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1 Introduction

Practical information. Prerequisites:

- Basic real analysis, e.g. MAT2400 (say compact subsets of \( \mathbb{R}^n \), uniform convergence of sequences of functions).
- Aquaintance with PDEs, e.g. MAT-INF3360 (some exposure to partial derivatives including Gauss-Green-Stokes theorem, examples from physics, chemistry, biology, social sciences, finance or whatever).

Strongly recommended (either before or in parallel):

- Basic measure theory and functional analysis: MAT4400.

Curriculum: Lawrence C. Evans *Partial Differential Equations*, Amer. Math. Soc., Graduate Studies in Mathematics. Selected topics in the following chapters:

- Chap. 2: Some explicit formulas, for the heat and Poisson equations.
- Chap. 5: Sobolev spaces, definition and some properties.
- Chap. 6: Applications of Sobolev spaces to linear elliptic PDEs.

Time permitting: the abstract theory of finite element methods.

Language. A comparison of formulations. Let \( I = [a,b] \) be a (closed and bounded) interval in \( \mathbb{R} \). Consider the following statements:

- Suppose \( f, g : I \to \mathbb{R} \) are two functions. Pick \( x \in I \). Suppose \( f \) and \( g \) are both differentiable at \( x \). Then \( f + g \) is differentiable at \( x \) and \( (f + g)'(x) = f'(x) + g'(x) \).

- Let \( \mathcal{F} \) denote the set of (real valued) functions on \( I \) and \( \mathcal{D} \) denote the set of differentiable functions. Then \( \mathcal{F} \) has a natural structure of vector space, and \( \mathcal{D} \) is then a subspace. Moreover differentiation defines a linear map from \( \mathcal{D} \) to \( \mathcal{F} \).

- Denote by \( \mathcal{C}^k(I) \) the space of functions which are \( k \) times differentiable on \( I \), the derivatives up to order \( k \) being continuous. It is a vector space, and the following defines a norm:

\[
\|u\|_k = \sup \{|u^{(l)}(x)| : l \leq k, \ x \in I\}.
\] (1)

Here \( u^{(l)} \) denotes the derivative of order \( l \) of \( u \). Then derivation defines a bounded linear map from \( \mathcal{C}^k(I) \) to \( \mathcal{C}^{k-1}(I) \) (\( k \geq 1 \)), by construction. Moreover \( \mathcal{C}^k(I) \) equipped with this norm, is complete.

For this course I am going to assume that the students think the first statement is a distant memory, that they are familiar with the second statement and that they can work through the last. This is mostly about reformulating key results from a second calculus course in the language of functional analysis.

Remarks:
• If the interval $I$ is not compact, one can restrict attention to functions that are bounded and have bounded derivatives. This guarantees that the above norm is still well defined.

• Similar statements hold for functions of several variables, as seen in class.

This defines some function spaces. Sobolev spaces are other function spaces, that turn out to be more adapted to the study of partial differential equations, for instance because Hilbert space techniques can be applied to them.

**Substance.** For most PDEs there is no “explicit formula” for the solution. One then studies questions such as: does the PDE have a solution, is it unique, and what are its properties. Consider the following reasoning:

Let $U$ be an open bounded subset of $\mathbb{R}^n$. Let $f : U \to \mathbb{R}$ be a function and $g : \partial U \to \mathbb{R}$ be another one. Consider the equation:

$$-\Delta u = f \text{ on } U,$$
$$u = g \text{ on } \partial U.$$  \hfill (2)  \hfill (3)

To show that this equation has at most one solution one writes:

Suppose that $u$ solves the homogeneous equation:

$$-\Delta u = 0 \text{ on } U,$$
$$u = 0 \text{ on } \partial U.$$  \hfill (4)  \hfill (5)

Then, by integration by parts:

$$\int |\nabla u|^2 = -\int u\Delta u = 0.$$  \hfill (6)

Hence $\nabla u = 0$, so $u$ is constant, and using the boundary condition again, we deduce $u = 0$. Since the equation is linear, this proves uniqueness.

There are many steps in this reasoning, and the course will justify each of them, and explore in which generality such results hold. For instance we need to define integration on the boundary of a set and justify integration by parts. In fact we will carry through this reasoning even for functions that are not differentiable! More precisely we will generalize the notion of derivative, so that it applies to relatively “rough” functions, including some discontinuous ones.

For a numerical method applied to some PDE, one wants to estimate the distance from the computed (approximate) solution to the exact solution, with respect to some norm. It turns out that this is easier in Sobolev space norms than in those, more elementary, defined by (1).

**Welcome!**
2 The heat equation

20 August 2014

I gave a derivation of the heat equation from “physical principles” and gave some vocabulary related to several interpretations:

- temperature in a body (heat equation).
- density of a chemical in a fluid (convection–diffusion equation).
- value of an option (Black and Scholes).

Explicit formulas for the transport equation and the heat equation in idealized scenarios.

Reading assignment, in [1]:

- Introduction.
- Chapter 2: §2.1 p. 18–19. §2.3 p. 44 – 48 (but the proof of Theorem 1 p. 47 is optional).
- Appendices: Notations p. 697 onwards (get used to some of them...). The definitions in §D.1 p. 719 (three first) and §D.3.a p. 721 (two first).
Let $U$ be an open subset of $\mathbb{R}^n$. 

**Exercise 3.1.** Check that if $u \in C^1(U)$ (scalar field) and $v \in C^1(U)^n$ (vector field):
\[ \text{div}(uv) = (\text{grad } u) \cdot v + u \text{ div } v. \]

**Exercise 3.2.** Choose reals $a_i < b_i$ for integer $i$ such that $1 \leq i \leq n$. Suppose $U = [a_1, b_1] \times \cdots \times [a_n, b_n]$. You may start with $n = 2$. What is the boundary of $U$? What is the unit outward pointing normal on $\partial U$? We denote it by $\nu$.

Give a direct proof of Stokes theorem in this case, namely that if $u \in C^1(\mathbb{R}^n)$ then:
\[ \int_U \text{div } u = \int_{\partial U} u(x) \cdot \nu(x) dx. \]

Deduce (informally) the formula for a finite union of such boxes, possibly sharing faces.

**Exercise 3.3.** For a function $u \in C^1(U)$, we define for any vector $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$:
\[ \partial_a u(x) = \lim_{\epsilon \to 0} \epsilon^{-1}(u(x + \epsilon a) - u(x)). \]

Check:
\[ \partial_a u(x) = \sum_{i=1}^n a_i \partial_i u(x). \]

Let $(e_i)_{1 \leq i \leq n}$ denote the canonical basis of $\mathbb{R}^n$:
\begin{align*}
e_1 &= (1, 0, 0, \ldots, 0), \\
e_2 &= (0, 1, 0, \ldots, 0), \\
&\vdots \\
e_n &= (0, 0, 0, \ldots, 1).
\end{align*}

Compared with standard notation we have:
\[ \partial_i u = \partial_{e_i} u. \]

Let $(f_i)_{1 \leq i \leq n}$ be another orthonormal basis of $\mathbb{R}^n$.

Prove that for $u \in C^1(U)$:
\[ \sum_i (\partial_i u) e_i = \sum_i (\partial_{f_i} u) f_i. \]

and that for $u \in C^1(U)^n$:
\[ \sum_i \partial_i u_i = \sum_i \partial_{f_i} (u \cdot f_i). \]

Prove also that for $u \in C^2(U)$:
\[ \sum_{i=1}^n \partial_i^2 u = \sum_{i=1}^n \partial_{f_i}^2 u. \]
Exercise 3.4. Let $A$ be an invertible real $n \times n$ matrix, considered also as a map $\mathbb{R}^n \to \mathbb{R}^n$, $x \mapsto Ax$. Check that for $u \in C^1(U)$ we have:

$$\text{grad}(u \circ A) = (A^T \text{grad } u) \circ A, \quad (19)$$

and that for $v \in C^1(U)^n$:

$$\text{div}(A^{-1}v \circ A) = (\text{div } v) \circ A. \quad (20)$$

Deduce that if $A$ is an orthogonal matrix:

$$\Delta(u \circ A) = (\Delta u) \circ A. \quad (21)$$

Whether $A$ is orthogonal or not, find a positive definite matrix $B$ such that:

$$\Delta(u \circ A) = (\text{div } B \text{grad } u) \circ A. \quad (22)$$
4 Continuous linear or bilinear maps


Not very much functional analysis is required for this course. Essentially we need the following background material, as well as some results related to completeness that will be introduced later on.

In the following all vectorspaces are vectorspaces over a field $\mathbb{K}$ which is either $\mathbb{R}$ or $\mathbb{C}$.

**Proposition 4.1.** $E$ and $F$ two normed vectorspaces. $A : E \rightarrow F$ a linear map. The following are equivalent:

- $A$ is continuous.
- $A$ is continuous at 0.
- There is $C > 0$ such that for all $u \in E$, $\|Au\| \leq C\|u\|$.

Such linear maps are also said to be *bounded*.

**Proof.** See Lemma 4.1 p. 88 in [7].\qed

For a continuous linear map $A : E \rightarrow F$ one defines:

\[
\|A\| = \inf\{C \geq 0 : \forall u \in E \ \|Au\| \leq C\|u\|\}, \quad (23)
\]

\[
= \sup\{\|Au\| : u \in E \ \|u\| \leq 1\}. \quad (24)
\]

The continuous linear maps $E \rightarrow F$ form a vector space denoted $\text{L}(E, F)$ and the above expression defines a norm on this space.

In the special case $F = \mathbb{K}$, the space $\text{L}(E, \mathbb{K})$ is called the dual space of $E$ and denoted $E^*$.

Very similarly one proves:

**Proposition 4.2.** $E$, $F$ and $G$ three normed vectorspaces. $a : E \times F \rightarrow G$ a bilinear map. The following are equivalent:

- $a$ is continuous.
- $a$ is continuous at 0.
- There is $C > 0$ such that for all $u \in E$, all $v \in F$, $\|a(u, v)\| \leq C\|u\|\|v\|$.

Such bilinear maps are also said to be *bounded*.

