THE TOPOLOGICAL SINGER CONSTRUCTION

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ABSTRACT. We study the continuous homology of towers of spectra, with emphasis on a tower with homotopy limit the Tate construction X^{tG} of a G-spectrum X. When $G = C_p$ is cyclic of prime order and $X = B^{\wedge p}$ is a p-fold smash power of a bounded below spectrum B with $H^*(B; \mathbb{F}_p)$ of finite type, we prove that $(B^{\wedge p})^{tC_p}$ is a topological model for the Singer construction $R_+(H^*(B; \mathbb{F}_p))$ on $H^*(B; \mathbb{F}_p)$.

There is a map $\gamma : B \to (B^{\wedge})^{tC_p}$ inducing the $\mathrm{Ext}_{\mathscr{A}}$-equivalence $\epsilon : R_+(H^*(B; \mathbb{F}_p)) \to H^*(B; \mathbb{F}_p)$. Hence γ and the canonical map $(B^{\wedge p})^{C_p} \to (B^{\wedge p})^{hC_p}$ are p-adic equivalences.

CONTENTS

1. Introduction

We are interested in the study of the singular homology and cohomology groups associated to towers

$$Y \to \ldots \to Y_n \to Y_{n+1} \to \ldots \to Y_0\,, \tag{1.1}$$

where each Y_n is a bounded below spectrum of finite type. In (1.1) the spectrum Y denotes the homotopy inverse limit of the tower.

By a result of Caruso, May and Priddy [7], there exists an *inverse limit of Adams spectral sequences*-spectral sequence that calculates the homotopy groups of the p-completion of Y, where p is a prime. The input for this spectral sequence is the direct limit of cohomology groups with $\mathbb{F}_p$-coefficients

$$\operatorname*{colim}_n H^*(Y_n;\mathbb{F}_p) \tag{1.2}$$

arising from the tower (1.1), considered as a module over the mod p Steenrod algebra $\mathscr{A}$.

In §2, we discuss towers of spectra and their associated limit systems obtained by applying singular homology or cohomology with $\mathbb{F}_p$-coefficients. We would like to switch between cohomology and homology by dualizing. To overcome the technical difficulties involved in dualizing twice an infinite dimensional vector space, we discuss dualization and filtrations in §2.3.

In the setting described above, the first natural question is: how well do we understand the structure of (1.2) as an $\mathscr{A}$-module? There is an interesting example of a tower of spectra where this question has the answer: very well. In fact, these questions appeared in the study of Segal's Burnside ring conjecture for cyclic groups of prime order. At the heart of W.H. Lin's proof of the case $p = 2$, published in [14], lies a careful study of the $\mathscr{A}$-module

$$\operatorname*{colim}_{n\to\infty} H^*(\mathbb{R}P^\infty_{-n};\mathbb{F}_2) = \Lambda$$

and it's associated Ext-groups $\operatorname{Ext}^{*,*}_{\mathscr{A}}(\Lambda,\mathbb{F}_2)$. It turns out that Λ is isomorphic to the socalled Singer construction $R_+(\mathbb{F}_2)$ on the trivial $\mathscr{A}$-module $\mathbb{F}_2$. The Singer construction has an explicit description as a module over $\mathscr{A}$ and, more importantly, has the property that there is an ismorphism

$$\epsilon^* : \operatorname{Ext}^{*,*}_{\mathscr{A}}(M,\mathbb{F}_2) \xrightarrow{\cong} \operatorname{Ext}^{*,*}_{\mathscr{A}}(R_+(M),\mathbb{F}_2)$$

for any bounded below and finite type $\mathscr{A}$-module M.

Our objective is to present a generalization to these results. Specifically, for a bounded below spectrum B of finite type, we will construct a tower of spectra

$$(B^{\wedge p})^{tC_p} \to \ldots \to (B^{\wedge p})^{tC_p}[n+1] \to (B^{\wedge p})^{tC_p}[n] \to \ldots$$

such that

$$\operatorname*{colim}_n H^*((B^{\wedge p})^{tC_p}[n];\mathbb{F}_p) \cong R_+(H^*(B;\mathbb{F}_p))\,.$$

Here p is a fixed prime and $B^{\wedge p}$ is a chosen equivariant model of the p-fold smash power with the cyclic group C_p of order p acting by permutation of the factors. The case of the C_2-Segal conjecture will be a special case of this, when $B = S$ is the sphere spectrum. As an application, we prove in Corollary 5.15 that the fixed points of $B^{\wedge p}$ and its homotopy fixed points are homotopy equivalent after p-completion, under bounded below and finite type restrictions on the spectrum B.

In §3, we define the algebraic Singer construction on an $\mathscr{A}$-module M and study its cohomological, as well as its dual, homological version.

Then, in §4, we define a certain choice of tower of spectra with homotopy inverse limit equivalent to the Tate construction X^{tG} on some G-spectrum X. We address the associated spectral sequences and discuss the differences between working with homology contrary to previous work where one focuses on homotopy directly. ([3], [11]).

In §5, we then specialize and consider the case when $X = B^{\wedge p}$. We also need to discuss the genuine equivariant model of the permutation spectrum $B^{\wedge p}$.

For any $\mathscr{A}$-module M the Singer construction $R_+(M)$ is an $\mathscr{A}$-module resembling the $\mathscr{A}$-module $\mathscr{A} \otimes M$ in some ways. There is an $\mathscr{A}$-linear evaluation map $\epsilon : R_+(M) \to M$ analogous to the evaluation map $\mathscr{A} \otimes M \to M$ sending $\mathrm{Sq}^n \otimes x \mapsto \mathrm{Sq}^n(x)$ for $p = 2$, and similarly for $p > 2$.

1.1. **Notation.** We denote by $\mathscr{A}$ the mod p Steenrod algebra and by $\mathscr{A}_*$ its $\mathbb{F}_p$-linear hom dual. We will work with modules over $\mathscr{A}$ and comodules over $\mathscr{A}_*$.

The spectra that occur in the present paper will usually be named Y, X and B. The latter will typically mean some bounded below, finite type spectrum or S-algebras. Spectra denoted X will be equipped with an equivariant structure. The main examples we have in mind are $X = THH(B)$, the topological Hochschild homology spectrum and $X = (B^{\wedge p})$. When dealing with generic towers of spectra, we will use Y. Again, our example of main interest is the Tate construction on some equivariant spectrum X.

1.1.1. *History and Notation of the Singer construction.* The Singer construction appeared originally in [16]. The work presented here concentrates on its relation to the calculations by Lin and Gunawardena and their work on the Segal conjecture for cyclic groups of prime order. A published account for the case of the cyclic group of order 2 is found in [14]. A further study appears in [1], where a more conceptual definition of the Singer construction is given.

In W. Singer's papers [16] and [17] the Singer construction on an $\mathscr{A}$-module M is denoted by $R_+(M)$. In [16], the following question is posed: Let M be an unstable $\mathscr{A}$-module and let $\Delta = \mathbb{F}_2\{\mathrm{Sq}^s \mid s \in \mathbb{Z}\}$. Then there is a map of $\mathbb{F}_2$ vector spaces

$$d : \Delta \otimes M \to M$$

given by $\mathrm{Sq}^s \otimes m \mapsto \mathrm{Sq}^s(m)$. Does there exist a natural $\mathscr{A}$-module structure on the source of d rendering this map an $\mathscr{A}$-linear homomorphism?

The problem is solved by considering a functor R on the category of unstable $\mathscr{A}$-modules to itself: The object $R(M)$ is a module over $R(\mathbb{F}_2)$. The latter is an $\mathscr{A}$-algebra isomorphic to the polynomial ring $P(e)$ on one generator e in degree 1, and by an idea of Wilkerson [18] the module $R(M)$ is localized with respect to the multiplicative subset $\{1, e, e^2, \ldots\} \in R(\mathbb{F}_2)$ to produce an (in general unstable) $\mathscr{A}$-module denoted $R_+(M)$. Then it is shown that there is a natural map $d : R_+(M) \to M$ of degree $+1$, and finally that there is an $\mathbb{F}_2$-linear isomorphism $\Delta \otimes M \cong R_+(M)$.

The connection between the continuous cohomology of (5.4) and the Singer construction can be found in [5, §5, Theorem 5.1]. Here, the Singer functor is still denoted by R_+, but it differs by a shift of degree one from the R_+ in Singer's work [16] and [17].

In connection with the Segal conjecture, Adams, Gunawardena and Miller [1] published an algebraic account of the Singer construction using the letter T' for what was previously denoted R_+. Precisely, in [1], the authors define a functor T'' which is the same as the R_+ used in [5]. For the trivial $\mathscr{A}$-module $\mathbb{F}_p$, the Singer construction is isomorphic to the Tate cohomology groups $\widehat{H}^*(\Sigma_p;\mathbb{F}_p)$ but with a shift of one degree. Since the cup product structure is important in their work, the authors of [1] would like to have this isomorphism to be of degree zero, and are thus led to define $T' = \Sigma^{-1}T''$ as their Singer construction. This T' is the same functor as the R_+ in Singer's original work [16] and [17]

In the following we will make use of the fact that the Singer construction on an $\mathscr{A}$-module M comes equipped with a homomorphism of $\mathscr{A}$-modules $\epsilon : R_+(M) \to M$. In Singer's work [17], this map has degree $+1$ (and was named d), whereas in [5]'s definition of $R_+(-)$ it has degree 0. We choose to follow the grading conventions of [5] mainly because this is the functor that, with no shift of degrees, describes our continuous (co-)homology groups. The homomorphism ϵ will also be realized by a map of spectra and will therefore be of degree zero.

Furthermore we choose to write $R_+(M)$ instead of $T''(M)$, $T'(M)$ or $T(M)$ because the letter T is heavily overloaded by the presence of THH, the Tate-construction and the circle group $\mathbb{T}$. To add to the confusion, the letter T is also used in Singer's [17] work, but with a different meaning than the T appearing in [1].

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2. Limits of spectra

We introduce our first definitions regarding towers of spectra and their associated homology groups and cohomology groups. These towers will consist of spectra such that the singular $\mathbb{F}_p$ (co-)homology at each stage will be bounded below and of finite type over $\mathbb{F}_p$.

The motivation for this definition is a result by Caruso-May-Priddy saying that that there is an inverse limit of Adams spectral sequences arising from such towers, calculating the homotopy of the inverse limit spectrum.

The input for this inverse limit of Adams spectral sequences will give us the definitions of the continuous (co-)homology groups.

We will always work at a fixed prime p.

2.1. **Inverse limits of Adams spectral sequences.** Let $\{Y_n\}_{n\in\mathbb{Z}}$ be a collection of spectra with stable maps $f_n : Y_n \to Y_{n+1}$ for all n and let Y be the homotopy inverse limit over this system.

$$Y \to \ldots \to Y_n \xrightarrow{f_n} Y_{n+1} \xrightarrow{f_{n+1}} Y_{n+2} \to \ldots \tag{2.1}$$

Definition 2.1. *We say that a spectrum Z is bounded below and of finite type over $\mathbb{F}_p$ if the homotopy groups $\pi_*(Z)$ are bounded below, and if the singular homology with $\mathbb{F}_p$-coefficients $H_*(Z;\mathbb{F}_p)$ is of degreewise finite type over $\mathbb{F}_p$.*

When p is understood we will say that a spectrum Z is bounded below and of finite type without reference to the prime p. We will also write $H(-)$ for $H(-;\mathbb{F}_p)$ to shorten notation.

We assume for the rest of this chapter that Y is the inverse limit of a chosen tower of spectra (2.1) such that each Y_n is bounded below and of finite type over $\mathbb{F}_p$.

For each n there is an Adams spectral sequence $\{E_r(Y_n)\}_r$ with E_2-term

$$E_2^{s,t}(Y_n) = \mathrm{Ext}^{s,t}_{\mathscr{A}}(H^*(Y_n),\mathbb{F}_p) \Rightarrow \pi_{t-s}((Y_n)^\wedge_p)$$

converging strongly to the homotopy groups of the p-completion of Y_n. Here $\mathscr{A}$ denotes the mod p Steenrod algebra.

The tower (2.1) induces maps of Adams spectral sequences $f_{n*} : \{E_r(Y_n)\} \to \{E_r(Y_{n+1})\}$. For every r let $E_r(\underline{Y}) = \lim E^r(Y_n)$ denote the inverse limit as $n \to -\infty$. It is clear that

$$E_2^{s,t}(\underline{Y}) \cong \mathrm{Ext}^{s,t}_{\mathscr{A}}(\operatorname*{colim}_{n\to-\infty} H^*(Y_n),\mathbb{F}_p)\,. \tag{2.2}$$

We now state and prove a slightly sharper version of [7, Proposition 7.1].

Proposition 2.2. *Given a tower of spectra $\{Y_n\}$ such that for every n, Y_n is bounded below and of finite type over $\mathbb{F}_p$.*

The sequence of bigraded groups $\{E_r(\underline{Y})\}_r$ are the terms of a spectral sequence converging strongly to the homotopy of the p-completion of Y.

The essential difference between this statement and the statement in [7] lies in the hypothesis on Y_n: we replace the condition that each Y_n should be finite type over $\mathbb{Z}_p^\wedge$ with the condition that $H_*(Y_n;\mathbb{F}_p)$ should be of (degreewise) finite type.

We will refer to $\{E_r(\underline{Y})\}$as *the inverse limit of Adams spectral sequences* associated to the tower $\{Y_n\}$.

Proof. For any connective spectrum X with $H_*(X;\mathbb{F}_p)$ of finite type, the Adams spectral sequence converge to $\pi_*(X_p^\wedge)$. Since the E_2 and E_∞-terms of this spectral sequence are of finite type, the (Abelian) groups $\pi_*(X_p^\wedge)$ are compact and Hausdorff in the topology given by the Adams filtration.

The category of compact Hausdorff Abelian groups is an Abelian category. There is a Pontryagin duality that assignes to an Abelian group G, the character group $\hom(G,S^1)$. This duality induces a contravariant equivalence of categories between the category of compact Hausdorff Abelian groups and the category of discrete groups.

