MAT2410: Exercise set 3

1. Let $f(z) = u(x, y) + iv(x, y)$ be a complex function defined for all $z = x + iy$. If f is entire, then show that $u_{xxx}(x, y)$ is continuous for all $(x, y) \in \mathbb{R}^2$.

   Soln: As f is entire, we know that f', f'' and f''' are also entire functions. Furthermore, as f' exists, it is given by

   $$f'(z) = u_x(x, y) + iv_x(x, y),$$

   using the Cauchy-Riemann equations. Similarly, as f'' exists, it is given by

   $$f''(z) = (f'(z))' = (u_x)_x + i(v_x)_x = u_{xx} + iv_{xx}.$$

   Reiterating this argument once more, we obtain that

   $$f'''(z) = (f''(z))' = (u_{xx})_x + i(v_{xx})_x = u_{xxx} + iv_{xxx}.$$

   Furthermore, analyticity of f'' also implies that u_{xxx} is continuous.

2. f is continuous in a domain D. Furthermore, for any two contours Γ_1 and Γ_2, lying inside D, that have the same initial and terminal points, we have that

   $$\int_{\Gamma_1} f(z)dz = \int_{\Gamma_2} f(z)dz.$$

   Then show that f is analytic in D.

   Soln: We know that the following statements are equivalent for a continuous f:

   (i.) f has an anti-derivative in D.

   (ii.) $\oint_C f(z)dz = 0$ for every loop $C \in D$.

   (iii.) for any two contours Γ_1 and Γ_2, lying inside D, that have the same initial and terminal points, we have that

   $$\int_{\Gamma_1} f(z)dz = \int_{\Gamma_2} f(z)dz.$$

   Since we are given (iii), we see that (i) also holds and f has an anti-derivative i.e, there exists a function F such that $F'(z) = f(z)$ for all $z \in D$. As f is continuous, this implies that F is analytic.

   For any analytic function, derivatives of any order exist and are continuous. Therefore $F'(z) = f(z)$ is analytic.

3. Prove using Taylor expansions, the DeMoivre formula:

$$e^{iz} = \cos(z) + i\sin(z),$$

in a neighborhood of the origin of the Complex plane.

Soln; As $e^z, \cos(z)$ and $\sin(z)$ are entire, Taylor expansions around origin clearly converge. A simple evaluation of the Taylor series around origin results in

$$\begin{aligned}
e^{iz} &= 1 + iz + \frac{i^2z^2}{2!} + \frac{i^3z^3}{3!} + \frac{i^4z^4}{4!} + \frac{i^5z^5}{5!} + \dots \\
&= 1 + iz - \frac{z^2}{2!} - i\frac{z^3}{3!} + \frac{z^4}{4!} + i\frac{z^5}{5!} + \dots \\
\cos(z) &= 1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \dots \\
\sin(z) &= z - \frac{z^3}{3!} + \frac{z^5}{5!} + \dots \\
\Rightarrow isin(z) &= iz - i\frac{z^3}{3!} + i\frac{z^5}{5!} + \dots
\end{aligned}$$

The above identities clearly show (by substitution) that

$$e^{iz} = \cos(z) + i\sin(z).$$

4. Using power series, find a function f that is analytic in a neighborhood of the origin and satisfies the ordinary differential equation:

$$\frac{d^2f(z)}{dz^2} + 16f(z) = 0,$$

and with initial conditions $f(0) = 0$ and $f'(0) = 4$.

Soln: As f is analytic in a neighborhood of the origin, the convergent Taylor expansion is given by

$$f(z) = \sum_{j=0}^{\infty} f^{(j)}(0)\frac{z^j}{j!}.$$

We know from the initial conditions that $f(0) = 0$ and $f'(0) = 4$. The differential equation says

$$f''(z) = -16f(z).$$

As f is analytic in a neighborhood of the origin, we can differentiate the above differential equation j times to conclude

$$f^{(j+2)}(z) = (-1)^j 4^{2j} f^j(z), \forall j = 0, 1, 2, \dots.$$

Substituting the initial conditions $f(0) = 0$ in the above, we conclude

$$f^{(2k)}(0) \equiv 0, \forall k = 0, 1, 2, \dots,$$

and

$$f^{(2k+1)}(0) = (-1)^{(2k+1)} 4^{(2k+1)}, \forall k = 0, 1, 2, \dots.$$

Therefore substituting both these identities in the Taylor expansion of f, we obtain that

$$f(z) = \sum_{k=0}^{\infty} (-1)^{(2k+1)} \frac{4z^{(2k+1)}}{(2k+1)!} = sin(4z).$$

Uniqueness follows from uniqueness of the Taylor expansion of an analytic function. The fact that $f(z) = sin(4z)$ satisfies both the differential equation and the initial conditions can be verified by direct calculation.

