Limited diffraction beams and a new decomposition of arbitrary annular beams.

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Focusing

- 4 transmit focal zones
- dynamic focusing on receive
Spherical focus - Unfocused

Spherical focus (60 mm)
f = 2.275 MHz, R = 7.5 mm

Flat piston, $4R^2/\lambda = 332$ mm
Outline

1. Background: Solutions to wave equation
2. Limited diffraction beams: Properties
3. Expansion in sums of Bessel beams
4. Example: 10 ring Bessel transducer
5. Example: Conical transducer
Wave equation - Cartesian coordinates

\[ \nabla \vec{s} = \frac{\partial^2 s}{\partial x^2} + \frac{\partial^2 s}{\partial y^2} + \frac{\partial^2 s}{\partial z^2} = \frac{1}{c^2} \frac{\partial^2 s}{\partial t^2} \]

- \( s \) is a general scalar field (electromagnetics: electric or magnetic field, acoustics: sound pressure ...).
- \( c \) is the speed of propagation.

Monochromatic solution in \( x \), plane wave:

\[ s(x, t) = A \exp\{j(\omega t - k_x x)\} \]

where \( k_x = 2/\lambda \) is the wave number.

General monochromatic plane wave:

\[ s(\vec{x}, t) = A \exp\{j(\omega t - \vec{k} \cdot \vec{x})\} \]
Wave equation in spherical coordinates

Solution exhibits spherical symmetry:

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial s}{\partial r} \right) = \frac{1}{c^2} \frac{\partial^2 s}{\partial t^2}$$

Monochromatic solution, spherical wave:

$$s(r, t) = \frac{A}{r} \exp \{ j(\omega t - kr) \}$$
Wave equation - cylindrical coordinates

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial s}{\partial r} \right) + \frac{\partial^2}{\partial z^2} s = \frac{1}{c^2} \frac{\partial^2 s}{\partial t^2}
\]

- One solution is the family of Bessel beams, where lateral distribution is given by Bessel-function:

\[
s(r, z, t) = J_n(\alpha r) e^{-j(\beta z - \omega t)}, \quad k^2 = \beta^2 + \alpha^2.
\]

- First alluded to by Stratton, 1941: “Undistorted progressive waves”

- Diffraction-free beam \((J_0)\) ’rediscovered’ and verified experimentally for optics by Durnin, 1987

- Limited diffraction beam applied to ultrasonics by Hsu, Margetan, Thompson, 1989 and Lu and Greenleaf, 1990.

- General \(J_n\) solution found by Lu and Greenleaf, 1992.
1. Background: Solutions to wave equation

2. Limited diffraction beams: Properties

3. Expansion in sums of Bessel beams

4. Example: 10 ring Bessel transducer

5. Example: Conical transducer
0. order Bessel beam is the only one with a well-defined mainlobe:

\[ s(r, z, t) = J_0(\alpha r)e^{-j(\beta z - \omega t)} \]

Note that \( \alpha = 0 \) gives a plane wave.
Properties of $J_0$ beam

\[ s(r, z, t) = J_0(\alpha r) e^{-j(\beta z - \omega t)}, \beta^2 = k^2 - \alpha^2 \]

For $0 < \alpha < k = \omega/c$:

- Plane waves traveling at angle $\phi = \sin^{-1}(\alpha/k)$ as from a cone

\[ \phi \]

- Beamwidth is given by $J_0(\alpha r)$ - independent of $z$
- No diffraction for infinite aperture source
- Broadband Bessel beam is dispersive
Properties of finite aperture $J_0$ beam

- Source of radius $R$
- Cone angle is $\phi$
- Depth of field is $DOF = \frac{R}{\tan \phi} \approx \frac{Rk}{\alpha}$
Flat piston - Bessel beam

Flat piston, $4R^2/\lambda = 332$ mm

$J_0$ beam, $\alpha = 736$ m$^{-1}$, DOF = 95 mm

f=2.275 MHz, R = 7.5 mm
Spherical focus - Bessel beam

Spherical focus (60 mm)  \( J_0 \) beam, \( \alpha = 736 \text{ m}^{-1}, \phi = 4.4^\circ \)

