THE COARRAY OF SPARSE ARRAYS WITH MINIMUM SIDELOBE LEVEL

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ABSTRACT

Sparse arrays provide a large aperture with few elements. Through an exhaustive search of all possible thinning patterns for small linear sparse arrays, the Fourier properties of arrays with optimal coarray properties are explored. In addition coarray properties of sparse arrays with minimum peak sidelobe level are given. Minimum hole arrays have optimal low peak sidelobe level, while minimum redundancy arrays have a peak sidelobe level that is close to the global minimum. Other arrays with minimum peak sidelobe level have coarray properties similar to those of minimum hole and minimum redundancy arrays. Sparse arrays with optimal coarray qualities also possess optimal Fourier qualities.

1. INTRODUCTION

Sparsely sampled arrays and random arrays have been used or proposed in several fields such as radar, sonar, ultrasound imaging and seismics. The main reason for their use is economy; they provide a way of getting a large aperture with fewer channels. The arrays have usually been designed with one of two objectives in mind: creation of beampatterns with low mainlobe width and small sidelobes [1], or best possible sampling of a random field. In the latter case the correlation function of the array (coarray) should be optimized and be as uniform as possible [2, 3]. Very few researchers have compared these two objectives to see the relationship between them. We have only found [4] where it was stated that arrays with an aperiodic correlation feature more equi-ripple-like sidelobes, and that holes and redundancies have the same potential for increasing sidelobe level.

In this paper we show through an exhaustive search of small arrays that there is a close connection between properties of the beampattern and those of the coarray. Such a search has also been done in [5] for a symmetric array with 19λ aperture quantized in $\lambda/4$ and 9 active elements giving 7770 possible thinning patterns, but better computer resources and knowledge about the correlation properties make a new search interesting. We will show that minimum hole arrays have the optimal low peak sidelobe level and a relative narrow mainlobe. Minimum redundancy arrays are near optimal. In addition we will give properties of coarrays, that are neither minimum redundancy nor minimum hole, that correspond to arrays with desirable properties.

2. FUNDAMENTALS

For thinned, regular arrays the coarray is defined as the autocorrelation of the element weights

$$c(l) = \sum_{m=0}^{N-|l|-1} w_m w_{m+|l|} \tag{1}$$

where $w_m \in \{0, 1\}$ and N is the number of elements in the full aperture. For an N element linear array with element distance d, the coarray is related to the beampattern with:

$$|W(k)|^2 = \sum_{l=-(N-1)}^{N-1} c(l) \exp(jkld) \tag{2}$$

where $k = 2\pi/\lambda$ is the wave-number (spatial frequency). Due to the symmetry of the coarray, this implies

$$|W(k)|^2 = c(0) + \sum_{l=1}^{N-1} 2c(l) \cos(kld) \tag{3}$$

which is a superposition of cosines.

If $c(l_1) > 1$ then l_1 is a redundant lag. Otherwise if $c(l_1) = 0$ then the coarray has a hole at lag l_1. A perfect array has a coarray with no holes or redundancies except for lag zero. Unfortunately, perfect arrays do not exist for $n > 4$. Therefore we study arrays that approximate perfect arrays; the minimum redundancy and the minimum hole arrays. They are defined by the number of redundancies, R, and holes, H. Minimum redundancy (MR) arrays are those element configurations that satisfy $\min(R|H = 0, n = \text{const.})$ for a given number of elements n. Minimum hole arrays (MH) minimize the number of holes in the coarray, and are also known as Golomb rulers [3]. These element configurations satisfy $\min(H|R = 0, n = \text{const.})$.

The number of elements N in the aperture is:

$$N = \frac{(n-1)n}{2} + 1 + H - R \tag{4}$$

as shown in [4]. This implies that an array with n elements and aperture N bounded by

$$N_{MR} < N < N_{MH} \qquad (5)$$

where N_{MR} and N_{MH} are the apertures of minimum redundancy and minimum hole arrays, must have a coarray with both holes and redundancies.

