Wavelet
Representations of
Graph $\mathcal{C}^*$-algebras

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Cand. Scient. thesis

21st March 2003
Introduction

The wavelet representations of the Cuntz algebras $\mathcal{O}_n$ on $L^2(\mathbb{T})$ have been defined, using the filter functions $m_1, \ldots, m_n \in L^2(\mathbb{T})$ from multiresolution wavelet analysis, by

$$S_i\psi(z) = m_i(z)\psi(z^n)$$

These representations are much discussed in the recent book by Bratteli and Jorgensen, see [1], and in earlier papers, see [2, 3]. It is proved in [2], using a key result from [3], that this representation has finite-dimensional commutant, and hence is the direct sum of finitely many irreducible representations, provided the filter functions are Fourier polynomials. In this thesis, a definition is given of wavelet representations of a graph $C^*$-algebra. The class of finite graphs to which the finite decomposition result can be extended for all polynomial filters is then precisely characterized.
Chapter 1

Preliminaries

We recall some facts about Cuntz algebras and graph algebras.

The Cuntz algebra $\mathcal{O}_n$ ($n \geq 2$) is a $C^*$-algebra generated by isometries $S_1, \ldots, S_n$ which satisfy the Cuntz relations, viz.

$$S_i^*S_j = \delta_{ij} 1, \quad \sum_{i=1}^n S_i S_i^* = 1.$$  

In the article [5] by J. Cuntz, where he introduces these algebras, he also shows that any such algebra is simple. Consequently, any two such algebras are isomorphic in a way that identifies the corresponding isometries.

A directed graph $E$ is a quadruple $E = (E^0, E^1, d, r)$ where $E^0$ and $E^1$ are sets and $d, r : E^1 \to E^0$ are maps. The elements of $E^0$ are called vertices, and those of $E^1$ edges. We call $r$ the range map and $d$ the domain map; an edge $e$ is said to begin at $d(e)$ and end at $r(e)$.

A path $\alpha$ in $E$ is a (finite or infinite) sequence of edges $(\alpha_1, \alpha_2, \ldots)$, such that $r(\alpha_i) = d(\alpha_{i+1})$ for all $i \geq 1$ for which $\alpha_{i+1}$ is defined. The set of all paths of length $n$ in $E$ is denoted $E^n$, and the set of all finite paths $E^*$. If $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a finite path, and $\beta$ is a path, the composition $\alpha \beta$ is a path if and only if $r(\alpha_n) = d(\beta_1)$. In any case the maps $r$ and $d$ are extended to such paths by $r(\alpha) = r(\alpha_n), d(\beta) = d(\beta_1)$, compositability becoming equivalent to $r(\alpha) = d(\beta)$. We shall denote the set of all cycles, i.e. elements $\alpha \in E^*$ with $r(\alpha) = d(\alpha)$, by $E^c$.

In what follows, we shall always assume that the graph $E$ is column-finite, i.e. the number of edges ending at a vertex $v$ is finite, and we shall call this number the incidence number of the vertex $v$, and denote it $n_v$. For an edge $e$, let $n_e = n_r(e)$. 

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If $p_v$ is a projection for each vertex $v$, and $s_e$ is a partial isometry for each edge $e$, all contained in a $C^*$-algebra $\mathcal{A}$, and

$$s_e^*s_e = p_{d(e)} \quad (e \in E^1), \quad \sum_{r(e)=v} s_e^*s_e = p_v \quad (v \in r(E^1)),$$

the family consisting of these projections and partial isometries is said to be a Cuntz-Krieger E-family.\(^1\)