For a small category I and an Abelian category A, the functor category $A^I = Fun(I,A)$ is again an Abelian category.

It follows that the inverse limit functor of filtered diagrams of compact Hausdorff Abelian groups is an exact functor, by the Pontryagin duality and the fact that taking colimits of filtered diagrams of discrete groups is an exact functor.

We now repeat the proof of [7, prop 7.1], using our version on the exactness of the inverse limit functor.

We can construct a diagram of spectra

$$
\begin{array}{ccccccc}
\cdots \longrightarrow & Z_{\infty,s} & \longrightarrow \cdots \longrightarrow & Z_{\infty,0} = Y_{\infty}{}^{\wedge}_{p} \\
& \downarrow & & \downarrow \\
& \vdots & & \vdots \\
& \downarrow & & \downarrow f_{n+1} \\
\cdots \longrightarrow & Z_{n,s} & \longrightarrow \cdots \longrightarrow & Z_{n,0} = Y_{n}{}^{\wedge}_{p} \\
& \downarrow & & \downarrow f_{n} \\
& \vdots & & \vdots \\
& \downarrow & & \downarrow f_{1} \\
\cdots \longrightarrow & Z_{0,s} & \longrightarrow \cdots \longrightarrow & Z_{0,0} = Y_{0}{}^{\wedge}_{p}
\end{array}
\tag{2.3}
$$

where each row involving a finite n is an Adams resolution, and such that each $Z_{n,s}$ is a bounded below spectrum of finite type over $\mathbb{F}_p$ for every finite n. The limiting terms in the upper row are the homotopy limits

$$Z_{\infty,s} = \underset{n\to\infty}{\operatorname{holim}} Z_{n,s} \, .$$

For every n, each resolution of Y_n gives then rise to an exact couple of homotopy groups, such that all groups involved are Abelian, compact and Hausdorff.

Exact functors between Abelian categories preserve kernels, images, cokernels and homology. Hence, taking the inverse limit of these groups then commutes with taking homology, and the sequence of spectra

$$\cdots \to Z_{\infty,s} \to \cdots \to Z_{\infty,0} = Y_\infty \tag{2.4}$$

gives rise to an exact couple with an associated spectral sequence with terms given by

$$E_r^{s,t} \cong \lim_{n\to\infty} E_r^{s,t}(Y_n)\, .$$

Hence, the inverse limit of Adams spectral sequences has the desired E^2-term.

We must now check the convergence of this spectral sequence. The Adams resolution for each Y_n is constructed so that

$$\lim_s \pi_*(Z_{n,s}) = \operatorname{Rlim}_s \pi_*(Z_{n,s}) = 0\, .$$

These two conditions ensure that the Adams spectral sequence for Y_n converges conditionally, in the language of Boardman. For the inverse limit resolution (2.4), the standard interchange of limits isomorphism gives

$$\lim_s \lim_n \pi_*(Z_{n,s}) \cong \lim_n \lim_s \pi_*(Z_{n,s}) = 0\, .$$

Moreover, the exactness of the inverse limit functor in this case implies that the derived limit

$$\operatorname{Rlim}_s \lim_n \pi_*(Z_{n,s}) = 0$$

vanishes too and we get that the inverse limit Adams spectral sequence is conditionally convergent to $\pi_*(Y_\infty{}_p^\wedge)$.

The inverse limit of Adams resolutions (2.4) give rise to an upper halfplane spectral sequence with entering differentials, in the sense of Boardman. For such a spectral sequence, strong convergence follows from conditional convergence together with the vanishing of the groups

$$RE_\infty = \operatorname{Rlim}_r Z_r^{s,t}(Y_\infty)\,.$$

See [2, 5.5, 5.6, 8.1]. Again, the vanishing of RE_∞ is secured by the exactness of *lim*. □

2.2. **Continuous (co-)homology.** The spectral sequence (2.2) is central to the proof of the Segal conjecture for groups of prime order and will be the foundation for the present work. Our work will, in analogy with Lin's proof of the Segal conjecture, focus on the calculation and properties of the E_2-term of the above spectral sequence.

Definition 2.3. *Let p be any prime. Let Y be the homotopy inverse limit of a tower of spectra as in* (2.1) *with each Y_n bounded below and of finite type over $\mathbb{F}_p$. Define the continuous cohomology of Y with $\mathbb{F}_p$-coefficients as the colimit*

$$H_c^*(Y;\mathbb{F}_p) = \operatorname*{colim}_{n\to-\infty} H^*(Y_n;\mathbb{F}_p)\,.$$

Dually, define the continuous homology of Y with $\mathbb{F}_p$-coefficients as the inverse limit

$$H_*^c(Y;\mathbb{F}_p) = \lim_{n\to-\infty} H_*(Y_n;\mathbb{F}_p)\,.$$

As already mentioned, we choose to suppress from the notation the tower of which Y is a homotopy inverse limit. However, we do so with a warning: Let $p=2$ and let $Y=(S^0)_2^\wedge$ be the 2-completed sphere spectrum. Since Y is bounded below and of finite type, we may express Y by the constant tower of spectra. But by W. H. Lin's theorem, $(S^0)_2^\wedge \simeq \operatorname{holim}_n \Sigma\mathbb{R}P^\infty_{-n}$, where each $\Sigma\mathbb{R}P^\infty_{-n}$ is also finite type and bounded below. But $\operatorname{colim}_n H^*(\Sigma\mathbb{R}P^\infty_{-n}) = \Sigma P(e,e^{-1}) = R_+(\mathbb{F}_2)$ which is vastly larger than $H^*(S^0)\cong\mathbb{F}_2$. The continuous cohomology groups are dependent on the choice of inverse system.

Since we are considering field coefficients and each spectrum Y_n is assumed to be of finite type over $\mathbb{F}_p$, we get that the $\mathbb{F}_p$-linear dual of $H_c^*(Y)$ is isomorphic to $H_*^c(Y)$. The continuous homology will typically be an infinite dimensional vector space over $\mathbb{F}_p$ in each degree, so the double dual is generally not isomorphic to the continuous cohomology. However, if we take into account the topology on the inverse limit given by the filtration from the tower, we do get that the continuous $\mathbb{F}_p$-linear dual of the continuous homology is isomorphic to the continuous cohomology. We discuss this in §2.4.

Note that the continuous cohomology is a direct limit of bounded below $\mathscr{A}$-modules. The direct limit may of course not exist in the category of bounded below modules, but we do get a natural $\mathscr{A}$-module structure on $H_c^*(Y)$ in the category of $\mathscr{A}$-modules with no boundedness restrictions.

Dually, the continuous homology is an inverse limit of bounded below $\mathscr{A}_*$-comodules, but the inverse limit may be neither bounded below nor an algebraic $\mathscr{A}_*$-comodule. In general we get a completed coaction of $\mathscr{A}_*$

$$H_*^c(Y) \to \mathscr{A}_* \hat{\otimes}\, H_*^c(Y)$$

where $-\hat{\otimes}-$ is the tensor product completed with respect to the topology of the continuous homology given by the kernel filtration induced by the inverse limit system.

2.3. **Filtrations.** For every $n \in \mathbb{Z}$, let A^n be a graded $\mathbb{F}_p$-vector space and assume that these vector spaces fit into a sequence

$$(2.5) \qquad 0 \longrightarrow \cdots \longrightarrow A^n \longrightarrow A^{n-1} \longrightarrow \cdots \longrightarrow A^{-\infty}$$

with trivial inverse limit and colimit denoted by $A^{-\infty}$. We assume further that each A^n is finite dimensional in each degree. Let A_n be the $\mathbb{F}_p$-linear dual of A^n. The sequential limit above dualizes to a limit system

$$(2.6) \qquad A_{-\infty} \longrightarrow \cdots \longrightarrow A_{n-1} \longrightarrow A_n \longrightarrow \cdots \longrightarrow 0$$

with inverse limit $A_{-\infty}$ isomorphic to the dual of $A^{-\infty}$ and trivial colimit. The last fact follows from the assumption that A^n is finite dimensional: Indeed,

$$(2.7) \qquad \lim_n A^n \cong \lim_n \hom_{\mathbb{F}_p}(A_n, \mathbb{F}_p) \cong \hom_{\mathbb{F}_p}(\operatorname{colim}_n A_n, \mathbb{F}_p)$$

and thus $\lim_n A^n = 0$ implies that $\operatorname{colim}_n A_n$ is trivial since the latter injects into its double $\mathbb{F}_p$-linear dual.

Following Boardman's notation [2, Lemma 5.4], we define filtrations of the colimit of (2.5) and the limit of (2.6) using the corresponding sequential limit systems.

Definition 2.4. *For each $n \in \mathbb{Z}$, let*

$$F^n A^{-\infty} = \operatorname{im}(A^n \to A^{-\infty})$$
$$F_n A_{-\infty} = \ker(A_{-\infty} \to A_n)\,.$$

Then

$$(2.8) \qquad \ldots \subset F^n A^{-\infty} \subset F^{n-1} A^{-\infty} \subset \ldots \subset A^{-\infty}$$

and

$$(2.9) \qquad \ldots \subset F_{n-1} A_{-\infty} \subset F_n A_{-\infty} \subset \ldots \subset A_{-\infty}$$

define a decreasing (resp. increasing) sequence of subspaces of $A^{-\infty}$ (resp. $A_{-\infty}$).

By definition, the filtration (2.8) exhausts $A^{-\infty}$. By our assumption that the inverse limit of (2.5) is trivial, the filtration is also Hausdorff (i.e. $\lim_n F^n A^{-\infty} = 0$.) Furthermore, it follows that each $F^n A^{-\infty}$ is also degreewise finite dimensional and thus the first right derived limit vanishes; $\mathrm{Rlim}_n F^n A^{-\infty} = 0$. This is equivalent to saying that the filtration is complete, i.e. that the canonical map $A^{-\infty} \to \lim_n A^{-\infty}/F^n A^{-\infty}$ is an isomorphism. Completeness is equivalent to saying that Cauchy sequences converge in the topology given by the filtration. That the filtration is Hausdorff is saying that Cauchy sequences have unique limits.

For the filtration (2.9), one gets without any hypotheses that the filtration is Hausdorff and complete (compare with the proof of [2, Lemma 5.4 (b)]). Further, it follows from the fact that the colimit of (2.6) is trivial that the filtration (2.9) is exhaustive.

We collect the above facts in the following lemma.

Lemma 2.5. *Assume that the limit of* (2.5) *is trivial and each* A^n *is degreewise finite dimensional. Then both filtrations given in Definition 2.4 (of* $A^{-\infty}$ *and its dual* $A_{-\infty}$*) are exhaustive, Hausdorff and complete.*

2.4. **Dualization.** We defined (2.6) and (2.5) to be $\mathbb{F}_p$-dual to eachother. We now discuss how the limits correspond via dualization.

First of all, the dual of the colimit of (2.6) is isomorphic to the inverse limit of (2.5) since $\hom(-, \mathbb{F}_p)$ is right-exact. However, the dual of the limit of (2.6) is isomorphic to the double dual of $A^{-\infty}$ which contains $A^{-\infty}$ in a canonical way, but is often strictly bigger.

To remedy this, we take into account the topology on the inverse limit induced by the inverse system and dualize by considering the continuous hom-dual.

The collection $\{F_n A_{-\infty}\}_n$ gives a basis for the open neighborhoods around $0 \in A_{-\infty}$ and a continuous homomorphism $f : A_{-\infty} \to k$ is thus an $\mathbb{F}_p$-linear function whose kernel contains $F_n A_{-\infty}$ for some n. We write $\hom^c(A_{-\infty}, \mathbb{F}_p)$ for the continuous dual of $A_{-\infty}$.

Lemma 2.6. *There is a natural isomorphism*

$$\hom_{\mathbb{F}_p}(A^{-\infty}, \mathbb{F}_p) \cong A_{-\infty} .$$

Let $A_{-\infty}$ *have the topology induced by the system of neighborhoods* $\{F_n A_{-\infty}\}$*. Then there is a natural isomorphism*

$$\hom^c_{\mathbb{F}_p}(A_{-\infty}, \mathbb{F}_p) \cong A^{-\infty} .$$

Proof. The first isomorphism has already been explained. For the second, we have a canonical inclusion of $A^{-\infty}$ into its double (continuous) dual

$$A^{-\infty} \hookrightarrow \hom^c(\hom(A^{-\infty}, \mathbb{F}_p), \mathbb{F}_p) = \hom^c(A_{-\infty}, \mathbb{F}_p).$$

By definition, $F^n A^{-\infty} = \operatorname{im}(A^n \to A^{-\infty})$. It follows that

$$\hom(F^n A^{-\infty}, \mathbb{F}_p) \cong \operatorname{im}(A_{-\infty} \to A_n) \cong A_{-\infty}/F_n A_{-\infty} .$$

Thus we can write the canonical map from $F^n A^{-\infty}$ into its double dual as

$$F^n A^{-\infty} \hookrightarrow \hom(A_{-\infty}/F_n A_{-\infty}, \mathbb{F}_p) .$$

This map is an isomorphism since $F^n A^{-\infty}$ is degreewise finite dimensional. Passing to the colimit as $n \to -\infty$, we get the desired isomorphism

$$A^{-\infty} \cong \operatorname*{colim}_{n \to -\infty} \hom(A_{-\infty}/F_n A_{-\infty}, \mathbb{F}_p) \cong \hom^c(A_{-\infty}, \mathbb{F}_p) .$$

□

2.5. **Limits of $\mathscr{A}_*$-comodules.** Until now, the objects of our discussion have been vectorspaces over $\mathbb{F}_p$. We will now add more structure and assume that (2.5) is a system of modules over $\mathscr{A}$. It follows that the dual tower (2.6) is a system of comodules over the dual Steenrod algebra $\mathscr{A}_*$.

We need to discuss in what sense this structure carries over to $A^{-\infty}$ and $A_{-\infty}$.