For a continuous bilinear map $a : E \times F \rightarrow G$ one defines:

\[
\|a\| = \inf\{C \geq 0 : \forall u \in E \ \forall v \in F \ \|a(u, v)\| \leq C\|u\|\|v\|\}, \quad (25)
\]

\[
= \sup\{\|a(u, v)\| : u \in E \ \|u\| \leq 1, \ v \in F \ \|v\| \leq 1\}. \quad (26)
\]

The continuous bilinear maps constitute a normed vector space.

Notice that a continuous linear map is uniformly continuous, whereas a (non–zero) continuous bilinear map is not.
5 Laplace equation

- Multi-index notation for partial derivatives.
- Differential operators.
- Support of a continuous function.
- Spherically symmetric solutions of the Laplace equation.
- Fundamental solution.
6 Harmonic functions

• Convolution.
• Solving the Poisson equation using the fundamental solution.
• Mean value property.
• Standard mollifier.
• Smoothness of harmonic functions.
• Liouville theorem.
• An existence and uniqueness result for solutions to Poisson ($n \geq 3$).

Reading assignment: Evans Section 2.2.1, 2.2.2, 2.2.3 (b,d). See also Appendix C.5, E.1, E.2, E.3.
7 Energy methods

7.1 Complements

PDE theory draws on many (all?) sources. I hope you will find references that please you. Here are some of my favorites.

Differential Intrinsic definition of the differential of a function from one normed vector space to another, compared with partial derivatives:

- [10] Chapter 17: Several variable differential calculus.

Lebesgue measure and integral

- [9] Chapter 1: Measure theory.

Convolution Convolution and Dirac sequences:

- [9] Chapter 3.2: Good kernels and approximations to the identity. See also: [8] Chapter 2.3: Convolutions, Chapter 2.4: Good kernels.

7.2 The Dirichlet principle

Already known to Gauss, but used and extended by Riemann under this name.
Reading assignment: §2.2.5 in Evans.

7.3 Inner products

In the following all vector spaces are real.

The following is the Cauchy-Schwarz inequality (we check that it doesn’t require positive definiteness).

Proposition 7.1. Let $X$ be a vector space and $a$ a symmetric bilinear form on $X$ with the property:

$$\forall u \in X \quad a(u, u) \geq 0.$$  \hfill (27)

Then:

$$\forall u, v \in X \quad |a(u, v)| \leq a(u, u)^{1/2} a(v, v)^{1/2},$$  \hfill (28)

and the function:

$$\begin{cases} X \to \mathbb{R}, \\ u \mapsto a(u, u)^{1/2} \end{cases}$$  \hfill (29)

defines a seminorm on $X$. 
Proof. (i) Pick \( u, v \in X \), for which we want to prove (28).

We first write for all \( \lambda \in \mathbb{R} \):

\[
0 \leq a(u + \lambda v, u + \lambda v) = a(u, u) + 2\lambda a(u, v) + \lambda^2 a(v, v). \tag{30}
\]

With \( \lambda = -\text{sgn} a(u, v) \) we get:

\[
2|a(u, v)| \leq a(u, u) + a(v, v). \tag{31}
\]

From this we deduce, for all \( t > 0 \):

\[
2|a(u, v)| = 2|a(tu, t^{-1}v)| \leq t^2 a(u, u) + t^{-2} a(v, v). \tag{32}
\]

If both \( a(u, u) = 0 \) and \( a(v, v) = 0 \), then \( a(u, v) = 0 \) and we are done.

Suppose then, that \( a(u, u) \neq 0 \). If \( a(v, v) = 0 \) we get \( a(u, v) = 0 \) by letting \( t \to 0 \), so we are also done. If \( a(v, v) \neq 0 \), choosing \( t \) so that:

\[
t^2 = a(v, v)^{1/2}/a(u, u)^{1/2}, \tag{33}
\]

gives the result.

(ii) We have:

\[
a(u+v, u+v)^{1/2} = (a(u, u) + 2a(u, v) + a(v, v))^{1/2}, \tag{34}
\]

\[
\leq (a(u, u) + 2a(u, u)^{1/2}a(v, v)^{1/2} + a(v, v))^{1/2}, \tag{35}
\]

\[
\leq a(u, u)^{1/2} + a(v, v)^{1/2}. \tag{36}
\]

This is the main ingredient in showing that (29) defines a seminorm.

In the above circumstaces, the seminorm is a norm iff:

\[
a(u, u) = 0 \Rightarrow u = 0. \tag{37}
\]

In this case we call \( a \) an inner product on \( X \) and we say that \((X, a)\) is an inner product space. When \( X \) is equipped with the norm defined by \( a \), \( a \) is a continuous bilinear form on \( X \).

Definition 7.1. A Hilbert space is an inner product space which is complete.

Recall that for any normed vector space \( X \), its dual is the space of continuous linear forms on \( X \). We denote the dual by \( X^* \). It is equipped with the norm:

\[
||l|| = \sup_{u \in X} \frac{|l(u)|}{||u||}, \tag{38}
\]

where it is understood that only non-zero \( u \) are used.

Proposition 7.2. Let \((X, a)\) be an innerproduct space and \( l \in X^* \). Let \( F : X \to \mathbb{R} \) denote the functional defined by:

\[
F(u) = \frac{1}{2}a(u, u) - l(u). \tag{39}
\]

For any \( u \in X \), the following are equivalent:

\[
\forall v \in X \quad a(u, v) = l(v). \tag{40}
\]

\[
\forall v \in X \quad F(u) \leq F(v). \tag{41}
\]

There is at most one such \( u \).
Remark 7.1. Let \((X,a)\) be an inner product space. Then, for \(u \in X\):

\[
\|u\| = \sup_{v \in X} |a(u,v)| = \|a(u,\cdot)\|_{X^*}. \tag{42}
\]

In other words the linear map:

\[
\begin{cases}
X \rightarrow X^*, \\
u \mapsto a(u,\cdot).
\end{cases} \tag{43}
\]

is an isometry.

The following result is called the Riesz representation theorem.

**Theorem 7.3.** Let \((X,a)\) be a Hilbert space. Then for any \(l \in X^*\) there exists a unique \(u \in X\) such that:

\[
\forall v \in X \quad a(u,v) = l(v). \tag{44}
\]

**Proof.** Let \(F\) be the functional defined in (39).

For any \(v \in X\) we have:

\[
F(v) \geq \frac{1}{2} \|v\|^2 - \|l\| \|v\|, \tag{45}
\]

\[
\geq \frac{1}{2} (\|v\| - \|l\|)^2 - \frac{1}{2} \|l\|^2 \geq -\frac{1}{2} \|l\|^2. \tag{46}
\]

Define:

\[I = \inf \{ F(v) : v \in X \}. \tag{47}\]

Let \((u_n)\) be a sequence in \(X\) such that:

\[F(u_n) \to I. \tag{48}\]

(one says that \((u_n)\) is a “minimizing sequence”).

We have (from the “parallelogram identity”):

\[
\frac{1}{4} \|u_n - u_m\|^2 = \frac{1}{2} \|u_n\|^2 + \frac{1}{2} \|u_m\|^2 - \frac{1}{2} \|u_n + u_m\|^2, \tag{49}
\]

\[
= F(u_n) - l(u_n) + F(u_m) - l(u_m) - 2F\left(\frac{u_n + u_m}{2}\right) + l\left(\frac{u_n + u_m}{2}\right), \tag{50}
\]

\[
= F(u_n) + F(u_m) - 2F\left(\frac{u_n + u_m}{2}\right) \leq F(u_n) - I + F(u_m) - I. \tag{53}
\]

It follows that \((u_n)\) is Cauchy (notice how the fact that \(I > -\infty\) enters the argument). Let \(u\) be its limit. By continuity of \(F\), \(F(u) = I\), so \(F\) achieves its minimum at \(u\). \(\square\)

**Remark 7.2.** Let \((X,a)\) be an inner product space. It follows from the completeness of \(\mathbb{R}\), that \(X^*\) is complete\(^1\), so in order for the map (43) to be onto, it is necessary that \((X,a)\) is complete. The theorem says that this condition is also sufficient.

\(^1\)More generally, if \(X\) is a normed vector space any \(Y\) is a Banach space, then the space \(L(X,Y)\) of continuous linear maps \(X \to Y\), equipped with the operator norm, is a Banach space.
8 Derivatives and completeness

8.1 Derivation and antiderivation

The following can be called the Lemma of bounded increments.

**Lemma 8.1.** Let $X$ be a normed vector space. Suppose that $f : [a, b] \to X$ is continuous on $[a, b]$ and differentiable on $]a, b[$, with:

$$\forall x \in ]a, b[ \quad \|f'(x)\| \leq M.$$  \hfill (54)

Then:

$$\|f(b) - f(a)\| \leq M|b - a|.$$  \hfill (55)

**Proof.** The trick is to prove that for any $\epsilon > 0$, the set of points $x \in [a, b]$ such that:

$$\|f(x) - f(a)\| \leq \epsilon + (M + \epsilon)|x - a|,$$  \hfill (56)

contains $b$.

**Corollary 8.2.** Let $X$ be a normed vector space. Suppose that $f : [a, b] \to X$ is continuous on $[a, b]$ and differentiable on $]a, b[$, with differential 0. Then $f$ is constant.

**Proposition 8.3.** Let $X$ be a complete normed vector space. Suppose $I$ is an interval and $a \in I$. Suppose $f_n : I \to X$ is a sequence of differentiable functions such that:

- $f_n(a)$ converges.
- $f'_n$ converges uniformly to a function $g : I \to X$.

Then $f_n$ converges pointwise to a function $f : I \to X$, and the convergence is uniform on any bounded part of $I$. Moreover $f$ is differentiable, with derivative $g$.

**Proposition 8.4.** Let $I$ be an interval and $X$ a complete normed vector space. Consider the space of differentiable functions $I \to X$, which are bounded with bounded derivative. Equip it with the norm:

$$\|f\| = \sup_{x \in I} \|f(x)\| + \sup_{x \in I} \|f'(x)\|.$$  \hfill (57)

This space is complete.

Moreover the continuously differentiable functions which are bounded with bounded derivative, constitute a closed subspace.

**Theorem 8.5.** Let $X$ be a complete normed vector space. Suppose $I$ is an interval and that $f : I \to X$ is continuous. Then there is a a function $F : I \to X$ whose derivative is $f$, and it is unique up to a constant.