5. Find the Laurent series for the function,

$$f(z) = \frac{1}{z^2 + (i-1)z - i},$$

in the neighborhood $|z| > 1$.

Soln: We have

$$f(z) = \frac{1}{z^2 + (i-1)z - i} = \frac{1}{1+i} \left(\frac{1}{z-1} - \frac{1}{z+i} \right),$$

As $|z| > 1$, we have $\left|\frac{1}{z}\right| < 1$, therefore the Laurent expansion,

$$\frac{1}{z-1} = \frac{1}{z\left(1 - \frac{1}{z}\right)} = \frac{1}{z} \sum_{j=o}^{\infty} \frac{1}{z^j} = \sum_{j=o}^{\infty} \frac{1}{z^{j+1}},$$

converges. As $|z| > 1$, we have $\left|\frac{-i}{z}\right| < 1$, therefore the Laurent expansion,

$$\frac{1}{z+i} = \frac{1}{z\left(1 - \frac{-i}{z}\right)} = \frac{1}{z} \sum_{j=o}^{\infty} \frac{(-i)^j}{z^j} = \sum_{j=o}^{\infty} \frac{(-i)^j}{z^{j+1}},$$

also converges. combining the above expansions, we obtain a Laurent expansion for

$$\frac{1}{z^2 + (i-1)z - i} = \frac{1}{1+i} \left(\frac{1}{z-1} - \frac{1}{z+i} \right) = \frac{1}{1+i} \sum_{j=o}^{\infty} \frac{1 - (-i)^j}{z^{j+1}}.$$

6. Find the Laurent expansion of the function $f(z) = ze^{\frac{-1}{z}}$ near $z = 0$.

Soln: A Laurent expansion of the exponential around 0 yields,

$$\begin{aligned} e^w &= \sum_{j=0}^{\infty} \frac{w^j}{j!}, \\ \Rightarrow e^{-\frac{1}{z}} &= \sum_{j=0}^{\infty} \frac{(-1)^j}{z^j j!}, \\ \Rightarrow ze^{-\frac{1}{z}} &= z\sum_{j=0}^{\infty} \frac{(-1)^j}{z^j j!} = \sum_{j=0}^{\infty} \frac{(-1)^j}{z^{j-1} j!}. \\ &= z - 1 + \frac{1}{2z} - \frac{1}{6z^2} + \dots. \end{aligned}$$

7. Classify the singularities of the function $f(z) = \frac{\cot(z)}{z^2}$.

Soln: As

$$f(z) = \frac{\cot(z)}{z^2} = \frac{\cos(z)}{\sin(z)z^2}.$$

Therefore, the singularities of f coincide the zeros of z^2 and $\sin(z)$ i.e, $z = k\pi, \quad k = 0, \pm 1, \pm 2, \dots$. Note that $\cos(z) \neq 0$ at these points.

Hence, the function $\sin(z)$ has simple zeros at $z = k\pi, \quad k = 0, \pm 1, \pm 2, \dots$. Therefore, there exists a function $h_k(z)$ such that h_k is analytic in a neighborhood of $z = k\pi$, $h(k\pi) \neq 0$ and

$$sin(z) = h_k(z)(z - k\pi).$$

We differentiate between two cases:

– Case 1. $k = 0$, in this case,

$$f(z) = \frac{g_0(z)}{z^3}, \quad g_0(z) = \frac{\cos(z)}{h_0(z)}.$$

As $cos(0) \neq 0$ and $h_0(0) \neq 0$, we have that $g_0(0) \neq 0$. Furthermore, g_0 is clearly analytic. Hence, 0 is a pole of order 3 for f.

– Case 2. $k \neq 0$, in this case,

$$f(z) = \frac{g_k(z)}{z - k\pi}, \quad g_k(z) = \frac{\cos(z)}{h_k(z)z^2}.$$

As $cos(k\pi) \neq 0$ and $h_k(k\pi) \neq 0$, we have that $g_k(k\pi) \neq 0$. Furthermore, g_k is clearly analytic. Hence, $k\pi$ is a pole of order 1 for f at any $k = \pm 1, \pm 2, \ldots$.

8. Compute the contour integral:

$$\oint_\Gamma \frac{(z^3 + 4z^2 + 4z)}{z^3(z+2i)(z-8i)} dz,$$

where Γ is the contour $|z-1| = 4$ with counter-clockwise orientation .

