Nonlinear acoustic wave equations with fractional loss operators

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Fractional derivatives are well suited to describe wave propagation in complex media. When introduced in classical wave equations, they allow a modeling of attenuation and dispersion that better describes sound propagation in biological tissues. Traditional constitutive equations from solid mechanics and heat conduction are modified using fractional derivatives. They are used to derive a nonlinear wave equation which describes attenuation and dispersion laws that match observations. This wave equation is a generalization of the Westervelt equation, and also leads to a fractional version of the Khokhlov–Zabolotskaya–Kuznetsov and Burgers' equations.
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I. INTRODUCTION

Fractional derivatives, whether the formal name is used or not, have been used for modeling heat transfer or diffusion,[1,2] seismic data,[3] and sound wave propagation,[4–6] only to name a few. They allow the description of the physics of complex media in solid and fluid mechanics. When modeling sound propagation, the use of fractional derivatives leads to models that better describe observations of attenuation and dispersion.[7] The wave equation for viscous losses involving integer order derivatives only, leads to an attenuation which is proportional to the square of the frequency. This does not always reflect reality. In, e.g., biological tissues[8] and marine sediments,[9] the frequency dependency of attenuation and dispersion is more complicated. Different forms of the wave equation have been proposed to reflect this complexity.[4,7,10–12]

Nonlinear effects in sound wave propagation, may also be taken into account during numerical simulation. This is the case for the Bergen code,[13,14] the KZKTexas code,[15–17] and the angular spectrum method of Christopher and Parker.[18,19] In the case of the angular spectrum approach, the attenuation is modeled as proportional to ω^y, with ω the angular frequency and y non-integer, allowing one to simulate attenuation in media like biological tissue. Time domain simulators can use multiple relaxation processes to approximate such attenuation both in the linear case[20] and in the nonlinear case.[21] Typically, this requires two or more relaxation processes to model a power law over a restricted frequency range. Each process requires two parameters to be found from a curve fit. These parameters describe the physics in the case of propagation in sea water or air. In more complex media, the link to the physics is not so direct.

Several simulators take a modified nonlinear wave equation as a starting point by replacing the traditional loss operator by fractional derivatives,[7,22,23] or a convolution in time.[24–26] Their justification for modifying the standard wave equations is the ability of fractional derivatives to lead to a dispersion equation that better describes attenuation and dispersion. A wave equation based on fractional constitutive equations gives an alternative to modeling absorption and dispersion in complex media like biological tissues.

In this article, we aim at finding the source of the fractional derivative in the nonlinear wave equation. We derive a nonlinear wave equation using constitutive equations as a starting point. The purpose of the article is to relate noninteger power absorption laws to more fundamental physical phenomena, rather than just the measured absorption characteristics. It also establishes a connection between fractional constitutive equations coming from different fields of physics describing mechanical stress or heat transfer. The constitutive equations come from the fractional Kelvin–Voigt model from solid mechanics,[27,28] and a fractional extension of the Gurtin–Pipkin model from heat conduction,[29,30] while the other building equations come from fluid mechanics.[31,32]

We start by briefly recalling the definition and properties of the fractional derivative. Then, we derive a modified version of Euler's equation, and of the entropy equation, introducing fractional derivatives. We explain what these modifications are based on, using solid mechanics and heat diffusion theory. Combining these two equations, we get a wave equation by following the steps and approximations done in fluid mechanics theory. Thereafter, we show that the obtained wave equation is a generalization of the Westervelt equation, and that the dispersion equation can describe attenuation and dispersion for propagation in complex media such as biological tissues. Finally, generalized forms of the Khokhlov–Zabolotskaya–Kuznetsov (KZK) and Burgers' equations using fractional derivatives are obtained.

II. FRACTIONAL DERIVATIVE

The fractional derivative is an extension to integer order derivatives, and is best understood by looking at its Fourier transform in the frequency domain. For any positive integer n, the temporal Fourier transform of the nth order derivative of a function $f(t)$ satisfies the relation

$$\mathcal{F}\left\{\frac{d^n f}{dt^n}, \omega\right\} = (j\omega)^n \mathcal{F}\{f\}. \quad (1)$$

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The fractional derivative of order γ, for γ real, can be seen as the operator whose Fourier transform satisfies Eq. (1), where n is replaced by γ. In the time domain, this corresponds to a convolution

$$\frac{d^\gamma f}{dt^\gamma} = \frac{1}{\Gamma(1-r)} \int_0^t \frac{1}{(t-\tau)^r} \frac{d^n}{dt^n} f(\tau) d\tau, \tag{2}$$

where $0 \leq n-1 < \gamma < n, r = \gamma - n + 1$, and $\Gamma(1-r)$ is the γ function. Equation (2) is the definition of the fractional derivative given by Caputo.[33,34] Fractional derivatives introduce a memory effect in the physical process they describe.[28,35] The nth order derivative is convolved with a memory function

$$\frac{1}{\Gamma(1-r)} \frac{1}{t^r}. \tag{3}$$

In the case where $r \to 1$ (no memory), the memory function tends toward a Dirac impulse function, and the order of the fractional derivative tends toward the integer n. In the case where $r \to 0$ (infinite memory), Eq. (2) tends toward an integration of the nth order derivative resulting in the $(n-1)$th order derivative.