**Proof.** (Sketch). Suppose the interval is closed and bounded and choose $a \in I$. Approximate $f$ uniformly by a sequence $f_n$ of continuous piecewise linear functions. Let $F_n$ be an antiderivative of $f_n$ which has value 0 at $a$ (it exists because we have an explicit formula). Then apply Proposition 8.3.
Remark 8.1. This gives a first possible approach to integration. For a continuous function \( f : [a, b] \to X \), choose an antiderivative \( F \) and define:

\[
\int_a^b f = F(b) - F(a).
\] (58)

This can be extended to piecewise continuous functions.

Exercise 8.1. Let \( I \) be a bounded closed interval. We have the space \( C^1(I) \) equipped with the standard norm. Another norm on this space is the one derived from the scalar product:

\[
(u, v) \to \int uv + \int u'v'.
\] (59)

Find a sequence in \( C^1(I) \) which is Cauchy with respect to the latter norm, but not the former.

The following is Weierstrass critique of the Dirichlet principle.

Exercise 8.2. Define:

\[
\mathcal{A} = \{ v \in C^1([-1, 1]) : v(-1) = -1, \ v(1) = 1 \}.
\] (60)

We define a functional \( J : \mathcal{A} \to \mathbb{R} \) by:

\[
J(v) = \int_{-1}^1 x^2 |v'(x)|^2 dx.
\] (61)

a) Prove that:

\[
\forall v \in \mathcal{A} \quad J(v) > 0.
\] (62)

b) Define a function \( v_n \) by:

\[
v_n(x) = \frac{\arctan(nx)}{\arctan(n)}.
\] (63)

Show that \( J(v_n) \to 0 \).

c) Deduce that there is no point in \( \mathcal{A} \) where \( J \) reaches its infimum.

8.2 Completions

Lemma 8.6. \( E \) and \( F \) two normed vectorspaces. Suppose \( E_0 \) a linear subspace of \( E \) which is dense. Suppose \( A_0 : E_0 \to F \) a linear map. Suppose that \( F \) is complete and that there is \( C > 0 \) such that:

\[
\forall u \in E_0 \quad \|A_0 u\| \leq C\|u\|.
\] (64)

Then there is a unique continuous map \( A : E \to F \) whose restriction to \( E_0 \) is \( A_0 \). It is linear and satisfies:

\[
\forall u \in E \quad \|A u\| \leq C\|u\|.
\] (65)

Proof. [5] Chapter X.1, Theorem 1.2 p. 247. \( \Box \)
**Definition 8.1.** Suppose $X$ is normed vector space. A completion of $X$ is a complete normed vectorspace $Y$ together with a linear isometry $i : X \to Y$ whose range is a dense subspace of $Y$.

**Example 8.1.** If $X$ is a dense subspace of some complete space $Y$, the norm of $X$ being the restriction of the norm of $Y$, take $i : X \to Y$ to be the canonical injection.

The following is a uniqueness result for completions.

**Lemma 8.7.** If, given $X$, we have two completions $(i, Y)$ and $(j, Z)$ there is a unique continuous map $h : Y \to Z$ such that $h \circ i = j$. Moreover $h$ is a linear bijective isometry.

**Theorem 8.8.** Any normed vector space has a completion.

*Proof.* (Sketch) Suppose we want to complete a normed vector space $X$. One takes the space of Cauchy sequences in $X$, equipped with the sup norm. This is a Banach space. Those sequences that converge to 0 constitute a closed subspace. The quotient space is our space $Y$ and we take for $i : X \to Y$ the map sending an element $u$ in $X$ to the equivalence class of the constant sequence with values $u$.

See [5] Chapter VII.4: Completion of a normed vector space. See also [10] Exercise 12.4.8. \qed

But this particular “canonical” completion of a space, if it is applied to a space of functions, does not yield a space of functions. In this case one would like to know if the completion can be identified with a space of functions or not.

### 8.3 Complements

**Integration of regulated functions** The following sketches a way of defining the integral based on completion. For details on this approach, see [5] Chapter X: The integral in one variable.

Let $[a, b]$ be an interval and $X$ a complete normed vector space. We take for granted the following notions:

- Step functions from $[a, b]$ to $X$.
- Integration of step functions (independence of subdivision).

**Definition 8.2.** A function $f : [a, b] \to X$ is said to be regulated if it is the uniform limit of step functions.

**Proposition 8.9.** The space of regulated functions, equipped with the sup-norm is complete. The step functions constitute a dense subspace.

**Proposition 8.10.** There is a unique continuous linear form on the space of regulated functions, whose restriction to the step functions is the integral.

**Proposition 8.11.** Any continuous function is regulated.

**Theorem 8.12.** Let $f : [a, b] \to X$ be a continuous function.
• Define $F : [a, b] \to X$ by:

$$F(x) = \int_a^x f(y)\,dy.$$  \hfill (66)

Then $F$ is differentiable and $F' = f$.

• Suppose $F$ is any antiderivative of $f$. Then:

$$\int_a^b f = F(b) - F(a).$$  \hfill (67)

**Remark 8.2.** For those who know the Riemann integral: Any regulated function is Riemann integrable (easy), but the converse is false (hard).

**Remark 8.3.** Consider a regulated function $f$. Can we extend Theorem 8.12 to such functions? For a complete answer, see [3] Chapter V.3.13: Extension du théorème fondamental aux fonctions réglées.

**An approach to the Lebesgue integral through completion** Let $S$ denote the space of step functions $[a, b] \to X$. Equip $S$ with the seminorm defined by:

$$|f| = \int_a^b \|f(x)\|\,dx.$$  \hfill (68)

It turns out that the completion of this space with respect to this semi-norm, has all the properties we would like for a nice space $L^1([a, b], X)$. In particular its elements can be identified with functions (defined up to a subset of measure zero).

Realizing this program requires some work of course, see [5] Chapter X.4 p. 262 – 267, and [6] Appendix A, p. 529 – 541. Remark that completing $S$ with respect to the sup-norm yields the space of regulated functions and was comparatively easy.
9 Integration and mollification

9.1 Measures and integration

Definition 9.1. Let $E$ be a set. A $\sigma$-algebra on $E$ is a subset $\mathcal{M}$ of $\mathcal{P}(E)$ such that:

- $\emptyset \in \mathcal{M}$,
- For all $F \in \mathcal{M}$ we have $E \setminus F \in \mathcal{M}$,
- For any sequence $(F_n)_{n \in \mathbb{N}}$ in $\mathcal{M}$ we have $\cup_{n \in \mathbb{N}} F_n \in \mathcal{M}$.

When $E$ is a topological space, there is a smallest $\sigma$-algebra containing the open subsets of $E$, called the Borel $\sigma$-algebra on $E$.

Definition 9.2. Let $E$ be a set and $\mathcal{M}$ a $\sigma$-algebra on $E$. A measure on $\mathcal{M}$ is a function $\mu : \mathcal{M} \to [0, \infty]$ such that for any sequence $(F_n)_{n \in \mathbb{N}}$ in $\mathcal{M}$ of two by two disjoint sets:

$$\mu(\bigcup_{n \in \mathbb{N}} F_n) = \sum_{n \in \mathbb{N}} \mu(F_n).$$

(69)

Let $E$ be a set, $\mathcal{M}$ a $\sigma$-algebra on $E$ and $\mu$ a measure on $\mathcal{M}$. Then the following is also a $\sigma$-algebra:

$$\overline{\mathcal{M}} = \{ F \cup G : F \in \mathcal{M}, \text{ and } \exists G' \in \mathcal{M} \quad \mu(G') = 0 \text{ and } G \subseteq G' \}. \quad (70)$$

The measure $\mu$ has a unique extension to a measure $\overline{\mu}$ on $\overline{\mathcal{M}}$. For any subset $F$ of $E$, if there is $F' \in \overline{\mathcal{M}}$ such that $\overline{\mu}(F') = 0$ and $F \subseteq F'$, we have $F \in \overline{\mathcal{M}}$.

One says that $(\overline{\mathcal{M}}, \overline{\mu})$ is the completion of $(\mathcal{M}, \mu)$.

Once a $\sigma$-algebra on $E$ has been fixed, as well as a measure, the elements of the $\sigma$-algebra will be referred to as the measurable subsets of $E$.

A property $P(x)$ defined for $x \in E$ is said to hold almost everywhere, if there is a set $F$ of measure 0 such that $P(x)$ is true for $x \in E \setminus F$.

Definition 9.3. A function $f : E \to \mathbb{R}$ is measurable if for any open interval $I$ in $\mathbb{R}$, $f^{-1}(I)$ is a measurable subset of $E$.

Definition 9.4. For a function $f : E \to \mathbb{R}$ the following are equivalent:

- $f$ is measurable, has finite range and for $a \neq 0$, $f^{-1}\{\{a\}\}$ has finite measure.
- $f$ is a (finite) linear combination of characteristic functions of measurable subsets with finite measure.

Such functions are called simple. They form a vector space.

Lemma 9.1. Suppose $f$ is a simple function represented as:

$$f = \sum_{i=1}^{n} a_i \chi_{F_i}, \quad (71)$$
with $F_i$ measurable, with finite measure. If $f(x) = 0$ for almost every $x \in E$, then:

$$\sum_{i=1}^{n} a_i \mu(F_i) = 0. \quad (72)$$

As a consequence, if a simple function $f$ is represented as in (71), the number:

$$\sum_{i=1}^{n} a_i \mu(F_i), \quad (73)$$

does not depend on the choice of representation. It is called the integral of $f$ and denoted $\int f$.

For a measurable function $f : E \to [0, \infty]$ we define:

$$\int f = \sup \{ \int \phi : \phi \text{ is simple and } \phi \leq f \} \in [0, \infty]. \quad (74)$$

We define $L^1(E)$ to consist of measurable functions $f : E \to \mathbb{R}$ such that $\|f\| = \int |f| < \infty$. It is a normed vector space and the simple functions are dense in it. The integral, defined a priori for simple functions, has a unique extension to a continuous linear form on $L^1(E)$ called the integral and still denoted $\int$.