We now, more or less following [8], prove the existence of a $C^*$-algebra $C^*(E)$, uniquely determined modulo $*$-isomorphism, which is generated by a Cuntz-Krieger E-family $\{s_e, p_v\}$, and which also enjoys the so-called universal property: Given any $C^*$-algebra $\mathcal{A}$ and a Cuntz-Krieger E-family $\{S_e, P_v\} \subset \mathcal{A}$, there exists a unique $*$-homomorphism $\Phi : C^*(E) \to \mathcal{A}$, such that

$$\Phi(s_e) = S_e, \quad \Phi(p_v) = P_v e \in E^1, \quad v \in E^0.$$

To this end, let $\mathcal{A}_E$ be the free $*$-algebra generated by $\{Q_v, F_v : v \in E^0, e \in E^1\}$, and let $I = \cap (\ker \pi)$, the intersection taken over all representations $\pi$ of $\mathcal{A}_E$ on Hilbert space $\mathcal{H}$ such that $\{\pi(Q_v), \pi(F_v) : v \in E^0, e \in E^1\}$ is a Cuntz-Krieger E-family. Let $*(E) = \mathcal{A}_E/I$

Define a real-valued function $||||$ on $*(E)$ by

$$||x|| = \sup\{||\pi(x)||_{B(\mathcal{H})} : \pi \text{ is a representation of } *(E)\}.$$ 

Then $||||$ is a $C^*$-norm on $*(E)$. Let $C^*(E)$ be the completion of $*(E)$ in this norm. Note that $C^*(E)$ is generated by the Cuntz-Krieger E-family $\{P_v := \overline{Q_v}, S_e := \overline{F_v} : v \in E^0, e \in E^1\}$ ( $\overline{\cdot}$ denoting quotient image), and has the universal property.

**Comment:** In [8] $C^*(E)$ is constructed by considering a $*$-algebra $\mathcal{Y}$, isomorphic to $\mathcal{A}_E/\mathcal{K}$, where

$$\mathcal{K} = \langle Q_w^* - Q_v, Q_vQ_w - \delta_{vw}Q_v, F_e^*F_e - Q_{d(e)}, \sum_{r(j)=v} F_jF_j^* - Q_v \rangle$$

($v, w$ of course ranging over all $E^0$, and likewise $e$ over $E^1$, and $\langle \cdots \rangle$ meaning the ideal generated by $\cdots$).

One then gives a seminorm on this algebra by

$$||a||_0 = \sup\{||\pi(a)|| : \pi \text{ is a nondegenerate representation of } \mathcal{Y}\}.$$ 

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\(^1\)There is a slight variation of convention in the literature on this point; some authors have the edge and the operator going in opposite directions, i.e. $S_e^*S_e = P_{r(e)}$. I have followed the convention used by Katsura in [7].
\( \mathcal{Y} \) is generated by a Cuntz-Krieger \( E \)-family \( \{p_v, s_e\} \), being the image under the quotient map \( A_E \to A_E/K \cong \mathcal{Y} \) of \( \{Q_v, F_e\} \). By letting

\[
B = \text{the completion of } \mathcal{Y}/\{b \in \mathcal{Y} : \|b\|_0 = 0\},
\]

one obtains a \( C^* \)-algebra, and \( B \cong C^*(E) \). Note however that the last quotient is generally not trivial. Indeed, if \( r^{-1}(v) = \{e, f\} \), \( s^*_v s_f \) will have to be zero in every representation \( \pi \) of \( \mathcal{Y} \); note for instance that \( (s^*_v s_f)^* (s^*_v s_f) = 0 \), since

\[
(s^*_v s_f)^* (s^*_v s_f) + (s^*_v s_f)^* (s^*_v s_f) = s^*_v (s_e s_v^* + s_f s_v^*) s_f
\]

\[
= s^*_v p_v s_f = s^*_v s_f = (s^*_f s_f)^* (s^*_f s_f).
\]

However, \( F^*_e F_f \notin K \); to see this, note that in the free \( * \)-algebra, all words (in the generators and their formal adjoints) are linearly independent. Moreover, an element in a free \( * \)-algebra is in the ideal generated by a finite list of elements such as the above if and only if it is in the linear span of all words containing one of the listed generators. Inspecting the listed generators for \( K \), we see that this cannot be the case for the word \( F^*_e F_f \).

