmetric.

[5] Our assumption of proportional adjustment fees, adjusted for the constant elasticity η_ν, yields tractable solutions. It also seems plausible. For example, in a labor contract, adjustment costs may reflect time spent on bargaining, and the real contract payment (i.e., the real wage) seems an appropriate measure of the costs of time.

by player A, this might induce a jump in the real adjustment outcome Z. To allow for such discontinuities, we assume that the real adjustment outcome and the aggregate price level are given by the exponential of a Lévy process, i.e., that $Z(s,\omega) = \exp(F(s,\omega))$ and $Q(s,\omega) = \exp(G(s,\omega))$ where F, G are Lévy processes. Lévy processes include Brownian motion, jump processes that follow a Poisson distribution and many other stochastic processes that are, e.g., asymmetric or have heavier tails, see the formal definition in Appendix A.

To ensure that the objective functions are finite, it is necessary to bound Z and Q relative to the discount rate β. This requires two additional assumptions, referred to as Definition A.2 in Appendix A, which we assume hold throughout the paper.

By assuming that payoff functions exhibit constant elasticity in the real contract payment R, and that the stochastic processes are given by the exponential of Lévy processes, we ensure that the situation is the same after each adjustment, subject to a constant $Z(t,\omega)$. This property is crucial for the analysis, as it implies that the same strategies are optimal after each adjustment.

The strategy of a player is defined as a description of the criteria that apply when the player requires an adjustment of the contract. In principle, strategies may depend on anything that has happened in the history of the game. However, we will follow the tradition in the differential games literature and restrict our attention to Markov strategies, where the players' choice of action only depend on the state of the game. Thus, players may condition their play on the real contract payment R, the real adjustment outcome Z, or any combination of these variables. We do not allow players to condition their play on the opponent's play, except for any effect via the state variables R and Z. For example, we do not consider strategies where players punish a rapid adjustment by the opponent by another adjustment, inflicting further adjustment costs on both players.

Let s_ν denote the Markov strategy of player $\nu = A, B$. The times of contract adjustment t_j for $j = 1, 2, \ldots$, are determined by the strategies s_ν, the value of the real contract payment $R(s,\omega)$, and the real adjustment outcome $Z(s,\omega)$. Then each player ν tries to maximize

$$u_\nu(s_A, s_B) = E\{U_\nu(t_1, \ldots, \omega)\}. \tag{2}$$

The theorem below states that if one of the players uses a Markov strategy, there exists no strategy for the other player that gives higher expected value of the objective function than having a critical threshold for the ratio R/Z, i.e., adjusting the contract whenever R/Z is equal to or passes a certain threshold value. Other variables like R or Z separately, calendar time or the time duration since the previous adjustment, need not be used in the strategy.

Theorem 2.1. *Assume that one player uses a Markov strategy. Then there exists no strategy for the other player that gives higher expected payoff than the payoff that can be obtained with a threshold strategy based on the ratio of the real contract payment R to the real adjustment outcome Z.*

Given Theorem 2.1, we will in the sequel restrict attention to threshold strategies.

2.1. **The renegotiation process.** As explained above, our assumption that each of the players may unilaterally adjust the contract payment can be given two interpretations. Either there exists a reopening clause in the contract, or a player may induce a renegotiation process by unilaterally disrupting trade, or by threatening to do so. The outcome of a renegotiation process of an existing contract is an interesting and complicated problem, cf. previous analysis in Haller and Holden [22], Fernandez and Glazer [19], MacLeod and Malcomson [31] and Holden [28]. In general, the bargaining outcome will depend on the size of the surplus that is to be shared, the players' preferences, the outside alternatives, and the costs associated with a dispute in the bargaining. Some or all of these variables may vary over time, and thus lead to changes in the bargaining outcome. However, a proper analysis of this is far beyond the scope of the present study. Thus, we restrict ourselves to a reduced form approach. Specifically, we assume that the renegotiation process is costly to both players, as specified above. Furthermore, we assume that there is a unique outcome Z to the renegotiation process.

Observe that our specification with an exogenous adjustment outcome Z does not affect our analysis of optimal threshold strategies. At each point in time, Z is the outcome if one of the players requires an adjustment of the contract. Players cannot affect Z, except by choosing when to adjust the contract. Thus, as seen from the point of view of the players when they make the decision of whether to induce an adjustment, which is the problem that we study, the adjustment outcome Z is exogenous. This might have been different if we had allowed for players being able to commit to their future threshold level. Then, one would expect that the future thresholds would affect the adjustment outcome, and it would have been necessary to incorporate the effect of the choice of thresholds on the adjustment outcome in the analysis. However, by the Markov assumption, players cannot commit to their future threshold values.

An important limitation of letting the adjustment outcome be exogenous is however that we do not capture how a change in a parameter in the model would affect the adjustment outcome. Thus, in our analysis in Theorem 4.1 below, and the subsequent numerical simulations, we do not capture that a change in the parameters might be expected to affect the renegotiation process, and thus the real adjustment outcome. Hence, these exercises should be interpreted as *ceteris paribus*-analyses, where we explore the direct effect of the change in the parameters on the thresholds, neglecting any possible effect on the real adjustment outcome.

3. The model when both players use thresholds strategies

In this section we derive formulas for the expected objective functions and their derivatives, given that players use threshold strategies. These formulas can be computed numerically, and possibly also analytically, for specific stochastic processes. To illustrate the model, Figure 1 shows a realization of the real contract payment R when both players use threshold strategies.

Let r_B and r_A denote the critical thresholds, where $r_A < 1 < r_B$, as player A requires adjustment if the real contract payment is low ($R/Z \leq r_A$), while player B requires adjustment if the real contract payment is high ($R/Z \geq r_B$). Denoting the threshold strategies by r_ν, equation (2) gets the form below. Define the expected discounted sum of flow payoffs[6]

$$u_\nu(r_A, r_B) = E\{U_\nu(r_A, r_B)\} \tag{3}$$

where $U_\nu(r_A, r_B)$, with a slight abuse of notation, is defined from (1) when the players have critical thresholds r_A and r_B. Let $T(r_A, r_B, \omega)$ be the time of the first adjustment given the thresholds r_A and r_B, i.e., the first time after $t = 0$ that the contract payment relative to adjustment payment is either equal or below r_A or equal or above r_B, viz.,

$$T(r_A, r_B, \omega) = \inf\{t > 0 \mid R(t,\omega)/Z(t,\omega) \notin (r_A, r_B)\}. \tag{4}$$

Note that, as the situation is the same after each adjustment, subject to the constant $Z(t,\omega)$, $T(r_A, r_B, \omega)$ is also the time of the next adjustment, as measured as the distance from the previous adjustment.

In part (ii) in the following theorem, as well as in section 4 below, we will assume $E\{T(r_A, r_B)\}$ is differentiable. This is satisfied if, for instance, $Z(t,\omega)$ and $Q(t,\omega)$ are geometric Brownian motions. In Appendix B we find the analytic expression for $E\{T(r_A, r_B)\}$ in this case. It is also satisfied if we include jumps in $Z(t,\omega)$ and $Q(t,\omega)$, where the jumps are according to a Poisson process and the size of the jump has a continuous and decreasing distribution, for example being exponentially distributed.

Given the thresholds, define the expected contribution to the objective function of player ν from the start at $t = 0$ to the first contract adjustment,

$$f_\nu(r_A, r_B) = E\{\int_0^{T(r_A, r_B)} R^{\eta_\nu}(s) \exp(-\beta s) ds\}.$$

The expected flow payoff just after the first adjustment, discounted down to time $t = 0$, is defined by

$$h_\nu(r_A, r_B) = E\{Z^{\eta_\nu}(T(r_A, r_B)) \exp(-\beta T(r_A, r_B))\}.$$

Note that in the special case where the real adjustment outcome Z is a constant, h_ν is a pure discount factor. Note also that the second inequality in Definition A.2 ensures that $h_\nu < 1$.

[6]Here and in the following we will write $E\{\Phi(a, b)\}$ for the expectation value of a stochastic variable $\Phi(a, b, \omega)$, rather then more cumbersome $E\{\Phi(a, b, \omega)\}$ or $E\{\Phi(a, b, \cdot)\}$.

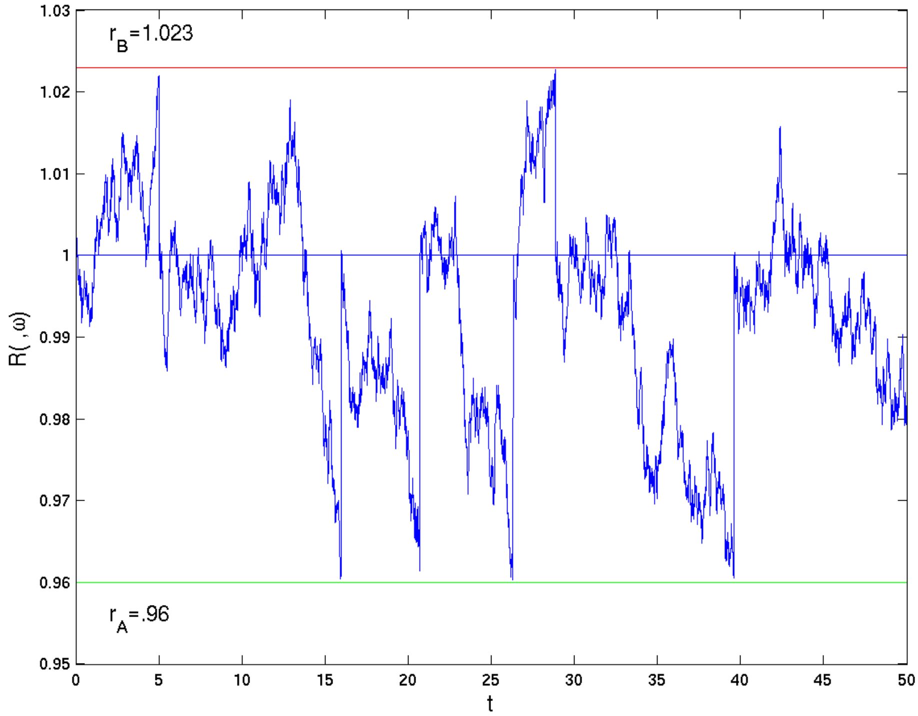

FIGURE 1. The figure shows one realization of the process $R(\,\cdot\,,\omega)$ in the case with $Z = 1$ and Q is geometric (or exponential) Brownian motion $Q = \exp((\alpha_q - a_q^2/2)t + a_q B_t)$ with drift $\alpha_q = .002$ and volatility $a_q = .01$. Here B_t denotes standard Brownian motion. When the unit of time is interpreted as one month, this corresponds to 2.4% annual inflation. The process is sampled at 5000 points.

Then we may formulate the following theorem.

Theorem 3.1. *Assume that both players use threshold strategies, implying that the contract is adjusted as soon as the contract payment relative to the adjustment outcome R/Z exits the interval (r_A, r_B). Then the following properties hold:*
(i) The expected values of the objective functions immediately after an adjustment satisfy

$$u_\nu(r_A, r_B) = \frac{f_\nu(r_A, r_B) - \tau_\nu h_\nu(r_A, r_B)}{1 - h_\nu(r_A, r_B)}, \quad \nu = A, B \tag{5}$$

and are defined for $0 \le r_A < 1 < r_B \le \infty$.
(ii) Assuming $E\{T(r_A, r_B)\}$ is differentiable, then f_ν, h_ν u_ν are differentiable and the derivatives satisfy

$$\frac{\partial u_\nu}{\partial r_\mu} = \frac{\frac{\partial f_\nu}{\partial r_\mu} + (u_\nu - \tau_\nu)\frac{\partial h_\nu}{\partial r_\mu}}{1 - h_\nu}, \quad \nu = A, B, \quad \mu = A, B. \tag{6}$$

Equation (6) captures the opposing effects of increasing the thresholds: For example, increasing r_A reduces the expected time until the next adjustment. This will reduce the expected payoff until the next adjustment, i.e., $\frac{\partial f_\nu}{\partial r_A} < 0$. Furthermore, reducing the expected time until the next adjustment raises the discount factor $\frac{\partial h_\nu}{\partial r_A} > 0$, reflecting that the adjustment cost τ_ν is incurred earlier, but also that the value of the objective function after an adjustment u_ν is received earlier.

4. Nash equilibrium

In this section we prove existence of a Nash equilibrium in thresholds, and we explore how the equilibrium depends on some of the parameters of the model.

Define the optimal thresholds for player A, as a function of the threshold for player B, $m_A(r_B)$, and similarly for player B, $m_B(r_A)$, as follows

$$\begin{aligned} m_A(r_B) &= \inf\{r_A \in [0,1) \mid u_A(r_A, r_B) = \sup_{r_A} u_A(r_A, r_B)\}, \\ m_B(r_A) &= \sup\{r_B \in (1,\infty] \mid u_B(r_A, r_B) = \sup_{r_B} u_B(r_A, r_B)\}. \end{aligned}$$

These definitions allow for the possibility that the optimal threshold is not unique, in which case they pick the most lenient value, i.e., the threshold farthest from unity. However, in Theorem 4.1 below, we show that with the assumptions mentioned above then the optimal threshold is indeed unique.

We may then state the following theorem regarding uniqueness of the optimal value and the existence of an equilibrium point.

Theorem 4.1. *Assume that if there are jumps in $Z(t,\omega)$ and $Q(t,\omega)$, then the density for the size of the jump is a continuous and decreasing distribution. Then $E\{T(r_A, r_B)\}$ is differentiable and the following properties hold:*

(i) The expected value of the objective function for player A, $u_A(r_A, r_B)$, is increasing in the threshold r_B of player B, i.e., $\frac{\partial u_A}{\partial r_B} > 0$. The expected value of the objective function for player B, $u_B(r_A, r_B)$, is decreasing in the threshold r_A of player A, i.e., $\frac{\partial u_B}{\partial r_A} < 0$.

