A New Class of Well-Balanced Finite Volume schemes for Conservation laws with source terms

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Joint Work with:

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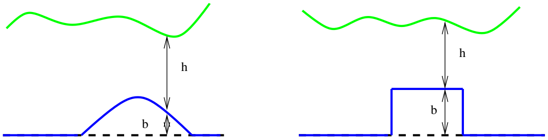
Basic Equations

$$U_t + (f(U))_x + (g(U))_y + (h(U))_z = S(x, U)$$

- System of Conservation laws in multi-D.
- Together with source terms.
- Source can be spatially dependent (maybe singular).
- Also termed Balance laws.

Flow on a non-trivial topography



Non−Trivial Smooth Bottom Topography

Discontinuous Bottom Topography

An Example

- Shallow water equations with Non-trivial Bottom Topography.

$$
\begin{aligned}
h_t + (hu)_x + (hv)_y &= 0 \\
(hu)_t + (hu^2 + \tfrac{1}{2}gh^2)_x + (huv)_y &= -ghb_x \\
(hv)_t + (huv)_x + (hv^2 + \tfrac{1}{2}gh^2)_y &= -ghb_y
\end{aligned}
$$

  - h is height of the free surface.
  - (u, v) is the velocity vector.
  - g - gravity constant.
  - b Topography function (can be discontinuous).

The Model Equation

- **Single conservation law in 1-d.**

$$u_t + f(u)_x = A(x, u)$$

- Unknown u, flux f and source A.
- Source can even be singular. (A can be a measure)

Special Cases

- *Autonomous source*

$$u_t + f(u)_x = g(u)$$

- *Scalar "Shallow Water" equations*

$$u_t + (f(u))_x = z'(x)b(u)$$

  - z is the topography function (possibly discontinuous)

- *Singular Sources*

$$u_t + (f(u))_x = z'(x)$$

- z Heaviside funtion $\Rightarrow$ RHS is a measure.

Weak Solutions

- Well Defined when $A(x,u) \in L^{\infty}$.

- $u \in L^{\infty}(\mathbb{R} \times \mathbb{R}_{+}) \cap L^{1}_{\mathrm{loc}}$ is a weak solution if for all test functions φ,

$$\int_{\mathbb{R}^{+}} \int_{\mathbb{R}} u\varphi_t + f(u)\varphi_x + A(x,u)\varphi \, dxdt + \int_{\mathbb{R}} u(x,0)\varphi(x,0) = 0 \quad (1)$$

- Special attention when $A \notin L^{\infty}$.
- Make sense of the non-conservative product

$$z'(x)b(u)$$

Entropy Solutions

- Well Defined when $A(x,u) \in L^{\infty}$.

- $u \in L^{\infty}(\mathbb{R} \times \mathbb{R}_{+}) \cap L^{1}_{\text{loc}}$ is a entropy solution if for all test functions $\varphi \geq 0$,

$$\int_{\mathbb{R}^{+}} \int_{\mathbb{R}} S(u)\varphi_t + Q(u)\varphi_x + S'(u)A(x,u)\varphi \, dxdt + \int_{\mathbb{R}} u(x,0)\varphi(x,0) \geq 0$$

- For any entropy-entropy flux pair (S, Q).
- Entropy solutions exist and are unique when $A \in L^{\infty}$.
- No general theory in the singular case except when $A(x,u) = z'(x)$

A Naive Numerical Scheme

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- Explicit Euler in Time.
- Godunov type numerical fluxes for the flux.
- Central differences for the source.

A Numerical Experiment

$$u_t + f(u)_x = z'(x)b(u)$$

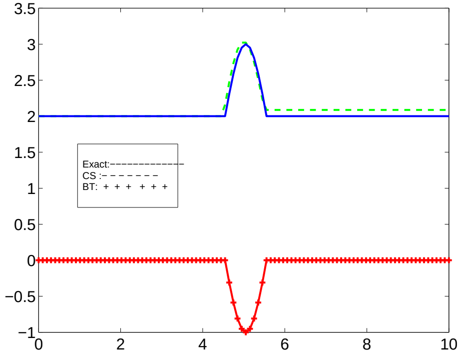
- With

$$\begin{aligned} f(u) &= \tfrac{1}{2}u^2 & b(u) &= u \\ -z(x) &= \begin{cases} \cos(\pi x) & \text{if } 4.5 < x < 5.5 \\ 0 & \text{Otherwise} \end{cases} \\ u(t,0) &= 2 & u(0,x) &= 0 \end{aligned}$$