Subsequently, the fractional integral of order α can also be defined as[34]

$$I^\alpha[f(t)] = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{(\alpha-1)} f(\tau) d\tau \quad \text{for} \quad 0 < \alpha. \tag{4}$$

Its Fourier transform satisfies the relation

$$\mathcal{F}\{I^\alpha[f(t)], \omega\} = (j\omega)^{-\alpha} \mathcal{F}\{f\}. \tag{5}$$

Hence fractional integrals and derivatives allow to model any power law in the frequency domain. Fractional integrals and derivatives can be combined, giving the property:

$$\frac{d^\gamma}{dt^\gamma}[I^\alpha] = \begin{cases} \frac{d^{\gamma-\alpha}}{dt^{\gamma-\alpha}} & \text{if} \quad 0 < \alpha < \gamma, \\ I^{\alpha-\gamma} & \text{if} \quad 0 < \gamma < \alpha. \end{cases} \tag{6}$$

Fractional derivatives have been introduced in solid mechanics to more appropriately describe the stress-strain relations,[28] or heat transfers[1] in viscoelastic media. This will be used here as a starting point to modify the constitutive equations.

III. FROM FRACTIONAL CONSTITUTIVE EQUATIONS TO FRACTIONAL EULER'S AND ENTROPY EQUATIONS

The basic equations that the nonlinear wave equation derived in this paper is built upon are: the equation of continuity, expressing the conservation of mass; the equation of state, expressing the thermodynamic state of the fluid; Euler's equation, that translates the conservation of momentum; and the entropy equation, expressing the conversion of energy in an irreversible process. The last two equations, Euler's equation, and the entropy equation, are the equations that we will modify by introducing fractional derivatives.

A. Euler's equation

In this section, we describe how the expression of the stress tensor can be described by the fractional Kelvin–Voigt model, and how this leads to a form of Euler's equation with fractional derivatives. Following the expression of Euler's equation in Eq. (15.5) of Landau and Lifshitz,[31] we have

$$\rho\left(\frac{\partial v_i}{\partial t} + v_k \frac{\partial v_i}{\partial x_k}\right) = \frac{\partial \sigma_{ik}}{\partial x_k} = -\frac{\partial p}{\partial x_i} + \frac{\partial \sigma'_{ik}}{\partial x_k}, \tag{7}$$

where ρ is the density, v_i the components of the particle speed vector, t and x_i the temporal and spatial coordinates, and p the total pressure. σ_{ik} and σ'_{ik} represent the stress tensor, and viscous stress tensor, respectively. Using Eqs. (15.2) and (15.3) of Ref. 31

$$\begin{aligned} \sigma_{ik} &= -p\delta_{ik} + \sigma'_{ik} \\ &= -p\delta_{ik} + \eta\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} - \frac{2}{3}\delta_{ik}\frac{\partial v_l}{\partial x_l}\right) + \zeta\delta_{ik}\frac{\partial v_l}{\partial x_l}, \end{aligned} \tag{8}$$

where η and ζ are the shear and bulk viscosity coefficients, respectively, and are independent of velocity. This is the same relation established by Markham *et al.*[27] in their Eq. (13.3). In their article, they refer to the physical model as the Stokes's model. Further on, approximating the static total pressure by the inviscid total pressure, they get the relation

$$p \approx K \frac{\rho_e}{\rho_0}, \tag{9}$$

where K is the Young's modulus, ρ_e the excess density, and ρ_0 the equilibrium density. And, finally, they get Eq. (14.2):

$$\begin{aligned} \sigma_{ik} = &- K\frac{\rho_e}{\rho_0}\delta_{ik} + \eta\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} - \frac{2}{3}\delta_{ik}\frac{\partial v_l}{\partial x_l}\right) \\ &+ \zeta\delta_{ik}\frac{\partial v_l}{\partial x_l}. \end{aligned} \tag{10}$$

Using a linear form of the equation of continuity (a full nonlinear form is presented in Sec. IV)

$$\frac{\partial \rho_e}{\partial t} = -\rho_0 \frac{\partial v_i}{\partial x_i}, \tag{11}$$

we get the constitutive equation

$$\begin{aligned} \sigma_{ik} = &K\delta_{ik}\frac{\partial u_i}{\partial x_k} + \eta\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} - \frac{2}{3}\delta_{ik}\frac{\partial v_l}{\partial x_l}\right) \\ &+ \zeta\delta_{ik}\frac{\partial v_l}{\partial x_l}, \end{aligned} \tag{12}$$

where u_i are the components of the displacement vector field and $v_i = \partial u_i/\partial t$. Comparing this relation to the Kelvin–Voigt model[11,28] (or Stokes's model as referred to by Markham *et al.*[27]) described by the stress-strain relation

$$\sigma = K\left[\varepsilon + \tau_\sigma \frac{\partial \varepsilon}{\partial t}\right], \tag{13}$$

where $\varepsilon = \partial u/\partial x$ is the strain, u the displacement, and τ_σ the creep time, we can match the component of the normal stress tensor $-p\delta_{ik} = K\varepsilon$, and of the viscous stress tensor $\sigma'_{ik} = K\tau_\sigma \partial\varepsilon/\partial t$.

The theory presented by Markham *et al.*[27] is remarkable in that it relates Eq. (8) to Eq. (13) via Eq. (12), bridging the gap between the stress-strain formulations used in fluid mechanics, and solid mechanics. The equivalence of both formulations for the stress-strain relation is not often found in the literature.