(ii) Given the threshold for player B, r_B, there exists a unique best response for player A, i.e., a unique value $0 \le r_A = m_A(r_B) < 1$ that maximizes $u_A(r_A, r_B)$. Correspondingly, given r_A, there exists a unique value $\infty \ge r_B = m_B(r_A) > 1$ that maximizes $u_B(r_A, r_B)$.

(iii) The functions $m_A(r_B)$ and $m_B(r_A)$ are both strictly increasing.

(iv) $m_A(r_B)$ is strictly decreasing in τ_A and $m_B(r_A)$ is strictly increasing in τ_B.

(v) There is at least one Nash equilibrium point (r_A^e, r_B^e), where

$$\begin{aligned} r_A^e &= \text{argmax}_{r<1}\{u_A(r, r_B^e)\}, \\ r_B^e &= \text{argmax}_{r>1}\{u_B(r_A^e, r)\}. \end{aligned}$$

Theorem 4.1 states a number of key results. First, part (ii) shows that both players have unique best response functions in the form of thresholds r_ν. The optimal threshold reflects two opposing concerns. A threshold very close to unity is costly due to frequent renegotiatons, while a threshold far from unity involves the risk of lengthy periods with a bad contract payment.

Second, and more importantly, part (v) shows that there exists a Nash equilibrium in thresholds. Thus, in equilibrium, player A will demand an adjustment whenever R/Z is below player A's threshold r_A, irrespective of whether this is caused by high inflation eroding the real value of the contract payment, or by an increase in the real adjustment outcome. Conversely, player B will require an adjustment whenever R/Z is above player B's threshold r_B. An adjustment will ensure that R is set equal to Z, implying that the ratio $R/Z = 1$.[7]

Third, Theorem 4.1 (iv) reveals that higher adjustment costs make a player more reluctant to require an adjustment, by pushing his threshold value further from unity, r_A down for player A, and r_B up for player B. This is as expected: Players weigh the gains from improving the contract against the costs of doing so, and higher costs make players more reluctant to require an adjustment.

Fourth, part (i) shows that if one player becomes more active (that is, has a threshold closer to unity), this reduces the expected value of the objective function for the opponent. Again, the result is intuitive: When one player becomes more active, the opponent loses from both more

[7]We have not been able to prove uniqueness of the Nash equilibrium in the general case, nor have we been able to construct cases with multiple Nash equilibria with the assumptions in Theorem 4.1. Thus, for each set of stochastic processes, it is necessary to verify that there is only one equilibrium point.

frequent costly adjustments and from the fact that the contract on average becomes less favorable for the opponent.

Fifth, and again more important, Theorem 4.1(iii), identifies strategic substitutability in the choice of thresholds. This follows from the optimal thresholds $m_A(r_B)$ and $m_B(r_A)$ being increasing functions. If, in equilibrium, one player becomes more active by choosing a threshold closer to unity, the other player becomes more passive by choosing a threshold further from unity. In other words, if, say, the adjustment fee of player B is reduced, making him more active, this will induce player A to become more passive.

At first glance, this result might be surprising. If the threshold of my opponent is close to unity, so that he is very active, demanding an adjustment whenever the contract is slightly favorable to me, surely I should also be more active, to prevent that the contract is almost always to my disadvantage? Yet this argument is misleading. If my opponent becomes more active, this is clearly a disadvantage to me. However, becoming more active would not help. The relevant issue is that as my opponent becomes more active, the expected time until he will demand an adjustment is reduced. This implies that if I were to induce a renegotiation, the expected duration of the novel contract payment will be shorter than before, as the opponent is more active. This makes it less attractive for me to require an adjustment, leading me to be more passive, with a threshold further from unity.

The strategic substitutability effect is in contrast to Andersen and Christensen [4], who find strategic complementarity in the choice of thresholds. Their finding of complementarity is due to the fact that they consider only one contract adjustment, implying an incentive for players to preempt the opponent. Thus, if one player is active, the opponent has an incentive to also be active, to increase the likelihood of being the player who obtains the advantage of asking for an adjustment at a suitable moment. Andersen and Christensen [3] consider the model with a finite, but large number of contract renewals, but it is not stated whether the strategic complementarity holds in that model.

The model may be generalized to the case where the adjustment costs τ_A and τ_B depend on which player that requires contract adjustment. In equations (5) and (6), this would require that τ_ν is replaced by the expected value of the contract adjustment fee, which again would be a function of r_A and r_B. Theorem 4.1 is also valid in the generalized model, but in equation (24) in the proof and the calculations leading to this equation, τ_A would be the adjustment fee when player A requires an adjustment. τ_B would be used in the corresponding equation for B. The model may also be generalized to allow for the adjustment fees being stochastic, where τ_ν is the expected value of the adjustment fee.

4.1. **Numerical simulations.** As will become clear below, the model involves several strongly non-linear relationships. Thus, to explore the properties of the model further, numerical simulations are necessary. Figure 2 illustrates the game in setting thresholds. The curves show the best response functions m_A and m_B for different values of adjustment fees τ_ν. Higher adjustment fee leads to more passive play, with critical values further from unity (curves down). The best response functions are downward-sloping in the $r_A, 1/r_B$ space, reflecting strategic substitutability in the sense that players will be more passive (lower values of r_A or $1/r_B$) if the opponent becomes more active (higher values of r_A or $1/r_B$). Note that the best-response functions are highly non-linear, implying that the effect of the strategic substitutability will vary sharply depending on the initial conditions.

The intersections of the best response functions indicate Nash equilibria. For example, point C in Figure 2 indicates the Nash equilibrium for $\tau_A = \tau_B = .35$. The equilibrium thresholds are $r_A = .950$ and $1/r_B = .960$, implying that player A requires an adjustment whenever he can increase the real contract payment by 5 percent, while player B requires an adjustment if the real contract payment can be reduced by at least 4 percent. In this case, the strategic substitutability is rather strong. For example, if the adjustment costs of player A, τ_A, is reduced down to .05, while τ_B is constant at .35, r_A increases to .985, and $1/r_B$ falls to .918, cf. point D. Thus, in the new equilibrium player A is more active, requiring an adjustment whenever he can increase the real contract payment by 1.5 percent, as opposed to 5 percent before the change. Then the

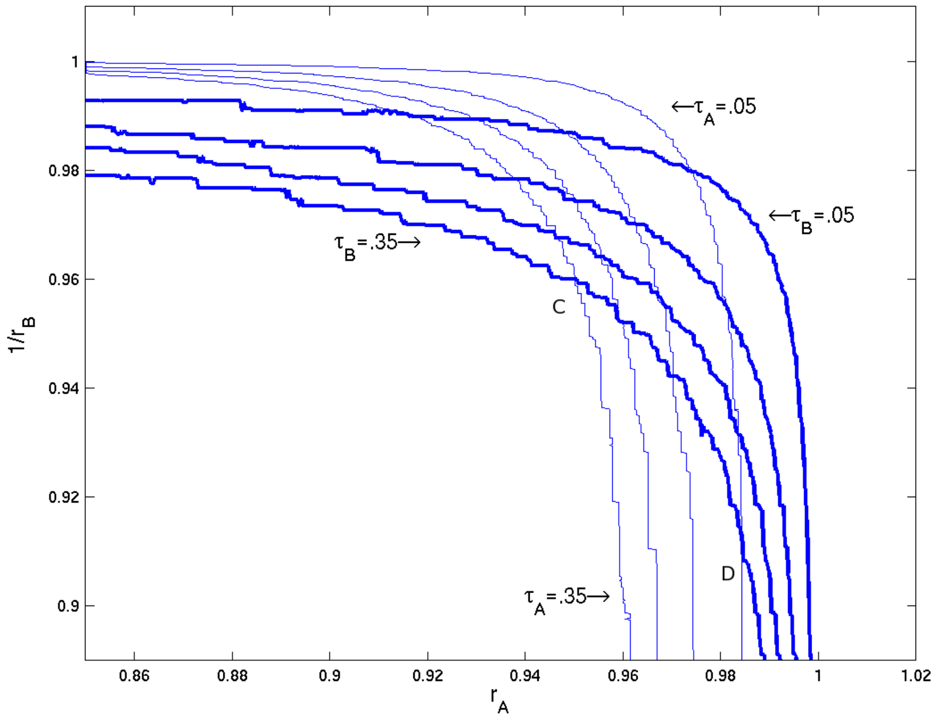

FIGURE 2. The best response threshold function m_A (thin curves) and m_B (thick curves) of players A and B, respectively, for values of τ_A and τ_B from .05 to .35. Intersection between m_A and m_B gives the Nash equilibrium point (r_A^e, r_B^e) for the particular set of (τ_A, τ_B). Other parameters and processes are as in Table 1.

TABLE 1. Each row presents the Nash equilibrium point for a combination of τ_A and τ_B. $P(r_A^e, r_B^e)$ is the expected fraction of times that adjustment is undertaken by player A, and $E\{T(r_A^e, r_B^e)\}$ is the expected time between adjustments. The process and parameters are as in Figure 1, including drift $\alpha_q = .002$ and volatility $a_q = .01$. Furthermore, $\beta = .005$, $\eta_A = 1$, and $\eta_B = -1.5$.

τ_A	τ_B	r_A^e	r_B^e	$u_A(r_A^e, r_B^e)$	$u_B(r_A^e, r_B^e)$	$P(r_A^e, r_B^e)$	$E\{T(r_A^e, r_B^e)\}$
.05	.05	.973	1.020	197.3	199.3	.64	5.3
.07	.05	.968	1.017	196.3	200.3	.58	5.9
.05	.07	.976	1.026	198.0	197.9	.74	5.7
.07	.07	.971	1.023	197.1	199.1	.69	6.7
.13	.13	.965	1.029	196.4	198.9	.75	9.9
.13	.23	.970	1.044	197.6	195.9	.87	10.9
.23	.13	.951	1.024	193.7	201.7	.64	12.1
.23	.23	.957	1.035	195.2	199.0	.78	13.7

strategic substitutability effect implies that player B becomes more passive, so that his critical threshold increases from 4 percent to 8.2 percent.

Given the complex effects with strong non-linearities, it is important to explore the properties of the model by use of simulations. We have undertaken extensive simulations, along several dimensions, and Tables 1 and 2 sum up the key results.

As noted above, the basic framework is symmetric in the sense that if the stochastic processes are symmetric (i.e. Z and Z^{-1} have the same properties, and so do Q and Q^{-1}), and $\eta_A = -\eta_B$,

TABLE 2. Each row present the Nash equilibrium points for a specific parameter combination given in the first four columns. In all simulations, $\tau_A = \tau_B = .07$, while the other parameters as in Table 1.

α_q	a_q	η_A	η_B	r_A^e	r_B^e	$u_A(r_A^e, r_B^e)$	$u_B(r_A^e, r_B^e)$	$P(r_A^e, r_B^e)$	$E\{T(r_A^e, r_B^e)\}$
.002	.01	1.0	-1.5	.971	1.023	197.1	199.1	.69	6.7
.002	.03	1.0	-1.5	.937	1.044	194.1	198.1	.44	3.2
0	.01	1.0	-1.5	.969	1.020	197.2	199.0	.38	6.7
.006	.01	1.0	-1.5	.969	1.028	195.6	199.5	.97	4.9
.002	.01	.7	-1.5	.964	1.020	197.0	200.5	.61	7.4
.002	.01	1.0	-1.0	.974	1.030	197.7	198.3	.77	7.2

then the behavior will also be symmetric, implying that $r_A = 1/r_B$. For example, this would be the case if Z and Q were geometric Brownian motions without drift. In the numerical simulations, we will therefore focus on the effects of asymmetries between the players. In almost all simulations, we include a positive drift $\alpha_q > 0$ in the aggregate price level Q, implying a tendency that the real value of the contract payment, R, falls over time, relative to the real adjustment outcome, Z.Thus, it will usually be player A (the seller) who demands an adjustment, unless the critical threshold of player B is close to unity. Note that the drift may be interpreted as anything that makes the real value of the contract payment fall relative to the adjustment outcome. For example, if one interprets the contract as a trade agreement between two countries, higher growth in one country inducing a gradual improvement in the country's bargaining position, might also be captured by such drift.

Table 1 displays Nash equilibria for various combinations of adjustment costs. We observe that an increase in the adjustment costs for one player makes this player less active, which increases the expected utility of the opponent, and in addition makes the opponent more active (strategic substitutability). An implication of the strategic substitutability is that asymmetries between the players may be exacerbated. Thus, if the adjustment cost of one player becomes higher, making this player more passive, the opponent becomes more active. However, when the opponent becomes more active, the first player responds by becoming even more passive. In some cases, the strategic effect may be substantial.

Table 2 shows the effect of variation in other parameters, where the first row gives the benchmark simulation.[8] Greater volatility a_q makes both players more reluctant to require an adjustment, with thresholds further away from unity, cf. row 2. The intuition is straightforward: With greater volatility, thresholds close to unity will imply too frequent adjustments, thus players are less active so as to reduce adjustment costs. This result is the same as derived by Andersen and Christensen [4]. Note, however, from the last column that the change in behavior is not so large that it prevents that the expected time between adjustments, $E\{T\}$, falls.