- Explicit steady state is given by

$$\overline{u}(x) = 2 + z(x)$$

At the steady state



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- $$f(u)_x \approx A(x, u)$$
- Numerical schemes have to preserve Flux-Source balance.
- Centered Source/Operator splitting doesn't respect it.
- Search for better schemes

- Greenberg, Leroux.
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- Gosse, Leroux.
- Botchorischvili, Perthame and Vasseur. (BPV)
- Bermudez, Vasquez
- Perthame, Bouchut, Bristeau, Klien, Audusse.
- Russo, Noelle, Kurganov, Levy and many others.

WBS (condensed)

- Consider Scalar Shallow water equations,

$$u_t + f(u)_x = z'(x)b(u)$$

- The steady state is formally,

$$\begin{array}{rrcl} & f(u)_x & = & z'(x)b(u) \\ \Rightarrow & f'(u)u_x & = & z'(x)b(u) \\ \Rightarrow & \frac{f'(u)}{b(u)}u_x & = & z'(x) \\ \Rightarrow & D(u)_x & = & z'(x) \\ & D(u) & = & \int^u \frac{f'(s)}{b(s)}ds \end{array}$$

- Steady State evaluated from

$$D - z = \text{Constant}$$

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- At nth time step let v_j^n be the cell-averages and z_j be averages of the topography, then define "local" steady states solving

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\begin{aligned}
D(v_j^n-) - z_j &= D(v_{j-1}^n) - z_{j-1} \\
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- Use the local steady states to define a Godonov type scheme with update
- $$v_j^{n+1} = v_j^n - \frac{\Delta t}{\Delta x}(F(v_j^n, v_j^n+) - F(v_j^n-, v_j^n))$$

• with F being Standard (Godunov, Enquist-Osher) flux corresponding to f

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- Shown to Converge to entropy solutions (via Kinetic formulation).
- Basis for WBS for systems.

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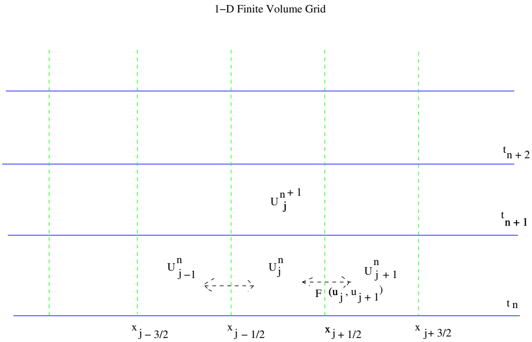
- Specialized: Difficult to extend when source is not in product form.
- Non-entropic: In some cases with discontinuous z.
- Possible loss of accuracy away from steady states.
- Subtle deficiences (see sequel)

Along the lines of

- Greenberg, Leroux, Baraille and Noussair. (Singular Sources)
- Noussair.
- LeVeque.
- Bale, LeVeque, Mitran, Rossmanith (Flux - Differencing)
- Adimurthi, Gowda, Mishra (Singular Sources)


1–D Finite Volume Grid
t n + 2
t n + 1
t n
U j n + 1
U j –1 n
U j n
U j + 1 n
F (u j , u j + 1)
x j – 3/2
x j – 1/2
x j + 1/2
x j+ 3/2

The Scheme: Design

- At time level n, let u_j^n be the cell averages,
  • Step 1: Freeze the source at t^n and define the piecewise constant

$$u^n(x) = \sum_j u_j^n \mathbf{1}_{\{I_j\}}(x)$$

with I_j being the *jth* cell. Formally ("local" in time) we have the equation

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  $$u_t + (f(u))_x = A(x, u^n(x))$$
- Primitive Reconstruction: Define the function
  $$\tilde{B}^n(x) = \int^x A(y, u^n(y))dy$$
  • We obtain the following discontinuous flux problem,