3. MINIMUM REDUNDANCY AND MINIMUM HOLE ARRAYS

For small arrays one can do a search of all element configurations and compute the beampattern in a few hours. For $n = 7$ there are 5 minimum redundancy arrays (+5 mirrored). All of them have 18 element apertures and when the two end elements are fixed, this gives a total of $\binom{16}{5} = 4368$ possible thinning patterns. Figure 1 shows the peak sidelobe level vs. -6 dB beamwidth for all arrays evaluated with a 4096-points FFT. One can see that there are

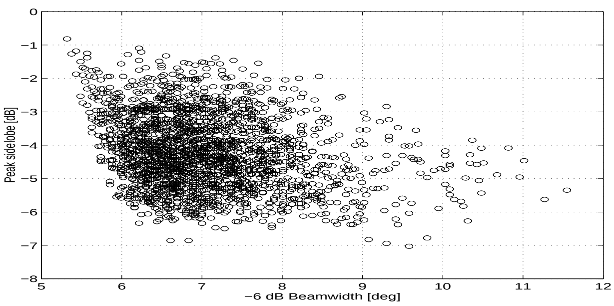

Figure 1: Peak sidelobe level vs. beamwidth for all arrays with $n = 7$ and $N = 18$. For linear arrays the beamwidth is a function of $k_x = \frac{2\pi}{\lambda}\sin\phi$. Here the wavelength $\lambda = 2\pi$.

only a few arrays with low peak sidelobe level. Figure 2 shows the locations of the minimum redundancy arrays relative to the optimal boundary, which is the lower bound of figure 1. Three minimum redundancy arrays have optimal low peak sidelobe level. They have $\max c(l) = 2$

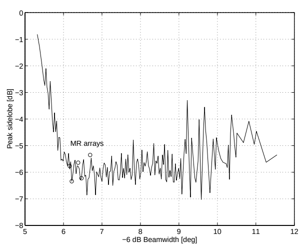

Figure 2: Optimal peak sidelobe level vs. beamwidth relative to the MR-arrays with $n = 7$ and $N = 18$.

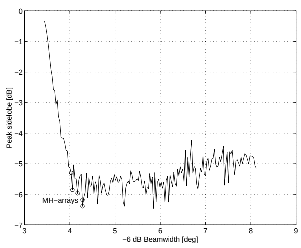

Figure 3: Optimal peak sidelobe level vs. beamwidth relative to the MH-arrays for $n = 7$ and $N = 26$.

while the other two have $\max c(l) = 3$. The one with the smallest peak sidelobe of the two is given as an example in figure 4. The peak sidelobe of this array is located at $\sin\phi = 0.5$, this is mainly due to the periodic redundancies.

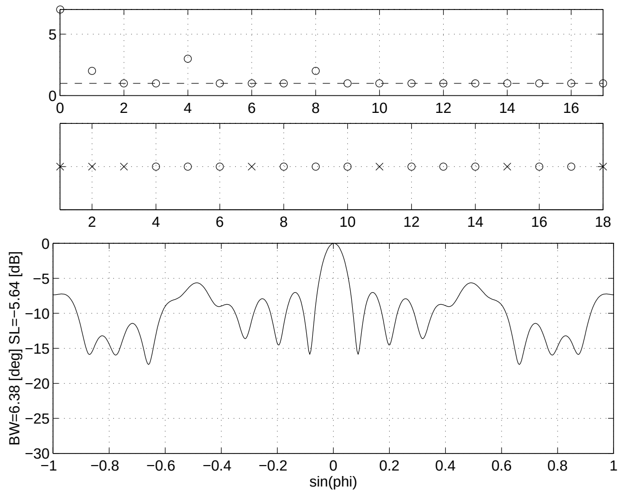

Figure 4: Coarray (right-hand side), aperture (active elements are marked with an x), and beampattern for $n = 7$ and $N = 18$.

For $n = 7$ elements there exists 5 minimum hole arrays. They have $N = 26$ element apertures which gives $\binom{24}{5} = 42504$ possible thinning patterns. Figure 3 shows the location of the minimum hole arrays relative to the optimal boundary. For this thinning problem these arrays are clearly the best solutions. They have a narrow mainlobe and optimally low peak sidelobe level. As an example, figure 5 shows the coarray and the beampattern of the minimum hole aperture with the lowest peak sidelobe level.

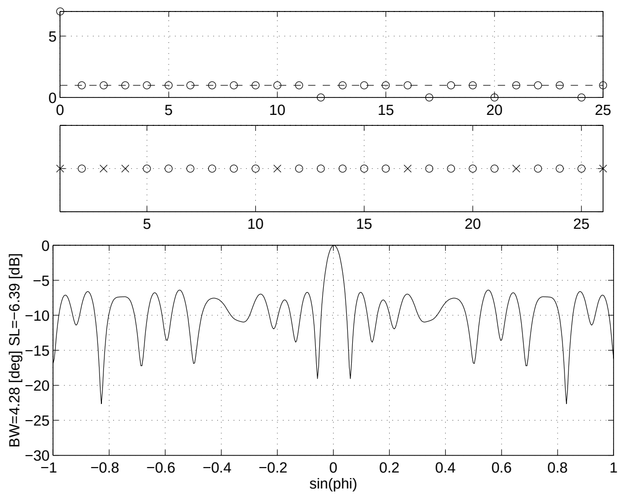

Figure 5: Coarray (right-hand side), aperture (active elements are marked with an x), and beampattern for $n = 7$ and $N = 26$.