Increased drift in the aggregate price level, which, as mentioned above, causes a reduction in the flow payoff of player A (the seller) over time, has a positive impact on the threshold of player B. As the drift increases from $\alpha_q = 0$ to $\alpha_q = .006$, r_B increases from 1.020 to 1.028, cf. rows 1, 3 and 4 in Table 2. This reflects that under high inflation (high drift), player B need not demand a adjustment even if he has been "unlucky" with the random movement, so that the contract payment is high relative to the adjustment payment. As the contract payment is set in nominal terms, high inflation will fairly soon reduce the real value of the contract payment, so player B need not incur the costs of an adjustment. In contrast, when there is little or no drift, player B cannot rely on this effect and must set the threshold level closer to unity to avoid a bad contract value. This result is in contrast to the findings of Andersen and Christensen [4], where increased drift makes player B (which corresponds to the principal in their model) more active. Their result

[8]When interpreting the effects of parameter changes, one should recall that Z is exogenous, implying that we do not capture any effect on this variable. As argued in section 2.1, endogenising Z would not affect the optimal threshold values, but in general it would affect the payoffs of the players. Thus, these effects should be treated more cautiously. The average real value of the contract would also be directly affected by endogenising Z, thus we do not calculate the effect of parameter changes on this variable.

is due to their assumption of only one renegotiation; if there is drift that is disadvantageous to the principal, there will be less reason for the principal to postpone an adjustment in the hope of a more favorable adjustment a later stage. Indeed, in Andersen and Christensen [3], it is shown that the effect of drift is ambiguous in the case where it is allowed for many renegotiations.

The threshold of player A is non-monotonic in the drift. Increasing the drift from $\alpha_q = 0$, via $\alpha_q = .002$ to $\alpha_q = .006$ yields r_A equal to .969, .971 and .969, respectively. The non-monotonicity reflects two opposing effects. On the one hand, stronger drift implies that for a given threshold, adjustments will be more frequent, so that the total costs incurred from adjustment increase. To reduce the rise in costs from frequent adjustments, player A will be more reluctant to demand an adjustment, thus the threshold is decreased. On the other hand, the strategic substitutability in the choice of thresholds implies that when higher drift increases the threshold of player B, making him less active, it also increases the threshold of player A. Intuitively, increasing the threshold of player B raises the expected duration of an beneficial adjustment by player A, making an adjustment more attractive to A.

Risk aversion for player A, i.e., reducing η_A below unity, has a negative effect on the thresholds of both players, player A's further away from unity and player B's closer to unity, cf. row 5. Thus, risk aversion makes player A more reluctant to require an adjustment, inducing player B to become more active. We also see that player B obtains higher expected utility when player A is more risk averse, corresponding to the well-known result that it is advantageous to bargain with a risk averse player. One possible interpretation of risk aversion for the seller is a union that is also concerned about the risk of unemployment, if wages become very high. Under this interpretation, our results suggest that unions that care about employment are less active (in the sense of having a threshold further from unity), thus improving the payoff for the firm.

Likewise, making player B less risk loving, by reducing the absolute value of η_B, improves the situation for player A, as the threshold of both players increases, making player A more active and player B more passive, resulting in an increase in the expected utility of player A, cf. row 6.

Combining the effect of the various parameters allows for interesting observations in relation to the literature on downward nominal wage rigidity. As observed by Akerlof, Dickens and Perry [2] and Lebow, Saks, and Wilson [30] (the latter paper also surveys more recent evidence), nominal wage cuts are fairly rare, even in a low inflation economy, suggesting that firms rarely require a downward adjustment in nominal wages. The third row shows that if adjustments costs are symmetric, and there is no drift (i.e., no inflation and no productivity growth, implying no trend increase in wages) player B, which corresponds to an employer, undertakes most of the adjustments (recall that $P(\,\cdot\,)$ shows the fraction of all adjustments that are induced by player A), implying that wage cuts would in fact be more common than wage increases. The reason for this asymmetry lies in the asymmetry in the payoff functions, where $\eta_A < |\eta_B|$. This property reflects that profits are a convex function of wages; by the envelope theorem, the derivative of the profit function with respect to wages is equal to the employment level, and the employment is itself a decreasing function of wages.

However, as noted above the evidence indicates that wage cuts are fairly rare, which implies that other mechanisms must be so strong that they more than overturns the effect of the profit function being convex. One effect in this direction is drift in wages, due to inflation, productivity growth or both. Holding first thresholds values constant, inflation erodes the real value of the contract payment, implying that the threshold of player A, the seller or worker, will bind more often, thus he will make most of the adjustments. As showed above, with optimal thresholds, the drift makes player B, the buyer or firm, less active, further reducing the incidence of wage cuts. The effect on player A's behavior is ambigous, as the direct effect of inflation is less active play by player A (active play and high inflation will lead to too frequent costly adjustment), while strategic substitutability effect implies that the less active player B makes player A more active.

A numerical example may be illustrative. If the drift equals $\alpha_q = .002$, which corresponds to annual wage growth of 2.4 percent if the unit of time is one month, this leads to an incidence of wage cuts of 31 percent ($1-.69 = .31$), cf. the first row in Table 2. In comparison, Lebow, Saks, and Wilson [30] report in the years with low median wage growth of an annual median of 2.3 percent (the years were 1987, 1993, and 1995), 18 percent of all wage changes in the US Employment cost index were negative. This suggests that there are also other mechanisms preventing nominal wage cuts, for example that firms believe that wage cuts may have an adverse effect on workers' morale, cf., e.g., Akerlof, Dickens, and Perry [2] or Bewley [7]. Note that if firms for some reason find it more costly to cut wages and thus become more reluctant to do so, the strategic substitutability effect will make workers more active, which will magnify the downward rigidity of wages.

5. Weaker assumptions on the stochastic processes

As noted above, in real life situations, payoff functions may often be discontinues when important new events occur, or when new information is revealed. Thus, it is of interest to see to what extent our results are robust to discontinuities in the stochastic processes. So far, we have assumed that if there are jumps in the Z and Q-processes, the size of the jumps is from a continuous decreasing distribution. In this section we also allow for jumps in Z where the distribution for the jumps has a mass point for a fixed size, implying that $E\{T(r_A, r_B)\}$ is not differentiable. We show that in this case a Nash equilibrium may in some specific cases only exists if one of the players uses a mixed strategy. The reason is that this may give discontinuities in the optimal responses $m_A(r_B)$ and $m_B(r_A)$ implying that they do not intersect. Then there will be no Nash equilibrium with constant threshold strategies. However, there will exist a Nash equilibrium in mixed strategies, in the sense that one player randomizes between two threshold values. From a theoretical point of view it is interesting to see that if we expand the class of stochastic processes Z and Q to include a wider class of jumps, then it is necessary to include randomization in the strategies in order to obtain Nash Equilibrium. To illustrate this, we construct a stylized example where there is a possibility that the real adjustment outcome Z may take a large fall, which may lead player B to require an adjustment to reduce the contract payment down to the new, low value of Z.

Example 5.1. Assume that the aggregate price level Q is constant, except for sudden increases according to a high intensity Poisson process, where the size of the increases are "small" and according to a continuous distribution. The real adjustment outcome Z is also constant except for "large" discrete decreases at a fixed rate $1+\rho$, according to a low intensity Poisson process. These assumptions imply that the ratio of the contract payment relative to the adjustment outcome R/Z is decreasing, except for sudden jumps where it increases with the percentage ρ.

Consider the situation if player B has chosen a threshold $r_B < 1 + \rho$. Then, if Z decreases immediately after an adjustment, R/Z will be above the critical threshold of player B, inducing an immediate adjustment by B. Thus, player A will not benefit from a period with high R/Z after the decrease in Z. On the other hand, if player A let R/Z fall below $r_B/(1 + \rho)$, R/Z will be below the threshold of the player B even after a decrease in Z. There will be no immediate adjustment, and player A will benefit from a period of high R/Z. This discontinuity at $r_B/(1+\rho)$ will imply a discontinuity in m_A, i.e., in the optimal threshold of player A. In this example $E\{T(r_A, r_B)\}$ is differentiable except for the curve $r_A = r_B/(1 + \rho)$ where there is a discontinuity in $\partial E\{T(r_A, r_B)\}/\partial r_\nu$.

The situation is illustrated in Figures 3 and 4. In equilibrium, player A randomizes between two thresholds $r_1 \approx .979$ and $r_2 \approx .971$. Figure 3 shows that R/Z decreases monotonically, except when it increases to unity at an adjustment, or increases above unity when Z decreases. Figure 4 illustrates the strategic effects. For values of r_B above $r_3 \approx 1.035 \approx 1/.966$, the optimal threshold of player A, r_A is increasing in r_B due to the strategic effect discussed in Section 4. For $r_B < r_3 \approx 1.035$, a fall in Z will always induce player B to adjust, implying that player A sets $r_A \approx .979$. However, for $r_B = r_3 \approx 1.035$, player A is indifferent between choosing a low threshold r_1, maintaining the possibility of benefiting from a fall in Z, and a high threshold r_2, which removes this possibility. In equilibrium, player A mixes between these two thresholds, with probabilities ensuring that it is indeed optimal for player B to choose the threshold $r_B = r_3 \approx 1.035$.

Let us now consider the consequences when the distribution for the jumps has a mass point for a fixed size more formally. Define S_ν as the class of strategies for a player ν, where the player randomizes between two thresholds r_1 and r_2, where r_1 is chosen with probability $1 - q$ and r_2 with probability q. Note that S_ν includes pure strategies, where $q = 0$. Let $s_\nu \in S_\nu$ denote a

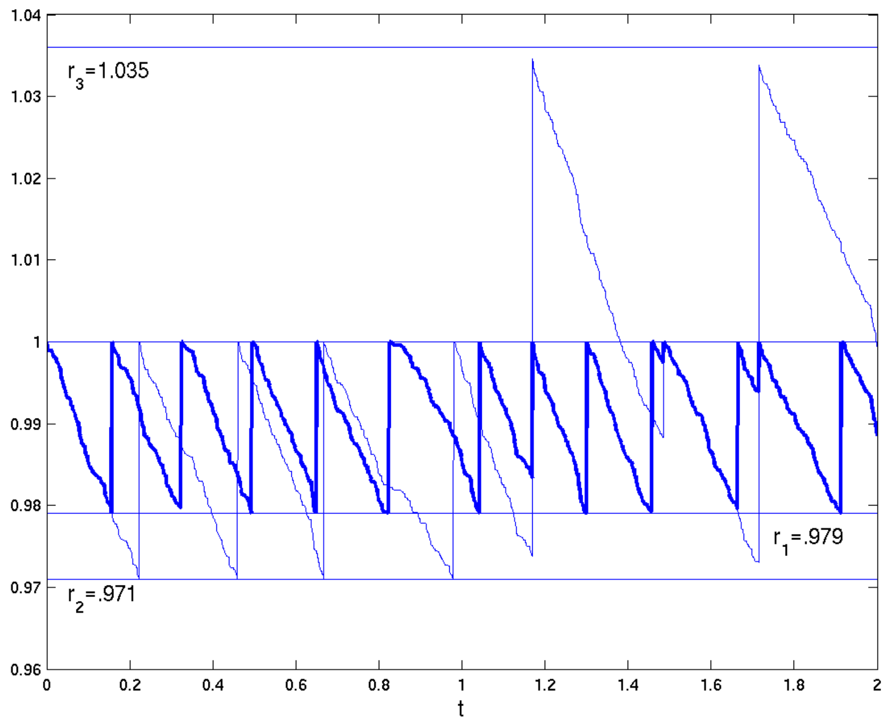

FIGURE 3. Two realization of R/Z for the same stochastic process, but corresponding to two different thresholds for player A, $r_2 = .971$ and $r_1 = .979$, in Example 5.1. Player B has threshold $r_B = r_3 = 1.035$. Threshold r_1 gives adjustment after every jump where R/Z increases, while the lower threshold r_2 may give periods with high R/Z values. In a Nash equilibrium, illustrated in Figure 4, player A randomizes between the thresholds r_2 and r_1. The two independent Lévy processes Z and Q are constants except for discrete changes according to a Poisson process. The Poisson process for Z has intensity .8 and in the changes, Z decreases with a factor 1.0625. The Poisson process for Q has intensity approximately 1200 and in the changes, Q increases according to a continuous distribution such that $E\{\ln(Q(t))\} \approx .13t$ and $\text{Var}\{\ln(Q(t,\omega))\} \approx .004t$. Furthermore, $\tau_A = .0017$ and $\tau_B = .0057$. The process is modeled with time step .001.

strategy. Furthermore, we assume that each time R/Z is equal to unity or jumps from one side of unity to the other side of unity, players select one of the two thresholds at random. This procedure ensures that past fluctuations of Z and Q have no impact on the probability each player perceives of the thresholds of the opponent.

We extend the definition of the expected values of the objective functions u_ν to allow for randomization by both players. Furthermore, we define the optimal threshold for each player when the opponent randomizes:

$$m_A^c(s_B) = \inf\{r_A \in [0,1) \mid u_A(r_A, s_B) = \sup_r u_A(r, s_B)\},$$
$$m_B^c(s_A) = \sup\{r_B \in (1,\infty] \mid u_B(s_A, r_B) = \sup_r u_B(s_A, r)\}.$$

Thus, the function $m_B^c(s_A)$ corresponds to the usual optimal response function for player B, $m_B(r_A)$, if player A uses a pure strategy. However, $m_B^c(s_A)$ is also defined if player A randomizes between thresholds r_1 and r_2, reflecting a discontinuity in $m_A(r_B)$. If $m_B^c(s_A)$ changes continuously from $m_B(r_1)$ to $m_B(r_2)$ when q changes from 0 to 1, we say that $m_B^c(s_A)$ is continuous. Continuity of the function $m_A^c(s_B)$ is defined similarly.