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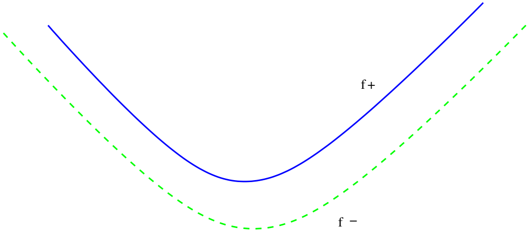
- Local Riemann problems at each interface

$$\begin{aligned} u_t + (f(u) - B^n_j)_x &= 0 \quad u(x,0) = u^n_j \quad x < x_{j+1/2} \\ u_t + (f(u) - B^n_{j+1})_x &= 0 \quad u(x,0) = u^n_{j+1} \quad x > x_{j+1/2} \end{aligned}$$

Shape of Adjacent fluxes



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  - with $F_{j+1/2}$ being the corresponding Godunov flux.
- Explicit formulas are available in most cases e.g (f convex) then

$$F_{j+1/2} = \max(f(\max(u_j, \theta) - B_j^n, f(\min(u_{j+1}, \theta)) - B_{j+1}^n))$$

Scheme: Properties

- Discrete steady states of the scheme

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  - Reflects Flux-Source balance.
- Rankine-Hugoniot Conditions + Jump entropy conditions $\Rightarrow$ *Entropic* Discrete steady states are preserved .
- Flexibility in the averaging steps to obtain equivalent discrete steady states.

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- Blow-up estimates on BV-norm $\Rightarrow$ Compactness of Approximations (Compensated Compactness).
- Jump entropy conditions + Structure of the scheme $\Rightarrow$ Convergence to entropy solutions if $A \in L^\infty$

Experiment 1 (Continuous Bottom)

$$u_t + f(u)_x = z'(x)b(u)$$

- With

$$f(u) = \tfrac{1}{2}u^2 \qquad b(u) = u$$

$$-z(x) = \begin{cases} \cos(\pi x) & \text{if } 4.5 < x < 5.5 \\ 0 & \text{if Otherwise} \end{cases}$$

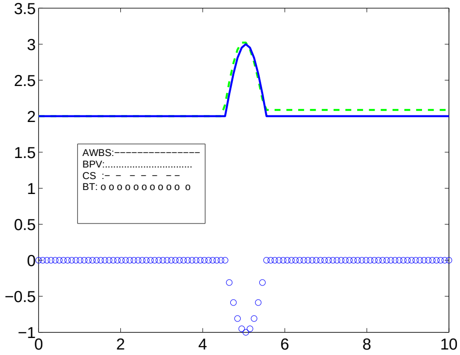
$$u(t,0) = 2 \qquad u(0,x) = 0$$

- Explicit steady state is given by

$$\overline{u}(x) = 2 + z(x)$$

- Comparision of Central Sources (CS), Existing Well-Balanced Scheme (BPV) and New Well-Balanced Scheme (AWBS)