The Kelvin–Voigt model is one of many stress-strain relations. Other models such as Maxwell's model, and the standard linear solid model, also called the Zener model, are often employed, each describing the material properties differently. A thorough review of those models was given by Rossikhin and Shitikova,[28] where they also discuss a generalization of each model using fractional derivatives. The Kelvin–Voigt model may be generalized using fractional derivatives:[11,28]

$$\sigma = K\left[\varepsilon + \tau_\sigma^\gamma \frac{\partial^\gamma \varepsilon}{\partial t^\gamma}\right] \quad \text{for} \quad 0 < \gamma \le 1, \tag{14}$$

where $\partial^\gamma/\partial t^\gamma$ describes the fractional time-derivative of order γ as defined in Eq. (2). Since the strain rate is $\partial\varepsilon/\partial t = \partial v/\partial x$, when considering a one-dimensional deformation, Eq. (14) becomes

$$\begin{aligned}\sigma &= K\left[\varepsilon + \tau_\sigma^\gamma I^{1-\gamma}\left(\frac{\partial v}{\partial x}\right)\right] \quad \text{for} \quad 0 < \gamma < 1\\ &= K\left[\varepsilon + \tau_\sigma \frac{\partial v}{\partial x}\right] \quad \text{for} \quad \gamma = 1,\end{aligned} \tag{15}$$

where $I^{1-\gamma}$ is the fractional integral of order $1-\gamma$ as defined in Eq. (4). Introducing the fractional integral in Eq. (7), Euler's equation can be generalized to

$$\rho\left(\frac{\partial v_i}{\partial t} + v_k \frac{\partial v_i}{\partial x_k}\right) = -\frac{\partial p}{\partial x_i} + \tau^{\gamma-1} I^{1-\gamma}\left(\frac{\partial \sigma'_{ik}}{\partial x_k}\right) \tag{16}$$

where τ is a time constant characteristic of the creep time. Replacing σ'_{ik} by its expression using Eq. (8), and assuming the viscosity coefficients η and ζ to be constant, Eq. (16) may, in vector notation, be written as

$$\begin{aligned}\rho\left[\frac{\partial v}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}\right] &= -\nabla p + \tau^{\gamma-1}\eta I^{1-\gamma}(\Delta\mathbf{v})\\ &\quad + \tau^{\gamma-1}\left(\zeta + \frac{1}{3}\eta\right) I^{1-\gamma}[\nabla(\nabla\cdot\mathbf{v})],\end{aligned} \tag{17}$$

which is a fractional integral generalization of Navier–Stokes equation. In this work, bold face symbols designate vectors. Equation (17) can be simplified the same way as Hamilton and Morfey[32] do in the case of thermoviscous fluids to get Eq. (32) in Ref. 32. We obtain the fractional Euler's equation

$$\begin{aligned}\rho_0 \frac{\partial \mathbf{v}}{\partial t} &= -\nabla p' + \left(\zeta + \frac{4}{3}\eta\right)\tau^{\gamma-1} I^{1-\gamma}(\Delta\mathbf{v})\\ &\quad - \frac{\rho_0}{2}\nabla v^2 - \rho' \frac{\partial \mathbf{v}}{\partial t},\end{aligned} \tag{18}$$

where $\rho' = \rho - \rho_0$ and $p' = p - p_0$ represent the dynamic density and pressure, which describe small disturbances relative to the equilibrium values ρ_0 and p_0.

B. The entropy equation

The constitutive equation linking heat flux to temperature gradient has evolved in a similar manner as the stress-strain relation. In 1958, Cattaneo[36] and Vernotte[37] modified the Fourier law to allow for a finite speed of propagation of disturbances. Indeed, as it is well explained in the introductions of Refs. 38 and 39, when very small time scales are considered, or when the materials have "a non-homogeneous inner structure,"[38] like biological tissues, the assumption of matter as a continuum fails. The classical descriptions for energy transport (e.g., Fourier law) are no longer applicable. The Cattaneo–Vernotte equation modifies the Fourier law "to account for the time lag between the temperature gradient and the heat flux induced by it."[39] The modification consists of the addition of the second term on the left-hand side in the following equation:

$$\mathbf{q} + \tau_{cv}\frac{\partial \mathbf{q}}{\partial t} = -\kappa\nabla T, \tag{19}$$

where $\mathbf{q}$ is the heat flux, T the absolute temperature, κ the thermal conductivity, and τ_{cv} is a relaxation time. In 1968, Gurtin and Pipkin[29] introduced a more general time-nonlocal relation (of which the Cattaneo–Vernotte equation is a particular case), linking heat flux transfer and temperature gradient

$$\mathbf{q}(t) = \int_0^\infty K(\tau)\nabla T(t-\tau)d\tau. \tag{20}$$

Assuming that the media is initially at constant temperature, that is $\nabla T(t) = 0$ for $t < 0$, Eq. (20) can be written

$$\mathbf{q}(t) = -\int_0^t K(t-\tau)\nabla T(\tau)d\tau. \tag{21}$$

This is the model that we adopt for the heat flux in the case of propagation in biological tissues whose structure is non-homogeneous. An extension of the model defined by Gurtin and Pipkin can be written using fractional integral notation.[30] Indeed if the heat flux relaxation function is defined as

$$K(t-\tau) = \frac{\kappa}{\Gamma(\alpha-1)}(t-\tau)^{\alpha-2} \quad \text{for} \quad 1 < \alpha \le 2, \tag{22}$$

the heat flux equation can be written

$$\mathbf{q}(t) = -\kappa I^{\alpha-1}\nabla T(t), \tag{23}$$

where $I^{\alpha-1}$ represents the fractional integral of order $\alpha-1$ as defined in Eq. (4). Equation (23) is thus the fractional constitutive equation describing the heat flux. Combined with the thermal energy equation

$$\nabla \mathbf{q}(t) = -\rho c_p \frac{\partial T}{\partial t}, \tag{24}$$

where c_p is the specific heat capacity at constant pressure, it leads to a fractional heat equation which was formulated around the nineties in Refs. 1 and 2. In one space dimension, it reads