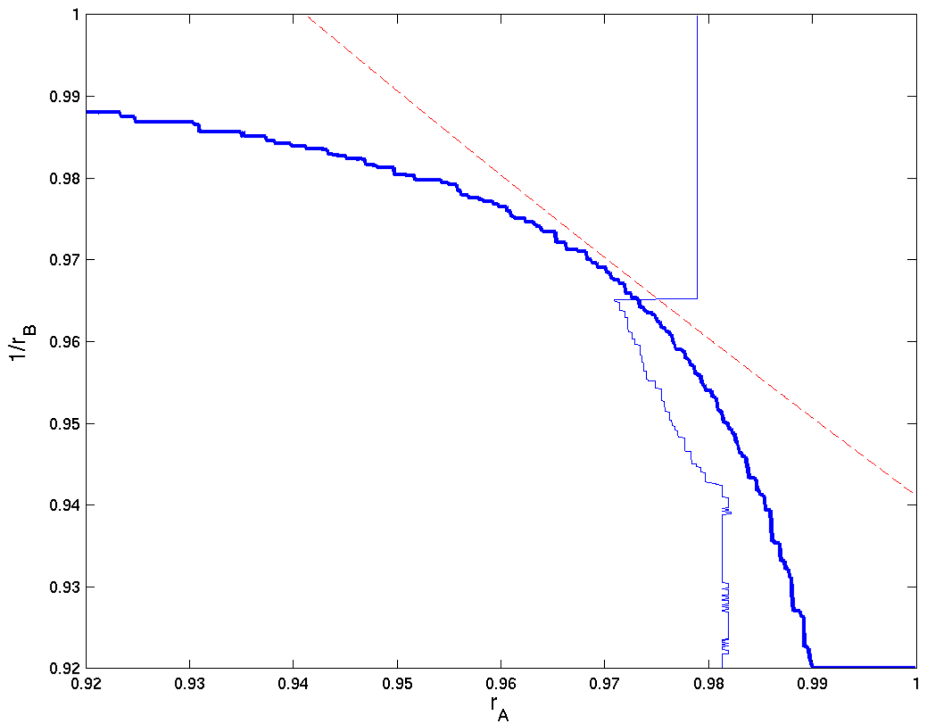

FIGURE 4. The best response functions $m_A(R_B)$ (thin curve) and $m_B(R_A)$ (thick curve) corresponding to Example 5.1, for $r_A, 1/r_B \in [.92, 1]$. The dashed curve is $r_B = 1.0625 r_A$. When the thresholds satisfy $r_B < 1.0625 r_A$, there is an adjustment after every jump in Z. Thus, within this interval r_B is immaterial, implying that $m_A(r_B)$ is vertical above the dashed curve. In equilibrium, player A's strategy s_A implies randomization between the thresholds $r_2 \approx .971$ and $r_1 \approx .979$, the endpoints of the horizontal line in m_A for $r_B \approx 1.035 \approx 1/.966$. The curve $(E\{r_A(s_A)\}, m_B^c(s_A))$ intersects the horizontal line in m_A at $r_A = r' \approx .973$. In a Nash equilibrium, player A selects the threshold r_1 with probability $(r' - r_2)/(r_1 - r_2) \approx .25$ and else r_2. The randomization makes the optimal threshold for player B equal to $r_B \approx 1.035 \approx 1/.966$. R and Z are as defined in Figure 3 and other constants are $\beta = .005$, $\eta_A = 1$, and $\eta_B = -1.5$. The plot is based on 10^5 realizations, each sampled at 10^5 points up to time 100.

A Nash-equilibrium point is a pair of strategies (s_A^e, s_B^e) with $s_A^e \in S_A$ and $s_B^e \in S_B$ where

$$s_A^e = \text{argmax}_{s_A \in S_A} \{u_A(s_A, s_B^e)\},$$
$$s_B^e = \text{argmax}_{s_B \in S_B} \{u_B(s_A^e, s_B)\}.$$

In order to prove existence of a Nash equilibrium, we assume that m_ν is piecewise continuous and that m_ν^c is continuous in each of the discontinuities in m_ν. This is a property of the stochastic processes Z and Q, but it will be fulfilled except in extreme cases. For example, it will not be fulfilled if Z or Q only take discrete values. We may then formulate the more general theorem for the existence of a Nash equilibrium.

Theorem 5.2. *Assume m_ν is piecewise continuous and m_ν^c is continuous in each of the discontinuities in m_ν. Then there exists at least one Nash-equilibrium point in which at most one player randomizes.*

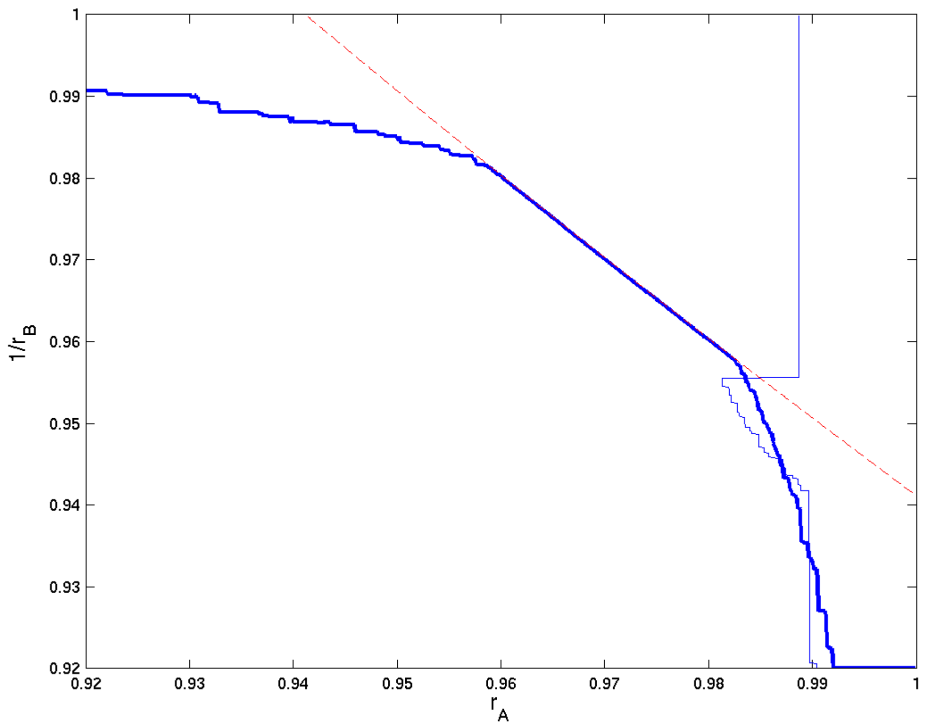

Figure 5. This figure shows non-uniqueness in the Nash equilibrium due to multiple crossings of the best response curves m_A (thin curve) and m_B (thick curve) for the process illustrated in Figure 4 with $\tau_A = .0005$ and $\tau_B = .0045$.

When the stochastic processes are discontinuous, we are able to construct examples where there exist multiple Nash equilibria. In Figure 5, there are two Nash equilibria with constant thresholds, and one with randomization.

6. Efficiency of the choice of adjustment costs

In this section, we extend the model by allowing an additional stage of the model, taking place ahead of the basic model, where each of the players may invest in adjustment capacity, reducing his costs of adjustment of the contract. For example, a firm may have a large salary department, taking care of the wage negotiations. Let the costs of obtaining adjustment fee τ_ν be given by the function $c_\nu(\tau_\nu)$, where we assume that c_ν is differentiable and strictly decreasing, and that c_ν converges to infinity when τ_ν converges to zero, and c_ν converges to zero when τ_ν converges to infinity. For simplicity, we assume that c_ν approaches zero sufficiently fast when τ_ν increases to ensure that equations (7) and (8) below have a solution.

With a slight abuse of notation, let $W_\nu(\tau_A, \tau_B)$ denote the expected value of the objective function of player ν, derived from Nash equilibrium in the basic model with adjustment fees τ_A and τ_B.[9]

When both players optimize their investment in adjustment capacity, the adjustment fees are given by the first order conditions

$$\frac{\partial W_\nu}{\partial \tau_\nu} - c'_\nu(\tau_\nu) = 0, \qquad \nu = A, B. \tag{7}$$

[9]We do not wish to go into issues of equilibrium selection here, so if there are multiple Nash equilibria, we assume that players observe a signal indicating which equilibrium applies. Associating probabilities with the various Nash equilibria, players take the expected value of the objective functions.

Assuming for simplicity that overall welfare can be measured by the sum of players' expected utility, the welfare maximizing levels of investment in adjustment capacity is given by

$$\frac{\partial W_A}{\partial \tau_\nu} + \frac{\partial W_B}{\partial \tau_\nu} - c'_\nu(\tau_\nu) = 0, \qquad \nu = A, B. \tag{8}$$

From Theorem 4.1, it follows that the expected value of the objective function for a player is increasing in the opponent's cost of adjusting the contract, i.e. that $\partial W_A/\partial \tau_B > 0$ and $\partial W_B/\partial \tau_A > 0$. This follows from Theorem 4.1, part (iv), showing that higher adjustment costs makes a player more passive (threshold further from unity), and part (i) showing that a player gains from his opponent becoming more passive. Thus, it follows that the values of τ_ν that satisfy (7) give positive values when put into the left-hand side of the equations (8). This implies that for each solution of (7), there exists a solution of (8) with higher values of τ_ν. This implies that when each player determines the adjustment fee from (7), there is an over-investment in adjustment capacity compared to a solution of equations (8).

This over-investment in adjustment capacity is due to the following. First, each of the players do not take into consideration that the contract payment in our setting is only a matter of a transfer between the players, so that what one player gains by adjusting the payment is directly linked to what the other player loses. Second, the first effect is exacerbated by the strategic substitutability in the choice of thresholds. By investing in adjustment capacity, thus reducing the adjustment fee, the threshold of the player is moved closer to unity, leading the other player to choose a threshold further away from unity. The player gains from both changes, i.e., both from lowering his own adjustment fee, and from inducing the opponent to set a threshold further away from unity.

7. Concluding remarks

The assumption that wages and prices are sticky in nominal terms plays a key role in macro and monetary economics. However, usually the timing of price adjustment is taken as exogenous. This has motivated a considerable literature studying the optimal adjustment of prices under stochastic evolution of money or aggregate prices. In this paper we extend this analysis by considering bilateral adjustment, where both parties to the trade, both the seller and buyer, are allowed to require adjustments of the contract.

We show that several of the key results from the literature on unilateral price adjustment also hold in the more general case of bilateral adjustment. Optimal behavior is characterized by threshold strategies, where players demand adjustment whenever they can improve the contract terms by a certain percentage, i.e., whenever the real contract payment, R, is too far away relative to the outcome of an adjustment of the contract, Z. Player A (the seller) adjusts whenever $R/Z \le r_A < 1$, while player B (the buyer) requires adjustment whenever $R/Z \ge r_B > 1$. As expected, higher volatility and larger adjustment costs make players more reluctant to demand an adjustment, implying threshold values further from unity.

Under rather general assumptions, even allowing for discontinuities in the stochastic processes, we prove the existence of a Nash equilibrium in thresholds. When we extend the class of stochastic processes further, it is necessary to include randomization in the strategies to obtain Nash equilibrium. A second main result is that in equilibrium, there is strategic substitutability in players' choice of threshold: If one player becomes more active, setting a threshold closer to unity, the other player becomes more passive, setting a threshold further from unity.

The basic economic setting we analyse has previously been studied by Andersen and Christensen [4]. However, they focus on only one adjustment, while we consider more general stochastic processes, allowing for an unlimited number of adjustments. Our allowing for an unlimited number of adjustments is important for several of the key results. For example, the strategic substitutability is in contrast to the finding of Andersen and Christensen [4].

The strategic substitutability may exacerbate asymmetries. If the adjustment costs of one player, say A, are reduced, this player will respond by raising the threshold closer to unity. However, this effect will be strengthened by player B raising his threshold, becoming more passive, further away from unity. Numerical simulations indicate that the strategic effect in some cases is substantial, depending on the initial conditions.

If there is drift in the stochastic processes, one player will make most of the adjustments. The drift will make the opponent more passive, in the form of a threshold further from unity, as any gain from adjustment is likely to be short-lived and thus small, due to the drift. As the opponent becomes more passive, the strategic substitutability will make the first player more active, amplifying the effect of the drift.

One application of these effects is that if there is both a positive drift in the form of a rising price level, and costs to the firm of cutting nominal wages (e.g., adverse effects on workers' morale), these effects will work to make the firm more passive. The workers will then respond by becoming more active in pushing wages up, thus magnifying the initial asymmetric effect. This magnifiying effect, which to our knowledge is novel in the literature, suggests that even small costs of cutting wages may be sufficient to ensure that only a fairly small part of all nominal wage changes are negative.

We also find that a risk averse player will be more passive, setting a threshold further from unity, thus benefiting the opponent. In contrast, a risk loving player will be more active, with adverse effects on the opponent.

Throughout the analysis, the outcome of an adjustment of the contract, Z, is taken as exogenous. This does not affect our analysis of the optimal adjustment behavior, i.e. players' choice of thresholds. By the Markov assumption, players' strategies can only depend on the state variables. Thus, while the players decide when the adjustment takes place, they cannot affect the outcome of the adjustment by their choice of thresholds.[10] However, by taking the adjustment outcome as exogenous when we vary parameter values in the numerical simulations, we do not capture the effect of these parameters on the adjustment outcome. Our simulations should consequently be viewed as *ceteris paribus*-exercises, showing the direct effect of parameter changes on optimal thresholds for adjustment, but keeping the adjustment outcome constant.

We extend the basic model by introducing a stage prior to the model, where players may invest in "adjustment ability", in the sense that they may reduce their own costs of undertaking an adjustment (e.g., by having a personnel department doing the wage negotiations). We then find that players will over-invest as compared to the socially efficient level. The over-investment arises for two reasons. First, players demand an adjustment too often from a social point of view, as they do not take into consideration that their own gain from better contract terms is reflected in a loss by the opponent. By investing to lower one's own adjustment costs, a player will demand an adjustment more often, thus hurting the other player. Secondly, the strategic substitutability mentioned above exacerbates the first effect. By reducing one's own adjustment costs, a player becomes more active. This makes the opponent more passive, which adds to the gain of the first player, as an adjustment induced by the opponent becomes less likely.

Our analysis is cast in a specific, but important setting, namely state-dependent adjustment of a nominal contract between two players, e.g., a labor contract or a tenancy agreement. This analysis seems relevant for research on macro models with sticky wages and prices. While a number of recent contributions have introduced state dependendent pricing (see references in the Introduction), contributions with state dependent wage setting, allowing both parties to require an adjustment, are still missing. Hopefully, our analysis could be helpful in future work on this issue.