The universal property implies that given a Hilbert space \( H \), the representations of \( C^*(E) \) on \( H \) are in canonical bijective correspondence with the Cuntz-Krieger \( E \)-families in \( B(H) \). In the case of a graph \( E \) with one vertex and \( n \) edges, \( C^*(E) \cong \mathcal{O}_n \), and in this case the correspondence is (a trivial extension of) the correspondence between nondegenerate representations of \( \mathcal{O}_n \) and families of isometries satisfying the Cuntz relations.
Chapter 2

Wavelet representations

Recall from [1] that, given functions $m_1, \ldots, m_n$, the operators defined on $L^2(\mathbb{T})$ by

$$S_i \psi(z) = m_i(z) \psi(z^n)$$

satisfy the Cuntz relations if and only if the identity

$$\sum_{\omega^n = z} m_i(\omega) \overline{m_j(\omega)} = \delta_{ij} n$$

(2.1)

holds for almost all $z \in \mathbb{T}$.

Let $H$ be the Hilbert space $L^2(\mathbb{T} \times E^0) = L^2(\mathbb{T} \times E^0, \mu \times #)$, # being the counting measure on $E^0$ and $\mu$ normalized Haar measure on $\mathbb{T}$. We shall occasionally suppress the canonical unitary map of this space onto the Hilbert space tensor product $L^2(\mathbb{T}) \otimes l^2(E^0)$.

Letting $H_v$ be the copy of $L^2(\mathbb{T})$ “sitting at $v$”, ie $H_v = \{ \psi \in H | \text{supp } \psi \subset \mathbb{T} \times \{v\} \}$, in what follows we will consider only those Cuntz-Krieger E-families $\{P_v, S_v\}$ for which $P_v$ is the orthogonal projection onto $H_v$.

For arbitrary vertices $v, w$, define an operator $D_{vw}$ in $B(H)$ by

$$(D_{vw} \psi)(z, u) = \delta_{uw} \psi(z, w)$$

for $\psi \in H$. We have the relations

$$D_{vw}^* = D_{wv},$$

$$D_{vw} D_{wz} = D_{vz},$$

and $D_{vw} = P_v$,

ie $D_{vw}$ is the canonical partial isometry going from $H_w$ to $H_v$. For any edge $e$, let $D_e = D_{r(e)} a(e)$. 

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2.1 Definition of the wavelet representations of $C^*(E)$

Assume we are given a family $\mathcal{R} = \{R_e | e \in E^1\}$ of isometries on $L^2(\mathbb{T})$ such that

$$\sum_{r(e)=v} R_e R_e^* = 1$$

for all $v \in r(E^1)$. A Cuntz-Krieger $E$-family in $B(H)$ is then given by

$$S_e = D_e (R_e \otimes 1) .$$

The family $\mathcal{R}$ is nothing but a family of isometries, indexed by the edges $E^1$, such that those isometries corresponding to the all of the edges that end at a specific vertex $v$ satisfy the Cuntz relations of order $n_v$. \(^1\) In order to produce such a family, it suffices then to produce a family $\mathcal{M} = \{m_e | e \in E^1\}$ of functions in $L^\infty(\mathbb{T})$ such that for any edges $e, f$ with $r(e) = r(f) = v$, the equality

$$\sum_{\omega n_v = z} m_e(\omega) \overline{m_f(\omega)} = \delta_{e,f} n_v$$

is valid for almost all $z \in \mathbb{T}$.

The resulting representation of $C^*(E)$ is explicitly given by

$$S_e \psi(z, v) = \delta_{r(e), v} \psi(z, d(e)) .$$

\(^1\)By the Cuntz relations of order 1 we shall mean: $S^*S = SS^* = 1$. Denote by $\mathcal{O}_1$ the $C^*$-algebra universal with respect to this relation, which is $^{*}$-isomorphic to $C(\mathbb{T})$. 
2.2 Polynomial representations

We first prove a “dilation theorem” for representations of graph algebras, strongly inspired by the proof of ([3], Theorem 5.1).