Let M_* be a complete topological graded vectorspace over $\mathbb{F}_p$. For a topological graded vectorspace V_* with the discrete topology, we often wish to consider the completed tensor product $V_* \hat{\otimes} M_*$, completed with respect to the induced topology on $V_* \otimes M_*$.

We say that M_* is a complete $\mathscr{A}_*$-comodule if there is a continuous graded homomorphism $\nu : M_* \to \mathscr{A}_* \hat{\otimes}\, M_*$ such that the diagram

$$(2.10)\qquad \begin{array}{ccccc} M_* & \xrightarrow{\nu} & \mathscr{A}_*\hat{\otimes}M_* & & \\ \downarrow{\scriptstyle \nu} & & & \searrow{\scriptstyle \psi\hat{\otimes}1} & \\ \mathscr{A}_*\hat{\otimes}M_* & \xrightarrow{1\hat{\otimes}\nu} & \mathscr{A}_*\hat{\otimes}(\mathscr{A}_*\hat{\otimes}M_*) & \xrightarrow{\cong} & (\mathscr{A}_*\otimes\mathscr{A}_*)\hat{\otimes}M_* \end{array}$$

commutes. Let N_* be another complete $\mathscr{A}_*$-comodule and let $f : M_* \to N_*$ be a continuous graded map. Then $f \in \hom^c_{\mathscr{A}_*}(M_*, N_*)$ if the obvious diagram commutes.

Suppose given a direct system of bounded below and finite type $\mathbb{F}_p$-vectorspaces as in (2.5). Suppose further that this system is one of $\mathscr{A}$-modules, i.e. we have a compatible system of action maps

$$(2.11)\qquad \mu[n] : \mathscr{A} \otimes A^n \to A^n\,.$$

Lemma 2.7. *In the setting above, $A^{-\infty}$ is an $\mathscr{A}$-module, and the topological vectorspace $A_{-\infty}$ is a complete $\mathscr{A}_*$-comodule.*

Proof. The category of $\mathscr{A}$-modules is closed under direct limits, so the first claim of the Lemma is immediate. We denote the $\mathscr{A}$-module with action map on the colimit by

$$\mu : \mathscr{A} \otimes A^{-\infty} \to A^{-\infty}\,.$$

For every n, we denote the $\mathbb{F}_p$-dual of (2.11) by

$$\nu[n] : A_n \to \mathscr{A}_* \otimes A_n$$

which gives A_n the structure of an $\mathscr{A}_*$-comodule. The $\mathbb{F}_p$-dual of μ is realized as the inverse limit of $\nu[n]$ as $n \to -\infty$:

$$(2.12)\qquad \nu = \lim_n \nu[n] : A_{-\infty} \to \lim_n \mathscr{A}_* \otimes A_n\,.$$

This coaction map is continuous since it is realized as the inverse limit of a homomorphism of towers. We identify the target of he coaction map ν in (2.12) in terms of the completed tensor product as follows:

$$\begin{aligned} \mathscr{A}_*\hat{\otimes}A_{-\infty} &= \lim_n \mathscr{A}_* \otimes A_{-\infty}/F_nA_{-\infty} \\ &\cong \lim_n \mathscr{A}_* \otimes \mathrm{im}(A_{-\infty} \to A_n) \hookrightarrow \lim_n \mathscr{A}_* \otimes A_n \end{aligned}$$

where the last inclusion is in fact an isomorphism since $\mathscr{A}_*$ is (degreewise) of finte type. Indeed, let $Z_n = A_n/\,\mathrm{im}(A_{-\infty} \to A_n)$. Then $\lim_n Z_n = 0$ and more importantly, $\lim_n \mathscr{A}_* \otimes Z_n = 0$ since $\mathscr{A}_*$ is of finite type. Taking the limit as $n \to -\infty$, the exact sequence

$$0 \to \mathscr{A}_* \otimes \mathrm{im}(A_{-\infty} \to A_n) \to \mathscr{A}_* \otimes A_n \to \mathscr{A}_* \otimes Z_n \to 0$$

induces a 6-term exact sequence which breaks down to the isomorphism

$$\lim_n \mathscr{A}_* \otimes \mathrm{im}(A_{-\infty} \to A_n) \cong \lim_n \mathscr{A}_* \otimes A_n$$

since the third term is trivial. Said in another way, the completion of the image of the natural inclusion

$$\mathscr{A}_* \otimes A_{-\infty} \hookrightarrow \lim_n \mathscr{A}_* \otimes A_n$$

is the whole of $\lim_n \mathscr{A}_* \otimes A_n$.

To summarize, $\mathscr{A}_{-\infty}$ is a complete topological graded $\mathbb{F}_p$-vectorspace with a continuous homomorphism

$$\nu : \mathscr{A}_{-\infty} \to \mathscr{A}_* \hat{\otimes} A_{-\infty} \,.$$

The commutativity of diagram (2.10) is immediate since it is obtained as the inverse limit of the corresponding diagrams involving A_n. Hence, $A_{-\infty}$ is a complete $\mathscr{A}_*$-comodule.

□

If we consider the category of complete, topological graded $\mathbb{F}_p$-vectorspaces and continuous homomorphisms, there is a subcategory of complete $\mathscr{A}_*$-comodules such that for two objects M_* and N_*, the morphism set is given by the equalizer diagram

$$\hom^c_{\mathscr{A}_*}(M_*, N_*) \to \hom^c_{\mathbb{F}_p}(M_*, N_*) \underset{f \mapsto \nu \circ f}{\overset{f \mapsto 1 \otimes f \circ \nu}{\rightrightarrows}} \hom^c_{\mathbb{F}_p}(M_*, \mathscr{A}_* \hat{\otimes} N_*) \,. \tag{2.13}$$

3. The algebraic Singer construction

Classically, the algebraic Singer contruction is presented as a functor on the category of modules over the Steenrod algebra. We recall its definition and some of its properties in §3.1. We then dualize the construction in §3.2.

Later, we will see how the algebraic Singer construction arises in its cohomological (and homological) form as the continuous cohomology (and continuous homology) of a certain tower of truncated Tate spectra. From this tower of spectra we will derive a filtration on the Singer construction which we will define algebraically in the present paragraph. When we come to §5.2, we will see how the filtration defined here relates to the natural filtration arising from topology.

3.1. The cohomological Singer construction.

Definition 3.1. *Let M be an $\mathscr{A}$-module. The Singer construction on M is a graded $\mathscr{A}$-module denoted $R_+(M)$ such that, for $p = 2$, additively we have*

$$\Sigma^{-1} R_+(M) = P(e, e^{-1}) \otimes M$$

with the action of the Steenrod squares given by the formula

$$\mathrm{Sq}^s(e^r \otimes m) = \sum_i \binom{r-i}{s-2i} e^{r+s-i} \otimes \mathrm{Sq}^i(m) \,. \tag{3.1}$$

For $p > 2$, the additive definition is

$$\Sigma^{-1} R_+(M) = E(e) \otimes P(x, x^{-1}) \otimes M$$

with the action of the Steenrod algebra is determined by the formulas

$$\begin{aligned} \mathrm{P}^s(x^r \otimes m) &= \sum_i (-1)^{s+i} \binom{r - i(p-1)}{s - pi} x^{r+(s-i)(p-1)} \otimes \mathrm{P}^i(m) \\ &\quad + \sum_i (-1)^{s+i} \binom{r - i(p-1) - 1}{s - pi - 1} e x^{r+(s-i)(p-1)-1} \otimes \beta\, \mathrm{P}^i(m) \\ \mathrm{P}^s(e x^{r-1} \otimes m) &= \sum_i (-1)^{s+i} \binom{r - 1 - i(p-1)}{s - pi} e x^{r-1+(s-i)(p-1)} \otimes \mathrm{P}^i(m) \,. \end{aligned}$$

and

$$\beta(e^i x^{r-i} \otimes m) = i(x^r \otimes m)$$

for $i = 0, 1$. *The grading is determined by letting* $deg(e) = 1$ *and* $deg(x) = 2$.

Note that the action of the Steenrod algebra is not the diagonal action gotten from the coproduct in the Hopf algebra structure of $\mathscr{A}$.

The Singer construction is related to the cohomology of the cyclic group C_p. In fact, let BC_p be the classifying space of the cyclic group of prime order p. Then $H^*(BC_{p+}; \mathbb{F}_p) \cong P(x, x^{-1}) \otimes E(e)$ with x and e of degree 2 and 1 as in the definition above. The structure of $H^*(BC_{p+}; \mathbb{F}_p)$ as an $\mathscr{A}$-module can be extended to an $\mathscr{A}$-module structure of the localization $H^*(BC_{p+}; \mathbb{F}_p)[x^{-1}]$. By letting $M = \mathbb{F}_p$ we get that $\Sigma^{-1} R_+(\mathbb{F}_p)$ is isomorphic, as an $\mathscr{A}$-module, to $H^*(BC_{p+}; \mathbb{F}_p)[x^{-1}]$.

When $p > 2$ there are two versions of the Singer construction: the one given here, which is related to the cyclic group C_p, and one which is related to the cohomology of Σ_p, the group permutations of p letters. The latter yields a smaller $\mathscr{A}$-module. The bigger version is, roughly speaking, $p-1$ times the size of the smaller version. In our work, we are only concerned with the version of the Singer construction related to the group C_p.

3.1.1. *The cohomological ϵ-map.* An important property of $R_+(M)$ is that there exists a natural homomorphism $\epsilon : R_+(M) \to M$ of $\mathscr{A}$-modules. In Singer's original definition for $p = 2$, the map is given as by the explicit formula

$$\Sigma e^{r-1} \otimes m \mapsto \mathrm{Sq}^r(m) \,. \tag{3.2}$$

For p odd, the sub $\mathscr{A}$-module spanned by elements on the form $\Sigma e^i x^{r(p-1)-i} \otimes m$ sits as a direct summand of $R_+(M)$. The homomorphism ϵ is given by first projecting onto this direct summand and then compose with the map

$$\Sigma e^i x^{r(p-1)-i} \otimes m \mapsto \beta^{1-i} \mathrm{P}^r(m) \,. \tag{3.3}$$

We recall the key property of ϵ. The following definition is due to Adams-Gunawardena-Miller [1].

Definition 3.2. *A map of $\mathscr{A}$-modules $L \to M$ is a Tor-equivalence if the induced map*

$$\mathrm{Tor}^A_{**}(\mathbb{F}_p, L) \to \mathrm{Tor}^A_{**}(\mathbb{F}_p, M) \tag{3.4}$$

is an isomorphism.

The relevance of this is the following.

Proposition 3.3 ([1]). *If $L \to M$ is a Tor-equivalence then for every bounded above right $\mathscr{A}$-module K the induced map*

$$\mathrm{Tor}^A_{**}(K, L) \to \mathrm{Tor}^A_{**}(K, M) \tag{3.5}$$

is an isomorphism. For every bounded below left $\mathscr{A}$-module N of finite type, the induced map

$$\mathrm{Ext}^{**}_{\mathscr{A}}(M, N) \to \mathrm{Ext}^{**}_A(L, N) \tag{3.6}$$

is an isomorphism.

And finally

Theorem 3.4 (Gunawardena, Miller [1])**.** *The evaluation map ϵ is a Tor-equivalence.*

In particular, we will in calculations encounter instances of $\mathscr{A}$-module homomorphisms $R_+(M) \to M$ induced by maps of spectra. In case M has a simple description as an $\mathscr{A}$-module, it is often possible to determine those maps by using the following corollary of Theorem 3.4.

Corollary 3.5. *Let M, N be any left $\mathscr{A}$-modules such that N is bounded below and of finite type over $\mathbb{F}_p$. Then any $\mathscr{A}$-linear map $f : R_+(M) \to N$ factors uniquely as $\bar{f} \circ \epsilon$ for some $\mathscr{A}$-linear homomorphism $\bar{f} : M \to N$.*

$$\begin{array}{ccc} R_+(M) & \xrightarrow{f} & N \\ \downarrow{\scriptstyle \epsilon} & \nearrow{\scriptstyle \bar{f}} & \\ M & & \end{array}$$

Proof. Since N is bounded below and of finite type, a special case of theorem 3.4 and proposition 3.3 says that $\epsilon^* : \mathrm{Hom}_A(M, N) \to \mathrm{Hom}_A(R_+(M), N)$ is an isomorphism, and the corollary follows. □

Remark 3.6. A special case of this occurs when $N = M$ is a cyclic $\mathscr{A}$-module. Then $\mathbb{F}_p \cong \mathrm{Hom}_{\mathscr{A}}(M, M) \cong \mathrm{Hom}_A(R_+(M), M)$, so any $\mathscr{A}$-linear map $R_+(M) \to M$ is equal to ϵ, up to a scalar multiple.

3.2. **The homological Singer construction.** Before we define the homological version of the Singer construction on an $\mathscr{A}_*$-comodule M_*, we need to discuss a filtration on the cohomological Singer construction. For a bounded below, finite type $\mathscr{A}$-module M, let

$$\begin{aligned} F^n R_+(M) &= \mathbb{F}_p\{\Sigma x^r e^i \otimes m \mid i \in \{0,1\}, r \in \mathbb{Z}, m \in M_k, \\ &\qquad\qquad 1 + 2r + i - k(p-1)/2 \geq n\} \\ F^n R_+(M) &= \mathbb{F}_2\{\Sigma e^j \otimes m \mid j \in \mathbb{Z}, m \in M_k, \\ &\qquad\qquad 1 + j - k \geq n\}\,. \end{aligned}$$

for $p > 2$ and $p = 2$ respectively.

Then

$$\ldots \subset F^n R_+(M) \subset F^{n-1} R_+(M) \subset \ldots \subset R_+(M) \tag{3.7}$$

is an exhaustive filtration of $R_+(M)$, which is clearly Hausdorff. Because M is bounded below and of finite type, each $F^n R_+(M)$ is of finite type, and the right derived limit is also trivial. Hence, the filtration is complete.