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{D}\frac{\partial^\alpha T}{\partial t^\alpha} \quad \text{with} \quad D = \frac{\kappa}{\rho c_p} > 0, \tag{25}$$

where D is the thermal diffusivity. In Refs. 1 and 2, Eq. (25) is defined for $0 < \alpha \leq 2$. For $0 < \alpha \leq 1$, it is a fractional diffusion equation and for $1 < \alpha \leq 2$, it is a fractional wave equation. In Ref. 40, Nigmatullin explained how fractional derivatives appeared when describing diffusion in a medium of fractal geometry. In this work, we consider Eq. (25) as a fractional wave equation expressed in three space dimensions:

$$\nabla^2 T = \frac{1}{D}\frac{\partial^\alpha T}{\partial t^\alpha}, \quad \text{with} \quad 1 < \alpha \leq 2. \tag{26}$$

Due to the fractional integral in Eq. (23), the unit of the thermal conductivity κ is $\mathrm{W\,s^{1-\alpha}/(K\,m)}$, and in Eq. (26), the unit of the thermal diffusivity D (Ref. 41) is $\mathrm{m^2/s^\alpha}$. Equation (26) can then be written as

$$\nabla^2 T = \frac{\tau_{\text{th}}^{\alpha-2}}{c_0^2}\frac{\partial^\alpha T}{\partial t^\alpha}, \quad \text{with} \quad 1 < \alpha \leq 2, \tag{27}$$

where τ_{th} is a relaxation time characteristic of the medium.[36]

In Eq. (33) of Ref. 32, Hamilton and Morfey use a simplified version of the entropy equation:

$$\rho_0 T_0 \frac{\partial s}{\partial t} = \kappa \nabla^2 T, \tag{28}$$

where T_0 and ρ_0 are the equilibrium temperature and density, respectively, and s the entropy per unit mass. This equation expresses the thermal losses in a thermoviscous fluid as a function of temperature, and is a valid approximation well away from solid boundaries.[32] In combination with Eq. (27) we obtain the fractional entropy equation:

$$\rho_0 T_0 \frac{\partial s}{\partial t} = \frac{\kappa \tau_{th}^{\alpha-2}}{c_0^2}\frac{\partial^\alpha T}{\partial t^\alpha}. \tag{29}$$

IV. FRACTIONAL WAVE EQUATION

In this section, we use the fractional versions of Euler's equation (18) and the entropy equation (29), to obtain a wave equation with fractional derivatives. Following the approximations to the second order[42] of Hamilton and Morfey[32] the fractional Euler's equation (18), can be simplified as follows:

$$\rho_0 \frac{\partial v}{\partial t} = -\nabla p - \frac{\tau^{\gamma-1}}{\rho_0 c_0^2}\left(\zeta + \frac{4}{3}\eta\right)\frac{\partial^\gamma}{\partial t^\gamma}\nabla p - \nabla \mathcal{L}, \tag{30}$$

where $\mathcal{L}$ is the second-order Lagrangian density defined as

$$\mathcal{L} = \frac{1}{2}\rho_0 v^2 - \frac{p^2}{2\rho_0 c_0^2}. \tag{31}$$

The prime notation for p' used in Eq. (18) has been dropped, but p still represents the dynamic pressure from this point on. Approximations to the second order as in Ref. 32 lead to the following form of the continuity equation:

$$\frac{\partial \rho'}{\partial t} + \rho_0 \nabla \cdot \mathbf{v} = \frac{1}{\rho_0 c_0^4}\frac{\partial p^2}{\partial t} + \frac{1}{c_0^2}\frac{\partial \mathcal{L}}{\partial t}. \tag{32}$$

This equation is nonlinear, and will be one of the contributors to the nonlinear term in the final wave equation.

We introduce the equation of state as a Taylor series of $P(\rho, s)$ about the equilibrium state (ρ_0, s_0), where terms of third order are neglected:[32]

$$p = c_0^2 \rho' + \frac{c_0^2}{\rho_0}\frac{B}{2A}\rho'^2 + \left(\frac{\partial P}{\partial s}\right)_{\rho,0} s', \tag{33}$$

with B/A the medium parameter of nonlinearity and $s' = s - s_0$ the dynamic entropy. This equation is also nonlinear and is the other contributor to the nonlinear term of the final wave equation. Introducing $T' = T - T_0$, and integrating Eq. (29) with respect to time gives

$$\rho_0 T_0 s' = \frac{\kappa \tau_{\text{th}}^{\alpha-2}}{c_0^2}\frac{\partial^{\alpha-1} T'}{\partial t^{\alpha-1}}, \tag{34}$$

which is used to eliminate s' in favor of T' in Eq. (33). Following the steps described by Hamilton and Morfey (see Ref. 32 for a detailed description), we get the following equation:

$$\rho' = \frac{p}{c_0^2} - \frac{1}{\rho_0 c_0^4}\frac{B}{2A}p^2 - \frac{\kappa \tau_{\text{th}}^{\alpha-2}}{\rho_0 c_0^4}\left(\frac{1}{c_v} - \frac{1}{c_p}\right)\frac{\partial^{\alpha-1} p}{\partial t^{\alpha-1}}, \tag{35}$$

where c_v and c_p are the heat capacity per unit of mass at constant volume and pressure, respectively. Subtracting the time derivative of Eq. (32) from the divergence of Eq. (30), and using Eq. (35) to eliminate ρ', we get