2.2.1 Theorem

Let $E$ be a finite\(^2\) directed graph. If $\mathcal{K}$ is a Hilbert space, and \{${V_e, Q_v : v \in E^0, e \in E^1}$\} $\subset B(\mathcal{K})$, where $Q_v$ are mutually orthogonal projections, such that

$$\sum_{\tau(e)=v} V_e V_e^* = Q_v$$  \hspace{1cm} (2.2)

for all $v \in E^0$, and

$$\bigvee_{v \in E^0} Q_v = 1_{\mathcal{K}},$$  \hspace{1cm} (2.3)

then there is a Hilbert space $\mathcal{H}$, an injection of $\mathcal{K}$ into $\mathcal{H}$, and a representation $\pi : s_e \mapsto S_e \in B(\mathcal{H})$ of $C^*(E)$, such that $\mathcal{K}$ is cyclic for $\pi$, and such that

$$Q_v = P_v|_{\mathcal{K}}, V_e^* = S_e^*|_{\mathcal{K}}.$$  \hspace{1cm} (2.4)

The system $(\mathcal{H}, \pi, \iota)$ is unique up to unitary equivalence. Let $\sigma : B(\mathcal{K}) \to B(\mathcal{K})$ be the map defined by

$$\sigma(A) = \sum_{e \in E^1} V_e A V_e^*.$$  

The map $\pi(C^*(E))' \ni A' \mapsto P A'|_{\mathcal{K}}$ (where $P = [\mathcal{K}]$) is a linear injection of the commutant of the representation into the fixed point set $B(\mathcal{K})^\sigma = \{A \in B(\mathcal{K}) : \sigma(A) = A\}$.

**Proof.** Let $H_n' = \{\sum_{j=0}^m c_j \alpha_j : m \in \mathbb{N}, c_j \in \mathbb{C}, \alpha_j \in \bigcup_{k \leq n} E^k\}$, $E^k$ still denoting the set of all paths of length $k$. Furthermore, let $H_n = H_n' \otimes \mathcal{K}$ (algebraic tensor product). For $\alpha \in E^*$, define a linear operator $S_\alpha$ on $H = \bigcup_n H_n$ by first defining

$$S_\alpha(\beta \otimes \xi) = \alpha \beta \otimes \xi$$

if $\alpha$ and $\beta$ are composable paths, and

$$S_\alpha(\beta \otimes \xi) = 0$$

\(^2\)This result can be extended to infinite graphs; we shall not need this in what follows.
if not, and then extending by linearity. Define a positive semidefinite sesquilinear form on $\mathcal{H}$ by first defining

$$\langle \alpha \otimes \xi | \alpha \beta \otimes \eta \rangle = \langle \xi | V_{\beta} \eta \rangle, \langle \alpha \beta \otimes \xi | \alpha \otimes \eta \rangle = \langle V_{\beta} \xi | \eta \rangle,$$

letting $\langle \alpha \otimes \xi | \beta \otimes \eta \rangle = 0$ if the pair $(\alpha, \beta)$ is not of the above form, and then extending by linearity. By the universal property of the tensor product, it is clear that this form is well defined. The following proves positive semidefiniteness: Let $\mathcal{T} \in H_n$. If $n = 0$, $\mathcal{T} \in \mathcal{K}$, and $(\mathcal{T}|\mathcal{T}) = (\mathcal{T}\mathcal{T})_\mathcal{K}$. If $n \geq 1$, we can write

$$\mathcal{T} = \sum_{e \in F} S_e \mathcal{T}_e + \mathcal{T}_0$$

for a finite set $F \subseteq E^1$, $\mathcal{T}_e \in H_{n-1}$, and $\mathcal{T}_0 \in \mathcal{K}$. We compute

$$\langle \mathcal{T}|\mathcal{T} \rangle = \left\langle \sum_{e \in F} S_e \mathcal{T}_e + \mathcal{T}_0 \right| \sum_{f \in F} S_f \mathcal{T}_f + \mathcal{T}_0 \right\rangle = \left\langle \mathcal{T}_0|\mathcal{T}_0 \right\rangle + \sum_{e \in F} \langle S_e \mathcal{T}_e|S_f \mathcal{T}_f \rangle + 2 \text{Re} \sum_{e \in F} \langle S_e \mathcal{T}_e|\mathcal{T}_0 \rangle = \|\mathcal{T}_0 + \sum_{e \in F} S_e \mathcal{T}_e\|^2 \geq 0.$$