**Theorem 9.2.** (dominated convergence) Suppose that $f : E \to \mathbb{R}$ is some function, $(f_n)_{n \in \mathbb{N}}$ is a sequence in $L^1(E)$ and $g : E \to [0, \infty]$ satisfies:

- $g$ is measurable and $\int g < \infty$,
- for almost every $x \in E$, for all $n \in \mathbb{N}$, $|f_n(x)| \leq g(x)$,
- for almost every $x \in E$, $f_n(x) \to f(x)$ as $n \to \infty$.

Then $f \in L^1(E)$ and:

$$\lim_{n \to \infty} \int f_n = \int f. \quad (75)$$

**Proposition 9.3.** $L^1(E)$ is complete.

Actually the norm on $L^1(E)$ is just a seminorm, since the $L^1(E)$ norm of a function is 0 iff the function is zero almost everywhere. One usually overlooks this problem and considers that such functions are 0.

**Proposition 9.4.** On the Borel $\sigma$-algebra of $\mathbb{R}^n$ there is a unique measure which:

- is translation invariant,
- and is such that the measure of the open unit cube is 1.

The completion of this measure is called the Lebesgue measure and the associated integral is called the Lebesgue integral. By default this is the measure and integral we will always consider.

If $S$ is an open subset of $\mathbb{R}^n$ we denote by $L^1(S)$ the space of functions on $S$ whose extension by 0 to $\mathbb{R}^n$ is in $L^1(\mathbb{R}^n)$. This is also a Banach space.

**Proposition 9.5.** For any open subset $S$ of $\mathbb{R}^n$, the space $C_c(S)$ is dense in $L^1(S)$.  

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We define:

\[ L^1(S)_{loc} = \{ u : \mathbb{R}^n \to \mathbb{R} : \forall K \subset S \text{ compact } \Rightarrow \chi_K u \in L^1(S) \}. \] (76)

For instance the fundamental solution to the Laplace operator is in \( L^1(\mathbb{R}^n)_{loc} \) but not \( L^1(\mathbb{R}^n) \).

### 9.2 The standard bump

Define a function \( \phi : \mathbb{R} \to \mathbb{R} \) by, for \( x > 0 \):

\[ \phi(x) = \exp(-1/x), \] (77)

and for \( x \leq 0 \), \( \phi(x) = 0 \).

**Exercise 9.1.** By induction on \( k \), show that for any \( k \in \mathbb{N} \) there exists a polynomial \( P \) and \( l \in \mathbb{N} \) such that the \( k \)-th order derivative of \( \phi \) takes the form, for \( x > 0 \):

\[ \phi^{(k)}(x) = \frac{P(x)}{x^l} \exp(-1/x). \] (78)

Deduce that \( \phi \in C^\infty(\mathbb{R}) \).

Define \( \Phi : \mathbb{R}^n \to \mathbb{R} \) by, for \( x \in \mathbb{R}^n \):

\[ \Phi(x) = \phi(1 - |x|^2). \] (79)

What is the support of \( \Phi \)? Why is its integral strictly positive?

Define then:

\[ \mu(x) = \Phi(x)/\int \Phi, \] (80)

and:

\[ \mu_\epsilon(x) = \epsilon^{-n} \mu(\epsilon^{-1} x). \] (81)

What is the integral of \( \mu_\epsilon \)? These functions will be called the standard mollifiers.

### 9.3 Convolution

Given two functions \( u, v : \mathbb{R}^n \to \mathbb{R} \) one defines \( u \ast v : \mathbb{R}^n \to \mathbb{R} \) by:

\[ u \ast v(x) = \int u(y)v(x-y)dy, \] (82)

whenever this makes sense.

**Proposition 9.6.** For \( u, v \in L^1(\mathbb{R}^n) \) we get a well defined \( u \ast v \in L^1(\mathbb{R}^n) \) and:

\[ \|u \ast v\|_{L^1(\mathbb{R}^n)} \leq \|u\|_{L^1(\mathbb{R}^n)} \|v\|_{L^1(\mathbb{R}^n)}. \] (83)

**Exercise 9.2.** Check that \( u \ast v = v \ast u \).

**Lemma 9.7.** If \( u \in C_c(\mathbb{R}^n) \), then \( u \) is uniformly continuous.

**Proposition 9.8.** If \( u \in L^1_{loc}(\mathbb{R}^n) \) and \( v \in C_c(\mathbb{R}^n) \) then \( u \ast v \in C(\mathbb{R}^n) \).
Proposition 9.9. If $u \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $v \in C^1_c(\mathbb{R}^n)$ then $u \ast v \in C^1(\mathbb{R}^n)$ and:
\[
\partial_i(u \ast v) = u \ast \partial_i v.
\] (84)

Proposition 9.10. If $u \in C_c(\mathbb{R}^n)$ then as $\epsilon \to 0$, $u \ast \Psi_\epsilon$ converges to $u$, uniformly on $\mathbb{R}^n$.

Proposition 9.11. If $u \in L^1(\mathbb{R}^n)$ then as $\epsilon \to 0$, $u \ast \Psi_\epsilon$ converges to $u$ in $L^1(\mathbb{R}^n)$.

Corollary 9.12. If $u \in L^1_{\text{loc}}(\mathbb{R}^n)$ satisfies, for all $v \in C^\infty_c(\mathbb{R}^n)$:
\[
\int uv = 0,
\] (85)
then $u = 0$. 

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10 Extension to $L^1(U)$

- Correction of the exercises.

10.1 Epsilon–neighborhoods and epsilon–interiors

For a subset $E$ of $\mathbb{R}^n$ and a point $x \in \mathbb{R}^n$ one defines:

$$d(x, E) = \inf \{|x - y| : y \in E\}. \quad (86)$$

**Exercise 10.1.** Check:

$$d(x, E) = 0 \iff x \in \overline{E}, \quad (87)$$

and:

$$|d(x, E) - d(y, E)| \leq |x - y|. \quad (88)$$

**Definition 10.1.** Let $E$ be a subset of $\mathbb{R}^n$:

- The $\epsilon$–neighborhood of $E$ is:
  $$\mathcal{N}_\epsilon(E) = \{x \in \mathbb{R}^n : d(x, E) < \epsilon\}. \quad (89)$$

- The $\epsilon$–interior of a set $E$ is:
  $$I_\epsilon(E) = \{x \in E : d(x, \mathbb{R}^n \setminus E) > \epsilon\}. \quad (90)$$

**Lemma 10.1.** If $\epsilon < \epsilon'$ then:

$$\mathcal{N}_\epsilon(E) \subseteq \mathcal{N}_{\epsilon'}(E). \quad (91)$$

**Lemma 10.2.** If $K \subseteq U \subseteq \mathbb{R}^n$, with $K$ compact and $U$ open, there exists $\epsilon > 0$ such that $\mathcal{N}_{\epsilon}(K) \subseteq U$.

**Exercise 10.2.** $\text{supp}(v * \mu_\epsilon) \subseteq \overline{\mathcal{N}_{\epsilon}(\text{supp}(v))}$.

**Exercise 10.3.** $\mathcal{N}_\epsilon(I_\epsilon(E)) \subseteq E$.

10.2 Two results

**Proposition 10.3.** Suppose $U$ is open in $\mathbb{R}^n$ and $u \in L^1_{\text{loc}}(U)$ satisfies:

$$\forall v \in C^\infty_c(U) \int uv = 0. \quad (92)$$

Then $u = 0$.

**Proposition 10.4.** Suppose $U$ is open in $\mathbb{R}^n$. Then $C^\infty_c(U)$ is dense in $L^1(U)$.
11 \( L^p \) spaces

22 september 2014

11.1 ... and convolution

• A reference for the curious: \cite{2} Section 6.1 (p. 181 – 186): Basic theory of \( L^p \) spaces.

**Proposition 11.1.** For functions on \( \mathbb{R}^n \):
\[
\|u * v\|_{L^p} \leq \|u\|_{L^1} \|v\|_{L^p}.
\] (93)

A useful variant is:

**Proposition 11.2.** For a function defined on \( U \) and the standard mollifier \( \mu_{\epsilon} \):
\[
\|u * \mu_{\epsilon}\|_{L^p(I_{\epsilon}(U))} \leq \|u\|_{L^p(U)}.
\] (94)

**Proposition 11.3.** Suppose \( U \) is open in \( \mathbb{R}^n \). Then \( C^\infty_c(U) \) is dense in \( L^p(U) \) for \( p \in [1, \infty[ \).

11.2 Exercises on Hölder’s inequality

**Exercise 11.1.** Suppose \( U \subseteq \mathbb{R}^n \) is open. Suppose we have reals \( p, q \geq 1 \) such that:
\[
\frac{1}{p} + \frac{1}{q} = 1.
\] (95)

Given \( u \in L^p(U) \), find a non-zero \( v \in L^q(U) \) such that:
\[
\int uv = \|u\|_{L^p(U)} \|v\|_{L^q(U)}.
\] (96)

Hint: find \( v \) such that \( uv = |u|^p \).

**Exercise 11.2.** Suppose \( U \subseteq \mathbb{R}^n \) is open and has finite measure (e.g. bounded). Suppose we have reals \( 1 \leq p < q \). Suppose \( u \in L^q(U) \). Show that \( u \in L^p(U) \) and:
\[
\|u\|_{L^p(U)} \leq \text{meas}(U)^{\frac{1}{q} - \frac{1}{p}} \|u\|_{L^q(U)}.
\] (97)

Hint: write \( |u|^p = 1 \cdot |u|^p \) and find a Hölder inequality that can be applied to this product.