\( f=2.275 \text{ MHz}, R = 7.5 \text{ mm} \)
Bessel beams with 2 or 3 lobes

\[ \alpha = 736 \text{ m}^{-1}, \text{DOF} = 95 \text{ mm} \]

\[ \alpha = 1149.8 \text{ m}^{-1}, \text{DOF} = 61 \text{ mm} \]

\[ f = 2.275 \text{ MHz}, R = 7.5 \text{ mm} \]
Experimental verification

d=2.5 mm, \( \Delta d = 10 \ \mu m \), f=305 mm, R = 3.5 mm, \( \lambda = 639 \ \text{nm} \) \( \Rightarrow \) \( Z_{\text{max}} = 85 \ \text{mm} \), width = 70 \( \mu m \)
**X waves**

- Integration over infinite number of Bessel beams:

\[
s(r, z, t) = \int_{0}^{\infty} B(k) J_0(kr \sin \phi) e^{-j(k \cos \phi - \omega t)} e^{-a_0 k}
\]

- \(\alpha = k \sin \phi, \beta = k \cos \phi\)

- All frequency components travel at angle \(\phi\)

- Non-dispersive

- Lu & Greenleaf 1992
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Orthogonality of Bessel function

\[ \int_0^a J_0(\alpha_k x) \cdot J_0(\alpha_l x) \, dx = 0, \quad k \neq l; \]

where \( \alpha_k = r_i/a \) and \( J_0(x_k) = 0, \quad k = 1, \ldots \)
and \( a \) is a modelling radius (\( a = x_1 = 2.405 \ldots \) in plot).
Fourier-Bessel series

For any absolutely integrable function, \( f(r) \):

\[
f(r) = \sum_{i=1}^{\infty} A_i \cdot J_0(\alpha_i r)
\]

where

\[
A_i = \frac{2}{a^2 J_1^2(x_i)} \int_{0}^{a} f(r) \cdot J_0(\alpha_i r) r \, dr
\]
Fourier-Bessel series for annular array beams

- Wave equation is linear

- $\alpha > k \Rightarrow$ evanescent wave ($k^2 = \alpha^2 + \beta^2$).

- Upper limit for index $l_{max} \approx ka/\pi + 1/4$

\[
f(r) = \sum_{l=1}^{l_{max}} A_l \cdot J_0(\alpha_l r)
\]

Higher index $\Rightarrow$ smaller DOF and narrower beam

- Any beam generated by a finite annular aperture can be described in terms of a finite sum of Bessel beams (Holm & Fox 1999, Fox & Holm, 2002)
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Example: 10-ring Bessel transducer

Quantised and first 10 Basis Bessel Functions

Ideal and quantized pressure profile
Field generated by 10-ring Bessel transducer.

$f = 2.5$ MHz, $R = 25.4$ mm, $\alpha = 1202.45$, DOF = 221 mm.

Notice irregularities up to about 70 mm.
10-ring Bessel transducer

Coefficients for 10-ring Bessel transducer. Large contributions from: 29, 30 (DOF=71, 68 mm), 49 (DOF=36 mm), 68, 69 (DOF = 19, 18 mm).
Ratio of squared undesired and desired: 114%
10-ring Bessel transducer

Field generated by 10-ring Bessel transducer
Strong components no. 29, 30 with DOF of 71, 68 mm.
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Example 2: Conical transducer

Conical transducer, $\phi = 6.77^\circ$, DOF = 215 mm, f=2.5 MHz, R=25.48 mm

$$s(\rho, z) \approx \sin \theta \sqrt{\frac{2\pi z \cos \theta}{k}} \cdot J_0(k \rho \sin \theta)$$
Conical transducer - Bessel beam

Conical transducer, $\phi = 6.77^\circ$  
$J_0$ beam, $\alpha = 1202.45 \text{ m}^{-1}$

f=2.5 MHz, R = 25.48 mm, DOF = 215 mm.
Conical transducer

254% energy outside of coefficient 10
Conical transducer with shift

14.3% energy outside of coefficient 10
Can geometry of cone be improved?

Bessel Design, $N = 250$, $R = 25.48$ mm
Conical transducer with rounded center

13.2% energy outside of coefficient 10
Conical transducer with shift and weighting

2.9% energy outside of coefficient 10
Relationship between Bessel beam and conical beam

\[ J_0(x) \approx \sqrt{2/\pi x} \cdot \cos(x - \pi/4) \]

- Shift by \( \pi/4 \) is essential in order to bring down error to 14.3%
- Best unweighted shape, with more pointed cone tip, gives 13.2%
- Weighting by \( 1/\sqrt{x} \) brings it further down to 2.9%

Very close relationship between Bessel beams and conical transducers
• Wave equation
• Limited diffraction beams
• Description of any annular source in terms of Bessel beams
• 10 ring Bessel transducer
• Conical transducer