Let $\mathcal{H}$ be the completion of $H/\{h \in H : \langle h|h \rangle = 0\}$, and let $\Lambda : H \rightarrow \mathcal{H}$ be the canonical map. $S_e$ induces an isometry on $\mathcal{H}$ for every $e \in E^1$, defined by

$$S_e \Lambda(\mathcal{R}) = \Lambda(S_e \mathcal{R})$$

and extension by linearity and continuity. In the following computations, the quotient map $\Lambda$ is suppressed.

The isometric property follows from

$$\langle S_e^* S_e(\alpha \otimes \xi)|\alpha \beta \otimes \eta \rangle = \langle e\alpha \otimes \xi|e\alpha \beta \otimes \eta \rangle = \langle \xi| V_{\beta} \eta \rangle = \langle \alpha \otimes \xi|\alpha \beta \otimes \eta \rangle,$$

$$\langle S_e^* S_e(\alpha \beta \otimes \xi)|\alpha \otimes \eta \rangle = \langle e\alpha \beta \otimes \xi|e\alpha \otimes \eta \rangle = \langle V_{\beta} \xi|\eta \rangle = \langle \alpha \beta \otimes \xi|\alpha \otimes \eta \rangle.$$ 

Now for $\xi, \eta \in \mathcal{K}, \alpha \in E^*$,

$$\langle S_e^* \xi|\eta \rangle = \langle \xi| e \otimes \eta \rangle = \langle \xi| V_{\alpha} \eta \rangle = \langle V_{e}^* \xi|\eta \rangle,$$

$$\langle S_e^* \xi|\alpha \otimes \eta \rangle = \langle \xi| e \alpha \otimes \eta \rangle = \langle \xi| V_{\alpha} \eta \rangle = \langle V_{e}^* \xi|\alpha \otimes \eta \rangle.$$ 

If $\alpha, \beta \in E^*, \xi, \eta \in \mathcal{K}$, we have

$$\langle S_e^*(\alpha \otimes \xi)|\beta \otimes \eta \rangle = \langle \alpha \otimes \xi|e\beta \otimes \eta \rangle.$$ 

This is zero unless $(\alpha, \beta)$ is of the form $(e\mu, \mu \lambda)$. In that case,

$$\langle S_e^*(e\mu \otimes \xi)|\mu \lambda \otimes \eta \rangle = \langle e\mu \otimes \xi|e\mu \lambda \otimes \eta \rangle = \langle \xi| V_{\lambda} \eta \rangle = \langle \mu \otimes \xi|\mu \lambda \otimes \eta \rangle.$$ 

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For $\xi, \eta \in \mathcal{K}, \alpha, \beta \in E^*, v \in E^0, f \in E^1, r(f) = v$, we have

$$
\langle \sum_{r(e) = v} S_e S_e^*(f \alpha \otimes \xi)(f \beta \otimes \eta) = \sum_{r(e) = v} \langle S_e^*(f \alpha \otimes \xi)|S_e^*(f \beta \otimes \eta)\rangle = \langle S_e^*(f \alpha \otimes \xi)|S_e^*(f \beta \otimes \eta)\rangle = \langle f \alpha \otimes \xi | f \beta \otimes \eta \rangle.
$$

Therefore, $\pi : s_e \mapsto S_e$ is a representation of $C^*(E)$. The subspace $\mathcal{K}$ is cyclic by construction, since $\Lambda \mathcal{H}$ is a dense subspace in $\mathcal{H}$, and all vectors of $\mathcal{H}$ are of the form $\sum \alpha_i \xi_i$, with $\alpha_i \in E^*$ and $\xi_i \in \mathcal{K}$.