**Exercise 11.3.** Suppose \( U \subseteq \mathbb{R}^n \) is open and that reals \( p, q, r \geq 1 \) satisfy:
\[
\frac{1}{p} + \frac{1}{q} = \frac{1}{r}.
\] (98)

Suppose \( u \in L^p(U) \) and \( v \in L^q(U) \). Show that \( uv \in L^r(U) \) and:
\[
\|uv\|_{L^r(U)} \leq \|u\|_{L^p(U)} \|v\|_{L^q(U)}.
\] (99)
Exercise 11.4. Suppose $U \subseteq \mathbb{R}^n$ is open and that reals $p, q, r \geq 1$ satisfy:

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1. \quad (100)$$

Suppose $u \in L^p(U)$, $v \in L^q(U)$ and $w \in L^r(U)$. Show that $uvw \in L^1(U)$ and:

$$\|uvw\|_{L^1(U)} \leq \|u\|_{L^p(U)} \|v\|_{L^q(U)} \|w\|_{L^r(U)}. \quad (101)$$

Exercise 11.5. Suppose $U \subseteq \mathbb{R}^n$ is open and that we have reals $p, q \geq 1$, as well as $\theta \in [0, 1]$. Define $r$ by:

$$\frac{1}{r} = \theta \frac{1}{p} + (1 - \theta) \frac{1}{q}. \quad (102)$$

Suppose $u \in L^p(U) \cap L^q(U)$. Show that $u \in L^r(U)$ and:

$$\|u\|_{L^r(U)} \leq \|u\|_{L^p(U)}^\theta \|u\|_{L^q(U)}^{(1-\theta)}. \quad (103)$$

Definition 11.1. Suppose $U \subseteq \mathbb{R}^n$ is open. For $u : U \rightarrow \mathbb{R}$ measurable, one defines

$$\|u\|_{L^\infty(U)} = \inf \{ C \in [0, \infty] : \text{for a.e. } x \in U \ |u(x)| \leq C \}. \quad (104)$$

One also defines:

$$L^\infty(U) = \{ u : U \rightarrow \mathbb{R} : u \text{ is measurable and } \|u\|_{L^\infty(U)} < \infty \}. \quad (105)$$

One checks that this is a vectorspace and that (104) defines a norm on this space. Moreover this normed vectorspace is complete.

Exercise 11.6. Try to extend the preceding exercises to the case where at least one of $p$, $q$ and $r$ is infinite.

### 11.3 Complement: integral operators

In the following (as always) we only consider integration with respect to Lebesgue measure.

Theorem 11.4. Suppose $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ are open. Suppose $K : U \times V \rightarrow \mathbb{R}$ is measurable, and that we have two constants $0 \leq C_0, C_1 < \infty$ such that:

$$\text{for a.e. } x \in U \quad \int |K(x, y)| \, dy \leq C_0. \quad (106)$$

and

$$\text{for a.e. } y \in V \quad \int |K(x, y)| \, dx \leq C_1. \quad (107)$$

Pick $p \in [1, \infty]$ and define $q \in [1, \infty]$ by:

$$\frac{1}{p} + \frac{1}{q} = 1. \quad (108)$$

Pick $u \in L^p(V)$. Then the following holds.
For a.e. $x \in U$ the function $y \mapsto K(x, y)u(y)$ is in $L^1(V)$. Moreover the expression:

$$(Ku)(x) = \int_V K(x, y)u(y)dy,$$  \hfill (109)

defines a function $Ku \in L^p(U)$ which satisfies:

$$\|Ku\|_{L^p(U)} \leq C_0^\frac{1}{p} C_1^\frac{1}{p} \|u\|_{L^p(V)}.$$  \hfill (110)

If one does not worry about which functions are measurable, this is just a clever application of Hölder’s inequality. The measure-theoretic details of the proof, which involves an appeal to the Fubini and Tonelli theorems, are beyond the scope of this course. Reference: [2] Theorem 6.18.
12 Weak derivatives

Weak derivatives. See [1] §5.2.
13 \( L^p \) spaces continued

- Correction of the exercises on Hölder.
14 Sobolev spaces

1 october 2014

15 Distributions

6 october 2014

Introduction to distributions (in the sense of L. Schwartz).
The canonical reference is now probably [4], but a bit too long for a course...

- Definition of distribution. Order.
- Examples: distribution associated with a $L^1_{loc}$ function. Dirac deltas.
  Principal value of $x \mapsto 1/x$.
- Multiplication by smooth functions. Associativity.
- Derivation. A Leibniz rule for products.
- Derivation of functions with a jump discontinuity.
- Convergence in the sense of distributions. Example: the standard mollifiers.
16 Distributions continued

8 october 2014

- Correction of exercise: $PV(x \mapsto 1/x)$ defines a distribution.
- Pathologies associated with attempts to multiply distributions.
- Partitions of unity.
- Restriction and support of a distribution.
17 Stokes’ theorem

- Domains
- Integration on boundaries
- Piola transform
- Stokes theorem
18  Approximation

Reference: [1] §5.3.

Evans presents three theorems on the approximation of elements in Sobolev spaces by smooth functions.

- Theorem 1 says that as long as one stays away from the boundary, smoothing by convolution will do the job.
- Theorem 2 shows how one can get global approximation by smooth functions defined on the whole of $U$.
- Theorem 3 shows how one can get global approximation by smooth functions defined on a neighborhood of $U$. Notice that contrary to Theorem 2, this uses some smoothness assumption on $\partial U$.

Of these three, the most useful one is the last, because it shows that there are dense subspaces of $W^{1,p}(U)$ for which integration by parts makes sense. This is used to define traces of functions on the boundary (See Evans §5.5).
19 No lecture

Work on the compulsory exercises (Appendix A).
20 No lecture

22 october 2014

Work on the compulsory exercises (Appendix A).
21 Correction of last year’s compulsory exercises

The following exercises constituted a subset of the compulsory exercises for autumn 2013. They will be corrected this day.

Some notation: For \( x \in \mathbb{R}^n \):

\[
|x| = (x_1^2 + \cdots + x_n^2)^{1/2}.
\] (111)

and for an open \( U \subseteq \mathbb{R}^n \) and a measurable vectorfield \( v : U \to \mathbb{R}^n \) we define:

\[
\|v\|_{L^p(U)} = \| |v| \|_{L^p(U)}.
\] (112)

Exercise 21.1. We define:

\[
\mathbb{R}_+ = \{ x \in \mathbb{R} : x > 0 \}.
\] (113)

For any continuous function \( w : \mathbb{R}^n \to \mathbb{R}_+ \) define the space:

\[
X_w = \{ u : \mathbb{R}^n \to \mathbb{R} : u \text{ is measurable and } \int u^2 w < \infty \}.
\] (114)

For \( u \in X_w \) we define:

\[
\|u\|_w = (\int u^2 w)^{1/2}.
\] (115)

(a) Fix a continuous function \( w : \mathbb{R}^n \to \mathbb{R}_+ \). Check that \( X_w \) is a vector space and that \( \| \cdot \|_w \) defines a norm on this space. In the following we always consider \( X_w \) as equipped with this norm. Show that for any open bounded subset \( U \) of \( \mathbb{R}^n \) we have a continuous linear surjection:

\[
X_w \to L^2(U),
\] (116)

defined by \( u \mapsto u|_U \).

(b) Pick two functions \( w_0 \) and \( w_1 \), which are continuous \( \mathbb{R}^n \to \mathbb{R}_+ \). For \( \theta \in [0,1] \) define:

\[
w_\theta = w_0^{1-\theta} w_1^\theta.
\] (117)

Show that for any \( \theta \in [0,1] \) and any \( u \in X_{w_0} \cap X_{w_1} \) we have \( u \in X_{w_\theta} \) and:

\[
\|u\|_{w_\theta} \leq \|u\|_{w_0}^{1-\theta} \|u\|_{w_1}^\theta.
\] (118)

(c) Consider the setup of question (b). Suppose we have, for each \( n \in \mathbb{N} \) a function \( u_n \in X_{w_0} \cap X_{w_1} \). Suppose that the sequence \( (u_n) \) converges in \( X_{w_0} \) to \( u \in X_{w_0} \). Suppose also that \( (u_n) \) is bounded in \( X_{w_1} \). Check that there exists a constant \( C \geq 0 \) such that for any open bounded subset \( U \) of \( \mathbb{R}^n \):

\[
\int_U u^2 w_1 \leq C.
\] (119)

Use the monotone convergence theorem to deduce that \( u \in X_{w_1} \). Show that \( (u_n) \) converges to \( u \) in \( X_{w_\theta} \) for any \( \theta \in [0,1] \).
Exercise 21.2. (a) Show that there exists a function $\theta \in C^\infty_c(\mathbb{R}^n)$ such that:

$$\forall x \in \mathbb{R}^n \quad |x| \leq 1 \Rightarrow \theta(x) = 1.$$  \hspace{1cm} (120)

(b) Let $\theta$ be a function as in (a). For non-zero $m \in \mathbb{N}$ define $\theta_m : \mathbb{R}^n \to \mathbb{R}$ by:

$$\theta_m(x) = \theta(x/m).$$  \hspace{1cm} (121)

Fix $p \in [1, +\infty[$.

- Show that for any $u \in L^p(\mathbb{R}^n)$, the functions $\theta_m u$ are in $L^p(\mathbb{R}^n)$ and converge to $u$ in $L^p(\mathbb{R}^n)$, as $m \to \infty$.

- Show that for any $u \in W^{1,p}(\mathbb{R}^n)$, the functions $\theta_m u$ are in $W^{1,p}(\mathbb{R}^n)$ and converge to $u$ in $W^{1,p}(\mathbb{R}^n)$, as $m \to \infty$.

(c) Use known results on smoothing by convolution to deduce that $C^\infty_c(\mathbb{R}^n)$ is dense in $W^{1,p}(\mathbb{R}^n)$ for all $p \in [1, +\infty[$.

Exercise 21.3. Pick a real $c > 0$. Given a function $v : \mathbb{R}^3 \to \mathbb{R}$ we consider the equation, on $\mathbb{R}^3$:

$$-\Delta u + c^2 u = v.$$  \hspace{1cm} (122)

Given $v \in C(\mathbb{R}^3)$, a strong solution to the equation (122) is a function $u \in C^2(\mathbb{R}^3)$ such that (122) holds in the sense of pointwise differentiation.

Given $v \in L^1_{loc}(\mathbb{R}^3)$, a weak solution to the equation (122) is a function $u \in L^1_{loc}(\mathbb{R}^3)$ such that for all $\phi \in C^\infty_c(\mathbb{R}^3)$:

$$\int u(-\Delta \phi + c^2 \phi) = \int v\phi.$$  \hspace{1cm} (123)

(a) Show that if $v \in C(\mathbb{R}^3)$, a function $u \in C^2(\mathbb{R}^3)$ is a strong solution to (122) if and only if it is a weak solution. Also show that the homogeneous equation (that is, with $v = 0$) always has infinitely many strong solutions.