We also believe that our analysis has wider applications. For example, the gain from trade between two countries may depend on the trade agreement that prevails between the countries. If one country perceives adjustment of the agreement as less costly than the other, the former country will be more active in the sense of being willing to adjust the agreement even when only a small improvement is possible. Alternatively, if the relative strength of the countries changes over time, the adverse drift will make the losing country more passive. Our finding of strategic substitutability in thresholds suggests that these forms of asymmetry will be exacerbated, in the sense that the active behavior of one country and the passive behavior of the other will reenforce each other. This exacerbating of asymmetries resembles the finding of Haller and Holden [23], where it is shown

[10]In contrast, if players could commit to a certain threshold in the future, this would in general affect the adjustment outcome, and it should consequently be treated as endogenous.

that asymmetries in the bargaining positions of two countries over an international treaty may be exacerbated by the country with the "stronger" bargaining position setting a stricter super-majority ratification requirement, thus magnifying the effect of the strong bargaining position.

An important restriction of our model is that the gains from trade are taken to be exogenous. If the gain from trade could be affected by player's investment or effort decisions, it would be important to design the contract so as to sustain efficiency. MacLeod and Malcomson [31] show that simple contracts with rigid contract payment may induce efficiency in some cases. An interesting avenue for future research would be to explore the efficiency properties of a simple nominal contract in a dynamic, stochastic environment, by an appropriate extension of the present model.

Appendix A. The stochastic processes

For the benefit of the reader we recall the definition of a Lévy process (see, e.g., Sato [33]).

Definition A.1 (Lévy process). A stochastic process X_t is a Lévy process provided the following conditions hold:
(i) For any n and any $0 \leq t_0 < \cdots < t_n$ the random variables $X_{t_0}, X_{t_1} - X_{t_0}, \ldots, X_{t_n} - X_{t_{n-1}}$ are independent.
(ii) $X_0 = 0$ almost surely.
(iii) The distribution of $X_{s+t} - X_s$ is independent of s.
(iv) The process is stochastically continuous, i.e., $\lim_{t \downarrow 0} \text{Prob}(|X_t| > \epsilon) = 0$ for all $\epsilon > 0$.
(v) The process is right-continuous with left limits.

For Lévy processes we have the Lévy–Khintchine formula for the characteristic function of X_t (see, e.g., Sato [33])

$$E\{\exp(i\lambda X_t)\} = \exp\Big(t\big(i\alpha\lambda - \tfrac{1}{2}\lambda^2 a^2 + \int_{-\infty}^{\infty}(e^{i\lambda x} - 1 - i\lambda x \chi_{\{|x|\leq 1\}}(x))d\sigma(x)\big)\Big), \tag{9}$$

where $d\sigma$ is a σ-finite measure, denoted the Lévy-measure, with $\int_{-\infty}^{\infty} \min(|x|^2, 1) d\sigma(x) < \infty$ and $\sigma(\{0\}) = 0$. The process is uniquely defined by the quantities $(\alpha, a, d\sigma)$. The measure σ describes the size and intensity of the jumps in the process. The process is Gaussian if and only if $\sigma = 0$, and in that case, α denotes the drift and a the volatility. If σ satisfies

$$\int_1^{\infty} e^{\eta x} d\sigma(x) < \infty,$$

we may conclude that

$$E\{\exp(\eta X_t)\} = \exp\Big(t\big(\alpha\eta + \tfrac{1}{2}\eta^2 a^2 + \int_{-\infty}^{\infty}(e^{\eta x} - 1 - \eta x \chi_{\{|x|\leq 1\}}(x))d\sigma(x)\big)\Big)$$

holds and is finite.

To ensure that the objective functions are finite, it is necessary to bound Z and Q relative to the discount rate β. This requires two additional assumptions. First, we assume that the volatility of the non-gaussian part is bounded, by assuming that the Lévy-measures satisfy

$$\int_1^{\infty} e^{\eta_h x} d\sigma_h(x) < \infty, \quad h = z, q \tag{10}$$

for some η_h. We may then define the drift in the processes for the real adjustment outcome Z and the aggregate price level Q by

$$\mu_{\nu,h} = \alpha_h \eta_\nu + \tfrac{1}{2}\eta_\nu^2 a_h^2 + \int_{-\infty}^{\infty}(e^{\eta_\nu x} - 1 - \eta_\nu x \chi_{\{|x|\leq 1\}}(x))d\sigma_h(x), \quad \eta_\nu \leq \eta_h,$$

for $\nu = A, B$ and $h = z, q$. We have that $E\{Z^{\eta_\nu}(t)\} = \exp(t\mu_{\nu,z})$ and $E\{Q^{\eta_\nu}(t)\} = \exp(t\mu_{\nu,q})$. For example, $\mu_{A,z}$ is the expected rate of increase in the real adjustment payment, adjusted for the relative rate of risk aversion of player A, η_A. The second assumption is that the drift parameters $\mu_{\nu,h}$ must be bounded by the discount rate β.

Definition A.2 (Property $\mathcal{F}$)**.** We say that the stochastic contract model has property $\mathcal{F}$ if the following properties hold:
(i) The real adjustment outcome $Z(s,\omega) = \exp(F(s,\omega))$ and $Q(s,\omega) = \exp(G(s,\omega))$ where F and G are Lévy processes given by $(\alpha_z, a_z, d\sigma_z)$ and $(\alpha_q, a_q, d\sigma_q)$, respectively.
(ii) There exists η_h such that

$$\int_1^\infty e^{\eta_h x} d\sigma_h(x) < \infty, \quad h = z, q,$$

and assume that $\eta_\nu \le \eta_h$ for $\nu = A, B$ and $h = z, q$.
(iii) We have that

$$\mu_{\nu,h} < \beta, \quad \nu = A, B, \, h = z, q.$$

Appendix B. Analytic expression—an example

In special cases it is possible to find analytic expressions for some of the variables. Assume the real adjustment outcome relative to the real contract payment is given by a geometric Brownian motion $Z(t,\omega)/R(t,\omega) = Z(t,\omega)Q(t,\omega) = \exp((\alpha - a^2/2)t + aB_t(\omega))$ where the notation is simplified by the normalization $Z(0,\omega)Q(0,\omega) = 1$, and B_t denotes the standard Brownian motion. Then (see Borodin and Salminen [8, p. 233, formula 3.0.1])

$$E\{\exp(-\beta T(r_A, r_B))\} = \big(r_A^\gamma(r_B^\sigma - r_B^{-\sigma}) - r_B^\gamma(r_A^\sigma - r_A^{-\sigma})\big)\big((r_B/r_A)^\sigma - (r_B/r_A)^{-\sigma}\big)^{-1}$$

with $\gamma = \alpha a^{-2} - 1/2$ and $\sigma = \sqrt{\gamma^2 + 2\beta a^{-2}}$. By differentiating this expression with respect to β at $\beta = 0$ we find, where $\tilde\sigma = \sqrt{a^2 + 8\beta}/(2a)$,

$$\begin{aligned} E\{T(r_A, r_B)\} = &\frac{1}{a^2\gamma(r_A r_B)^{1/2}}\Big(\Big(\frac{r_B}{r_A}\Big)^{\tilde\sigma} - \Big(\frac{r_B}{r_A}\Big)^{-\tilde\sigma}\Big)^{-1} \\ &\times \Big[\ln(r_A)(r_A^{\tilde\sigma+1/2} - r_A^{-\tilde\sigma+1/2}) - \ln(r_B)(r_B^{\tilde\sigma+1/2} - r_B^{-\tilde\sigma+1/2}) \\ &- \ln(r_B/r_A)\frac{\big(\frac{r_B}{r_A}\big)^{\tilde\sigma} + \big(\frac{r_B}{r_A}\big)^{-\tilde\sigma}}{\big(\frac{r_B}{r_A}\big)^{\tilde\sigma} - \big(\frac{r_B}{r_A}\big)^{-\tilde\sigma}} \\ &\times \Big((r_A^{\tilde\sigma+1/2} - r_A^{-\tilde\sigma+1/2}) + (r_B^{\tilde\sigma+1/2} - r_B^{-\tilde\sigma+1/2})\Big)\Big] \end{aligned}$$

is the expected time to the first adjustment.

Appendix C. Approximation formulas

Given the weak assumptions we impose on the stochastic processes, explicit formulas are difficult to obtain. However, we can derive some approximate formulas that may provide useful intuition for how the model works, and to get some sense of the numerical magnitudes that are involved.

We will first explore the effect on the payoff of a player from a marginal reduction in his threshold. Let players A and B have thresholds $r_1 > 1$ and $r_B < \infty$. Consider the situation at T_1 when $R(T_1,\omega)/Z(T_1,\omega) = r_1$. If player A sticks to the threshold r_1, there will be an immediate adjustment at T_1. In contrast, if player A adopts a new threshold $r_2 < r_1$, there will be an adjustment at T_2, where T_2 denotes the first time after T_1 where R/Z has decreased at least by a factor r_2/r_1, or increased at least by a factor r_B/r_1. Formally

$$T_2 = \inf\{s > 0 \mid R(T_1 + s,\omega)/Z(T_1 + s,\omega) \notin (r_2, r_B)\}.$$

Considering the payoffs associated with selecting the threshold r_2, when we let r_2 converge to r_1 from below, we obtain the effect of a marginal reduction in r_1. The limit of the ratio of R/Z is then

$$v_A(r_1, r_B) = \lim_{r_2 \to r_1 -} \frac{E\{\int_0^{T_2} R^{\eta_A}(T_1 + s)\exp(-\beta s)ds\}}{E\{\int_0^{T_2} Z^{\eta_A}(T_1 + s)\exp(-\beta s)ds\}}. \tag{11}$$

Define $v_B(r_A, r_1)$ correspondingly.

Assume there is a Gaussian component in either Z or Q, i.e., that either $a_z > 0$ or $a_q > 0$. Then a well-known property of Gaussian processes implies that when r_1 is reached, the probability of reaching r_B before r_2, converges to zero when $r_2 \to r_1-$. Furthermore, the expected time until r_2 is reached converges to zero, i.e.,

$$\lim_{r_2 \to r_1-} E\{T_2(r_2/r_1, r_B/r_1)\} = 0. \tag{12}$$

Equation (12) implies that

$$v_A(r_1, r_B) \approx r_1^{\eta_A}. \tag{13}$$

Thus, v_A is quite insensitive with respect to variation in r_B. (The more volatile the ratio R/Z is, and the closer r_1 and r_B are to 1, the more $v_A(r_1, r_B)$ is sensitive with respect to variation in r_B.) We do not have equality in the limit when $r_2 \to r_1$, since with probability zero, the time $T_2(r_2/r_1, r_B/r_1, \omega)$ is positive and in this time period we have that $R(s,\omega)/Z(s,\omega) > r_2$ and $R(s,\omega)/Z(s,\omega)$ may reach r_B before r_2. In this approximation we neglect the possibility that T_2 does not vanish in the limit. By a similar argument, we have $v_B(r_A, r_B) \approx r_B^{\eta_B}$.

In the proof of Theorem 4.1 below, equation (24), it is shown that the optimal threshold satisfies

$$v_A(m_A(r_B), r_B) = (\beta - \mu_{A,z})(u_A(m_A(r_B), r_B) - \tau_A) \tag{14}$$

if $u_A(m_A(r_B), r_B) > \tau_A$. Correspondingly, if $u_B(r_A, m_B(r_A)) > \tau_B$, then

$$v_B(r_A, m_B(r_A)) = (\beta - \mu_{B,z})(u_B(r_A, m_B(r_A)) - \tau_B).$$

When combining (13) and (14) we get the approximations

$$u_A(m_A(r_B), r_B) \approx \frac{1}{\beta - \mu_{A,z}} m_A^{\eta_A}(r_B) + \tau_A, \tag{15}$$

$$u_B(r_A, m_B(r_A)) \approx \frac{1}{\beta - \mu_{B,z}} m_B^{\eta_B}(r_A) + \tau_B. \tag{16}$$

To obtain some intuition for these expressions, consider the case with time invariant adjustment outcome, $Z = 1$, implying that $\mu_{\nu,z} = 0$. If in addition, $\eta_A = 1$ and $\eta_B = -1$, then (15) and (16) read $u_A \approx r_A/\beta + \tau_A$ and $u_B \approx 1/(r_B\beta) + \tau_B$, which can be rearranged to $r_A \approx (u_A - \tau_A)\beta$ and $1/r_B \approx (u_B - \tau_B)\beta$.

The following heuristic argument explains these expressions: By renegotiating the contract, a player incurs the adjustment fee, and then obtains the expected utility after an adjustment, u_ν. Multiplying by the discount rate β, we obtain the equivalent flow payoff. A player should demand an adjustment when the real contract payment equals the equivalent flow payoff from requiring an adjustment, i.e., the critical thresholds are given by these formulas.

These approximations imply that the volatility only influences the thresholds through the expected objective functions u_ν. These relations may be useful in order to find the optimal thresholds. Comparing with the numerical simulations in Section 4, these approximations underestimate u_A and u_B by about 2 percent. The approximation is better the smaller the volatility in Z and Q.

Appendix D. Proofs

We have the following three technical results that are proved at the end of this section.

Lemma D.1. *The real adjustment outcome model Z satisfies*

$$E\{1 - Z^{\eta_\nu}(t)\exp(-\beta t)\} = (\beta - \mu_{\nu,z})E\{\int_0^t Z^{\eta_\nu}(s)\exp(-\beta s)ds\}.$$

We need the following definitions before we may state the next lemma. Let

$$F_\nu(r_1, r_A, r_B) = \frac{E\{\int_0^{T_1} R^{\eta_\nu}(s)\exp(-\beta s)ds\}}{E\{\int_0^{T_1} Z^{\eta_\nu}(s)\exp(-\beta s)ds\}}$$

where $R(0,\omega)/Z(0,\omega) = r_1$ and

$$T_1 = \inf\{s > 0 \mid R(s,\omega)/Z(s,\omega) \notin (r_A, r_B)\}.$$

In this definition we assume $Z(0,\omega) \neq 1$ in order to simplify the expressions. Further, define P_τ as the probability for passing the boundary r_τ at $t = T_1$, i.e.,

$$P_A = \text{Prob}(R(T_1,\omega)/Z(T_1,\omega) \leq r_A) \tag{17}$$

and $P_B = 1 - P_A$.