If both $(\mathcal{H}, \{S_e : e \in E^1\}, \iota)$ and $(\mathcal{J}, \{R_e : e \in E^1\}, \kappa)$ satisfy the condition (2.2.1), one defines the unitary operator $U \in B(\mathcal{H}, \mathcal{J})$ implementing the equivalence of the representations by letting

$$
U(P(S_{e_1}, S_{e_1}^*, \ldots, S_{e_m}, S_{e_m}^*)\iota(\xi)) = P(R_{e_1}, R_{e_1}^*, \ldots, R_{e_m}, R_{e_m}^*)\kappa(\xi)
$$

for every ordered polynomial in $2m$ variables.

Let $A \in \pi(C^*(E))'$. Then,

$$
\sigma(PA|_{\mathcal{K}}) = \sum_{e \in E^1} V_e PA|_{\mathcal{K}} V_e^* = \sum_{e \in E^1} V_e PA S_e^*|_{\mathcal{K}} = \sum_{e \in E^1} V_e PA S_e^*|_{\mathcal{K}} = \sum_{e \in E^1} V_e PS_e^* A|_{\mathcal{K}} = \sum_{e \in E^1} PS_e S_e^* A|_{\mathcal{K}} = PA|_{\mathcal{K}}.
$$

Assume now that $PA|_{\mathcal{K}} = 0$. Since

$$
PA \alpha = PS \alpha A \xi = PS \alpha PA \xi = 0
$$

for $\xi \in \mathcal{K}, \alpha \in E^*$, $PA = 0$, so

$$
\langle AS \alpha \xi | S \beta \eta \rangle = \langle AS \alpha \xi | S \beta \xi \rangle = 0,
$$

so $A = 0$.

**Remark** In [3], where the above result is proved for the Cuntz algebra case, one also shows (see Proposition 4.1) that the map

$$
\pi(C^*(E))' \ni A \mapsto PA|_{\mathcal{K}} \in B(\mathcal{K})^*\quad
$$

is onto, under the additional assumption of the existence of a cyclic vector $\Omega$ for the representation $\pi$. The proof is directly applicable in the present setting; indeed, if $\Omega$ is a cyclic vector and $D \in B(\mathcal{K})^*$, with $0 \leq D \leq 1$, the linear functional on $C^*(E)$ defined by

$$
\tilde{\omega}(s_\alpha s_\beta^*) = \langle V_\alpha^* \Omega | D V_\beta \Omega \rangle
$$

is positive, and, applying the same argument to $1 - D$, we find that

$$
0 \leq \tilde{\omega} \leq \langle \cdot | \Omega \rangle|\Omega\rangle,
$$

which implies the existence of an $X \in \pi(C^*(E))'$ (with $0 \leq X \leq 1$), satisfying

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\[ \tilde{\omega}(s_\alpha s_\beta^*) = \langle \Omega | XS_\alpha S_\beta^* \Omega \rangle \]

(See [4], Theorem 2.3.19). But since \( X \in \pi(C^*(E))' \), we have

\[ \langle \Omega | XS_\alpha S_\beta^* \Omega \rangle = \langle V_\alpha^* \Omega | XV_\beta^* \Omega \rangle = \langle V_\alpha^* \Omega | DV_\beta^* \Omega \rangle, \]

ie \( PX|_E = D \). It is unclear to me how one is to prove this surjectivity without the cyclic vector \( \Omega \), although Theorem 5.1 of [3] seems to assert (without explicitly proving) that this is possible, at least in the Cuntz algebra case.
2.2.2 Theorem

If $E$ is a finite graph, the following two statements are equivalent.

(i) Every cycle has an entrance, and every vertex has an incident edge, i.e. $r$ is surjective, and for every $\alpha = (\alpha_1, \ldots, \alpha_N) \in E^*$ there is an $i$ with $n_{\alpha_i} \geq 2$.