(b) Find all functions $f \in C^2([0, +\infty[)$ such that the function $u_f : \mathbb{R}^3\setminus\{0\} \to \mathbb{R}$ defined by,

$$\forall x \in \mathbb{R}^3\setminus\{0\} \quad u_f(x) = f(|x|),$$  \hspace{1cm} (124)

satisfies (in the sense of pointwise differentiation):

$$-\Delta u_f + c^2 u_f = 0 \text{ on } \mathbb{R}^3\setminus\{0\}.$$  \hspace{1cm} (125)

(c) Define the function $E : \mathbb{R}^3 \to \mathbb{R}$ by:

$$E(x) = \frac{1}{4\pi|x|} \exp(-c|x|).$$  \hspace{1cm} (126)

Check that $E$ is locally integrable and show that for all $\phi \in C^\infty_c(\mathbb{R}^3)$:

$$\int E(-\Delta \phi + c^2 \phi) = \phi(0).$$  \hspace{1cm} (127)

Deduce that for any $v \in C^\infty_c(\mathbb{R}^3)$, $E * v$ is a strong solution to (122).

(d) Show that $E \in W^{1,2}(\mathbb{R}^3)$. Deduce that for $v \in L^2(\mathbb{R}^3)$, $E * v \in H^1(\mathbb{R}^3)$ and is a weak solution to (122).
(e) For this question you may use the result of question (c) in Exercise 21.2. Pick \( v \in L^2(\mathbb{R}^3) \) and \( u \in H^1(\mathbb{R}^3) \). Show that \( u \) is a weak solution to (122) if and only if for any \( w \in H^1(\mathbb{R}^3) \):

\[
\int \text{grad} \, u \cdot \text{grad} \, w + c^2 uw = \int vw. \tag{128}
\]

Deduce that, given \( v \in L^2(\mathbb{R}^3) \), equation (122) has a unique weak solution in \( H^1(\mathbb{R}^3) \).

**Exercise 21.4.** For any function \( u : \mathbb{R}^n \to \mathbb{R} \) and \( x \in \mathbb{R}^n \) we define \( \tau_x u : \mathbb{R}^n \to \mathbb{R} \) by:

\[
\forall y \in \mathbb{R}^n \quad (\tau_x u)(y) = u(y - x). \tag{129}
\]

Show that for any \( u \in H^1_0(\mathbb{R}^n) \) and any \( x \in \mathbb{R}^n \):

\[
\|u - \tau_x u\|_{L^2(\mathbb{R}^n)} \leq |x| \|\text{grad} \, u\|_{L^2(\mathbb{R}^n)}. \tag{130}
\]

You may start by noticing that for \( u \in C^\infty_c(\mathbb{R}^n) \):

\[
u(y) - u(z) = \int_0^1 \text{grad} \, u(z + t(y - z)) \cdot (y - z) \, dt. \tag{131}\]
22 Traces

Reference: [1] §5.5.

22.1 The trace operator $W^{1,p}(U) \to L^p(\partial U)$

Notice that there are essentially three ingredients in its definition:

- A lemma on the construction of linear operators (extension by continuity from dense subspaces): Lemma 8.6 in these notes.
- A so-called “estimate” on functions, proved for smooth enough functions defined on a half-space. In this setting integration by parts, Leibniz rules and Hölder’s inequality can all be safely applied.
- The technique of partitions of unity and straightening of the boundary.

22.2 Characterizations of $W^{1,p}_0(U)$

Theorem 22.1. Let $U$ be bounded, open and of class $C^1$. Then for any $u \in W^{1,p}(U)$ the following are equivalent:

- $u$ can be approximated by elements of $C^\infty_c(U)$ in the $W^{1,p}(U)$ norm.
- The trace of $u$ on $\partial U$ (defined in $L^p(\partial U)$) is zero.
- The extension of $u$ by zero outside $U$ is in $W^{1,p}(\mathbb{R}^n)$.

In the proof of this theorem we came across the following fact. For an element $u$ of $W^{1,p}(U)$, extended by zero outside $U$, its partial derivative in direction $i$, defined in the sense of distributions, contains two terms:

- the weak partial derivative in direction $i$ of $u$, defined in $U$, and extended by 0 outside $U$.
- a boundary term on $\partial U$ stemming from the integration by parts. This is a distribution with support in $\partial U$. 

23 Poincaré inequality and Dirichlet principle

3 November 2014

23.1 Poincaré

Proposition 23.1. Let $U \subseteq \mathbb{R}^n$ be open and bounded. There exists $C > 0$ such that for all $u \in H^1_0(U)$ we have:

$$\int_U |u|^2 \leq C \int_U |\nabla u|^2. \quad (132)$$

Theorem 23.2. Let $U \subseteq \mathbb{R}^n$ be open and bounded. For any $f \in L^2(U)$ there is a unique $u \in H^1_0(U)$ such that:

$$\forall u' \in H^1_0(U) \quad \int \nabla u \cdot \nabla u' = \int fu'. \quad (133)$$

Remark 23.1. • Equation (133) is called (by definition) the weak formulation of the equation:

$$-\Delta u = f. \quad (134)$$

• The boundary condition $u|_{\partial U} = 0$ is encoded by the condition $u \in H^1_0(U)$.

• $u$ is the unique point where the Dirichlet functional reaches its minimum on $H^1_0(U)$. Any minimizing sequence converges to $u$ in $H^1_0(U)$.

23.2 Lax-Milgram

The following defines orthogonal projection in a Hilbert space.

Proposition 23.3. Let $X$ be a Hilbert space and $Y$ a closed subspace. Then for any $u \in X$, there is a unique $v \in Y$ such that $u - v \perp Y$.

Proof. Let $\langle \cdot, \cdot \rangle$ denote the scalar product of $X$. The vector space $Y$, equipped with the restriction of $\langle \cdot, \cdot \rangle$ is a Hilbert space. Also, $\langle u, \cdot \rangle$ is a continuous linear form on $Y$. We may apply the Riesz representation theorem. □

The following theorem, proposition or lemma (according to taste) is known as Lax-Milgram.

Proposition 23.4. Let $X$ be a Hilbert space, and let $a$ be continuous bilinear form on $X$, which is coercive in the sense that for some $\alpha > 0$:

$$\forall u \in X \quad a(u, u) \geq \alpha \|u\|^2. \quad (135)$$

Then for any $l \in X^*$ there exists a unique $u \in X$ such that:

$$\forall v \in X \quad a(u, v) = l(v). \quad (136)$$

Proof. (sketch)

Let $\langle \cdot, \cdot \rangle$ denote the scalar product of $X$.

For any $u \in X$ denote by $Au \in X$ the unique solution to the problem:

$$\forall v \in X \quad \langle Au, v \rangle = a(u, v), \quad (137)$$
given by the Riesz representation theorem.

Then $A : X \rightarrow X$ is linear and continuous. Moreover:

$$\|Au\| \geq \alpha \|u\|.$$  \hspace{1cm} (138)

One deduces that $\text{ran } A$ is closed.

If $\text{ran } A \neq X$ we may find, by the above corollary, a non-zero $u \in X$ such that $u \perp \text{ran } A$. We would get:

$$\alpha \|u\|^2 \leq a(u, u) = (Au, u) = 0.$$ \hspace{1cm} (139)

Therefore $A$ is an isomorphism.

$\Box$

\textit{Remark 23.2.} Consider the hypotheses of the proposition and suppose in addition that $a$ is symmetric. Then $a$ is an inner product on $X$. It defines a norm on $X$ which is equivalent to the original one. Therefore $(X, a)$ is a Hilbert space and Lax-Milgram reduces to the Riesz representation theorem.
24 Second order elliptic PDEs

24.1 Variational formulation

[1] §6.1, 6.2.1, 6.2.2.

24.2 Galerkin method

Céa’s lemma.
25 The space $W^{1,1}$ of an interval and extension

10 November 2014

25.1 The space $W^{1,1}$ of an interval

Proposition 25.1. Let $a < b$ be two reals and consider the interval $I = [a, b[$.

(i) Suppose $v \in L^1(I)$ and define $u : I \to \mathbb{R}$ by:

$$u(x) = \int_a^x v(y) dy.$$  \hfill (140)

Then:

a. $u$ is continuous.

b. $u$ is differentiable almost everywhere, (in the classical sense) with $u'(x) = v(x)$.

c. $v$ is the weak derivative of $u$.

(ii) We have:

a. There exists $C > 0$ such that for all $u \in W^{1,1}(I)$:

$$\|u\|_{L^\infty(I)} \leq C \|u\|_{W^{1,1}(I)}. \hfill (141)$$

b. Any $u \in W^{1,1}(I)$ coincides a.e. with an element of $C^1(T)$.

c. If $u \in W^{1,1}(I) \cap C^1(T)$ and $v \in L^1(I)$ denotes the weak derivative of $u$, we have:

$$\forall x \in I \quad u(x) = u(a) + \int_a^x v(y) dy.$$  \hfill (142)

Proof. (i)

a. by the Lebesgue dominated convergence theorem

b. by the Lebesgue differentiation theorem.

c. We write, for any $\phi \in C^\infty_0(I)$:

$$- \int_a^b u(x) \phi'(x) dx = - \int_a^b \left( \int_a^x v(y) \phi'(x) dy \right) dx,$$  \hfill (143)

$$= - \int_a^b \int_a^b v(y) \phi'(x) \chi(x, y) dy dx,$$  \hfill (144)

$$= - \int_a^b ( \int_y^b v(y) \phi'(x) dx ) dy,$$  \hfill (145)

$$= \int_a^b v(y) \phi(y) dy,$$  \hfill (146)

where $\chi$ is defined by:

$$\chi(x, y) = \begin{cases} 1 & \text{for } y \leq x, \\ 0 & \text{for } x < y, \end{cases} \hfill (147)$$

and Fubini was used.
For $u \in C^1(I)$ we have:
\[
|u(x)| = |u(y)| + \left| \int_y^x u'(z) \, dz \right|,
\]  
\[
\leq |u(y)| + \int_a^b |u'(z)| \, dz.
\]  
Hence:
\[
(b - a)|u(x)| \leq \int_a^b |u(y)| \, dy + (b - a) \int_a^b |u'(z)| \, dz.
\]  
So:
\[
\|u\|_{L^\infty(I)} \leq \frac{1}{b-a} \|u\|_{L^1(I)} + \|u'\|_{L^1(I)}.
\]  
For $u \in W^{1,1}(I)$, we may find a sequence $u_n \in C^1(I)$, such that $u_n \to u$ in $W^{1,1}(I)$. By the inequality (151) it is Cauchy in $C(T)$. The limit in $C(T)$ must be equal a.e. to the limit in $W^{1,1}(I)$ (why?). So we may consider that $u \in C^1(T)$. Then (151) holds for each $u_n$ and we may pass to the limit $n \to \infty$.