Lemma D.2. *Assume that if there are jumps in $Z(t,\omega)$ or $Q(t,\omega)$, then the jumps are from a continuous and decreasing distribution. Then $E\{T(r_A,r_B)\}$ and $F_\nu(r_1,r_A,r_B)$ for $\nu = A,B$ are differentiable. In addition $F_A(r_1,r_A,r_B)$ is increasing in all variables and $F_B(r_1,r_A,r_B)$ is decreasing in all variables.*

Lemma D.3. *Assume that if there are jumps in $Z(t,\omega)$ or $Q(t,\omega)$, then the jumps are from a continuous and decreasing distribution. Then $v_A(r_A,r_B)$ is increasing in both variables and differentiable where*

$$\frac{\partial v_A(r_A,r_B)}{\partial r_B} < (\beta - \mu_{A,z})\frac{\partial u_A(r_A,r_B)}{\partial r_B}.$$

Correspondingly, $v_B(r_A,r_B)$ is decreasing in both variables and differentiable where

$$\frac{\partial v_B(r_A,r_B)}{\partial r_A}(\beta - \mu_{B,z})\frac{\partial u_B(r_A,r_B)}{\partial r_A}.$$

Proof of Theorem 2.1. The objective function may be written

$$\begin{aligned} U_\nu(t_1,\dots,\omega) &= \sum_{j=0}^{\infty}\Big(\int_{t_j}^{t_{j+1}} R^{\eta_\nu}(s,\omega)\exp(-\beta s)ds - \tau_\nu Z^{\eta_\nu}(t_{j+1},\omega)\exp(-\beta t_{j+1})\Big) \\ &= \tau_\nu + \sum_{j=0}^{\infty} Z^{\eta_\nu}(t_j,\omega)\exp(-\beta t_j)\Big(\int_{t_j}^{t_{j+1}} \frac{Q^{\eta_\nu}(t_j,\omega)}{Q^{\eta_\nu}(s,\omega)}\exp(-\beta(s-t_j))ds - \tau_\nu\Big). \end{aligned}$$

The expected value of integral in the last expression above is bounded due to property $\mathcal{F}$. Then $E\{U_\nu\}$ is bounded if the number of adjustments is finite.

If there is an infinite number of adjustments, it is in addition necessary to bound

$$E\{\sum_{j=0}^{\infty} Z^{\eta_\nu}(t_j)\exp(-\beta t_j)\}.$$

This expression is bounded due to property $\mathcal{F}$.

When both $\tau_A, \tau_B > 0$, neither player benefits from requiring adjustment immediately all the time, e.g., have a critical threshold equal to 1. Hence the problem is well-defined.

Let T satisfy $t_i < T \leq t_{i+1}$. Define C_T as the contribution to the objective function for $t < T$ that cannot be changed when $t \geq T$, that is,

$$\begin{aligned} C_T = \sum_{j=0}^{i-1}\Big(\int_{t_j}^{t_{j+1}} R^{\eta_\nu}(s,\omega)\exp(-\beta s)ds - \tau_\nu Z^{\eta_\nu}(t_{j+1},\omega)\exp(-\beta t_{j+1})\Big) \\ + \int_{t_i}^{T} R^{\eta_\nu}(s,\omega)\exp(-\beta s)ds \end{aligned} \tag{18}$$

and $H_\nu(t_{i+2}-t_{i+1},\dots,\omega)$ as the contribution to the object function after t_{i+1}, that is,

$$\begin{aligned} H_\nu(t_{i+2}-t_{i+1},\dots,\omega) = \sum_{j=i+1}^{\infty}\Big(\int_{t_j}^{t_{j+1}} R^{\eta_\nu}(s,\omega)\exp(-\beta s)ds \\ - \tau_\nu Z^{\eta_\nu}(t_{j+1},\omega)\exp(-\beta t_{j+1})\Big). \end{aligned}$$

The function $H_\nu(t_{i+2}-t_{i+1},\dots,\omega)$ has the same distribution as $U_\nu(t_0,\dots,\omega)$. Then we may write the objective function as

$$U_\nu(t_1,\dots,\omega)$$

$$= C_T + Z^{\eta_\nu}(T,\omega)\exp(-\beta T)\Big(\frac{R^{\eta_\nu}(T,\omega)}{Z^{\eta_\nu}(T,\omega)}\int_T^{t_{i+1}}\frac{R^{\eta_\nu}(s,\omega)}{R^{\eta_\nu}(T,\omega)}\exp(-\beta(s-T))ds$$
$$+\frac{Z^{\eta_\nu}(t_{i+1},\omega)}{Z^{\eta_\nu}(T,\omega)}\exp(-\beta(t_{i+1}-T))(H_\nu(t_{i+2}-t_{i+1},\ldots,\omega)-\tau_\nu)\Big).$$

The ratios $R(s,\omega)/R(T,\omega)$ and $Z(t_{i+1},\omega)/Z(T,\omega)$ are independent of $R(T,\omega)$ and $Z(T,\omega)$ due to the Markov properties. The future contribution to the object function depends on $R(T,\omega)$ and $Z(T,\omega)$, but the optimal strategy is only a function of the ratio $R(T,\omega)/Z(T,\omega)$ and there is no memory in the game, i.e., dependencies on $t_j < T$, $R(s,\omega)$ for $s < T$ or $Z(s,\omega)$ for $s < T$.

Let s_B and s_A denote the strategies of player B and player A, respectively. With a slight abuse of notation, let $U_A(s_A, s_B, \omega)$ denote the objective function with the strategies s_A and s_B, respectively. Then $\sup_{s_A} E\{U_A(s_A, s_B)\}$ is well-defined and there is a sequence $s_{A,i}$ such that

$$\lim_{i\to\infty} E\{U_A(s_{A,i}, s_B)\} = \sup_{s_A} E\{U_A(s_A, s_B)\}. \tag{19}$$

Define the sequence of sets S_i where $r \in S_i$ if player A with strategy $s_{A,i}$ requires contract adjustment for any interval for any price $Q(\,\cdot\,,\omega)$ at time t where $R(t,\omega)/Z(t,\omega) = r$. To ensure that the sets are non-empty, add the number 0 to S_i. If the adjustment is not the first time t when $R(t,\omega)/Z(s,\omega) = r$, this is not critical, since it is the contribution to the objective function of the player in the future that is critical. Since all $r \in S_i$ satisfies $0 \le r < 1$, then for any sequence $\{r_i\}_i$ with $r_i \in S_i$, there is an accumulation point r' (if several, take the largest). Consider a strategy s' with a critical threshold r'. Since the expected value of the future contribution to the objective function at time t only is a function of the present $R(t,\omega)/Z(t,\omega)$, and equation (19), then

$$E\{U_A(s', s_B)\} = \sup_{s_A} E\{U_A(s_A, s_B)\}.$$

If the adjustment outcome ZQ does not only change in discrete jumps, then the adjustments will come with shorter and shorter time intervals if $r_A \to 1$. Assuming $\tau_A > 0$, then the adjustment cost dominates the objective function which implies that the accumulation point $r' < 1$. If the price ZQ only changes in discrete jumps, then the relative flow payoff can only take discrete values and $r = 1$ cannot be an accumulation point for the chain where all elements in the chain satisfies $r_i < 0$. This implies that the critical threshold may be set equal to the accumulation point $0 \le r' < 1$.

Correspondingly, if player A has the same strategy in each time interval, then there is a corresponding argument showing that there cannot be a better strategy for player B than what is possible to obtain with a critical threshold r_B. □

Proof of Theorem 3.1. Let H'_ν be defined as H_ν in the proof of Theorem 2.1 but with $T < t_1$ and with parameters r_A and r_B instead of $t_1 - t_0, \ldots$. The definition of $U_\nu(r_A, r_B, \omega)$ in (1) and (3) implies

$$U_\nu(r_A, r_B, \omega) = \int_{t_0}^{t_1} R^{\eta_\nu}(s,\omega)\exp(-\beta s)ds + Z^{\eta_\nu}(t_1,\omega)\exp(-\beta t_1)(H'_\nu(r_A, r_B, \omega) - \tau_\nu).$$

The stochastic variables U_ν and H'_ν have similar distribution and have expectation equal to u_ν. The time for the end of the first interval t_1 is independent of what is happening after t_1 due to the Markov property. Hence $Z^{\eta_\nu}(t_1,\omega)\exp(-\beta t_1)$ is independent of $H'_\nu(r_A, r_B, \omega)$. This implies that

$$u_\nu(r_A, r_B) = f_\nu(r_A, r_B) - \tau_\nu h_\nu(r_A, r_B) + h_\nu(r_A, r_B)u_\nu(r_A, r_B),$$

leading to

$$u_\nu(r_A, r_B) = \frac{f_\nu(r_A, r_B) - \tau_\nu h_\nu(r_A, r_B)}{1 - h_\nu(r_A, r_B)},$$

which can be rewritten as

$$u_\nu(r_A, r_B) = \frac{f_\nu(r_A, r_B) - \tau_\nu}{1 - h_\nu(r_A, r_B)} + \tau_\nu. \tag{20}$$

It follows from Appendix A that $E\{Z(t)\}$ is differentiable. Assuming $E\{T(r_A, r_B)\}$ is differentiable, we may do the following calculation

$$\frac{\partial h_\nu}{\partial r_\tau} = E\{(\eta_\nu - \beta Z(T-))Z^{\eta_\nu - 1}(T-)\exp(-\beta T)\}\frac{\partial E\{T\}}{\partial r_\tau}$$

and

$$\frac{\partial f_\nu}{\partial r_\tau} = E\{R^{\eta_\nu}(T-)\exp(-\beta T)\}\frac{\partial E\{T\}}{\partial r_\tau}$$

where $T-$ denote the left limit and $\tau = A$ or $\tau = B$. This implies that f_ν and h_ν are differentiable and the derivatives satisfy

$$\frac{\partial u_\nu}{\partial r_\mu} = \frac{\frac{\partial f_\nu}{\partial r_\mu}(1-h_\nu) + (f_\nu - \tau_\nu)\frac{\partial h_\nu}{\partial r_\mu}}{(1-h_\nu)^2} = \frac{\frac{\partial f_\nu}{\partial r_\mu} + (u_\nu - \tau_\nu)\frac{\partial h_\nu}{\partial r_\mu}}{1-h_\nu}$$

for $\mu = A, B$. □

Proof of Theorem 4.1. We first prove that the assumption that if there are jumps in $R(t,\omega)$ or $Z(t,\omega)$ then the jumps are from a continuous and decreasing distribution implies that $E\{T(r_A, r_B)\}$ is differential. Let $P_A(r_A, r_B)$ be defined from (17). Then

$$E\{T(r_A - \Delta r, r_B)\} - E\{T(r_A, r_B)\} = P_A E\{T(\frac{r_A - Deltar}{r_A}, \frac{r_B}{r_A})\}.$$

The Markov property and the additional assumption on the jumps, implies that in the limit when Δr vanish, the influence of the second argument r_B/r_A reduces, implying that $E\{T(r_A, r_B)\}$ is differential. (i) From equation (20) we have

$$u_\nu(r_A, r_B) = \frac{f_\nu(r_A, r_B)}{1 - h_\nu(r_A, r_B)} - \frac{\tau_\nu}{1 - h_\nu(r_A, r_B)} + \tau_\nu. \tag{21}$$

Lemma D.1 gives

$$\frac{f_A}{1-h_A} = \frac{E\{\int_0^T R^{\eta_A}(s)\exp(-\beta s)ds\}}{(\beta - \mu_{A,z})E\{\int_0^T Z^{\eta_A}(s)\exp(-\beta s)ds\}}. \tag{22}$$

Combining these implies

$$u_\nu(r_A, r_B) = \frac{F_\nu(1, r_A, r_B)}{\beta - \mu_{A,z}} - \frac{\tau_\nu}{1 - h_\nu(r_A, r_B)} + \tau_\nu. \tag{23}$$

This expression is used in order to show that u_A is increasing in r_B and that u_B is decreasing in r_A. Lemma D.2 states that $F_A(r_1, r_A, r_B)$ increases in r_B and $F_B(r_1, r_A, r_B)$ decreases in r_A. The second term has the same sign since h_A is decreasing in r_B and h_B is increasing in r_A.

(ii) Theorem 3.1 states that f_ν, h_ν, and u_ν are continuous. To simplify the expressions below, define the expected value of the objective functions of each of the players, given optimal play by this player, as

$$u_{m,A}(r_B) = \sup_{r_A} u_A(r_A, r_B),$$
$$u_{m,B}(r_A) = \sup_{r_B} u_B(r_A, r_B).$$

Then $u_{m,A}$ is continuous and m_A is well-defined, piecewise continuous and $u_A(m_A(r_B), r_B) = u_{m,A}(r_B)$ and $u_B(r_A, m_B(r_A)) = u_{m,B}(r_A)$.

Let $r_B > 1$ be fixed. We will first prove that there is a value $0 \leq r_A < 1$ that maximizes $u_A(r_A, r_B)$. The function $u_A(r_A, r_B)$ is defined for $0 \leq r_A < 1$. We will give an argument that the maximum value is attained for r_A in the closed interval $[0, 1-\epsilon]$ for $\epsilon > 0$ sufficiently small. Since the interval is closed, the maximum value will be attained for a value $r_A = m_A(r_B)$.