For reasons to appear later, we will refer to this filtration as the *Tate filtration*. When M is the cohomology of a bounded below and finite type spectrum, we will see how (3.7) is induced from topology. In this case, it will also be immediate that the filtration is one of $\mathscr{A}$-modules. For a general $\mathscr{A}$-module M, this can be checked directly using the explicit formulas in Definition 3.1.

We are now in the situation we discussed in the previous paragraphs, with $A^n = F^n R_+(M)$ and $A^{-\infty} = R_+(M)$. Consider the $\mathbb{F}_p$-dual of the direct system (3.7). We let $F^n R_+(M)_* = \hom_{\mathbb{F}_p}(F^n R_+(M), +F_p)$ and get an inverse system

$$\cdots \to F^n R_+(M)_* \to F^{n+1} R_+(M)_* \to \cdots \tag{3.8}$$

as in (2.6) with $A_n = F^n R_+(M)_*$.

Definition 3.7. *Let M_* be a left $\mathscr{A}_*$-comodule which is bounded below and of finite type. Denote by M the $\mathscr{A}$-module $\hom_{\mathbb{F}_p}(M_*, \mathbb{F}_p)$, dual to M_*.*

The homological Singer construction on M_* is the complete $\mathscr{A}_*$-comodule given by

$$R_+(M_*) = \hom_{\mathbb{F}_p}(R_+(M), \mathbb{F}_p)\,.$$

Thus, $R_+(M_*)$ is isomorphic to the inverse limit of (3.8).

A more explicit description can be given: Let $p = 2$. We choose the monomial basis for $\Sigma P(e, e^{-1})$. Then the $\mathbb{F}_2$-dual is isomorphic to the ring of Laurent polynomials $P(u, u^{-1})$, where u^n is of degree $-n$ and is dual to Σe^{-n-1}.

Likewise, for $p > 2$, we let $E(u) \otimes P(t, t^{-1})$ be the dual of $\Sigma E(e) \otimes P(x, x^{-1})$ where t has degree -2, u has degree -1 and $u^i t^n$ is dual to $\Sigma e^{1-i} x^{-n-1}$.

Using our notation for the dual of the Laurent polynomials, we have the following identifications

$$\begin{aligned} F^n R_+(M)_* &\cong \mathbb{F}_p\{u^i t^r \otimes \alpha \mid i \in \{0,1\}, r \in \mathbb{Z}, \alpha \in (M_*)_k, \\ &\qquad\qquad\qquad\qquad i + 2r + k(p-1)/2 \leq -n\} \\ F^n R_+(M)_* &\cong \mathbb{F}_2\{u^j \otimes \alpha \mid j \in \mathbb{Z}, \alpha \in (M_*)_k, \\ &\qquad\qquad\qquad\qquad j + k \leq -n\}\,. \end{aligned}$$

for $p > 2$ and $p = 2$ respectively. Moreover, the $\mathbb{F}_p$-dual of the maps of (3.7) is given by the obvious projections. The $\mathbb{F}_p$ functional dual of $R_+(M)$ is isomorphic to the inverse limit of the dual of (3.7):

$$\dots \to F^{n-1} R_+(M)_* \to F^n R_+(M)_* \to \dots \tag{3.9}$$

Thus, $R_+(M_*)$ is isomorphic to the vectorspace of formal series

$$\sum_{i \in \mathbb{Z}} u^i \otimes \alpha_i \qquad \text{for } p = 2, \text{ and} \tag{3.10}$$

$$\sum_{i \in \mathbb{Z}} t^i \otimes \alpha_i + \sum_{j \in \mathbb{Z}} u t^j \otimes \alpha_j \qquad \text{for } p > 2. \tag{3.11}$$

Using the topology on $R_+(M)_*$ given by the kernel filtration coming from (3.9), we may reformulate this as follows: Let

$$\Lambda = \begin{cases} P(t, t^{-1}) \otimes E(u) & ; p > 2 \\ P(u, u^{-1}) & ; p = 2\,. \end{cases}$$

Consider $\Lambda \otimes M_* \subset R_+(M)_*$ inside the dual of $R_+(M)$. Then for every n, the composition $\Lambda \otimes M_* \subset R_+(M)_* \to F^n R_+(M)_*$ is surjective, so the completion $\Lambda \hat{\otimes} M_*$ is easily seen to be canonically isomorphic to $R_+(M)_*$.

3.2.1. *The homological ϵ_*-map.* Let

$$\epsilon_* : M_* \to R_+(M_*)$$

be the dual of $\epsilon : R_+(M) \to M$. Then ϵ_* is a continuous homomorphism of complete $\mathscr{A}_*$-comodules. Continuity is trivially satisfied since the source of ϵ_* has the discrete topology.

Dualizing (3.2) and (3.3), we see that ϵ_* is given by the formulas

$$\epsilon_*(\alpha) = \textstyle\sum_i u^{-i} \otimes \mathrm{Sq}_*^i(\alpha)\,. \tag{3.12}$$

for $p=2$, and

$$\epsilon_*(\alpha) = \textstyle\sum_i u^{-i} t^{-n(p-1)} \otimes (\beta^i \mathrm{P}^n)_*(\alpha)\,. \tag{3.13}$$

for p odd.

Lemma 3.8. *Let M and N be bounded below, finite type $\mathscr{A}$-modules and let M_* and N_* be their respective $\mathbb{F}_p$-dual $\mathscr{A}_*$-comodules.*

Then,

$$\epsilon_* : \hom^c_{\mathscr{A}*}(N_*, M_*) \to \hom^c_{\mathscr{A}*}(N_*, R_+(M_*))$$

is an isomorphism.

Notice that continuous homomorphisms are the same as ordinary homomorphisms since N_* is discrete.

Proof. Let $\epsilon[n]$ be the composition

$$F^n R_+(M) \subset R_+(M) \xrightarrow{\epsilon} M\,,$$

and $\epsilon[n]_* : M_* \to F^n R_+(M)_*$ be its $\mathbb{F}_p$-dual. Then both source and target of $\epsilon[n]$ and $\epsilon[n]_*$ are bounded below and of finite type. Consider the commutative diagram

$$\begin{array}{ccc}
\hom_{\mathscr{A}}(F^n R_+(M), N) & \xleftarrow[\epsilon[n]^*]{} & \hom_{\mathscr{A}}(M, N) \\
\downarrow \cong & & \downarrow \cong \\
\hom^c_{\mathscr{A}_*}(N_*, F^n R_+(M)_*) & \xleftarrow[\epsilon[n]_*]{} & \hom^c_{\mathscr{A}_*}(N_*, M_*)\,,
\end{array}$$

where the vertical homomorphisms are the standard isomorphism between $\mathscr{A}$-module homomorphisms and $\mathscr{A}_*$-comodule homomorphisms. Again, the functor $\hom^c_{\mathscr{A}_*}(N_*, -)$ is the same as $\hom_{\mathscr{A}_*}(N_*, -)$ since N_* is discrete.

By passing to inverse limits, we get the commutative square

$$\begin{array}{ccc}
\hom_{\mathscr{A}}(R_+(M), N) & \xleftarrow[\epsilon^*]{\cong} & \hom_{\mathscr{A}}(M, N) \\
\downarrow \cong & & \downarrow \cong \\
\lim_n \hom^c_{\mathscr{A}_*}(N_*, F^n R_+(M)_*) & \longleftarrow & \hom^c_{\mathscr{A}_*}(N_*, M_*)\,.
\end{array}$$

By commutativity of the diagram, the lower horizontal homomorphism is an isomorphism. The lemma follows from the fact that $\hom^c_{\mathscr{A}_*}(N_*, -)$ is a right-exact functor on the category of continuous $\mathscr{A}_*$-comodules, and thus commutes with inverse limits.

By (2.13), it suffices to check that $\hom^c_{\mathbb{F}_p}(N_*, -)$ is right-exact. Indeed, let V_* be a complete $\mathbb{F}_p$-vectorspace and let $\{U_\alpha\}$ be a system of open neighborhoods containing $0 \in V_*$. Then

$$\begin{aligned}
\hom^c_{\mathbb{F}_p}(V_* \hat{\otimes} N_*, W_*) &\cong \operatorname*{colim}_\alpha \hom_{\mathbb{F}_p}(V_*/U_\alpha \otimes N_*, W_*) \\
&\cong \operatorname*{colim}_\alpha \hom_{\mathbb{F}_p}(V_*/U_\alpha, \hom_{\mathbb{F}_p}(N_*, W_*)) \\
&\cong \hom^c_{\mathbb{F}_p}(V_*, \hom_{\mathbb{F}_p}(N_*, W_*))\,.
\end{aligned}$$

Thus, $\hom_{\mathbb{F}_p}(N_*, -) = \hom^c_{\mathbb{F}_p}(N_*, -)$ is right adjoined to $-\hat{\otimes} N_*$ and the claim follows. □

3.2.2. *Various remarks on the homological Singer construction.* The following remarks are not necessary for our immediate applications, but we include them to shed some light on the coaction map and on the ϵ-homomorphism and their relations to the completions introduced so far. For simplicity, we let $p = 2$ for the rest of the paragraph.

Consider the $\mathscr{A}_*$-coaction map

$$\nu : R_+(M_*) \to \mathscr{A}_* \hat{\otimes} R_+(M_*)\,.$$

Let $\alpha \in M_*$ and $n \in \mathbb{Z}$. Dualizing (3.1), we get that the dual Steenrod operations on $R_+(M_*)$ are given by

$$\mathrm{Sq}^s_*(u^n \otimes \alpha) = \sum_i \binom{-1-n-s}{s-2i} u^{n+s-i} \otimes \mathrm{Sq}^i_*(\alpha)\,. \tag{3.14}$$

The sum (3.14) is finite since M_* is assumed to be bounded below, so we have the following commutative diagram:

$$\begin{array}{ccc} R_+(M_*) & \xrightarrow{\ \nu\ } & \mathscr{A}_* \hat{\otimes} R_+(M)_* \\ \uparrow & & \uparrow \\ P(u,u^{-1}) \otimes M_* & \longrightarrow & \mathscr{A}_* \hat{\otimes} [P(u,u^{-1}) \otimes M_*]\,. \end{array} \tag{3.15}$$

Two remarks are in order. First, the subspace topology on $P(u, u^{-1}) \otimes M_*$ is not the discrete topology, nor is it complete with respect to this topology. Indeed, any topological space is complete with respect to the discrete topology. On the other hand, we saw earlier that the completion of $P(u, u^{-1}) \otimes M_*$ was isomorphic to the entire homological Singer construction $R_+(M_*)$. Hence the subspace $P(u, u^{-1}) \otimes M_*$ is not a complete $\mathscr{A}_*$-comodule in the sense explained above.

Second, note that there are elements $m \in P(u, u^{-1}) \otimes M_*$ with the property that $\mathrm{Sq}^i_*(m)$ is nontrivial for arbitrary many indices i. Take for example $m = u \otimes \alpha$ for any $\alpha \in M_*$. Then, according to (3.14), letting $s = 2^r - 1$ yields

$$\mathrm{Sq}^{2^r-1}_*(u \otimes \alpha) = \sum_i \binom{-2^r-1}{2^r-1-2i} u^{2^r+i} \otimes \mathrm{Sq}^i_*(\alpha)$$

where each binomial coefficient is nonzero. Hence, for every $r \geq 0$, this sum contains at least one nonzero term for $i = 0$.

Next, we will identify the image of the homological version of the Singer evaluation map

$$\epsilon_* : M_* \to R_+(M_*) \cong P(u, u^{-1}) \hat{\otimes} M_*\,.$$

We know already that ϵ_* is injective. The statement is, loosely speaking, that the image consists of those elements of $P(u, u^{-1}) \otimes M_* \subset R_+(M_*)$ with *finite co-action.* We will now explain this.

Even though ϵ_* takes values in the completed tensor product, its image is in fact contained in the algebraic tensor product. This follows from our assumption that M_* is bounded below. I.e. there is a commutative diagram

$$\begin{array}{ccc} M_* & \xrightarrow{\ \epsilon_*\ } & R_+(M_*) \\ & \searrow & \uparrow \\ & & P(u,u^{-1}) \otimes M_*\,. \end{array} \tag{3.16}$$

In this diagram, we interpret M_* as having the discrete topology and $P(u, u^{-1}) \otimes M_*$ to have the subspace topology. As noted above, the latter is not a *complete* $\mathscr{A}_*$-comodule, so only the homomorphism ϵ_* lives in the category of complete $\mathscr{A}_*$-comodules.

Let $\mathscr{F} \subset P(u, u^{-1}) \otimes M_*$ be the maximal algebraic sub $\mathscr{A}_*$-comodule. By this we mean the the maximal subspace $\mathscr{F}$ such that the following extension of (3.15) diagram commutes

$$\begin{array}{ccc} P(u,u^{-1}) \otimes M_* & \xrightarrow{\ \nu\ } & \mathscr{A}_* \hat{\otimes} [P(u,u^{-1}) \otimes M_*] \\ \uparrow & & \uparrow \\ \mathscr{F} & \xrightarrow{\ \nu'\ } & \mathscr{A}_* \otimes \mathscr{F}\ , \end{array}$$

and such that ν' gives $\mathscr{F}$ the structure of an $\mathscr{A}_*$-comodule. Note that $\mathscr{F}$ is nonempty: Let $\alpha \in M_*$ be an $\mathscr{A}_*$-comodule primitive class. Such an α exists since M_* is assumed to be bounded below. Then the class $1 \otimes \alpha \in P(u, u^{-1}) \otimes M_*$ is itself $\mathscr{A}_*$-comodule primitive and is thus contained in $\mathscr{F}$. This fact is easily checked by using the explicit formula (3.14). Notice also that $1 \otimes \alpha$ is the image under ϵ_* of the class α. The complete description of $\mathscr{F}$ generalizes this nicely and is due to Marcel Bökstedt and John Rognes:

Proposition 3.9. *For $p = 2$, the injective image of the homomorphism ϵ_* is equal to the sub $\mathscr{A}_*$-comodule $\mathscr{F}$.*

Proof. Since M_* is assumed to be a bounded below $\mathscr{A}_*$-comodule and ϵ_* is a homomorphism of complete $\mathscr{A}_*$-comodules, we conclude that diagram (3.16) can be further sharpened to say that the image of ϵ_* is contained in $\mathscr{F} \subset R_*(M_*)$. I.e. we have the following commutative diagram

$$\begin{array}{ccc} M_* & \overset{\epsilon_*}{\hookrightarrow} & R_+(M_*) \\ & \searrow^{\epsilon'_*} & \uparrow i \\ & & \mathscr{F}\ , \end{array} \tag{3.17}$$

where i is the composite inclusion $\mathscr{F} \subset P(u, u^{-1}) \otimes M_* \subset P(u, u^{-1}) \hat{\otimes} M_* = R_+(M)$. We will show that ϵ'_* is surjective, and thus an isomorphism, by producing an inverse. To accomplish this, we would like to apply Lemma 3.8 with $K_* = M_*$ and $L_* = \mathscr{F}$. However, we do not know a priori that $\mathscr{F}$ is bounded below and finite type. Working around this, we will write $\mathscr{F}$ as a colimit of bounded below $\mathscr{A}_*$-comodules of finite type, apply the Lemma and finally pass to the limit.