$$\begin{aligned}&\Box^2 p + \frac{\tau^{\gamma-1}}{\rho_0 c_0^2}\left(\zeta + \frac{4}{3}\eta\right)\frac{\partial^\gamma}{\partial t^\gamma}\nabla^2 p \\ &+ \frac{\kappa \tau_{\text{th}}^{\alpha-2}}{\rho_0 c_0^4}\left(\frac{1}{c_v} - \frac{1}{c_p}\right)\frac{\partial^{\alpha+1} p}{\partial t^{\alpha+1}} \\ &= -\frac{\beta}{\rho_0 c_0^2}\frac{\partial^2 p^2}{\partial t^2} - \left(\nabla^2 + \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\right)\mathcal{L},\end{aligned} \tag{36}$$

where

$$\Box^2 = \nabla^2 - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2} \tag{37}$$

is the d'Alembertian operator and

$$\beta = 1 + \frac{B}{2A} \tag{38}$$

is the medium coefficient of nonlinearity. The expression for β regroups contributions to nonlinear propagation coming both from the equation of state, and the equation of continuity. Discarding the term containing $\mathcal{L}$, we get a fractional wave equation:

$$\Box^2 p + \frac{\tau^{\gamma-1}}{\rho_0 c_0^2}\left(\zeta + \frac{4}{3}\eta\right)\frac{\partial^\gamma}{\partial t^\gamma}\nabla^2 p + \frac{\kappa\tau_{\text{th}}^{\alpha-2}}{\rho_0 c_0^4}\left(\frac{1}{c_\upsilon} - \frac{1}{c_p}\right)\frac{\partial^{\alpha+1}p}{\partial t^{\alpha+1}} = -\frac{\beta}{\rho_0 c_0^4}\frac{\partial^2 p^2}{\partial t^2}. \tag{39}$$

For clarity, the following notations are introduced:

$$L_\upsilon = \frac{\tau^{\gamma-1}}{\rho_0 c_0^2}\left(\zeta + \frac{4}{3}\eta\right), \tag{40}$$

$$L_t = -\frac{k\tau_{\text{th}}^{\alpha-2}}{\rho_0 c_0^2}\left(\frac{1}{c_\upsilon} - \frac{1}{c_p}\right), \tag{41}$$

$$L_\upsilon > L_t.$$

Equation (39) may then be expressed as a fractional form of the Westervelt equation

$$\Box^2 p + L_\upsilon \frac{\partial^\gamma}{\partial t^\gamma}\nabla^2 p - \frac{L_t}{c_0^2}\frac{\partial^{\alpha+1}p}{\partial t^{\alpha+1}} = -\frac{\beta}{\rho_0 c_0^4}\frac{\partial^2 p^2}{\partial t^2}, \tag{42}$$

which we will call the fractional Westervelt equation of the first form. The first term on the left hand side of Eq. (42) characterizes diffraction. The second and third terms characterize attenuation coming from the fractional Euler's equation and the fractional entropy equation, respectively. The term on the right hand side characterizes nonlinearity, and comes from the continuity equation and the equation of state.

In order to get a fractional form of the Westervelt equation with a non-integer frequency power attenuation law, we note that $\gamma = 1$ and $\alpha = 2$ in Eq. (42) leads to the Westervelt equation (see next section). Thus, in that case, the fractional orders are linked. We generalize this link by setting $\gamma = \alpha - 1 = y - 1$ with $1 < y \leq 2$. The same assumption is implicitly done in the derivation of the fractional forms of the Westervelt and Burgers' equations.[22,26] This leads to the fractional Westervelt equation of the second form:

$$\Box^2 p + L_\upsilon \frac{\partial^{y-1}}{\partial t^{y-1}}\nabla^2 p - \frac{L_t}{c_0^2}\frac{\partial^{y+1}p}{\partial t^{y+1}} = -\frac{\beta}{\rho_0 c_0^4}\frac{\partial^2 p^2}{\partial t^2}. \tag{43}$$

Even if it is difficult to find physical data on material properties to justify this assumption, except for $\gamma = 1$, it is reasonable to assume that γ and α are strongly linked. Indeed, the fractional integrals in the stress-strain relation, or the heat flux equation are both due to internal structures or inhomogeneities in the material. The nature of the medium dictates the order of the fractional integrals in both equations. Equation (43) is the form of the fractional wave equation we use in the rest of the article.

V. COMPARISON WITH WESTERVELT EQUATION

In the case of propagation in classical thermoviscous fluids, the stress tensor is described by Eq. (8). This leads to the non-fractional form of the Euler's equation: Eq. (18), where $\gamma = 1$. The entropy equation has also its non-fractional form. It is obtained by setting $\alpha = 2$ in Eq. (29). Equation (39) then becomes

$$\Box^2 p + \frac{1}{\rho_0 c_0^2}\left(\zeta + \frac{4}{3}\eta\right)\frac{\partial}{\partial t}\nabla^2 p + \frac{\kappa}{\rho_0 c_0^4}\left(\frac{1}{c_\upsilon} - \frac{1}{c_p}\right)\frac{\partial^3 p}{\partial t^3} = -\frac{\beta}{\rho_0 c_0^4}\frac{\partial^2 p^2}{\partial t^2}. \tag{44}$$

The low-frequency regime covers the applications involving compressional waves, while the high-frequency regime applies mostly to shear waves.[11] This article is mainly oriented toward applications of compressional waves, and low frequencies. In this regime, $\nabla^2 p$ can be approximated by $c_0^{-2}\partial^2 p/\partial t^2$. Equation (44) then gives the Westervelt equation

$$\Box^2 p + \frac{\delta}{c_0^4}\frac{\partial^3 p}{\partial t^3} = -\frac{\beta}{\rho_0 c_0^4}\frac{\partial^2 p^2}{\partial t^2}, \tag{45}$$

where

$$\delta = \frac{1}{\rho_0}\left(\zeta + \frac{4}{3}\eta\right) + \frac{\kappa}{\rho_0}\left(\frac{1}{c_\upsilon} - \frac{1}{c_p}\right) \tag{46}$$

is the diffusivity of sound. This shows that Eqs. (39), (42), and (43) are fractional generalizations of the Westervelt equation.