Soln: We have,

$$\oint_\Gamma \frac{(z^3 + 4z^2 + 4z)}{z^3(z+2i)(z-8i)} dz = \oint_\Gamma \frac{(z+2)^2}{z^2(z+2i)(z-8i)} dz.$$

The singularities of the integrand inside $|z-1| = 4$ are 0 and $-2i$ (Note $8i$ lies outside the contour Γ). Furthermore, the singularity of the integrand at $z = 0$ is pole of order 2 and the singularity at $z = -2i$ is a simple pole. Therefore, we compute the residues as

$$\begin{aligned} Res(0) &= \lim_{z \to 0} \frac{d}{dz} \left(\frac{(z+2)^2}{(z+2i)(z-8i)} \right) \\ &= \lim_{z \to 0} \left(\frac{2(z+2)}{z^2 - 6iz + 16} - \frac{(2z-6i)(z+2)^2}{(z^2 - 6iz + 16)^2} \right) \\ &= \frac{1}{4} + \frac{3i}{32}. \\ Res(-2i) &= \lim_{z \to -2i} \frac{(z+2)^2}{(z)(z-8i)} = \frac{2i}{5}. \end{aligned}$$

By Cauchy Residue theorem,

$$\oint_\Gamma \frac{(z^3 + 4z^2 + 4z)}{z^3(z+2i)(z-8i)} dz = 2\pi i(Res(0) + Res(-2i)) = -\frac{79\pi}{80} + i\frac{\pi}{2}.$$

9. Compute the contour integral:

$$\oint_\Gamma \left(ze^{\frac{-1}{z}} + \frac{\cos(z)}{z^2 + (i-1)\pi z - i\pi^2} \right) dz,$$

where Γ is the contour $|z-1| = 3$ with clockwise orientation .

Soln. Clearly,

$$\oint_\Gamma \left(ze^{\frac{-1}{z}} + \frac{\cos(z)}{z^2 + (i-1)\pi z - i\pi^2} \right) dz = \oint_\Gamma ze^{\frac{-1}{z}} dz + \oint_\Gamma \frac{\cos(z)}{z^2 + (i-1)\pi z - i\pi^2} dz.$$

For $f_i(z) = ze^{\frac{-1}{z}}$, the singularity at $z = 0$ clearly lies inside the contour. Therefore, using the Laurent series computed in problem 6, we see that

$$Res(f_1, 0) = \frac{1}{2}.$$

Hence, by Cauchy Residue theorem (note the clock-wise orientation),

$$\oint_\Gamma ze^{\frac{-1}{z}} dz = -2\pi i Res(f_1, 0) = -\pi i.$$

Define

$$f_2(z) = \frac{\cos(z)}{z^2 + (i-1)\pi z - i\pi^2} = \frac{\cos(z)}{(z-\pi)(z+\pi i)}.$$

The singularities of f_2 are $z = \pi$ and $z = -\pi i$. We check that only $z = \pi$ lies inside Γ and is a simple pole. Therefore, the residue is

$$Res(f_2, \pi) = \lim_{z\to\pi} \frac{\cos(z)}{z - \pi i} = \frac{-1}{\pi(1+i)}.$$

Hence, by Cauchy Residue theorem,

$$\oint_\Gamma f_2(z)dz = -2\pi i Res(f_2, \pi) = \frac{2i}{1+i}.$$

Therefore,

$$\oint_\Gamma \left(ze^{\frac{-1}{z}} + \frac{\cos(z)}{z^2 + (i-1)\pi z - i\pi^2} \right) dz = \frac{2i}{1+i} - \pi i.$$

10. Prove that the equation

$$\frac{z^4}{40} + z^2 - 1 = 0,$$

has exactly two roots inside the circle $|z| = 2$.
Soln: Let $f(z) = z^2 - 1$. Clearly, f is entire and has exactly two simple zeros inside $|z| \le 2$, located at $z = 1$ and $z = -1$.

Define $h(z) = \frac{z^4}{40}$. Again, h is entire. On the circle $|z| = 2$, we have

$$|h(z)| = \left|\frac{z^4}{40}\right| = \frac{|z|^4}{40} = \frac{16}{40} = \frac{2}{5}.$$

Furthermore, on the circle,

$$|f(z)| = |z^2 - 1| = |z-1||z+1|.$$

clearly $|z-1| \geq |z|-1 \geq 2-1 \geq 1$. Similarly, we can show that on the circle, $|z+1| \geq 1$. Hence, on the circle $|z|=2$, we have

$$|f(z)| \geq 1 > \frac{2}{5} = |h(z)|.$$

Therefore, by Rouche's theorem, $f+h$ has exactly the same number of zeros as f. In other words, $\frac{z^4}{40}+z^2-1=0$, has exactly two roots inside $|z|=2$.