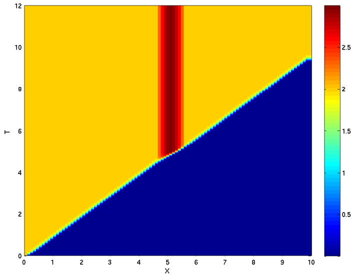
At Steady State: $\Delta x = 0.1$



Errors at Steady State

	L^{∞}	L^{1}
CS	0.1652	0.4824
AWBS	4.37×10^{-14}	2.22×10^{-13}
BPV	8.45×10^{-14}	2.26×10^{-13}

Transients: $\Delta x = 0.1$



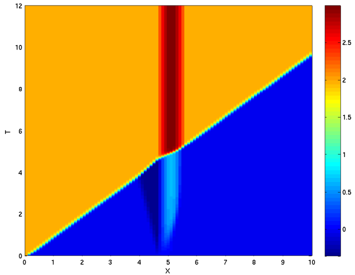
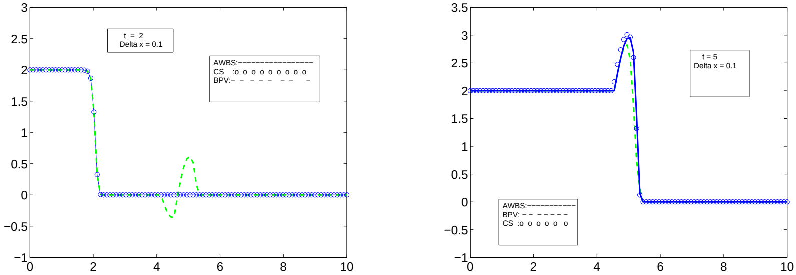
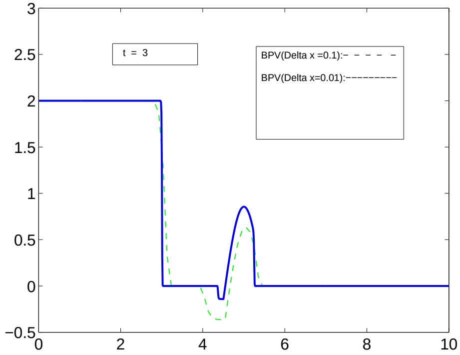


Figure: Left:AWBS, Right:BPV

Transient snapshots



BPV at high resolution



Experiment 2 (Discontinuous Bottom)

$$u_t + f(u)_x = z'(x)b(u)$$

- With

$$\begin{aligned} f(u) &= \tfrac{1}{2}u^2 & b(u) &= u \\ -z(x) &= \begin{cases} \cos(\pi x) & \text{if} \quad 5 < x < 6 \\ 0 & \text{if} \quad \text{Otherwise} \end{cases} & & \\ u(t,0) &= 2 & u(0,x) &= 0 \end{aligned}$$

- Explicit steady state is given by

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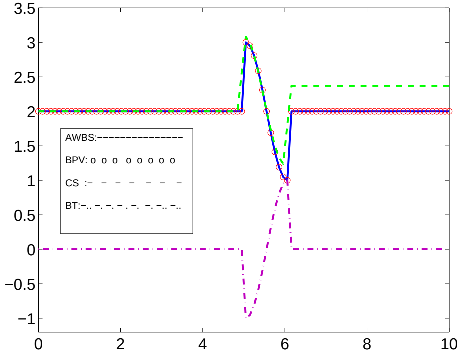
At Steady State: $\Delta x = 0.1$

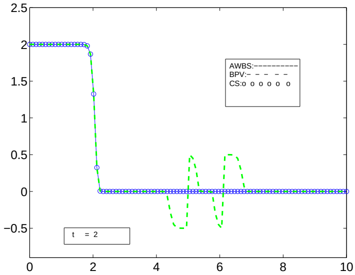


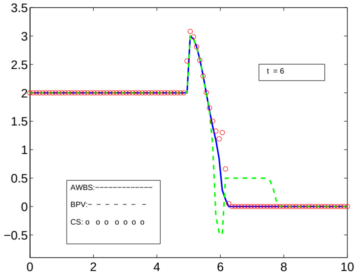
Errors at Steady State

	L^∞	L^1
CS1	0.8027	1.6449
AWBS	1.87×10^{-12}	8.12×10^{-13}
BPV	2.53×10^{-9}	6.34×10^{-10}

Table: Errors at the steady state for the three schemes with $\Delta x = 0.1$ in Experiment 2

A Transient snapshot





Experiment 3 (Another Discontinuous Bottom)

$$u_t + f(u)_x = z'(x)b(u)$$

- With

$$f(u) = \frac{1}{2}u^2 \qquad b(u) = u$$

$$-z(x) = \begin{cases} -\cos(\pi x) & \text{if} \quad 5 < x < 6 \\ 0 & \text{if} \quad \text{Otherwise} \end{cases}$$

$$u(t,0) = 2 \qquad u(0,x) = 0$$

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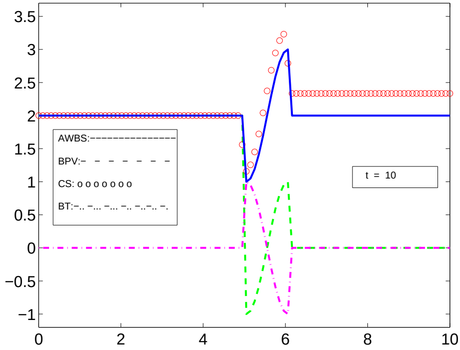
At Steady State: $\Delta x = 0.1$