Exercise 25.1. Let $I$ be a bounded open interval. Show that if $p \in ]1, \infty[$, then for any $u \in W^{1,p}(I)$ we have for a.e. $x, y \in I$:
\[
|u(x) - u(y)| \leq |x - y|^{1-1/p} \|u'\|_{L^p(I)}.
\]  
Exercise 25.2. Let $U \subseteq \mathbb{R}^n$ be open. Pick $u \in L^1(U)$. Show that for any $\epsilon > 0$ there exists $\delta > 0$ such that for any open $V \subseteq U$:
\[
\text{meas}(V) \leq \delta \Rightarrow \int_V |u| \leq \epsilon.
\]  
Hint: You may check it first when $u$ is a step function, or a continuous function with compact support.

Definition 25.1. A function $u : [a, b] \to \mathbb{R}$ is said to be absolutely continuous if for any $\epsilon > 0$ there exists $\delta > 0$ such that for any $n \in \mathbb{N}$, any family of two by two disjoint intervals $]a_i, b_i[ \subseteq [a, b]$ for $i \in [0, n]$, we have:
\[
\sum_{i=0}^n |b_i - a_i| \leq \delta \Rightarrow \sum_{i=0}^n |u(b_i) - u(a_i)| \leq \epsilon.
\]  
Exercise 25.3. Check that functions in $W^{1,1}(I)$ are absolutely continuous, and that absolutely continuous functions are continuous.
25.2 The Cantor - Lebesgue function

The Cantor set $C$ is a certain compact subset of $[0, 1]$ with measure 0.

The Cantor - Lebesgue function $f$ is a certain continuous function $[0, 1] \rightarrow [0, 1]$ with the following two rather paradoxical properties:

- $f|_C: C \rightarrow [0, 1]$ is onto.
- on the complement of $C$, $f$ is differentiable with differential 0.

In particular the Cantor - Lebesgue function is a continuous and a.e. differentiable function, such that the a.e. differential is in $L^1([0, 1])$, but such that, for $x > 0$:

$$f(x) \neq f(0) + \int_0^x f'(y)dy,$$  \hfill (156)

where we denote by $f'$ the a.e. differential of $f$ (which is 0). Notice that $f$ is not in $W^{1,1}([0, 1])$.

References: page 38 – 39 in [2], or page 37 – 38 in [9].

25.3 Extension

26  Gagliardo-Nirenberg-Sobolev inequality

13 november 2014

See also the scanned notes.
27 Rellich compactness

18 november 2014

Complements: see also the scanned notes for more information on compact operators.
A Compulsory exercises

To be handed in 29 October 2014.

Exercise A.1. Let $u$ and $v$ be two continuous maps $\mathbb{R}^n \to \mathbb{R}$. Suppose $\text{supp } u \subseteq E$ and $\text{supp } v \subseteq F$, with $E$ and $F$ closed subsets of $\mathbb{R}^n$, $F$ being, in addition, bounded.

(a) Show that $E + F$ is closed, where we denote:

$$E + F = \{ x + y : x \in E, y \in F \}.$$  \hfill (157)

(b) Show that $\text{supp}(u * v) \subseteq E + F$.

(c) (optional) Discuss whether this inclusion is strict or not.

Exercise A.2. Let $X$ be a real Hilbert space, with scalar product $\langle \cdot, \cdot \rangle$. Let $E$ be a closed convex subset of $X$. Suppose that $f$ is a continuous linear form on $X$. Let $J : X \to \mathbb{R}$ be the functional defined by:

$$J(u) = \frac{1}{2} \langle u, u \rangle - f(u).$$  \hfill (158)

(a) Show that there is a unique point $u$ in $E$ such that:

$$\forall v \in E \quad J(u) \leq J(v).$$  \hfill (159)

(b) Show that $u$ is uniquely characterized by the properties $u \in E$ and:

$$\forall v \in E \quad (u, v - u) - f(v - u) \geq 0.$$  \hfill (160)

(c) Deduce from the above, that given $v \in X$ there is a unique point $u \in E$, where $\|u - v\|$ is minimal.

Exercise A.3. In this exercise we identify $\mathbb{C}$ with $\mathbb{R}^2$ in the standard way. We say that $f : \mathbb{C} \to \mathbb{C}$ is an entire function if, for each $x \in \mathbb{C}$, the following limit exists in $\mathbb{C}$:

$$f'(x) = \lim_{h \to 0, h \in \mathbb{C} \setminus \{0\}} \frac{f(x + h) - f(x)}{h},$$  \hfill (161)

and moreover $f' : \mathbb{C} \to \mathbb{C}$ is a continuous function.

(a) Show that $f$ is entire iff $f$ is of class $\mathcal{C}^1$ and satisfies the partial differential equation:

$$\frac{\partial_1 f}{1} + i \frac{\partial_2 f}{2} = 0.$$  \hfill (162)

(b) Show that complex polynomials are entire functions.

(c) Show that if $f$ is entire and does not take the value 0, then $1/f$ is entire and $(1/f)' = -f'/f^2$.

(d) Show that if $f$ is entire and of class $\mathcal{C}^2$, then the real and imaginary parts of $f$ are harmonic functions.

(e) Use Liouville’s theorem and the above, to show that any non–constant complex polynomial has a root.

48
Exercise A.4. (a) On \(\mathbb{R}^4\) we denote the coordinates of \(x \in \mathbb{R}^4\) by \(x = (x_0, x_1, x_2, x_3)\), so that the partial derivatives become \(\partial_0, \partial_1, \partial_2, \partial_3\). Pick \(c > 0\). The wave equation, for a function \(u : \mathbb{R}^4 \to \mathbb{R}\) is:

\[
\partial_0^2 u = c^2 (\partial_1^2 + \partial_2^2 + \partial_3^2) u. \tag{163}
\]

Assume there is smooth function \(v : \mathbb{R}^3 \to \mathbb{C}\) and \(\omega > 0\) such that, for all \(x \in \mathbb{R}^4\):

\[
u(x_0, x_1, x_2, x_3) = \Re \{\exp(i\omega x_0) v(x_1, x_2, x_3)\}. \tag{164}\]

Show that \(u\) solves (163) if and only if \(v\) solves the Helmholtz equation on \(\mathbb{R}^3\):

\[
\Delta v + k^2 v = 0. \tag{165}
\]

with \(k = \omega/c\).

(b) For \(y \in \mathbb{R}^3\) define the plane wave \(v_y : \mathbb{R}^3 \to \mathbb{C}\) by, for all \(x \in \mathbb{R}^3\):

\[
v_y(x) = \exp(iy \cdot x) \tag{166}
\]

Show that \(v_y\) satisfies the Helmholtz equation (165) if and only if \(|y| = k\).

(c) A smooth spherically symmetric function \(v : \mathbb{R}^3 \setminus \{0\} \to \mathbb{C}\) is one such that there exists a smooth function \(w : [0, +\infty[ \to \mathbb{C}\) such that:

\[
\forall x \neq 0 \quad v(x) = w(|x|). \tag{167}
\]

Find all smooth functions \(w : [0, +\infty[ \to \mathbb{C}\) such that \(v\) defined by (167) solves the Helmholtz equation (165) on \(\mathbb{R}^3 \setminus \{0\}\).

Hint: look at the last question first.

(d) Show that for all \(\psi \in \mathcal{C}^\infty(\mathbb{R}^3)\) and all \(\phi \in \mathcal{C}^\infty_c(\mathbb{R}^3)\):

\[
\int_{\mathbb{R}^3} (\Delta \psi + k^2 \psi) \phi = \int_{\mathbb{R}^3} \psi(\Delta \phi + k^2 \phi). \tag{168}
\]

(e) Define \(w : [0, +\infty[ \to \mathbb{C}\) by:

\[
w(r) = \frac{\exp(ikr)}{4\pi r}, \tag{169}
\]

and \(v\) by (167). Check that for any \(R > 0\), \(w \in L^1(B(0, R))\).

Show that for all \(\phi \in \mathcal{C}^\infty_c(\mathbb{R}^3)\):

\[
\int_{\mathbb{R}^3} v(\Delta \phi + k^2 \phi) = -\phi(0). \tag{170}
\]

Restate this formula in terms of the Dirac delta.
B Various exercises

Exercise B.1. Choose \(a < b\) in \(\mathbb{R}\) and let \(I = [a, b]\). Show that there exists a constant \(C\) such that for all \(u \in H^1(I) = W^{1,2}(I)\):

\[
\|u\|_{L^\infty(I)} \leq C\|u\|_{L^2(I)}^{1/2}\|u\|_{H^1(I)}^{1/2}.
\]  

(171)

\(Hint:\) use that for smooth functions:

\[
u(y)^2 = \int_x^y 2u(s)u'(s)ds + u(x)^2.
\]  

(172)

Exercise B.2. Choose \(a < b\) in \(\mathbb{R}\) and let \(I = [a, b]\). Show that there exists a \(C > 0\) such that for all \(u \in H^1(I)\):

\[
\|u - \frac{1}{b-a} \int_I u\|_{L^2(I)} \leq C\|u'\|_{L^2(I)}.
\]  

(173)

\(Hint:\) Use the compactness of the injection \(H^1(I) \to L^2(I)\), or that smooth functions on an interval attain their average.

Exercise B.3. 1. Let \(I\) be the interval \([a, b]\). Define for any smooth function \(u \in C^\infty(T)\), the affine function \(P_Tu : I \to \mathbb{R}\) by:

\[
\forall x \in I \quad (P_Tu)(x) = u(a) + \frac{x-a}{b-a}(u(b) - u(a)).
\]  

(174)

Check that \(P_T\) is a projection. Show that \(P_T\) has a unique extension to a continuous operator \(H^1(I) \to H^1(I)\).