(ii) Every wavelet representation of $C^*(E)$, corresponding to filters which are Fourier polynomials, has finite-dimensional commutant.

Proof. Let $m_e, e \in E^1$, denote the filter functions, and $R_e$ the corresponding isometries on $L^2(\mathbb{T})$.

$(i) \Rightarrow (ii)$: By the uniqueness part of Theorem 2.2.1 it suffices to produce a finite-dimensional subspace $K \subset H$ which is cyclic for the representation, and invariant under all the $S^*_e$.

For $k \in \mathbb{Z}$, let $\exp_k(z) = z^k$, i.e. $\exp_k$ is the $k$'th standard basis vector in $L^2(\mathbb{T})$. Let $m$ be the smallest natural number such that for all $e \in E^1$ can write $m_e = \sum_{|j| \leq m} a_{ej} \exp_j$ with $a_{ej} \in \mathbb{C}$. Compute

$$R_e^* \exp_k(z) = \sum_{\omega^x=1} \sum_{|j| \leq m} a_{ej} \omega^{k-j} = \sum_{|j| \leq m} a_{ej} \omega_0^{-j} \sum_{l=0}^{n_e-1} e^{\frac{2\pi i l(k-j)}{n_e}}$$

$$= n_e \sum_{|j| \leq m, j \equiv k \pmod{n_e}} a_{ej} \omega^{(k-j)/n_e}, \tag{2.5}$$

where $\omega_0$ is a primitive $n_e$'th root of $z$. If $n_e = 1$, the polynomial $m_e$ is necessarily, by the identity $m_e(z)m_e(z) = 1$, equal to one of the monomials $\exp_k$, and the operator $R_e^*$ is just multiplication by $z^k$.

Given an edge $e$ and an integer $p$ we define the affine function $\sigma_{e,p}$ by

$$\sigma_{e,p}(x) = \frac{x - p}{n_e}. \tag{2.6}$$

Now (2.5) can be written as

$$R_e^* \exp_k = n_e \sum_{|j| \leq m, j \equiv k \pmod{n_e}} a_{ej} \exp_{\sigma_{e,j}(k)}. \tag{2.7}$$

The result of applying to $\exp_k$ a product $R_\alpha = R_{\alpha_N} \cdots R_{\alpha_1}$, where $\alpha$ is a path, is a polynomial of the form

$$R_\alpha^* \exp_k = \sum C_{j_1, \ldots, j_N} \exp_{\sigma_{\alpha_{j_1}} \cdots \sigma_{\alpha_{j_N}}(k)} \tag{2.8}$$
where the sum is taken over suitable multi-indices \((j_1, \ldots, j_N)\), all of course satisfying \(|j_i| \leq m\).

Each of the composed maps \(\sigma_{\alpha_1, j_1} \circ \cdots \circ \sigma_{\alpha_N, j_N}\) which occur in the nonzero terms of the above sum can be written in a unique way as a composition of a number \(M\) of maps of the type \(\varkappa_{e, p, s}\) defined by \(\varkappa_{e, p, s}(x) = \sigma_{e, p}(x) - s\), where \(n_e \geq 2\) and \(s \in \mathbb{Z}\), \(|s| \leq \sum_{k=1}^{N} \deg m_{e_k}\). Observe that the condition (i) implies that there exists a constant \(L\), dependent only on the graph, such that we always have \(N \leq LM\). For \(|p| \leq m\), estimate

\[
|\varkappa_{e, p, s}(x)| \leq \frac{|x| + m}{2} + \sum_{e_k=1} \deg m_{e_k}.
\]

If the sequence of \(\varkappa\)-maps is the one corresponding to a path \(\alpha\) of length \(N\), and \(j_i\)'s such that the coefficient \(C_{\alpha_1, \ldots, \alpha_N}\) can possibly be nonzero, we have \(l \geq \frac{N}{L}\), and

\[
|\sigma_{\alpha_1, j_1} \circ \cdots \circ \sigma_{\alpha_N, j_N}(k)| \leq \frac{|k|}{2^N L} + m + 2 \sum_{e_k=1} \deg m_{e_k}.
\]

This implies that the space \(\mathcal{F}_\theta\) of all Fourier polynomials of degree at most \(\theta\) is mapped into the space \(\mathcal{F}_{m + 2 \sum_{e_k=1} \deg m_{e_k}}\) by \(R_{\alpha}^*\) provided only that \(|\alpha| \geq L \log_2(\theta)\).