VI. DISPERSION EQUATION

To find the frequency dependence of attenuation and propagation velocity, a dispersion equation can be derived from Eq. (43). Using the principle of superposition, the Fourier transform in space and time of a harmonic plane wave solution, $\upsilon(x,t) = \exp[j(\omega t - kx)]$, gives the dispersion equation for any wave. Since the principle of superposition assumes linearity, the nonlinear term has to be excluded in Eq. (43). Using the Fourier transform's property of fractional derivatives given in Eq. (1), we get

$$k = \frac{\omega}{c_0}\sqrt{\frac{1 + L_t e^{j(y-1)\pi/2}\omega^{y-1}}{1 + L_\upsilon e^{j(y-1)\pi/2}\omega^{y-1}}} = \frac{\omega}{c(\omega)} - i\alpha(\omega), \tag{47}$$

with $\alpha(\omega)$ and $c(\omega)$ the frequency dependent attenuation and phase velocity, respectively. Simplifications in the low frequency regime give the following expressions:[11]

$$\alpha(\omega) = \frac{1}{c_0}\left(\frac{L_t}{2} - \frac{L_\upsilon}{2}\right)\cos\left(\frac{y\pi}{2}\right)|\omega|^y,$$
$$c(\omega) \approx c_0\left[1 - \left(\frac{L_t}{2} - \frac{L_\upsilon}{2}\right)\sin\left(\frac{y\pi}{2}\right)\omega^{y-1}\right]. \qquad (48)$$

These expressions lead to a velocity dispersion relation that satisfies the Kramers–Kronig relation,[43] and confirm that it fulfils the causality requirement. The expressions for $\alpha(\omega)$ and $c(\omega)$ describe observations in biological tissues[8] that the Westervelt equation fails to explain. For values of y between 1 and 2, the attenuation is proportional to ω^y, which covers the vast majority of attenuation laws for propagation in biological tissues.[8] For illustration, the frequency dependency for the attenuation and the velocity dispersion are shown on Figs. 1 and 2, respectively, for different values of y. There is a singularity for $y = 1$ as discussed in Ref. 11 which is why values for y in the range 1.1 to 2 are plotted. In these examples, the time constants involved in L_υ and L_t have been chosen identical. The following values were set for the plots: $\tau = 10^{-10}$ s, $c_0 = 1500$ m/s, $\alpha = 1.2$ dB/cm at 1 MHz for $y = 1.1$, and $\omega \leq 3 \times 10^7$ rad/s. The figures show the low-frequency regime, that is, $\omega\tau \ll 1$. Figures 1 and 2 show attenuation and phase velocity dispersion for $y = 1.1$ that is comparable to the measurements made by Kremkau *et al.*[44] on human brains and to previous illustrations of attenuation and dispersion frequency dependencies.[12,23,45]

VII. GENERALIZED KZK AND BURGERS' EQUATIONS

A. KZK equation

Again, starting from Eq. (43), when approximating $\nabla^2 p$ by $c_0^{-2}\partial^2 p/\partial t^2$ in the low-frequency domain, we get

$$\Box^2 p + \frac{L_\upsilon - L_t}{c_0^2}\frac{\partial^{y+1} p}{\partial t^{y+1}} = -\frac{\beta}{\rho_0 c_0^4}\frac{\partial^2 p^2}{\partial t^2}. \qquad (49)$$

Introducing the retarded time coordinate $\tau_r = t - z/c_0$, and following the same steps as Hamilton and Morfey in Ref. 32, Eq. (49) can be approximated to

$$\nabla_\perp^2 p - \frac{2}{c_0}\frac{\partial^2 p}{\partial z \partial \tau_r} + \frac{L_\upsilon - L_t}{c_0^2}\frac{\partial^{y+1} p}{\partial \tau_r^{y+1}} = -\frac{\beta}{\rho_0 c_0^4}\frac{\partial^2 p^2}{\partial \tau_r^2}, \qquad (50)$$

where $\nabla_\perp^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the Laplacian that operates in the plane perpendicular to the axis of the beam. This approximation is valid for well collimated sound beams satisfying the relation $ka \gg 1$, where k is the wave number and a the characteristic radius of the source. Equation (50) is a fractional derivative generalization of the KZK (Khokhlov–Zabolotskaya–Kuznetsov) equation. For $y = 2$, $\delta = c_0^2(L_\upsilon - L_t)$, and we obtain the KZK equation

$$\frac{\partial^2 p}{\partial z \partial \tau_r} - \frac{c_0}{2}\nabla_\perp^2 p - \frac{\delta}{2c_0^3}\frac{\partial^3 p}{\partial \tau_r^3} = \frac{\beta}{2\rho_0 c_0^3}\frac{\partial^2 p^2}{\partial \tau_r^2}. \qquad (51)$$