If Z or Q have a Brownian term, then $T(r_A, r_B)$ vanishes with probability 1 when $r_A \to 1$. Then $h_A(r_A, r_B) \to 1$ and $f_A(r_A, r_B) \to 0$ when $r_A \to 1$. Furthermore the expression (5) for u_A implies that $u_A(r_A, r_B) \to -\infty$ when $r_A \to 1$. If there are only jumps in ZQ, then $T(r_A, r_B)$ does not necessarily vanishes with probability 1 when $r_A \to 1$. But for $1 - \varepsilon < r_A < 1$ for ε sufficiently small, then either u_A is constant for r_A in the interval or the change in middle term in equation (23) is dominating. This term is negative. Hence for each value for r_B, there is a value $r_A = m_A(r_B) < 1$ where $u_A(r_A, r_B) = u_{m,A}(r_B)$.

We will prove that the value $r_A = m_A(r_B)$ is unique, i.e., if $r_A \neq m_A(r_B)$, then $u_A(r_A, r_B) < u_{m,A}(r_B)$. Let t_1 be the time for the first contract adjustment. Furthermore, let W_1 and W_2 be two new stochastic variables that are identical to U_A except that a different strategy (a different limit for r_A) is used before t_1. After t_1, we set $r_A = m_A(r_B)$. The first contract adjustment in W_2 and W_1, if required by player A, is required when R/Z reaches the values $r_2 < r_1 < 1$, respectively. Let P_{r_1} denote the probability that $R(t,\omega)/Z(t,\omega)$ reaches r_1 before it reaches r_B. In case the ratio reaches r_B first, there is no difference between W_1 and W_2. Further, let $T_1 = T(r_1, r_B, \omega)$ denote the time of the first contract adjustment for W_1. Finally, let

$$E_{r_1} = E\{\exp(-\beta T_1(r_1, r_B))\}$$

given that r_1 is reached. For the strategy associated with W_2 there is a contract adjustment when R/Z reaches r_2 or r_B. Assuming r_1 is reached, there is contract adjustment when either the ratio decreases with a factor r_2/r_1 or increases with a factor r_B/r_1. Let $T_2 = T(r_2/r_1, r_B/r_1, \omega)$ denote the time between r_1 is reached and either r_2 or r_B is reached. Let w_1 and w_2 be the expected values of W_1 and W_2, respectively. Assuming that r_1 is reached before r_B, then

$$w_1 = E\{C_{T_1}\} + E_{r_1} E\big\{(u_{m,A}(r_B) - \tau_A) Z^{\eta_A}(T_1) \exp(-\beta T_2)\big\}$$

and

$$\begin{aligned} w_2 = E\{C_{T_1}\} + E_{r_1} E\big\{ &\int_0^{T_2} R^{\eta_A}(T_1 + s) \exp(-\beta s) ds \\ &+ (u_{m,A}(r_B) - \tau_A) Z^{\eta_A}(T_1 + T_2) \exp(-\beta T_2)\big\} \end{aligned}$$

where C_{T_1} is defined in (18). As noted above, $W_1 = W_2$ are identical except if r_1 is reached. Since the probability that r_1 is reached before r_B is P_{r_1}, the difference is

$$\begin{aligned} w_2 - w_1 &= P_{r_1} E_{r_1} E\big\{ \int_0^{T_2} R^{\eta_A}(T_1 + s) \exp(-\beta s) ds \\ &\qquad - (u_{m,A}(r_B) - \tau_A)(Z^{\eta_A}(T_1) - Z^{\eta_A}(T_1 + T_2) \exp(-\beta T_2))\big\} \\ &= P_{r_1} E_{r_1} E\big\{ \int_0^{T_2} R^{\eta_A}(T_1 + s) \exp(-\beta s) ds \\ &\qquad - (\beta - \mu_{A,z})(u_{m,A}(r_B) - \tau_A) \int_0^{T_2} Z^{\eta_A}(T_1 + s) \exp(-\beta s) ds\big\} \\ &= P_{r_1} E_{r_1} \Big(\frac{E\{\int_0^{T_2} R^{\eta_A}(T_1 + s) \exp(-\beta s) ds\}}{E\{\int_0^{T_2} Z^{\eta_A}(T_1 + s) \exp(-\beta s) ds\}} - (\beta - \mu_{A,z})(u_{m,A}(r_B) - \tau_A)\Big) \\ &\qquad \times E\{ \int_0^{T_2} Z^{\eta_A}(T_1 + s) \exp(-\beta s) ds\}. \end{aligned}$$

Lemma D.1 is used in the second equality. Define the limit of the discounted real adjustment outcome when the lower threshold is slightly reduced by

$$L_A(r_1, r_B) = \lim_{r_2 \to r_1 -} \frac{E\{\int_0^{T_2} Z^{\eta_A}(T_1 + s) \exp(-\beta s) ds\}}{r_2 - r_1}.$$

Due to the Markov property and property $\mathcal{F}$, the numerator is monotone and hence the limit is well-defined. Letting $r_2 \to r_1$, we have

$$\frac{\partial w_1}{\partial r_1} = -P_{r_1} E_{r_1} (v_A(r_1, r_B) - (\beta - \mu_{A,z})(u_{m,A}(r_B) - \tau_A)) L_A(r_1, r_B)$$

where v_A is defined by (11). Hence, $\frac{\partial w_1}{\partial r_1} = 0$ when

$$v_A(r_1, r_B) = (\beta - \mu_{A,z})(u_{m,A}(r_B) - \tau_A). \tag{24}$$

According to Theorem 3.1 and Lemma D.3, $u_{m,A}(r_B)$ and $v_A(r_1, r_B)$ are continuous and $v_A(r_1, r_B)$ is increasing in r_1. Consider the function $w_1(r_1, r_B)$ with r_B fixed. Then $w_1(r_1, r_B)$ reaches its maximum with respect to r_1 either for $r_1 = 0$ or for a value $r_1 > 0$ when $\frac{\partial w_1(r_1,r_B)}{\partial r_1} = 0$. Since $v_A(r_1, r_B)$ is increasing in r_1, then $\frac{\partial w_1}{\partial r_1}$ changes sign when (24) is satisfied. Hence, the maximum is unique. Since the periods between contract adjustments are independent then the value r_1 that maximizes w_1 also maximizes $u_A(r_A, r_B)$. Hence, there is a unique value $r_A = m_A(r_B)$ that maximizes $u_A(r_A, r_B)$.

The corresponding argument may be applied for $u_B(r_A, r_B)$. However, since r_B varies in an unbounded interval we should consider u_B as a function of $1/r_B$ instead of r_B when applying the argument. This is possible since $u_B(r_A, r_B)$ is well-defined as r_B approaches ∞.

(iii) Above it is proved that the optimal value $m_A(r_B)$ satisfies the equation

$$v_A(m_A(r_B), r_B) = (\beta - \mu_{A,z})(u_A(m_A(r_B), r_B) - \tau_A).$$

Differentiating both sides with respect to r_B gives

$$\frac{\partial v_A}{\partial r_A}\frac{dm_A}{dr_B} + \frac{\partial v_A}{\partial r_B} = (\beta - \mu_{A,z})(\frac{\partial u_A}{\partial r_A}\frac{dm_A}{dr_B} + \frac{\partial u_A}{\partial r_B}).$$

Since

$$\frac{\partial v_A}{\partial r_B} < (\beta - \mu_{A,z})\frac{\partial u_A}{\partial r_B}$$

from Lemma D.3 and $\partial u_A/\partial r_A = 0$ since $m_A(r_B)$ is the optimal value of r_A, this implies that

$$\frac{\partial v_A}{\partial r_A}\frac{dm_A}{dr_B} > 0.$$

Since $\partial v_A/\partial r_A > 0$, then also $dm_A/dr_B > 0$, i.e., $m_A(r_B)$ is a strictly increasing function. The proof that $m_B(r_A)$ is strictly increasing is similar.

(iv) Equation (6) may be used in order to prove that $m_A(r_B)$ decreases when τ_A increases, assuming $m_A(r_B) > 0$. The function $u_A(r_A, r_B)$ has an optimal value for $r_A = m_A(r_B) > 0$. Since u_A is differentiable, there exists an $\varepsilon > 0$ such that $\frac{\partial u_A}{\partial r_A}(r_A, r_B) > 0$ for $m_A(r_B) - \varepsilon < r_A < m_A(r_B)$ and $\frac{\partial u_A}{\partial r_A}(r_A, r_B) < 0$ for $m_A(r_B) < r_A < m_A(r_B) + \varepsilon$. If τ_A is increased, then $\frac{\partial u_A}{\partial r_A}$ is decreased which implies a reduction in the r_A value where $\frac{\partial u_A}{\partial r_A} = 0$. This implies that increasing the adjustment fee reduces the optimal threshold value $m_A(r_B)$. Correspondingly, it is proved that $m_B(r_A)$ strictly increases when τ_B increases assuming $m_B(r_A) > 0$.

(v) Since $u_{m,\nu}$ and v_ν are continuous, we infer that the functions $m_A(r_B)\colon (1,\infty] \to [0,1)$ and $m_B(r_A)\colon [0,1) \to (1,\infty]$ are continuous. In the infinite rectangle defined by $0 \le r_A < 1$ and $r_B > 1$, $m_A(r_B)$ gives a continuous path between the lines defined by $r_B = 0$ and $r_B = \infty$. Similarly, $m_B(r_A)$ gives a path in the same rectangle between the lines defined by $r_A = 0$ and $r_A = 1$. Hence, these two curves must intersect at least once, giving an equilibrium point. □

Proof of Theorem 5.2. The existence of at least one equilibrium point (r_A^e, r_B^e) is proved similarly as in Theorem 4.1 where it is assumed that the price $E\{T(r_A, r_B)\}$ is differentiable, i.e., the equilibrium point is the intersection between $m_A(r_B)$ and $m_B(r_A)$. But in this case, these curves are not necessarily continuous which implies that there might not be an intersection.

Define graphs M_A and M_B by extending the curves $m_A(r_B)$ and $m_B(r_A)$ by continuity as follows: Wherever $m_A(r_B)$ or $m_B(r_A)$ make jumps, connect the two sides across the jump by straight lines with constant r_B and r_A, respectively. (See Figure 4.) Since M_A and M_B are continuous, they must intersect. If the intersection is on the straight lines, then randomization is necessary as illustrated in Section 5. Assume $m_B(r_A)$ intersects a straight line in M_A connecting the two points (r_1, r_B^e) and (r_2, r_B^e). Then $m_B(r_1)$ and $m_B(r_2)$ give values of r_B on opposite side of r_B^e. We may then define a one parameter family of strategies s_A where the probability for choosing r_1 varies in the interval $0 \le q \le 1$. Since $m_B(r_1)$ and $m_B(r_2)$ give values of r_B on opposite side of r_B^e, then also the endpoints $m_B^c(s_A)$ when s_A varies in the one-parameter family give values on the opposite side of r_B^e. The continuity of $m_B^c(s_A)$ ensures that there is a strategy s_A^e that randomizes r_A between r_1 and r_2 such that $m_B^c(s_A^e) = r_B$. There is a corresponding argument if $m_A(r_B)$ intersects a straight line in M_B.

The continuity of $m^c_A(s_B)$ and $m^c_B(s_A)$ implies that it is not necessary that both players randomize at the same time. If M_A and M_B intersect with two straight lines, then there may be two Nash equilibria defined by using $m^c_A(s_B)$ and $m^c_B(s_A)$, respectively. □

Proof of Lemma D.1. Let $t_i = it/n$ and $Z(0,\omega) = 1$. Then

$$\begin{aligned} E\{1 - Z^{\eta_\nu}(t)\exp(-\beta t)\} &= E\{\sum_{i=0}^{n} \big(Z^{\eta_\nu}(t_i)\exp(-\beta t_i) - Z^{\eta_\nu}(t_{i+1})\exp(-\beta t_{i+1})\big)\} \\ &= E\{\sum_{i=0}^{n} Z^{\eta_\nu}(t_i)\exp(-\beta t_i)(1 - \frac{Z^{\eta_\nu}(t_{i+1})}{Z^{\eta_\nu}(t_i)}\exp(-\beta(t_{i+1} - t_i))\} \\ &= \frac{1}{t_1}E\{1 - Z^{\eta_\nu}(t_1)\exp(-\beta t_1)\}E\{\sum_{i=0}^{n} Z^{\eta_\nu}(t_i)\exp(-\beta t_i)t_1\}. \end{aligned}$$

We have that

$$\begin{aligned} \lim_{t\to 0} E\{\frac{1 - Z^{\eta_\nu}(t)\exp(-\beta t)}{t}\} &= \lim_{t\to 0} \frac{1 - E\{Z^{\eta_\nu}(t)\}\exp(-\beta t)}{t} \\ &= \lim_{t\to 0} \frac{1 - \exp(t(\mu_{\nu,z} - \beta))}{t} \\ &= \beta - \mu_{\nu,z} \end{aligned}$$

and

$$\lim_{n\to\infty} E\{\sum_{i=0}^{n} Z^{\eta_\nu}(t_i)\exp(-\beta t_i)t_1\} = E\{\int_0^t Z^{\eta_\nu}(s)\exp(-\beta s)ds\}.$$

Combining these three calculations proves the lemma. □

Proof of Lemma D.2. The assumption that if there are jumps in $Q(t,\omega)$ and $Z(t,\omega)$, then the distribution for the jumps is from a continuous deceasing distribution implies that the distribution $R(t,\omega)/Z(t,\omega)$ is a continuous distribution with one mode. Then also the conditional distribution for $R(t,\omega)/Z(t,\omega)$ conditioned on $t < T$ also is a continuous distribution with one mode. This implies that $E\{T(r_A, r_B)$ is differentiable. When $E\{T(r_A, r_B)$ is differentiable it follows directly that $F_\nu(r_1, r_A, r_B)$ is differentiable.