Set \(\Theta = m + 2 \sum_{e_k=1} \deg m_{e_k}\). Define a new subspace \(\mathcal{F}\) by

\[
\mathcal{F} = \{ \psi \in H | \psi(\cdot, v) \in \bigvee_{|\alpha| \leq \Theta} R_{\alpha}^* \mathcal{F}_\theta \forall v \in E_0^1 \}.
\]

\(\mathcal{F}\) is clearly finite-dimensional. Moreover, \(S_{e}^* \mathcal{F} \subset \mathcal{F}\), for every \(e \in E_1^1\).

To show that \(\mathcal{F}\) is cyclic, pick a vertex \(v\) and let \(\psi \in H\) be such that \(\psi(\cdot, v)\) is a Fourier polynomial, while \(\psi(\cdot, w) \equiv 0\) for \(w \neq v\). Let \(\alpha \in E^*, |\alpha| \geq \Theta\). Then \(R_{\alpha}^* \psi \in \mathcal{F}\), and since \(\psi = R_{\alpha} R_{\alpha}^* \psi\) we see that \(\mathcal{F}\) is cyclic.

\((ii) \Rightarrow (i)\): Suppose \(\alpha = (\alpha_1, \ldots, \alpha_N) \in E^*\) such that \(n_{\alpha_i} = 1\) for all \(i\). Let \(V\) be the set of all vertices visited by the path \(\alpha\), ie

\[
V = d(\{\alpha_i | q \leq i \leq N\}) \cup r(\{\alpha_i | 1 \leq i \leq N\}).
\]

Choose a family of filter functions \(\{m_e | e \in E_1^1\}\) which are Fourier polynomials in \(z\), and make sure that whenever \(n_e = 1\), the monomial \(m_e\) has degree 0, ie \(m_e \equiv 1\). Let \(m\), as before, be the bound on the degrees of the filter polynomials.
For \( q \geq m \), let \( \mathcal{H}_q \) be the closed subspace of \( L^2(\mathbb{T}) \) spanned by all the monomials of order at least \( q \), and let \( H_q = \{ \psi \in H | \psi(\cdot, v) \in \mathcal{H}_q \ \forall \ v \in V \} \). Observe three things:

(i) \( S_e^* H_q = 0 \) unless \( e = \alpha_i \) for some \( i \).

(ii) \( S_e \psi(\cdot, v) = 0 \) for all \( \psi \in H_q \) and \( v \in V \), unless \( e = \alpha_i \) for some \( i \).

(iii) If \( 1 \leq i \leq N \), and \( \psi \in H_q \), then \( S_{\alpha_i} = D_{\alpha_i} \), and this operator clearly leaves \( H_q \) invariant.

Clearly, \( H_q \) is invariant under the representation for every \( q \geq m \). But we have \( H_m \supset H_{m+1} \supset \ldots \), all inclusions strict. This shows that the commutant of the representation cannot be finite-dimensional.

Assume now that there is an element \( v \in E^0 - r(E^1) \). Pick Fourier polynomial filter functions \( \{ m_e | e \in E^1 \} \) such that all the monomials corresponding to edges \( l \) with \( n_l = 1 \) have positive degree. If \( N \) is a positive integer dominating the degrees of all the filter polynomials, the subspace \( \{ \psi \in H | \psi(\cdot, v) \in \mathcal{F}_N \} \) is clearly invariant. This proves that the commutant cannot be finite-dimensional. \( \square \)
Bibliography