B. Burgers' equation

Likewise, rewriting Eq. (49) in one spatial dimension, we get

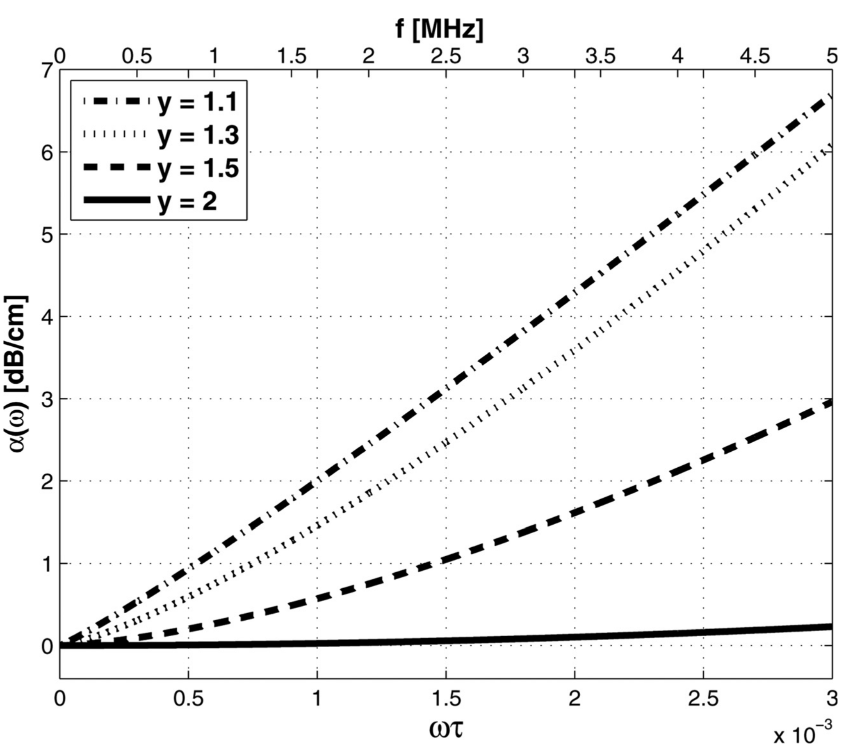

FIG. 1. Attenuation $\alpha(\omega)$ as a function of $\omega\tau$ for different values of y.

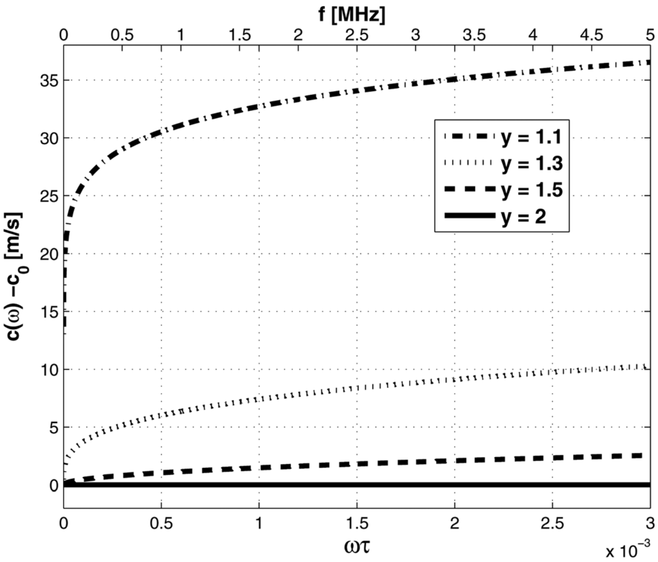

FIG. 2. Velocity dispersion $c(\omega) - c_0$ as a function of $\omega\tau$ for different values of y.

$$\left(\frac{\partial^2}{\partial z^2} - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\right)p + \frac{L_\upsilon - L_t}{c_0^2}\frac{\partial^{y+1}p}{\partial t^{y+1}} = -\frac{\beta}{\rho_0 c_0^4}\frac{\partial^2 p^2}{\partial t^2}. \quad (52)$$

Introducing the retarded time coordinate τ_r, it becomes

$$\frac{\partial^2 p}{\partial z^2} - \frac{2}{c_0}\frac{\partial^2 p}{\partial z \partial \tau_r} + \frac{L_\upsilon - L_t}{c_0^2}\frac{\partial^{y+1}p}{\partial \tau_r^{y+1}} = -\frac{\beta}{\rho_0 c_0^4}\frac{\partial^2 p^2}{\partial \tau_r^2}. \quad (53)$$

The first term on the left can be neglected when using second-order approximations as done by Hamilton and Morfey in Ref. 32. An integration with respect to τ_r gives

$$\frac{\partial p}{\partial z} - \frac{L_\upsilon - L_t}{2c_0}\frac{\partial^y p}{\partial \tau_r^y} = \frac{\beta}{2\rho_0 c_0^3}\frac{\partial p^2}{\partial \tau_r} = \frac{\beta p}{\rho_0 c_0^3}\frac{\partial p}{\partial \tau_r}. \quad (54)$$

Equation (54) is a fractional derivative generalization of the Burgers' equation. For $y=2$, we obtain the Burgers' equation

$$\frac{\partial p}{\partial z} - \frac{\delta}{2c_0^3}\frac{\partial^2 p}{\partial \tau_r^2} = \frac{\beta p}{\rho_0 c_0^3}\frac{\partial p}{\partial \tau_r}. \quad (55)$$

VIII. CONCLUSION

Fractional derivatives are introduced in the constitutive equations by use of previous studies on stress-strain relations in solid mechanics and heat conduction mechanisms linking two different areas of physics. The nonlinear wave equation obtained from these building equations leads to a frequency-dependent attenuation and dispersion that fit observations.[8,44] This work justifies why the modifications of the wave equation using fractional derivative, as done in the existing literature, are legitimate. These modifications have been done empirically in view of the observed attenuation or dispersion power laws. Such modifications do not always guarantee the causality of the solution. Since our fractional wave equation is derived from constitutive equations, its causality is assured.