Given an (algebraic) $\mathscr{A}_*$-comodule $\mathscr{E}$ with coaction map $\nu : \mathscr{E} \to \mathscr{A}_* \otimes \mathscr{E}$. We do not assume that $\mathscr{E}$ is bounded below nor finite type. Let $b\mathscr{E} \subset \mathscr{E}$ be a sub vectorspace which is finite dimensional, and let $\bar{b}\mathscr{E}$ the $\mathscr{A}_*$-comodule generated by $b\mathscr{E}$. Then obviously, $\bar{b}\mathscr{E}$ is also finite dimensional as a $\mathbb{F}_p$-vectorspace. In particular, $\bar{b}\mathscr{F}$ is bounded below and of finite type.

We will now use this to filter $\mathscr{F}$ by bounded below $\mathscr{A}_*$-comodules of finite type. Let n be any integer. Then the sub vectorspace

$$b\mathscr{F}_n = \mathbb{F}_p\{u^j \otimes \alpha \in \mathscr{F} \mid n \geq j \geq -n, \deg(\alpha) \leq n\}$$

is bounded below and of finite type. By letting $\bar{b}\mathscr{F}_n$ be the $\mathscr{A}_*$-comodule generated by $b\mathscr{F}_n$, we have a direct limit system

$$\cdots \subset \bar{b}\mathscr{F}_{n+1} \subset \bar{b}\mathscr{F}_n \cdots$$

of bounded below $\mathscr{A}_*$-comodules of finte type, with direct limit isomorphic to $\mathscr{F}$. Thus, by Lemma 3.8, we have a commutative diagram

$$\begin{array}{ccc} M_* & \stackrel{\epsilon_*}{\rightarrowtail} & R_+(M_*) \\ & {\scriptstyle j_n}\nwarrow & \uparrow \\ & & \bar{b}\mathscr{F} \end{array}$$

for every $n \geq 0$. Passing to the colimit as $n \to \infty$, we get

$$\begin{array}{ccc} M_* & \stackrel{\epsilon_*}{\rightarrowtail} & R_+(M_*) \\ & {\scriptstyle j}\nwarrow & \uparrow {\scriptstyle i} \\ & & \mathscr{F}\,. \end{array} \tag{3.18}$$

Combining (3.17) and (3.18), we conclude that ϵ'_* is an isomorphism of $\mathscr{A}_*$-comodules. □

4. The Tate construction

We define the Tate construction and set up the relation with homotopy orbit- and homotopy fixed point spectra. We show that the Tate spectrum can be expressed as the homotopy inverse limit of bounded below spectra. We will then focus on the continuous (co-)homology groups of the Tate construction.

The first section is concerned with the general setup of the Tate construction and its relatives; the homotopy orbit spectrum and the homotopy fix point spectrum.

We then move on to describing the homological Tate spectral sequences. There are two types, one converging to the continuous homology of the Tate construction and one converging to the continuous cohomology. These spectral sequences will be dual to each other but, as already noted in § 2.2, their target groups will generally not be dual.

Propositions 4.10, 4.11 and 4.13 state the properties of the (co-)homological Tate spectral sequences converging to the (co-)homology of the Tate construction of a G-spectrum X.

4.1. **Equivariant spectra and various fixed point constructions.** We review some notions from stable equivariant homotopy theory. We work within the framework of [13]. Let G be a compact Lie group and X be an equivariant G-spectrum indexed on a complete G-universe $\mathscr{U}$. In the notation of [13], X is an object in the category $G\mathscr{S}\mathscr{U}$. By the assumption that $\mathscr{U}$ is complete, we get that we have countably many copies of the trivial representation contained in $\mathscr{U}$. We denote the sum of these by $\mathbb{R}^\infty$. Then $\mathbb{R}^\infty = \mathscr{U}^G$ and we let $i : \mathbb{R}^\infty \hookrightarrow \mathscr{U}$ denote the inclusion.

A G-spectrum X indexed on $\mathbb{R}^\infty$ is called a naive G-spectrum. For such spectra, we have the notion of the associated fixed point spectrum X^G given by

$$X^G(\mathbb{R}^n) = X(\mathbb{R}^n)^G\,,$$

for $\mathbb{R}^n \subset \mathbb{R}^\infty$. This gives a non-equivariant spectrum X^G indexed on $\mathbb{R}^\infty$.

We may also form the associated orbit spectrum under the action of G. We first define a pre-spectrum by letting the $\mathbb{R}^n$th space be

$$X(\mathbb{R}^n)/G\,.$$

We then have to spectrify to get an honest spectrum. See [13, p. 12-13], or [10, Appendix A] for further information about the spectrification functor.

Choose EG to be a free contractible G-space. Define $\widetilde{EG}$ to be the unreduced suspension of EG. This is a G-space with exactly two fixed points. Choosing one of these as a base point makes $\widetilde{EG}$ into a based G-space and we have a fundamental cofiber sequence of based G-spaces

$$EG_+ \to S^0 \to \widetilde{EG} \tag{4.1}$$

defined by letting the first map collapse EG onto the non-basepoint of S^0.

Definition 4.1. *For a genuine G-spectrum X indexed on a complete universe $\mathscr{U}$, we let*

$X_{hG} = EG_+ \wedge_G i^*X$	homotopy orbit spectrum of X
$X^{hG} = \mathrm{F}(EG_+, X)^G$	homotopy fixed point spectrum of X
$X^{tG} = [\widetilde{EG} \wedge \mathrm{F}(EG_+, X)]^G$	Tate spectrum of X

For further details, see the introduction in [9].

Let $N \subset G$ be a normal subgroup and let $X \in G\mathscr{S}\mathscr{U}^G$ be a naive G-spectrum that is N-free. Then there is a natural transfer equivalence of G/N-spectra

$$\tau : X/N \xrightarrow{\simeq} \left(\Sigma^{-adN} i_* X\right)^N. \tag{4.2}$$

Here adN is the adjoint representation of N.

The forgetful functor $i^* : G\mathscr{S}\mathscr{U} \to G\mathscr{S}\mathbb{R}^\infty$ has a left adjoint i_*. For any genuine G-spectrum $X \in G\mathscr{S}\mathscr{U}$, the counit

$$\epsilon : i_* i^* X \to X$$

induces a non-equivariant equivalence. Hence, if X is a free G-CW spectrum then ϵ is a G-equivalence. In our applications, $G = N$ and G will be a closed subgroup of $\mathbb{T}$. In particular G will be Abelian, so the adjoint representation will be trivial of dimension 1 if and only if $G = \mathbb{T}$ and zero otherwise. In this situation the transfer equivalence takes its most simple form; for a free G-CW-spectrum X there is a natural homotopy equivalence of spectra

$$(i^*X)/G \xrightarrow{\simeq} \Sigma^{-dimG}(i_* i^* X)^G \xrightarrow{\simeq} \Sigma^{-dimG} X^G \tag{4.3}$$

in $\mathscr{S}\mathbb{R}^\infty$.

4.2. **The Norm-Restriction diagram.** The map $c : EG_+ \to S^0$ induces a map of G-spectra

$$\mathrm{F}(c, 1) : X \to \mathrm{F}(EG_+, X) \tag{4.4}$$

for any G-spectrum X. Smashing the cofiber sequence (4.1) with the map (4.4), we get the following map of G-cofiber sequences:

$$
\begin{array}{ccccc}
EG_+ \wedge X & \longrightarrow & X & \longrightarrow & \widetilde{EG} \wedge X \\
\downarrow \simeq & & \downarrow & & \downarrow \\
EG_+ \wedge \mathrm{F}(EG_+, X) & \longrightarrow & \mathrm{F}(EG_+, X) & \longrightarrow & \widetilde{EG} \wedge \mathrm{F}(EG_+, X)
\end{array}
\tag{4.5}
$$

The left hand vertical map is a G-homotopy equivalence by the equivariant Whitehead theorem. Moreover, the transfer equivalence (4.3) gives a natural homotopy equivalence

$$\Sigma^{\dim G} EG_+ \wedge_G i^* X \xrightarrow{\simeq} [EG_+ \wedge X]^G . \tag{4.6}$$

By applying G-fixed points to and using (4.6), we can rewrite (4.5) as

$$
\begin{array}{ccccc}
\Sigma^{\dim G} X_{hG} & \xrightarrow{N} & X^G & \xrightarrow{R} & [\widetilde{EG} \wedge X]^G \\
\| & & \downarrow \Gamma_G & & \downarrow \hat{\Gamma}_G \\
\Sigma^{\dim G} X_{hG} & \xrightarrow{N^h} & X^{hG} & \xrightarrow{R^h} & X^{tG} .
\end{array}
\tag{4.7}
$$

The spectra in the lower row have been studied by means of spectral sequences converging to the their homotopy groups. These spectral sequences arise in the case of the homotopy orbit and fixed point spectra by choosing a filtration of EG, and by a filtration of $\widetilde{EG}$ introduced by Greenlees [8] in the case of the Tate spectrum X^{tG}.

We will return to the case of the Tate filtration in §4 and the resulting Tate spectral sequence, but we will be concerned with the spectral sequence that arises from applying homology with $\mathbb{F}_p$-coefficients instead of homotopy.

4.3. **Tate cohomology and the Greenlees filtration of $\widetilde{EG}$.** We recall the definition of the Tate (co-)homology groups from [6]. Let G be a finite group and let $\{P_*, d_*\}$ be a complete resolution of the trivial $\mathbb{F}_pG$-module $\mathbb{F}_p$ by free $\mathbb{F}_pG$-modules. This is a commutative diagram

$$
\begin{array}{ccccccccccc}
\cdots & \longrightarrow & P_1 & \xrightarrow{d_1} & P_0 & & \xrightarrow{d_0} & & P_{-1} & \xrightarrow{d_{-1}} & P_{-2} \xrightarrow{d_{-2}} \cdots \\
 & & & & & \searrow & & \nearrow & & & \\
 & & & & & & \mathbb{F}_p & & & &
\end{array}
$$

where the P_n's are free $\mathbb{F}_pG$-modules and the horizontal sequence is exact.

Definition 4.2. *Given an $\mathbb{F}_pG$-module M the Tate homology and cohomology groups are defined by*

$$\widehat{H}_n(G; M) = H_n(P_* \otimes_{\mathbb{F}_pG} M)$$

and

$$\widehat{H}^n(G; M) = H^n(\mathrm{Hom}_{\mathbb{F}_pG}(P_*, M))$$

where $\{P_*\}$ is a complete $\mathbb{F}_pG$-resolution. These groups are independent of the chosen complete $\mathbb{F}_pG$-resolution and there are isomorphisms

$$\widehat{H}^n(G; M) \cong \widehat{H}_{-n-1}(G; M)$$

for all n.

Complete resolutions in algebra have topological analogues introduced by Greenlees [8]. We recall briefly the construction. Let EG be equipped with a G-CW structure and consider the associated skeleton filtration.

To form a complete resolution of $\widetilde{EG}$ we will have to work in the stable category of G-spectra. Let $\widetilde{E}_n$ be the cofiber of $EG_+^{(n-1)} \to S^0$ for $n > 0$ and for $n = 0$ let $\widetilde{E}_0 = S^0$ be the fixed points of $\widetilde{EG}$. For $n < 0$ we let $\widetilde{E}_n = D(\widetilde{E}_{-n})$, the G-equivariant Spanier-Whitehead dual of $\widetilde{E}_{-n}$. This gives a sequential system of maps with homotopy direct limit equivalent to $\widetilde{EG}$, and non-equivariantly contractible homotopy inverse limit $D(\widetilde{EG}) \simeq *$:

$$* \to \ldots \to \widetilde{E}_{-1} \to \widetilde{E}_0 = S^0 \to \widetilde{E}_1 \to \ldots \to \widetilde{EG} \tag{4.8}$$

Applying $\mathbb{F}_p$-homology to the filtration $\{\widetilde{E}_n\}$ gives a spectral sequence with $E^1_{st} = H_{s+t}(\widetilde{E}_s/\widetilde{E}_{s-1};\mathbb{F}_p)$ and differential $d^r_{st} : E^r_{st} \to E^r_{s-r,t-r+1}$ that converges to $H_*(\widetilde{EG};\mathbb{F}_p) = 0$. The spectral sequence collapses at the E^2-term since $\widetilde{E}_n/\widetilde{E}_{n-1}$ is a G-free wedge of n-spheres.