We will first prove that $F_A(r_1, r_A, r_B)$ is increasing in r_1. Let R_1, Z_1 and R_2, Z_2 be two different independent realizations of $R(t,\omega)$ and $Z(t,\omega)$ where $R_1(0,\omega)/Z_1(0,\omega) = r_1$ and $R_2(0,\omega)/Z_2(,\omega) = r_2$ and $r_2 < r_1$. The Markov property implies that

$$P(R_1(t,\omega)/Z_1(t,\omega) < r) < P(R_2(t_1,\omega)/Z_2(t_1,\omega) < r)$$

for $t > 0$. The limitation on the distribution for the jumps in $Z(t,\omega)$ and $Q(t,\omega)$ implies that this property also is satisfied if we condition on $r_A < R_i(t,\omega)/Z_i(t,\omega) < r_B$, i.e.

$$P(R_1(t,\omega)/Z_1(t_1,\omega) < r|t < T_1) < P(R_2(t,\omega)/Z_2(t,\omega) < r|t_2 < T_2)$$

where T_2 is defined similarly as T_1. Since small jumps are more likely than large, we do not have the possibility that is illustrated in Example 5.1. Define $\tilde{r}_i$ by the equation

$$E\{\int_0^{T_i} R_i^{\eta_\nu}(s)\exp(-\beta s)ds\} = \tilde{r}_i^{\eta_\nu} E\{\int_0^{T_i} Z_i^{\eta_\nu}(s)\exp(-\beta s)ds\}$$

for $i = 1, 2$. Then $\tilde{r}_2 < \tilde{r}_1$. This implies that $F_A(r_2, r_A, r_B) < F_A(r_1, r_A, r_B)$. It is similarly for $r_2 > r_1$ implying that $F_A(r_1, r_A, r_B)$ is increasing in r_1.

Then we will prove that $F_A(r_1, r_A, r_B)$ is increasing in r_A. Let $0 < r_s < r_A$ and define $T_s = T(r_s, r_B)$. Then

$$F_A(r_1 r_s, r_B) = \frac{E\{\int_0^T R^{\eta_A}(s)\exp(-\beta s)ds\} + E\{\int_T^{T_s} R^{\eta_A}(s)\exp(-\beta s)ds\}}{E\{\int_0^T Z^{\eta_A}(s)\exp(-\beta s)ds\}E\{\int_T^{T_s} Z^{\eta_A}(s)\exp(-\beta s)ds\}}.$$

The interval (T, T_s) is either empty or $R(T,\omega)/Z(T,\Omega) \leq r_A$. Since $F_A(r_1, r_A, r_B)$ is increasing in r_1 we have

$$F_A(r_A, r_s, r_B) = \frac{E\{\int_{T_1}^{T_s} R^{\eta_A}(s)\exp(-\beta s)ds\}}{E\{\int_{T_1}^{T_s} Z^{\eta_A}(s)\exp(-\beta s)ds\}} < F_A(r_A, r_s, r_B)$$

This implies that $F_A(r_1, r_A, r_B)$ is increasing in r_A. It is proved similarly that $F_A(r_1, r_A, r_B)$ is increasing in r_A and the corresponding properties for $F_B(r_1, r_A, r_B)$. Note that $\eta_A > 0 > \eta_B$. $\square$

Proof of Lemma D.3. We have

$$v_\nu(r_A, r_B) = \lim_{r_1 \to r_\nu} r_1^{\eta_\nu} F_\nu(r_A/r_1, r_B/r_1).$$

Lemma D.2 states that F_ν is differentiable and $F_A(r_1, r_A, r_B)$ increases in r_A and r_B while $F_B(r_1, r_A, r_B)$ decreases in r_A and r_B. Hence also v_ν is differentiable.

It is left to prove

$$\frac{\partial v_A}{\partial r_B} < (\beta - \mu_{A,z})\frac{\partial u_A}{\partial r_B}$$

and the corresponding result for v_B. Using equation (23) it is sufficient to prove that

$$\frac{\partial v_A}{\partial r_B} = \frac{\partial}{\partial(r_B/r_1)} \lim_{r_1 \to r_A} r_1^{\eta_A} F_A(r_A/r_1, r_B/r_1) \leq \frac{\partial F_A(1, r_A, r_B)}{\partial r_B}.$$

The above result follows from

$$\frac{\partial F_A(r_1, r_A, r_B)}{\partial r_B} \leq \frac{\partial F_A(1, r_A, r_B)}{\partial r_B} \tag{25}$$

for $r_1 < 1$. We have $R(0,\omega)/Z(0,\omega) = r_1$ and $R(s,\omega)/Z(s,\omega) \in (r_A, r_B)$ for $s < T_1$. We may split in two cases; that there exists $s < T_1$ such that $R(s,\omega)/Z(s,\omega) \geq r'$ for $s < T_1$ or $R^{\eta_\nu}(s,\omega)/Z^{\eta_\nu}(s,\omega) < r'$ for all $s < T_1$. Let P' denote the probability for the first case, and let $T' < T_1$ denote the corresponding time if this happens. We define r' such that

$$E\{\int_{T'}^{T_1} R^{\eta_\nu}(s)\exp(-\beta s)ds\} = E\{\int_0^T R^{\eta_\nu}(s)\exp(-\beta s)ds\} \tag{26}$$

with T defined by (4). If R/Z is continuous then $r' = 1$. If R/Z has jumps, then $r' < 1$ such that the $E\{R(T')\}/E\{Z(T')\}$ is close to 1. Equation (26) implies

$$E\{\int_0^{T_1} R^{\eta_\nu}(s)\exp(-\beta s)ds\} = E\{\int_0^{T'} R^{\eta_\nu}(s)\exp(-\beta s)ds\} + P'E\{\int_{T'}^{T_1} R^{\eta_\nu}(s)\exp(-\beta s)ds\}.$$

Correspondingly, we have

$$E\{\int_0^{T_1} Z^{\eta_\nu}(s)\exp(-\beta s)ds\} = E\{\int_0^{T'} Z^{\eta_\nu}(s)\exp(-\beta s)ds\} + P'E\{\int_{T'}^{T_1} Z^{\eta_\nu}(s)\exp(-\beta s)ds\}.$$

We also have

$$E\{\int_0^{T'} R^{\eta_\nu}(s)\exp(-\beta s)ds\} = \tilde{r}^{\eta_\nu} E\{\int_0^{T'} Z^{\eta_\nu}(s)\exp(-\beta s)ds\}$$

where $r_1 < \tilde{r} < 1$ since $r_1 < R(s,\omega)/Z(s,\omega) < 1$ in these integrals. Then

$$\begin{aligned}
F_\nu(r_1, r_A, r_B) &= \frac{E\{\int_0^{T'} R^{\eta_\nu}(s)\exp(-\beta s)ds\} + P'E\{\int_{T'}^{T_1} R^{\eta_\nu}(s)\exp(-\beta s)ds\}}{E\{\int_0^{T'} Z^{\eta_\nu}(s)\exp(-\beta s)ds\} + P'E\{\int_{T'}^{T_1} Z^{\eta_\nu}(s)\exp(-\beta s)ds\}} \\
&= \frac{\tilde{r}^{\eta_\nu} E\{\int_0^{T'} Z^{\eta_\nu}(s)\exp(-\beta s)ds\} + P'E\{\int_{T'}^{T_1} R^{\eta_\nu}(s)\exp(-\beta s)ds\}}{E\{\int_0^{T'} Z^{\eta_\nu}(s)\exp(-\beta s)ds\} + P'E\{\int_{T'}^{T_1} Z^{\eta_\nu}(s)\exp(-\beta s)ds\}} \\
&= \frac{\tilde{r}^{\eta_\nu} E\{\int_0^{T'} Z^{\eta_\nu}(s)\exp(-\beta s)ds\} + P'F_\nu(1, r_A, r_B)E\{\int_{T'}^{T_1} Z^{\eta_\nu}(s)\exp(-\beta s)ds\}}{E\{\int_0^{T'} Z^{\eta_\nu}(s)\exp(-\beta s)ds\} + P'E\{\int_{T'}^{T_1} Z^{\eta_\nu}(s)\exp(-\beta s)ds\}} \\
&= \tilde{r}^{\eta_\nu} d + F_\nu(1, r_A, r_B)(1-d)
\end{aligned}$$

for $0 < d < 1$. This implies equation (25). We prove

$$\frac{\partial v_B}{\partial r_A} < (\beta - \mu_{B,z})\frac{\partial u_B}{\partial r_A}$$

correspondingly.

□

References

[1] P. Aghion, M. Dewatripont, and P. Rey. Renegotiation design with unverifiable information. *Econometrica* **62**, (1994) 257–282.
[2] G. Akerlof, W. Dickens, and W. Perry. The macroeconomics of low inflation. *Brookings Papers on Economic Activity*, (1996) 1–76.
[3] T. Andersen and M. Stampe Christensen. Contract renegotiation under uncertainty. Working paper No 1997-5, Dep. of Economics, School of Economics and Management, University of Aarhus.
[4] T. Andersen and M. Stampe Christensen. Contract renewal under uncertainty. *J. Economic Dynamics & Control* **26**, (2002) 637–652.
[5] H. Bakhshi, H. Kahn, and B. Rudolf. The Phillips curve under state-dependent pricing. Working paper 227, Bank of England (2004).
[6] O. Bandiera. Contract duration and investment incentives. Evidence from land tenancy agreements. *J. European Economic Assocititation* **5**, (2007) 693-986.
[7] T. Bewley. *Why Wages Do Not Fall During a Recession.* Harvard University Press, Boston (1999).
[8] A. N. Borodin and P. Salminen. *Handbook of Brownian Motion — Facts and Formulae.* Birkhäuser, Basel (1996).
[9] A. Caplin and J. Leahy. Aggregation and optimization with state-dependent pricing. *Econometrica* **65**, (1997) 601–625.
[10] A. Caplin and D. F. Spulber. Menu costs and the neutrality of money. *Quarterly Journal of Economics* **CII**, (1987) 703–725.
[11] L. Danziger. Price adjustment with stochastic inflation. *International Economic Review* **24**, (1983) 699–707.
[12] L. Danziger. Contract reopeners. *Journal of Labor Economics* **13**, (1995) 62–87.
[13] L. Danziger. Output and welfare effects of inflation with costly price and quantity adjustments. *American Economic Review* **91**, (2001) 1608–1620.
[14] L. Danziger. Delays in renewal on labor contracts. Theory and evidence. IZA Discussion Paper 709, (2003).
[15] L. Danziger. Output effects of inflation with fixed price and quantity adjustment costs. *Economic Inquiry*, to appear.
[16] E. Dockner, S. Jørgensen, N. Van Long, and G. Sorger. *Differential Games in Economics and Management Science.* Cambridge University Press, Cambridge (2000).
[17] M. Dotsey, R. G. King, and A.L. Wolman. State-dependent pricing and the general equilibrium dynamics of money and output. *Quarterly Journal of Economics* **114**, (1999) 655–690.
[18] C. J. Erceg, D. W. Henderson, and A. T. Levin. Optimal monetary policy with staggered wage and price contracts. *Journal of Monetary Economics* **46**, (2000) 281–313.
[19] R. Fernandez and J. Glazer. Striking for a bargain between two completely informed agents. *American Economic Review* **81**, (1991) 240–252.
[20] M. Gertler and J. Leahy A Phillips curve with an Ss foundation. NBER Working Paper 11971, (2006).
[21] P. Grout. Investment and wages in the absence of binding contracts. A Nash bargaining approach. *Econometrica* **52**, (1984) 449–460.
[22] H. Haller and S. Holden. A letter to the Editor on wage bargaining. *Journal of Economic Theory* **52**, (1990) 232–236.
[23] H. Haller and S. Holden. Ratification requirement and bargaining power. *International Economic Review* **38**, (1998) 825–851.
[24] O. Hart and J. Moore. Incomplete contracts and renegotiation. *Econometrica* **56**, (1988) 755–785.
[25] H. Holden, L. Holden. and S. Holden. Contract adjustment under uncertainty. *CESifo Working paper* **1472**, (May 2005).
[26] S. Holden. Wage bargaining and nominal rigidities. *European Economic Review* **38**, (1994) 1021–1039.
[27] S. Holden. Wage bargaining, holdouts and inflation. *Oxford Economic Papers* **49**, (1997) 235–255.
[28] S. Holden. Renegotiation and the efficiency of investments. *Rand Journal of Economics* **30**, (1999) 106–119.
[29] R. Isaacs. *Differential Games.* Wiley, New York (1965).
[30] D. E. Lebow, R. E. Saks, and B. A. Wilson. Downward nominal wage rigidity: Evidence from the employment cost index. *Advances in Macroeconomics* **3**, (2003), Issue 1, Article 2.
[31] W. B. MacLeod, and J. M. Malcomson. Investment, holdup, and the form of market contracts. *American Economic Review* **37**, (1993) 343–354.
[32] J. M. Malcomson. Contracts, holdup and labor markets. *Journal of Economic Literature* **35**, (1997) 1916–1957.
[33] K.-I. Sato. *Lévy Processes and Infinitely Divisible Distributions.* Cambridge University Press, Cambridge (1999).

[34] E. Sheshinski and Y. Weiss. Inflation and costs of price adjustment. *Review of Economic Studies* **50**, (1983) 513–529.

(H. Holden) Department of Mathematical Sciences, Norwegian University of Science and Technology, NO–7491 Trondheim, Norway, and
Centre of Mathematics for Applications, University of Oslo, P.O. Box 1053, Blindern, NO–0316 Oslo, Norway

E-mail address: holden@math.ntnu.no
URL: www.math.ntnu.no/~holden/

(L. Holden) Norwegian Computing Center, P.O. Box 114 Blindern, NO–0314 Oslo, Norway
E-mail address: lars.holden@nr.no
URL: www.nr.no/~holden/

(S. Holden) Department of Economics, University of Oslo, P.O. Box 1095 Blindern, NO–0317 Oslo, Norway
E-mail address: steinar.holden@econ.uio.no
URL: http://folk.uio.no/sholden/