2. Deduce from Exercise B.2 that there exists a \(C > 0\) such that for all \(u \in H^1(I)\):

\[
\|u - P_Tu\|_{L^2(I)} \leq C\|u'\|_{L^2(I)}.
\]  

(175)

3. In the preceding question \(a\) and \(b\) were considered given – we now study the dependence of the constant \(C\) on \(a\) and \(b\). Show that there exists a \(C > 0\) such that for all \(a < b\) and all \(u \in H^1(I)\) (with \(I = [a, b]\)):

\[
\|u - P_Iu\|_{L^2(I)} \leq C(b-a)\|u'\|_{L^2(I)}.
\]  

(176)

\(Hint:\) Use the preceding result in the case \(I = [0, 1]\) and make a change of variable from \([0, 1]\) to \([a, b]\).

4. (Optional) By similar arguments show that there exists a \(C > 0\) such that for any \(h > 0\), any interval \(I\) of length \(h\), we have for any \(u \in H^2(I)\):

\[
\|(u - P_Iu)'\|_{L^2(I)} \leq Ch\|u''\|_{L^2(I)}.
\]  

(177)

and for any \(u \in H^1(I)\):

\[
\|(P_Iu)'\|_{L^2(I)} \leq C\|u'\|_{L^2(I)}.
\]  

(178)
Exercise B.4. Let $I$ be the interval $[0,1]$.

1. Pick $n > 0$ an integer. Define for integer $i \in [0,n]$: 
   \[ x_i^n = i/n \in I. \]  
   \[ \text{(179)} \]

   Define the function $\lambda_i^n : I \to \mathbb{R}$ by:
   \[ \lambda_i^n(x) = \begin{cases} 
   0 & \text{for } x \leq x_{i-1}^n, \\
   n(x - x_{i-1}^n) & \text{for } x_{i-1}^n < x \leq x_i^n, \\
   1 - n(x - x_i^n) & \text{for } x_i^n < x \leq x_{i+1}^n, \\
   0 & \text{for } x_{i+1}^n < x. 
   \end{cases} \]  
   \[ \text{(180)} \]

   Check that $\lambda_i$ is continuous on $I$, affine on every sub-interval $[x_j^n, x_{j+1}^n]$ and that:
   \[ \lambda_i^n(x_j^n) = \delta_{ij}. \]  
   \[ \text{(181)} \]

   Define an operator $P_n : H^1(I) \to H^1(I)$ by:
   \[ P_n u = \sum_{i=0}^{n} u(x_i) \lambda_i^n. \]  
   \[ \text{(182)} \]

   Show that $P^n$ is a continuous projector in $H^1(I)$.

   Show, using (176) and (177) on each subinterval $[x_j^n, x_{j+1}^n]$, that for any $u \in H^2(I)$ there exists $C > 0$ such that for all $n$:
   \[ \|u - P_n u\|_{H^1(I)} \leq C/n. \]  
   \[ \text{(183)} \]

2. Define $X_n$ by:
   \[ X_n = \text{span}\{\lambda_i^n : 0 < i < n\}. \]  
   \[ \text{(184)} \]

   Check that for any $u \in H^1_0(I)$ we have that $P_n u \in X_n$.

   Show using (176) and (178) that $P_n$ is uniformly bounded in $H^1_0(I) \to H^1(I)$, that is, there exists $C > 0$ such that for all $u \in H^1_0(I)$ and all $n$:
   \[ \|P_n u\|_{H^1(I)} \leq C\|u\|_{H^1(I)}. \]  
   \[ \text{(185)} \]

   Then show that for any $u$ in $H^1_0(I)$, $P_n u$ converges to $u$ in $H^1(I)$, as $n \to \infty$.

3. Pick $\delta > 0$ and $\gamma \in \mathbb{R}$. For a given $f \in L^2(I)$ we want to solve the equation:
   \[ -\delta u'' + \gamma u' = f. \]  
   \[ \text{(186)} \]

   with boundary conditions $u(0) = u(1) = 0$.

   Define the bilinear form $a$ on $H^1_0(I)$ by, for all $u,v \in H^1_0(I)$:
   \[ a(u, v) = \delta \int u'v' + \gamma \int u'v. \]  
   \[ \text{(187)} \]

   Show that there exists $C > 0$ such that for all $u \in H^1_0(I)$:
   \[ a(u, u) \geq 1/C\|u\|_{H^1(I)}^2. \]  
   \[ \text{(188)} \]

   Apply the Lax-Milgram theorem to show that there is a unique solution $u_* \in H^1_0(I)$ to the weak formulation of (186). Check that $u_* \in H^2(I)$.
4. Check that there is a unique $u_n \in X_n$ solving:

$$\forall v \in X_n \quad a(u_n, v) = \int fv. \quad (189)$$

5. Show using (188) that there is a constant $C$ such that for all $n$:

$$\|u_* - u_n\|_{H^1(I)} \leq C\|u_* - P_n u_*\|_{H^1(I)}. \quad (190)$$

Deduce that $u_n$ converges to $u_*$ in $H^1(I)$, at least at the rate $O(1/n)$.

**Exercise B.5.** (a) Let $U \subseteq \mathbb{R}^n$ be open and bounded. Fix $p \in [1, +\infty]$. Show that there exists $C > 0$ such that for all $u \in W^{1,p}_0(U)$:

$$\|u\|_{L^p(U)} \leq C \|\text{grad } u\|_{L^p(U)} \quad (191)$$

(b) Fix $u \in C_0^\infty(\mathbb{R}^n)$ and consider for each $\lambda > 0$ the function $u_\lambda : \mathbb{R}^n \to \mathbb{R}$ defined by:

$$\forall x \in \mathbb{R}^n \quad u_\lambda(x) = u(\lambda x). \quad (192)$$

Express $\|u_\lambda\|_{L^p(\mathbb{R}^n)}$ and $\|\text{grad } u_\lambda\|_{L^p(\mathbb{R}^n)}$ in terms of $\lambda$, $\|u\|_{L^p(\mathbb{R}^n)}$ and $\|\text{grad } u\|_{L^p(\mathbb{R}^n)}$.

(c) Let $U \subseteq \mathbb{R}^n$ be open. Fix $p \in [1, +\infty]$. Suppose that for each $R > 0$ there exists $x \in U$ such that $B(x, R) \subseteq U$. Show that there does not exist $C > 0$ such that for all $u \in W^{1,p}_0(U)$:

$$\|u\|_{L^p(U)} \leq C \|\text{grad } u\|_{L^p(U)}. \quad (193)$$

**Exercise B.6.** Choose $p, q \in [1, \infty]$ and $\theta \in [0, 1]$. Define $r$ by:

$$1/r = \theta/p + (1 - \theta)/q. \quad (194)$$

Let $U$ be an open subset of $\mathbb{R}^n$.

a) Show that for all $u \in L^p(U) \cap L^q(U)$, $u \in L^r(U)$ and:

$$\|u\|_{L^r(U)} \leq \|u\|_{L^p(U)}^{\theta} \|u\|_{L^q(U)}^{1-\theta}. \quad (195)$$

b) Deduce that if a sequence of functions $U \to \mathbb{R}$ is bounded in $L^p(U)$ and converges in $L^q(U)$ then it converges also in $L^r(U)$.

**Exercise B.7.** Let $U = ]0, 2\pi[$. For $n > 0$ define $u_n : U \to \mathbb{R}$ by:

$$u_n(x) = \sin(nx). \quad (196)$$

a) Show that for all $p \in [1, \infty]$, the sequence $(u_n)_{n \in \mathbb{N}}$ is bounded in $L^p(U)$.

b) Show that for any $\phi \in C_0^\infty(U)$:

$$\lim_{n \to \infty} \int u_n \phi = 0. \quad (197)$$

c) Deduce that if a subsequence of $(u_n)_{n \in \mathbb{N}}$ converges in $L^1(U)$ to some function $v$, then $v = 0$.

d) Show that actually no subsequence of $(u_n)_{n \in \mathbb{N}}$ converges in $L^1(U)$.

e) Is the canonical injection $L^p(U) \to L^1(U)$ compact for some $p$?
Exercise B.8. Let $U = [0, 1]$. For $n > 0$ define $u_n : U \to \mathbb{R}$ by:

$$u_n(x) = \begin{cases} n & \text{for } x \in [0, 1/n], \\ 0 & \text{for } x \in [1/n, 1]. \end{cases}$$

(198)

a) Remark that $u_n$ is bounded in $L^1(U)$.
b) Show that for any $\phi \in C^0_c(U)$:

$$\lim_{n \to \infty} \int u_n \phi = 0.$$  

(199)
c) Is there a $u \in L^1(U)$ such that for all $v \in L^\infty(U)$ we have:

$$\lim_{n \to \infty} \int u_n v = \int uv?$$

(200)

Exercise B.9. Fix an integer $n > 0$. Let $\phi : \mathbb{R}^n \to \mathbb{R}$ be a smooth function such that:

$$\supp \phi \subset B(0, 1),$$

(201)

$$\int \phi = 1,$$  

(202)

$$\forall x \in \mathbb{R}^n \phi(x) \geq 0.$$  

(203)

$$\forall x \in \mathbb{R}^n \phi(x) = \phi(-x).$$  

(204)

For any $\epsilon > 0$ we define $\phi_\epsilon$ by:

$$\phi_\epsilon(x) = \epsilon^{-n} \phi(x/\epsilon).$$

(205)

a) Pick $\psi \in C^\infty_c(\mathbb{R}^n)$ and define $\hat{\psi}$ by $\hat{\psi}(x) = \psi(-x)$. Show that for all $u, v \in L^2(\mathbb{R}^n)$ we have:

$$\int u (v * \psi) = \int (u * \hat{\psi}) v.$$  

(206)

b) We suppose that we have a function $u \in L^2(\mathbb{R}^n)$ for which there exists a $M > 0$ such that for each $i \in \{1, \cdots, n\}$ and all $\epsilon > 0$:

$$\|\partial_i (\phi_\epsilon * u)\|_{L^2(\mathbb{R}^n)} \leq M.$$  

(207)

Show that there exists a $C > 0$ such that for all $\psi \in C^\infty_c(\mathbb{R}^n)$:

$$| \int u \partial_i \psi | \leq C \|\psi\|_{L^2(\mathbb{R}^n)}.$$  

(208)

c) Deduce that $u \in H^1(\mathbb{R}^n)$. 

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References