Hence, we get a long exact sequence

$$\begin{array}{ccccccccc} \cdots \longrightarrow & H_2(\widetilde{E}_2/\widetilde{E}_1) & \longrightarrow & H_1(\widetilde{E}_1/\widetilde{E}_0) & & \longrightarrow & H_0(\widetilde{E}_0/\widetilde{E}_{-1}) & \longrightarrow \cdots \\ & & & \downarrow & & \nearrow & & \\ & & & H_0(S^0) \cong \mathbb{F}_p & & & & \end{array} \tag{4.9}$$

of free $\mathbb{F}_pG$-modules. Letting $P_n = H_{n+1}(\widetilde{E}_{n+1}/\widetilde{E}_n)$ for all n, yields a complete resolution of $\mathbb{F}_p$ by free $\mathbb{F}_pG$-modules.

4.4. **Continuous homology of the Tate construction.** Let G be a finite subgroup of the circle $\mathbb{T}$ and let X be a G-spectrum. For the current work, we are interested in the case when $G = C_p \subset \mathbb{T}$ is the cyclic subgroup of prime order.

By means of the Greenlees filtration (4.8), we may filter the Tate construction. For $n \in \mathbb{Z}$, let $\widetilde{EG}/\widetilde{E}_{n-1}$ be the cofiber of the map $\widetilde{E}_{n-1} \to \widetilde{EG}$, and let

$$X^{tG}[n,\infty] = [\widetilde{EG}/\widetilde{E}_{n-1} \wedge \mathrm{F}(EG_+, X)]^G . \tag{4.10}$$

We will often abbreviate further by writing $X^{tG}[n]$ instead of $X^{tG}[n,\infty]$.

For all $n \in \mathbb{Z}$ we have maps $X^{tG}[n,\infty] \to X^{tG}[n+1,\infty]$, and we will study the continuous (co-)homology of X^{tG} with respect to this filtration. To make sense of the continuous (co-)homology groups, we need the following.

Lemma 4.3. *The homotopy inverse limit of $X^{tG}[n,\infty]$ as $n \to -\infty$ is equivalent to X^{tG}. The homotopy colimit as $n \to \infty$ is contractible.*

Proof. Let $\{\widetilde{E}_n\}_{n\in\mathbb{Z}}$ be the system of G-spectra (4.8). Consider the stable G-equivariant (co-)fibration sequence $\widetilde{E}_n \to \widetilde{EG} \to \widetilde{EG}/\widetilde{E}_n$ for $n \in \mathbb{Z}$. It is still a (co-)fibration sequence after smashing with $\mathrm{F}(EG_+, X)$, taking G-fixed points and passing to the homotopy inverse limit over k. In other words we have a fibration sequence

$$\operatorname*{holim}_{n\to-\infty} [\widetilde{E}_n \wedge \mathrm{F}(EG_+, X)]^G \to X^{tG} \to \operatorname*{holim}_{n\to-\infty} X^{tG}[n+1,\infty] .$$

When n is negative, $\widetilde{E}_n = D(\widetilde{E}_{-n})$ is the Spanier-Whitehead dual of $\widetilde{E}_{-n}$, so the fiber is equivalent to

$$\begin{aligned}
\operatorname*{holim}_{k\to-\infty}[D(\widetilde{E}_{-n}) \wedge \mathrm{F}(EG_+, X)]^G &\simeq \operatorname*{holim}_{n\to-\infty} \mathrm{F}(\widetilde{E}_{-n}, \mathrm{F}(EG_+, X))^G \\
&\cong \mathrm{F}(\operatorname*{hocolim}_{n\to\infty} \widetilde{E}_n, \mathrm{F}(EG_+, X))^G \\
&\simeq \mathrm{F}(\widetilde{EG}, \mathrm{F}(EG_+, X))^G \\
&\cong \mathrm{F}(\widetilde{EG} \wedge EG_+, X)^G \\
&\simeq *
\end{aligned}$$

Here we are using that $\widetilde{E}_n$ is dualizable in the stable category of G-spectra, and that $\widetilde{EG} \wedge EG_+$ is G-equivariantly contractible.

To show the last part of the lemma we use that, for $n \geq 0$, $\widetilde{EG}/\widetilde{E}_n$ is a free G-CW complex. Indeed, for n positive,

$$\widetilde{EG}/\widetilde{E}_n \simeq \Sigma EG/EG^{(n-1)}\,.$$

Thus, by the transfer equivalence we have

$$\begin{aligned}
X^{tG}[n,\infty] &= [\widetilde{EG}/\widetilde{E}_{n-1} \wedge \mathrm{F}(EG_+, X)]^G \\
&\simeq \Sigma^{\dim G} \widetilde{EG}/\widetilde{E}_{n-1} \wedge_G i^* \mathrm{F}(EG_+, X)\,.
\end{aligned}$$

Since colimits commute with orbits and smash products, the result follows since $\operatorname{hocolim}_{n\to\infty} \widetilde{EG}/\widetilde{E}_n$ is G-equivariantly contractible. □

For $n \in \mathbb{Z}$, we let $X^{tG}[-\infty, n] = [\widetilde{E}_n \wedge \mathrm{F}(EG_+, X)]^G$. We have the following diagram in which the columns are fibration sequences.

(4.11)
$$\begin{array}{ccccccccc}
* & \to \cdots \to & X^{tG}[-\infty, n-1] & \to & X^{tG}[-\infty, n] & \to \cdots \to & X^{tG} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow{=} \\
X^{tG} & \xrightarrow{=} \cdots \to & X^{tG} & \xrightarrow{=} & X^{tG} & \xrightarrow{=} \cdots \to & X^{tG} \\
\downarrow{=} & & \downarrow & & \downarrow & & \downarrow \\
X^{tG} & \to \cdots \to & X^{tG}[n, \infty] & \to & X^{tG}[n+1, \infty] & \to \cdots \to & *
\end{array}$$

When X is bounded below and of finite type over $\mathbb{F}_p$ each of the spectra $X^{tG}[n,\infty]$ will be bounded below and of finite type as well. Thus, it makes sense to talk about the continuous homology and cohomology of X^{tG}, defined in § 2.1, with respect to the Tate filtration $\{X^{tG}[n,\infty]\}_{n\in\mathbb{Z}}$.

We end this section by giving a useful reformulation of the Tate construction, known as Warwick duality. The following two lemmas will be used in § 5.2 when making a topological model for the Singer construction.

Proposition 4.4 (Greenlees-May [9], proposition 2.6)**.** *There is a natural chain of equivalences* $\Sigma\,\mathrm{F}(\widetilde{EG}, EG_+ \wedge X) \simeq \widetilde{EG} \wedge \mathrm{F}(EG_+, X)$ *of G-spectra.*

Proof. We have a commutative diagram of G-spectra

$$
\begin{array}{ccc}
\mathrm{F}(S^0, EG_+ \wedge X) & \longrightarrow & \mathrm{F}(EG_+, EG_+ \wedge X) \\
\simeq \uparrow & \nearrow & \\
EG_+ \wedge \mathrm{F}(S^0, EG_+ \wedge X) & & \\
\downarrow \simeq & & \downarrow \simeq \\
EG_+ \wedge \mathrm{F}(S^0, X) & & \\
\downarrow \simeq & & \\
EG_+ \wedge \mathrm{F}(EG_+, X) & \longrightarrow & \mathrm{F}(EG_+, X)\,.
\end{array}
\tag{4.12}
$$

The proposition follows by taking horizontal cofibers and fixed points. □

We chose to express X^{tG} as the homotopy inverse limit of the tower in the lower row of diagram (4.11). In light of Proposition 4.4 there is another obvious tower, namely

$$X^{tG} \to \ldots \to \Sigma\,\mathrm{F}(\widetilde{E}_n, EG_+ \wedge X)^G \to \Sigma\,\mathrm{F}(\widetilde{E}_{n-1}, EG_+ \wedge X)^G\,. \tag{4.13}$$

This tower is again a tower with homotopy inverse limit equivalent to the Tate construction on X. By the transfer equivalence and the fact that $\widetilde{E}_n$ is dualizable for every $n \in \mathbb{Z}$, we also get that the homotopy colimit

$$\operatorname*{hocolim}_{n \to -\infty} \Sigma\,\mathrm{F}(\widetilde{E}_n, EG_+ \wedge X)^G$$

is contractible. The next comparison result says that the two towers introduced so far are in fact equivalent (as towers).

Lemma 4.5. *For $n < 0$ there is a chain of equivalences*

$$X^{tG}[n] \simeq \Sigma\,\mathrm{F}(\widetilde{E}_{-n+1}, EG_+ \wedge X)^G\,,$$

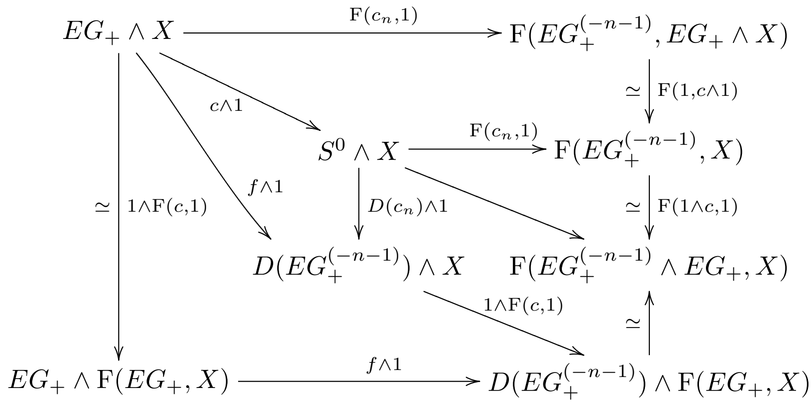
natural in n and the G-spectrum X.

Proof. For $n \leq 0$, consider the composite map of spectra

$$f : EG_+ \xrightarrow{\;c\;} S^0 \simeq D(S^0) \xrightarrow{D(c_n)} D(EG_+^{(-n-1)})\,.$$

We then have the following commutative diagram, analogous to (4.12).

(4.14)



From the cofibration sequence

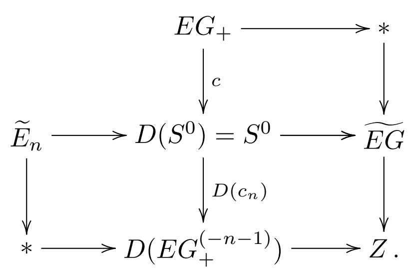
$$EG_+^{(-n-1)} \xrightarrow{c_n} S^0 \xrightarrow{j_n} \widetilde{E}_{-n} \,. \tag{4.15}$$

we get that the cofiber of the upper horizontal map of (4.14) is homotopy equivalent to $\Sigma\,\mathrm{F}(\widetilde{E}_{-n}, EG_+ \wedge X)$.

For the lower row, consider the homotopy cofibration sequence gotten by taking the Spanier-Whitehead dual of (4.15):

$$\widetilde{E}_n = D(\widetilde{E}_{-n}) \to D(S^0) \to D(EG_+^{(-n-1)})\,.$$

We get the following commutative diagram where the squares are stable homotopy pushouts



From this, we see that

$$cofiber(f) = Z = \widetilde{EG}/\widetilde{E}_n\,,$$

and so the lower horizontal map $f \wedge 1$ of diagram (4.14) has cofiber $\widetilde{EG}/\widetilde{E}_n \wedge \mathrm{F}(EG_+, X)$.

All vertical maps in the outer part of the diagram are G-equivalences, so the there is a chain of weak G-equivalences between the horizontal cofibers

$$\Sigma\,\mathrm{F}(\widetilde{E}_{-n}, EG_+ \wedge X) \simeq \widetilde{EG}/\widetilde{E}_n \wedge \mathrm{F}(EG_+, X)$$

Diagram (4.14) is natural in both X and n and the lemma now follows by taking G-fixed points. □

4.5. **The homological Tate spectral sequence.** We assume that X is a bounded below G-spectrum of finite type. Applying cohomology with $\mathbb{F}_p$-coefficients to the lower tower of spectra in diagram (4.11), we get a sequential system of cohomology groups with direct limit equal to the continuous cohomology groups of X^{tG} with respect to the tower $\{X^{tG}[n]\}_{n\in\mathbb{Z}}$.

Likewise, applying homology to the same tower of spectra, we get an inverse system of homology groups with inverse limit equal to the continuous homology of X^{tG}. Remember that this inverse limit is generally not the homology of X^{tG}.

Lemma 4.6. *Applying cohomology to the lower tower of* (4.11) *gives a system*

$$0 \to \ldots \to H^*(X^{tG}[n{+}1]) \to H^*(X^{tG}[n]) \to \ldots \to H^*_c(X^{tG})\,, \tag{4.16}$$

with trivial inverse, and derived inverse limit.

Applying homology to the same tower of spectra gives a system

$$H_*^c(X^{tG}) \to \ldots \to H_*(X^{tG}[n]) \to H_*(X^{tG}[n{+}1]) \to \ldots \to 0\,, \tag{4.17}$$

with trivial colimit and trivial derived inverse limit.

Proof. In cohomology, we have a Milnor lim-$\lim^1$ exact sequence

$$0 \to \operatorname{Rlim}_n H^{*-1}(X^{tG}[n]) \to H^*(\operatorname{hocolim}_n X^{tG}[n]) \to \lim_n H^*(X^{tG}[n]) \to 0\,.$$

By lemma 4.3 the homotopy colimit of $X^{tG}[n]$ as $n \to \infty$ is homotopy trivial. Thus the middle term, and hence the inverse limit on the right, vanishes.

In the homological case we use that direct limits and homology commute, so by lemma 4.3, the colimit $H_*(X^{tG}[n,\infty])$ as $n \to \infty$ is trivial. The assumption that X is bounded below and of finite type, also secures that each group in the limit system is bounded below and of finite type. Hence the derived inverse limit vanishes in the homological case. $\square$

Remark 4.7. These two limit systems should be compared to (2.8) and (2.9) respectively.

Remark 4.8. In our applications, all the groups in the limit systems will be bounded below and finite type. Therefore, by the discussion in §2.4, the lemma follows from the fact that $\operatorname{colim}_n H_*(X^{tG}[n,\infty]) = 0$.