We have shown that the Westervelt equation is a particular case of our wave equation, which may be simplified into a fractional KZK equation. Therefore, fractional derivatives offer an alternative to multiple relaxations used for time-domain simulators like the KZKTexas code[15] to approximate attenuation and dispersion in biological tissue. The fractional derivative can, for instance, be solved in the time domain by using a backward difference power series[10] or the Grünwald–Letnikov formulation.[23] Using the constitutive equations as a starting point instead of an approximated wave equation could also be an alternative to simulate sound propagation in non-homogeneous media.

Using a fractional wave equation as a starting point, two constants can be used for describing attenuation and dispersion: a pre-factor determined by measurement and the order of the fractional derivative. If fractional constitutive equations are used, they require nine constants: two for the relaxation times, two for the order of the fractional derivatives, two for the viscosities, two for the heat capacities, and one for the thermal conductivity. A model using two relaxation processes requires five constants:[21] two for the relaxation frequencies, two for the relaxation dispersion, and one for the thermoviscous pre-factor. However, the thermoviscous pre-factor can be expressed using the diffusivity of sound which is a combination of five constants: two for the viscosities, two for the heat capacities, and one for the thermal conductivity, bringing the total number of constants to nine. The complexity of a model using fractional constitutive equations should therefore not exceed the complexity of existing models based on multiple relaxation processes. In addition relaxation processes approximate attenuation and dispersion in tissue for a reduced frequency range, the fractional constitutive equations give a more broadband model.

This article also calls for a better determination of the time constants involved in the fractional loss operators, and maybe measurements establishing a link between the order of the fractional integrals in the stress-strain relation and the heat flux equation. This, in turn, would lead to a better understanding of the attenuation and dispersion mechanisms that happen in hybrid media, like biological tissues, which fall between solids and fluids.

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Addendum to [Nonlinear acoustic wave equations with fractional loss operators, JASA, 2011]

October 23, 2012

Due to the presence of some unfortunate errors in [1], the follow-up paper [2] should be consulted for the correct forms of:

- Eq. (29), the fractional entropy equation
- Eq. (35), density as a function of pressure
- Eq. (42), the fractional Westervelt equation of the first form
- Eq. (43), the fractional Westervelt equation of the second form
- Eq. (44), the Westervelt eq. with loss terms expressed by the Laplacian
- Eq. (47), the dispersion relation

In addition the link between constants should be changed from $\gamma = \alpha - 1 = y - 1$ (before Eq. (43) in [1]) to $\gamma = 2 - \alpha = y - 1$ (before Eq. (32) in [2]).

The abstract of the follow-up paper, [2] is:

> A corrected derivation of nonlinear wave propagation equations with fractional loss operators is presented. The fundamental approach is based on fractional formulations of the stress-strain and heat flux definitions but uses the energy equation and thermodynamic identities to link density and pressure instead of an erroneous fractional form of the entropy equation as done in Prieur and Holm [Nonlinear acoustic wave equations with fractional loss operators, J. Acoust. Soc. Am. 130(3), 11251132 (2011)]. The loss operator of the obtained nonlinear wave equations differs from the previous derivations as well as the dispersion equation, but when approximating for low frequencies the expressions for the frequency dependent attenuation and velocity dispersion remain unchanged.

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Addendum 2 to [Nonlinear acoustic wave equations with fractional loss operators, JASA, 2011]

September 7, 2015

The correct forms of the fractional Westervelt equation of the first and second forms are also given in the review paper [1]. The first form is:

$$\nabla^2 p - \frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2} + \tau_\sigma^\alpha \frac{\partial^\alpha}{\partial t^\alpha}\nabla^2 p + \tau_{th}^{2-\gamma}\frac{\partial^{2-\gamma}}{\partial t^{2-\gamma}}\nabla^2 p = -\frac{\beta_{NL}}{\rho_0 c_0^4}\frac{\partial^2 p^2}{\partial t^2}. \tag{1}$$

Here α is the fractional order due to mechanical effects and γ is the order due to thermal effects. In the second form, the fractional orders are coupled: $\gamma = 2-\alpha$:

$$\nabla^2 p - \frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2} + \left(\tau_\sigma^\alpha + \tau_{th}^\alpha\right)\frac{\partial^\alpha}{\partial t^\alpha}\nabla^2 p = -\frac{\beta_{NL}}{\rho_0 c_0^4}\frac{\partial^2 p^2}{\partial t^2} \tag{2}$$

and where the mechanical and thermal time constants are:

$$\tau_\sigma = \frac{1}{\rho_0 c_0^2}\left(\zeta + \frac{4}{3}\eta\right), \quad \tau_{th} = \frac{\kappa}{\rho_0 c_0^2}\left(\frac{1}{c_v} - \frac{1}{c_p}\right) \tag{3}$$

The non-fractional case is found for $\alpha = \gamma = 1$.

Note that α and γ have changed roles compared to [2, 3]. Also the constants L_v and L_t are now expressed by mechanical and thermal time constants, τ_σ and τ_{th} instead. This makes the terminology consistent with common usage.

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