Transients

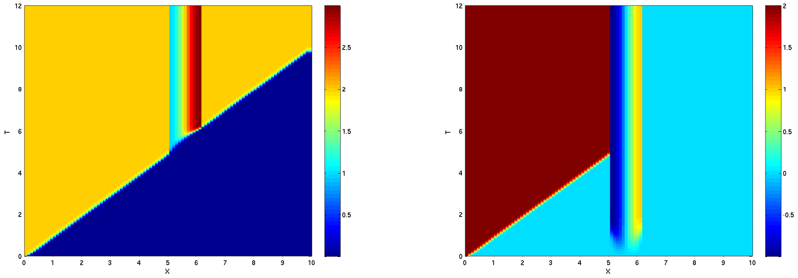


Figure: Left:AWBS, Right:BPV

Experiment 4 (Non-Monotone D)

$$u_t + f(u)_x = z'(x)b(u)$$

- With

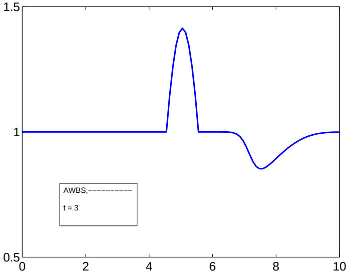
$$f(u) = \frac{1}{3}u^3 \qquad b(u) = u$$

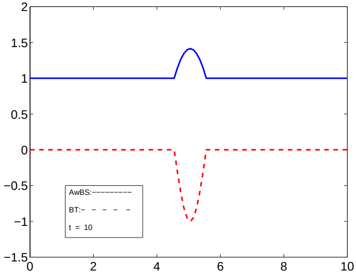
$$-z(x) = \begin{cases} \cos(\pi x) & \text{if} \quad 4.5 < x < 5.5 \\ 0 & \text{if} \quad \text{Otherwise} \end{cases}$$

$$u(t,0) = 2 \qquad u(0,x) = 0$$

- Difficult to define BPV as $D = u^2$ is not monotone. No problems with AWBS

Numerical Results at : $\Delta x = 0.1$





Experiment 5 (Source not in Product form)

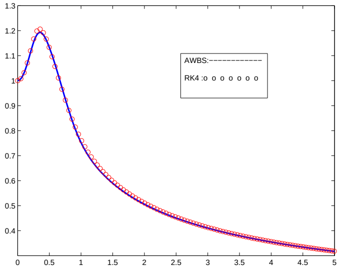
$$u_t + f(u)_x = A(x, u)$$

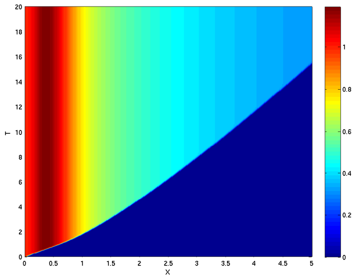
- With

$$\begin{aligned} f(u) &= \tfrac{1}{2}u^2 \qquad & A(x,u) &= \sin(2\pi x u^2) \\ u(t,0) &= 1 & u(0,x) &= 0 \end{aligned}$$

- Unclear how to define BPV in this case (other than using ODE solvers at each mesh point) whereas AWBS is well-defined

Numerical results with : $\Delta x = 0.1$





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- Problems at resonance $u = 0$

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  - Robust and proved to be convergent.
  - Very General: Work with different type of fluxes and sources.
  - Tailormade for discontinuous and singular sources.

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- New class of Well-Balanced Schemes are
  - Very Simple to implement (Explicit formulas, No extra equations)
  - Robust and proved to be convergent.
  - Very General: Work with different type of fluxes and sources.
  - Tailormade for discontinuous and singular sources.
  - Numerically efficient at both transients and steady states.

Ongoing and Future Work

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- Discontinuous and Singular A.

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- Systems: Shallow Water, Euler.
- Stiff Source terms.