For every n, we have a fiber sequence

$$[\widetilde{E}_n/\widetilde{E}_{n-1} \wedge \mathrm{F}(EG_+, X)]^G \longrightarrow X^{tG}[n,\infty] \longrightarrow X^{tG}[n{+}1,\infty]\,. \tag{4.18}$$

The fiber is equivalent to $[\widetilde{E}_n/\widetilde{E}_{n-1} \wedge X]^G$ since $\widetilde{E}_n/\widetilde{E}_{n-1}$ is a free G-spectrum.

Applying homology with $\mathbb{F}_p$-coefficients, these fibration sequences yield an exact couple

$$\begin{array}{ccc} A_{n-1} & \xrightarrow{\;i\;} & A_n \\ {\scriptstyle k}\uparrow & \swarrow{\scriptstyle j} & \\ \hat{E}_n & & \end{array} \tag{4.19}$$

with

$$\begin{aligned} A_n &= H_*(X^{tG}[n{+}1]) \\ \hat{E}_n &= H_*([\widetilde{E}_n/\widetilde{E}_{n-1} \wedge X]^G)\,. \end{aligned}$$

The maps i, k in the exact couple is degree-preserving, while j has degree -1.

By Lemma 2.5 and Lemma 4.6, we get an exhaustive, complete Hausdorff filtration of $H_*^c(X^{tG})$ by the subgroups

$$F^n H_*^c X^{tG} = \ker[H_*^c(X^{tG}) \to H_*^c(X^{tG}[n{+}1, \infty])] \,. \tag{4.20}$$

Therefore, the spectral sequence derived from the exact couple converges conditionally in the sense of Boardman ([2] Definition 5.10) to the inverse limit $H_*^c(X^{tG})$.

To identify the $\hat{E}^2$-term, we use the natural isomorphisms

$$\begin{aligned} H_{s+t}([\widetilde{E}_s/\widetilde{E}_{s-1} \wedge X]^G) &\cong H_{s+t}(\widetilde{E}_s/\widetilde{E}_{s-1} \wedge_G X) \\ &\cong H_s(\widetilde{E}_s/\widetilde{E}_{s-1}) \otimes_{\mathbb{F}_p G} H_t(X) \,. \end{aligned}$$

With these identifications the $\hat{d}^1$-differential is

$$\hat{d}_{s-1} \otimes \mathrm{id} : H_s(\widetilde{E}_s/\widetilde{E}_{s-1}) \otimes_{\mathbb{F}_p G} H_t(X) \to H_{s-1}(\widetilde{E}_{s-1}/\widetilde{E}_{s-2}) \otimes_{\mathbb{F}_p G} H_t(X),$$

where $\hat{d}_*$ is the differential in the complete resolution (4.9). From what we have now, we make the following definition.

Definition 4.9. *For a bounded below G-spectrum X of finite type over $\mathbb{F}_p$, we define the homological Tate spectral sequence associated with X to be the spectral sequence induced from the exact couple* (4.19)*.*

Under no assumptions on X, the homological Tate spectral sequence associated with X has

$$\hat{E}^2_{s,t}(X) \cong \hat{H}_{s-1}(G; H_t(X)) \cong \hat{H}^{-s}(G; H_t(X)) \Rightarrow H^c_{s+t}(X^{tG}) \,, \tag{4.21}$$

and converges conditionally to the continuous homology of X^{tG} with $\mathbb{F}_p$-coefficients.

Under the assumption that X has bounded below homology, the homological Tate spectral sequence is concentrated above some horizontal line. Thus the spectral sequence is one with entering differentials in the sense of Boardman.

Since X is of finite type the $\hat{E}^2$-term is finitely generated in each bidegree, so Boardman's derived limit RE^∞ vanishes as well, and the spectral sequence converges strongly to its target. See [2, Theorem 7.1] for further details.

The spectral sequence arises by applying homology with $\mathbb{F}_p$-coefficients. Hence, the exact couple (4.19) is equipped with a (algebraic) coaction of the dual mod p Steenrod algebra $\mathscr{A}_*$ and the resulting spectral sequence is an $\mathscr{A}_*$-comodule spectral sequence. As noted in §2.1, the continuous homology is a completed $\mathscr{A}_*$-comodule. This coaction induces an $\mathscr{A}_*$-comodule structure on the associated graded with respect to the filtration (4.20) and convergence of the spectral sequence means that the associated graded is isomorphic to the $\hat{E}^\infty$-term as comodules over $\mathscr{A}_*$.

We summarize these results in the following proposition.

Proposition 4.10. *Let $G \subset \mathbb{T}$ be a finite subgroup and let X be a bounded below G-spectrum of finite type over $\mathbb{F}_p$.*

Then X^{tG} is equivalent to the homotopy inverse limit of a tower of bounded below spectra of finite type. There is an $\mathscr{A}_*$-comodule Tate spectral sequence of homological type converging strongly to the continuous homology of X^{tG} as a completed $\mathscr{A}_*$-comodule. The homological Tate spectral sequence has $\hat{E}^2$-term

$$\hat{E}^2_{s,t}(X) = \widehat{H}^{-s}(G; H_t(X; \mathbb{F}_p)) \Rightarrow H^c_{s+t}(X^{tG}; \mathbb{F}_p) \,. \tag{4.22}$$

By dualizing the exact couple (4.19), we obtain a cohomological Tate spectral sequence. This produces a filtration of the colimit cohomology groups, i.e., the continuous cohomology $H_c^*(X^{tG})$, by the image subgroups

$$F_n H_c^*(X^{tG}) = \mathrm{im}[H^*(X^{tG}[n]) \to H_c^*(X^{tG})] \,.$$

Again, by Lemma 2.5 and Lemma 4.6, the filtration is exhaustive, complete and Hausdorff.

Thus the Tate filtration of X^{tG} gives rise to a conditionally convergent spectral sequence of cohomological type with target equal to the continuous cohomology $H_c^*(X^{tG})$.

Again, since X is assumed to be bounded below, we get a half-plane spectral sequence concentrated above some horizontal line. This is now a spectral sequence with exiting differentials and by [2, Theorem 6.1] the spectral sequence converges strongly to the continuous cohomology $H_c^*(X^{tG})$.

Since the cohomological exact couple comes from applying cohomology with $\mathbb{F}_p$-coefficients, we get a natural $\mathscr{A}$-module structure on the spectral sequence, with differentials being $\mathscr{A}$-module homomorphisms.

Proposition 4.11. *Let $G \subset \mathbb{T}$ be a finite subgroup and let X be a bounded below G-spectrum of finite type over $\mathbb{F}_p$.*

Then X^{tG} is equivalent to the homotopy inverse limit of a tower of bounded below spectra of finite type. There is an $\mathscr{A}$-module, Tate spectral sequence of cohomological type converging strongly to the continuous cohomology of X^{tG} as an $\mathscr{A}$-module. The cohomological Tate spectral sequence has $\hat{E}_2$-term

$$\hat{E}_2^{s,t}(X) = \widehat{H}_{-s}(G; H^t(X;\mathbb{F}_p)) \Rightarrow H_c^{s+t}(X^{tG};\mathbb{F}_p) \,. \tag{4.23}$$

The cohomological Tate spectral sequence is dual to the homological Tate spectral sequence in the sense that $\hat{E}^r_{*,*}$ is dual to $\hat{E}_r^{*,*}$ in each bidegree for all r and that the cohomological differential $d_r : \hat{E}_r^{s,t} \to \hat{E}_r^{s+r,t-r+1}$ is dual to the homological differential $d^r : \hat{E}^r_{s+r,t-r+1} \to \hat{E}^r_{s,t}$ for all s,t and $r \geq 1$.

4.5.1. *Homotopy vs. Homology.* We used the tower

$$X^{tG} \longrightarrow \cdots \longrightarrow X^{tG}[n] \longrightarrow X^{tG}[n+1] \longrightarrow \cdots \longrightarrow * \tag{4.24}$$

to define our homological Tate spectral sequence by applying homology with $\mathbb{F}_p$-coefficients.

When studying the *homotopy groups* of the Tate construction, it has been customary to apply $\pi_*(-)$ to the upper tower in (4.11):

$$* \longrightarrow \cdots \longrightarrow X^{tG}[-\infty, n-1] \longrightarrow X^{tG}[-\infty, n] \longrightarrow \cdots \longrightarrow X^{tG} \tag{4.25}$$

See e.g. [3] or [11]. Applying a homology functor to these two towers of spectra, gives different exact couples with isomorphic spectral sequences. If we are working with homotopy, we can choose either one and get spectral sequences converging to the same groups: Using (4.25) yields a spectral sequence converging to the colimit

$$\operatorname*{colim}_n \pi_*(X^{tG}[-\infty, n]) \cong \pi_*(X^{tG}) \,,$$

while using (4.24) yields an isomorphic spectral sequence converging to the inverse limit

$$\lim_n \pi_*(X^{tG}[n]) \cong \pi_*(X^{tG}) \,. \tag{4.26}$$

The latter isomorphism uses that X is bounded below and of finite type.

When working with singular homology with $\mathbb{F}_p$-coefficients, instead of homotopy groups, the failure of the isomorphism we made use of in (4.26) makes the situation more interesting. Applying $H_*(-;\mathbb{F}_p)$ to the tower (4.25) will produce a sequential limit sequence of homology groups where the inverse limit is not trivial in general. This means that the spectral sequence associated with the exact couple will not be conditionally convergent to the direct limit

$$\operatorname*{colim}_n H_*(X^{tG}[-\infty, n]; \mathbb{F}_p) \cong H_*(X^{tG}; \mathbb{F}_p)\,.$$

In fact, we have seen that the (isomorphic) homological Tate spectral sequence, arising from (4.24), converges strongly to

$$\lim_n H_*(X^{tG}[n]; \mathbb{F}_p) = H_*^c(X^{tG}; \mathbb{F}_p)\,,$$

which is seldom isomorphic to $H_*(X^{tG}; \mathbb{F}_p)$ since inverse limits and homology does not commute in general.

We end this discussion by noticing that the continuous homology groups of the Tate construction on X can be thought of as the homotopy of a certain Tate spectrum. In other words, continuous homology of X^{tG} is a special case of homotopy.

Proposition 4.12. *Let H denote the $\mathbb{F}_p$ Eilenberg-MacLane spectrum with trivial G-action. For any bounded below, finite type G-spectrum X, there is a natural isomorphism $\pi_*(H \wedge X)^{tG} \to H_*^c(X^{tG})$.*

Proof. For $n < 0$, we have

$$\begin{aligned}(H \wedge X)^{tG}[n] &\simeq \Sigma\, \mathrm{F}(\widetilde{E}_{-n+1}, EG_+ \wedge H \wedge X)^G \\ &\simeq \Sigma H \wedge \mathrm{F}(\widetilde{E}_{-n+1}, EG_+ \wedge X)^G \\ &\simeq H \wedge X^{tG}[n, \infty]\,.\end{aligned}$$

The first and last equivalences follow from lemma 4.5 and the middle equivalence follows from the fact that $\widetilde{E}_{-n+1}$ is G-equivariantly dualizable. Thus, we have that $\pi_*(H \wedge X)^{tG}[n] \cong H_*(X^{tG}[n])$ for all $n < 0$. For a general G-spectrum X, we then have the following surjective maps for any k:

$$\pi_k(H \wedge X)^{tG} \to \lim_{n\to-\infty} \pi_k(H \wedge X)^{tG}[n] \tag{4.27}$$

$$\cong \lim_{n\to-\infty} H_k(X^{tG}[n]) = H_k^c(X^{tG})\,. \tag{4.28}$$

Since X was assumed to be bounded below and of finite type, then the groups in the first inverse limit system are all of finite type, so $\lim^1$ vanishes and the map in (4.27) is an isomorphism. □

The previous lemma tells us that the continuous homology of X^{tG} can be considered both as the colimit $\operatorname{colim}_n \pi_*(H \wedge X)^{tG}[-\infty, n]$ by applying homotopy to the tower (4.25), or as the inverse limit $\lim_n \pi_*(H \wedge X)^{tG}[n, \infty]$. In both cases, the filtration of the two groups given by their defining towers are the same.

4.6. **Multiplicative structure.** In the discussion of the Tate spectral sequence, we have so far not taken into account the presence of a multiplicative structure. Assume that X is a bounded below, finite type G-equivariant ring spectrum. We assume that the unit $\eta : S \to X$ and the multiplication map $\mu : X \wedge X \to X$ are equivariant with respect to the G-action.

By [9, Proposition 3.5] both the homotopy fixed point spectrum X^{hG} and the Tate spectrum X^{tG} are ring spectra. In the Tate case the product is defined in the following way: There is a composition

$$
\begin{array}{c}
[\widetilde{EG} \wedge \mathrm{F}(EG_+, X)]^G \wedge [\widetilde{EG} \wedge \mathrm{F}(EG_+, X)]^G \\
\downarrow \\
[\widetilde{EG} \wedge \widetilde{EG} \wedge \mathrm{F}(EG_+ \wedge EG_+, X \wedge X)]^G \\
\downarrow {\scriptstyle 1\wedge 1\wedge \Delta^*} \\
[\widetilde{EG} \wedge \widetilde{EG} \wedge \mathrm{F}(EG_+, X \wedge X)]^G \\
\downarrow {\scriptstyle 1\wedge 1\wedge \mu_*} \\
[\widetilde{EG} \wedge \widetilde{EG} \wedge \mathrm{F}(EG_+, X)]^G \,.
\end{array}
\tag{4.29}
$$

Up to homotopy, there is a unique G-equivalence $\widetilde{EG} \wedge \widetilde{EG} \xrightarrow{\simeq} \widetilde{EG}$. Taking the composition above followed by this homotopy equivalence, we get a product $X^{tG} \wedge X^{tG} \to X^{tG}$. The unit comes from the unit of X together with the canonical maps $EG_+ \to S^0$ and $S^0 \to \widetilde{EG}$, by the composition

$$
S \xrightarrow{\eta} X \to \mathrm{F}(EG_+, X) \to \widetilde{EG} \wedge \mathrm{F}(EG_+, X)\,. \tag{4.30}
$$

The homotopy fixed point spectrum also has a product coming from the product on X and the diagonal map $\Delta : EG_+ \to EG_+ \wedge EG_+$. The first two maps of (4.30) compose to give a unit for X^{hG} after taking G-fixed points.