Table 1: Properties of the arrays with minimum number of redundancies (R) and minimum number of holes (H) for N=18,19,…,26. The array with the lowest peak sidelobe (SL) for a relatively small -6 dB beamwidth is presented in each case. The mean sidelobe level and the mean peak sidelobe level have been calculated from the linear values.

N	R	H	# arrays	$N\sin\phi$	peak(SL) [dB]	mean(SL) [dB]	peak $SL-$ mean(peak SL) [dB]	Comment
18	4	0	5	1.94	-6.34	-9.75	1.88	Min. redundancy array
	5	1	112	2.06	-6.86	-9.99	0.79	Lowest peak SL, but not a min. redundancy array.
19	4	1	40	2.18	-6.51	-10.00	1.33	
20	3	1	10	2.02	-6.76	-9.69	1.60	
21	2	1	1	1.90	-5.72	-9.53	1.92	The highest peak SL and the largest peak SL deviation
	3	2	68	2.18	-6.77	-9.92	0.94	Larger value of $R+H$, but better peak SL
22	2	2	13	2.09	-6.70	-9.66	1.01	
23	2	3	49	2.10	-6.28	-9.67	1.37	
24	1	3	10	1.93	-6.19	-9.57	1.14	
25	1	4	46	1.91	-6.09	-9.62	1.06	
26	0	4	5	1.94	-6.39	-9.40	0.62	Min. hole array which has the lowest peak SL with this mainlobe width. The smallest peak SL deviation

A search of all n element arrays, with apertures between those of the minimum redundancy and the minimum hole arrays, shows that arrays with a low number of redundancies and holes are optimal or close to optimal. The arrays with the lowest peak sidelobe level with $n = 7$ elements for each of the apertures $N = 18, 19, \ldots, 26$ are presented in table 1. The beamwidths are normalized to $N\sin\phi$, since the beamwidth of a full array is $\sin\phi = 4\pi/Nd$ where d is the element spacing. The minimum redundancy and the minimum hole arrays have a small normalized beamwidth and a low peak sidelobe level. No other array has any lower peak sidelobe level for this beamwidth. The minimum hole array has a lower peak sidelobe than the minimum redundancy array, but one array with the same aperture as the latter and $R = 5, H = 1$ has the best peak sidelobe level of all of the arrays. Thus for $N = 18$ one could achieve a better peak sidelobe level by allowing a hole in the coarray. The difference between the peak sidelobe level and the mean peak sidelobe level expresses the deviation from a Dolph-Chebyshev sidelobe response. The minimum hole array has the smallest deviation from the average peak sidelobe level, while the minimum redundancy array has one of the largest deviations. This becomes more evident for larger arrays, where the minimum hole array's sidelobe level is almost totally uniform.

4. COARRAYS WITH LOW PEAK SIDELOBE LEVEL

From eq. 4 one knows that $H - R =$ const. for given n and N. Almost all of the arrays which give a coarray with the minimum value of H and R have a very low sidelobe level and a narrow mainlobe. For an array with $n = 7$ elements and $N = 25$ elements in the aperture there exists $\binom{23}{5} = 33649$ possible thinning patterns. There are 46 arrays with $H = 4$ and $R = 1$, all with narrow mainlobe and low peak sidelobe level. For large values of H and R the arrays are spread out on a diagonal from small beamwidth and high peak sidelobe level to large beamwidth and low peak sidelobe level. The rate of optimal arrays is decreasing with increasing H and R, and for values larger than $H = 7, R = 4$ no arrays are of interest.

Only 0.85% of all arrays have peak sidelobe level below -5.5 dB. Figure 6 shows the average number of holes as a function of the element lags for these arrays relative to the same function for all arrays. For small lags the arrays with low peak sidelobe level have fewer holes than the average. For large lags the situation is opposite. In

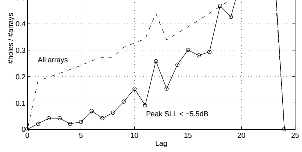

Figure 6: Mean(H) vs. element lag for the 286 arrays with peak SLL < -5.5 dB for the arrays with $n = 7$ and $N = 25$.

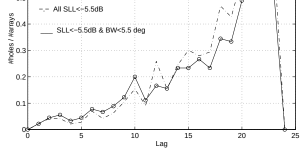

Figure 7: Mean(H) vs. element lag for the 180 arrays with BW < 5.5 [deg] and peak SLL < -5.5 dB for the arrays with $n = 7$ and $N = 25$.

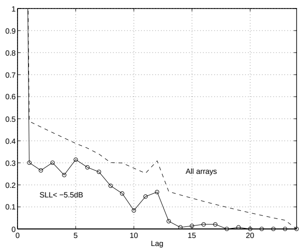

Figure 8: Mean(R) vs. element lag for the 286 arrays with peak SLL < -5.5 dB for the arrays with $n = 7$ and $N = 25$.

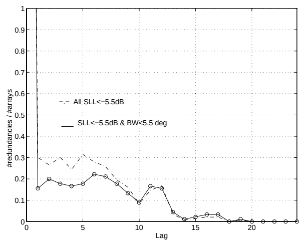

Figure 9: Mean(R) vs. element lag for the 180 arrays with BW < 5.5 [deg] and peak SLL < -5.5 dB for the arrays with $n = 7$ and $N = 25$.

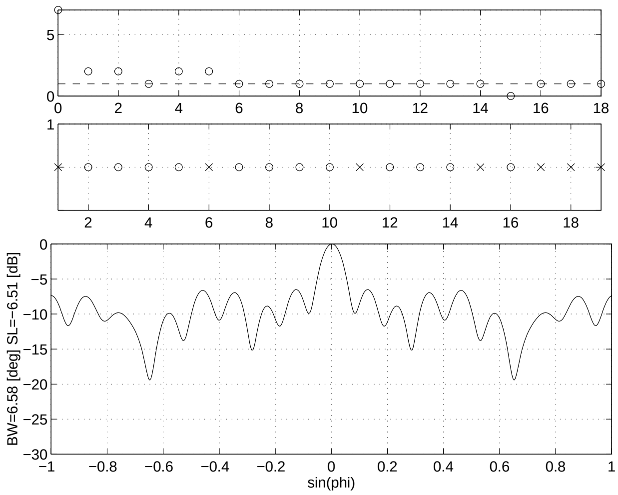

Figure 10: Coarray (right-hand side), aperture (active elements are marked with an x), and beampattern for $n = 7$ and $N = 19$.

figure 7 the same selection is also sorted on beamwidth, but the difference is insignificant. The holes for large lags are equivalent to missing rapidly oscillating cosines in the beampattern (eq 3). These cosines would reinforce the sidelobes due to the slowly oscillating cosines, giving high narrow sidelobes.

Figure 8 shows the average number of redundancies as a function of the element lags for the set with peak sidelobe level below -5.5 dB. The selection with low peak sidelobe level has fewer redundancies for all lags than the average, and for large arrays the average number of redundancies is close to zero. Once again the selection with low peak sidelobe level is also sorted on beamwidth as shown in figure 9. The selection with small beamwidth only deviates for small lags where they have fewer redundancies. For arrays with optimal low peak sidelobe level, the coarray should have few redundancies and they should appear for small lags. This means reinforcing the contribution from slowly oscillating cosines which will increase the beamwidth. Eq. 3 gives that the contribution from a redundant lag l_1 is $c(l_1)$ times that of a non-redundant one. This is why periodic redundancies will give a severe increase in the peak sidelobe level (gratinglobe-like sidelobes). This search has shown that the average number of redundancies should decrease and the average number of holes increase with increasing element lag. An example of such an array is given in figure 10 which has the optimal low peak sidelobe level for $N = 19$. This array has $R = 4$ and $H = 1$. The redundancies are located at relatively low lags ($l \leq 5$) and the hole at a high lag ($l = 15$).

5. CONCLUSION

We have shown through an exhaustive search of an array with $n = 7$ active elements in an aperture between $N = 18$ and $N = 26$ elements, that minimum hole and minimum redundancy arrays have a narrow mainlobe and optimal or close to optimal peak sidelobe level. However the set of possible apertures where minimum hole and minimum redundancy solutions can be found are restricted, and therefore we have also studied arrays with both holes and redundancies. They also have near-optimal properties for small values of H and R. It is important that they do not have periodicities in the redundancies or holes. Further the redundancies should be located at relatively low lags and the holes at high lags in order to achieve low sidelobes for acceptable mainlobe widths. We believe these properties can be generalized to larger arrays. Larger examples would be prohibitive in computer requirements, as for instance an increase in the number of elements by one, would generate more than 26 times as many configurations to search.

6. REFERENCES

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