The ring structures are compatible in that the maps in the right hand square of (4.7) are maps of ring spectra.

Up to homotopy there is a unique homotopy equivalence $EG_+ \wedge EG_+ \xrightarrow{\simeq} EG_+$ as well. Using this we may define a product on the homotopy orbit spectrum X_{hG}. However, this spectrum lacks a unit, so it is not a ring spectrum.

The above facts can be found in [9, §3] or [11, §4.4].

Using the Greenlees filtration, we may now filter $(H \wedge X)^{tG}$ by the tower (4.25). This produces a homotopical Tate spectral sequence with $\hat{E}^2$-term isomorphic to

$$
\hat{E}^2_{s,t} \cong \hat{H}^{-s}(G; \pi_t(H \wedge X)) \cong \hat{H}^{-s}(G; H_t(X)) \tag{4.31}
$$

converging to the homotopy $\pi_{s+t}(H \wedge X)^{tG} \cong H^c_{s+t}(X^{tG})$.

When X is a G-equivariant ring spectrum, then so is $H \wedge X$ and we have seen that $(H \wedge X)^{tG}$ is also a ring spectrum. The induced ring structure on the homotopy of $(H \wedge X)^{tG}$ then gives a ring structure on the continuous homology by the isomorphism of Proposition 4.12.

Moreover, the homotopical Tate spectral sequence (4.31) is an algebra spectral sequence with differentials being derivations with respect to the product. The argument for this fact is done by constructing a bifiltration of the Tate construction and comparing it to the product structure. See [11, Proposition 4.4.4].

Proposition 4.13. *Let $G \subset \mathbb{T}$ be a finite subgroup and let X be a bounded below G-ring spectrum of finite type over $\mathbb{F}_p$.*

Then X^{tG} is equivalent to a homotopy inverse limit of a tower of bounded below spectra of finite type. There is an $\mathscr{A}_*$-comodule algebra, Tate spectral sequence *of homological type converging strongly to the continuous homology of X^{tG} as an $\mathscr{A}_*$-comodule algebra.*

The homological Tate spectral sequence has $\hat{E}^2$-term

$$\hat{E}^2_{s,t}(X) = \widehat{H}^{-s}(G; H_t(X;\mathbb{F}_p)) \Rightarrow H^c_{s+t}(X^{tG};\mathbb{F}_p) \tag{4.32}$$

and the differentials are derivations with respect to the product from the Tate cohomology groups.

5. The topological Singer construction

We follow [5, II §5] and describe a particular inverse system of bounded below, finite type spectra. Its continuous cohomology was described by Miller by means of the Singer construction on modules over the Steenrod algebra. We quickly review this calculation and define the Singer construction.

5.1. Realizing the Singer construction as continuous cohomology.

Let B be a non-equivariant bounded below spectrum of finite type over $\mathbb{F}_p$. Then consider

$$D_\pi(B) = E\pi \ltimes_\pi B^{(j)} ,$$

the extended power construction [5, I §5] on B with respect to some subgroup $\pi \subseteq \Sigma_j$ of the permutation group on j letters.

We shall come back to the definition of the extended power construction in §5.2 where a more detailed description is needed.

Let W be a free π-CW complex and let X be a CW spectrum with cellular action of π. Then by [5, Theorem I.1.3] we have that $W \ltimes_\pi X$ is a CW spectrum with cellular chains

$$C_*(W \ltimes_\pi X) \cong C_*W \otimes_\pi C_*X \tag{5.1}$$

The following is an immediate corollary.

Corollary 5.1. *Let π be a finite group and let X be as above. Then if X is bounded below and of finite type, then so is $E\pi \ltimes_\pi X$.*

Proof. Since π is finite, the usual model for $E\pi$ is a π-free CW complex with finitely many π-cells in each degree. Thus, both W and X are bounded below and of finite type with cellular chains given by equation (5.1), so the claim follows. □

As stated in [5, Corollary 2.1], we have that the external j-fold smash power $B^{(j)}$ is a CW spectrum with cellular chains $C_*(B)^{\otimes j}$. The action of π given by permuting the tensor factors and we have

$$C_*(W \ltimes_\pi B^{(j)}) \cong C_*W \otimes_\pi C_*(B)^{\otimes j} . \tag{5.2}$$

Finally, since we are working over a field there is a π-equivariant chain homotopy equivalence

$$H_*(B)^{\otimes j} \xrightarrow{\simeq} C_*(B)^{\otimes j}$$

given by choosing cycle representatives for homology classes. Via this equivalence, (5.2) can be rewritten as

$$C_*(W \ltimes_\pi B^{(j)}) \simeq C_*W \otimes_\pi H_*(B)^{\otimes j} . \tag{5.3}$$

Thus the homology of $D_\pi(B)$ is computed by group homology:

Corollary 5.2 ([5], Corollary 2.3)**.**

$$H_*(D_\pi(B)) \cong H_*(\pi; H_*(B)^{\otimes j})$$

We are interested in the special case where $j = p$ is our fixed prime and $\pi = C_p$ is the cyclic group of order p. We view C_p as the subgroup of Σ_p generated by the cyclic permutations $(1\,2\,\ldots\,p)$. Using the diagonal map $\Delta : S^1 \to S^1 \wedge \ldots \wedge S^1 \cong S^p$, we get an inverse system of maps

$$(5.4) \qquad \ldots \to \Sigma^{n+1} D_{C_p}(\Sigma^{-n-1}B) \to \Sigma^n D_{C_p}(\Sigma^{-n}B) \to \ldots \to D_{C_p}(B)\,.$$

This is a tower of bounded below, finite type spectra, so it makes sense to talk about the associated continuous cohomology.

Still following [5, II $\mathscr{S}$5], we describe this cohomological system explicitly. The homology of $D_{C_p}(B)$ is given by Corollary 5.2. A standard calculation which can be found in [6] gives that additively we have

$$(5.5) \qquad H_*(D_{C_p}(B)) \cong \mathbb{F}_p\{e_0 \otimes \alpha_1 \otimes \ldots \otimes \alpha_p\} \oplus \mathbb{F}_p\{e_i \otimes \alpha^{\otimes p} \big| i \geq 0\}$$

where the elements α_i and α run through a basis for $H_*(B)$ and $\{\alpha_1 \otimes \ldots \otimes \alpha_p, \alpha^{\otimes p}\}$ run through a set of C_p-generators for $H_*(B)^{\otimes p}$. The grading is determined by letting e_i have degree i.

Dualizing to cohomology, we denote the dual of e_j by w_j. Then for $a \in H^{q+n}(B)$, we have $\Sigma^n w_j \otimes (\Sigma^{-n}a)^{\otimes p} \in H^{n+j+pq}(\Sigma^n D_{C_p}(\Sigma^{-n}B))$. Quoting [5, Lemma II.5.6] we have that the maps in (5.4) are given in cohomology as

$$(5.6) \qquad (\Sigma^n\Delta)^*(\Sigma^n w_j \otimes (\Sigma^{-n}a)^{\otimes p}) = \Sigma^{n+1}(-1)^{j+1}\tau(q) w_{j+p-1} \otimes (\Sigma^{-n-1}a)^{\otimes p}$$

where $a \in H^{q+n}(B)$, $m = (p-1)/2$ and $\tau(q) = -(-1)^{mq} m!$. The terms of the cohomology involving $w_0 \otimes a_1 \otimes \ldots \otimes a_p$ map to zero.

The action of the Steenrod algebra on $H^*(D_{C_p}(B))$ is well known to be given by the Nishida relations, and described in [5, Theorem 5.5]. Together with the explicit formulas for the maps in the direct system, this determines the direct limit of cohomology groups as an $\mathscr{A}$-module.

Miller observed that this direct limit is described in a closed form by the Singer construction on the $\mathscr{A}$-module $H^*(B)$.

Theorem 5.3 ([5, Theorem 5.1])**.** *For any spectrum B that is bounded below and of finite type, there is a natural isomorphism of $\mathscr{A}$-modules*

$$\omega : \operatorname*{colim}_{n\to\infty} H^*(\Sigma^n DC_p(\Sigma^{-n}B)) \xrightarrow{\cong} \Sigma^{-1}R_+(H^*(B))\,.$$

Proof. For $a \in H^{q-n}(B)$, the isomorphism ([5, page 47, proof of II.5.1]) is given by

$$(5.7) \qquad \omega(\Sigma^n w_{j+n} \otimes (\Sigma^{-n}a)^{\otimes 2}) = e^{j+q} \otimes a\,.$$

when $p = 2$. When p is odd, the isomorphism is given by

$$\omega(\Sigma^n w_{2(j+km)} \otimes (\Sigma^{-n}a)^{\otimes p}) = (-1)^{[(k+q-n)/2]+q+n}\nu(q-n)^{-1}x^{j+m(k+q-n)} \otimes a$$

and

$$\omega(\Sigma^n w_{2(j+km)-1} \otimes (\Sigma^{-n}a)^{\otimes p}) = (-1)^{[(k+q-n)/2]+q+2n}\nu(q-n)^{-1}ex^{j+m(k+q-n)-1} \otimes a\,.$$

Here $m = (p-1)/2$ and $0 \leq j < m$. The coefficient ν is given by $\nu(2j+\epsilon) = (-1)^j(m!)^\epsilon$ for $\epsilon = 0, 1$. See [5, p. 47-48] for details. □

That the spectrum B is bounded below and of finite type is used in the determination of the cohomology of $H^*(D_{C_p}(B))$ by dualization from H_*, and in the determination of the Steenrod operations.

The inverse system (5.4) was used by Jones [12] to relate the root invariant and the quadratic construction.

5.2. **The relationship between the Tate and Singer constructions.** We return to the inverse system of bounded below finite type spectra (5.4). We have already given a recollection about how the continuous cohomology associated to this tower is identified with the Singer construction on the cohomology of B. We will now see that this inverse system can be rewritten as an inverse limit of truncated Tate spectra. Our main result is Lemma 5.9. In order to do this, we need to be precise about our genuinely equivariant models for the spectrum which has underlying, non-equivariant homotopy type equal to the p-fold smash power $B \wedge \ldots \wedge B$. Thus, we will need to recall some details from [13] in what follows.

From now on, we let $G = C_p$. Let $\mathscr{U}$ be the complete C_p-universe

$$\mathscr{U} = \mathbb{R}^\infty \oplus \ldots \oplus \mathbb{R}^\infty = (\mathbb{R}^\infty)^p$$

. Here, C_p acts on $\mathscr{U}$ by permuting the summands.

We let $\mathfrak{A}$ be the standard indexing set of $\mathbb{R}^\infty$, consisting of all the finite dimensional subspaces of $\mathbb{R}^\infty$. We then have the "diagonal" set $\mathfrak{A}^p = \{V^{\oplus p} \mid V \in \mathfrak{A}\}$. This set is then an indexing set for $\mathscr{U}$.

Let $B \in \mathscr{S}\mathbb{R}^\infty$ and let $\chi : EC_p \to \mathscr{I}(\mathscr{U}, \mathbb{R}^\infty)$ be a chosen C_p-equivariant map into the space of linear isometries from $\mathscr{U}$ to $\mathbb{R}^\infty$.

The p-extended power construction, sending B to $D_{C_p}B$, is constructed from the composite of three functors

$$\mathscr{S}\mathfrak{A} \xrightarrow{(-)^{(p)}} C_p\mathscr{S}\mathfrak{A}^p \xrightarrow{\chi \ltimes -} C_p\mathscr{S}\mathfrak{A} \xrightarrow{(-)/C_p} \mathscr{S}\mathfrak{A}\,,$$

where the first functor is the formation of the external p-fold smash power. The homotopy type of $\chi \ltimes (-)$ does not depend on the choice of χ. For this reason, the functor $\chi \times (-)$ is often written $EC_p \ltimes (-)$ when we are working in the homotopy category.

For the purposes of defining the p-extended power construction, the diagonal indexing set $\mathfrak{A}^p$ is all we need to give meaning to the equivariant structure of the p-fold smash product.

For the purposes of the current section, we would like to extend the indexing set to all finite dimensional representations in $\mathscr{U}$.

For a general compact Lie group G, let $\mathscr{V}$ be a G-universe, and let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be two indexing sets of $\mathscr{V}$, such that $\mathfrak{A}_1 \subset \mathfrak{A}_2$. Then there is a forgetful functor $\phi : G\mathscr{S}\mathfrak{A}_2 \to G\mathscr{S}\mathfrak{A}_1$ which is the right adjoint to a functor that builds in the missing representations:

$$\psi : G\mathscr{S}\mathfrak{A}_1 \to G\mathscr{S}\mathfrak{A}_2\,.$$

Let $V \in \mathfrak{A}_2$. For a G-spectrum E indexed on $\mathfrak{A}_1$, this left adjoint is defined by the colimit

$$\psi E(V) = \operatorname*{colim}_{W \in \mathfrak{A}_1} \Omega^{W-V} E(W)\,.$$

The unit and counit of this adjunction are both equivalences and define an equivalence of categories. See [13, Proposition I.2.4].

In the proof of Lemma 5.9, we need the following

Lemma 5.4. *For a general compact Lie group G, let $\mathscr{V}$ and $\mathscr{V}'$ be G-universes, and let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be two indexing sets of $\mathscr{V}$, such that $\mathfrak{A}_1 \subset \mathfrak{A}_2$.*

Let W be a CW-complex which is a colimit of its compact subspaces, and let $\chi : W \to \mathscr{I}(\mathscr{V}, \mathscr{V}')$ be a filtered G-map.

For $E \in G\mathscr{S}\mathfrak{A}_1$, there is a chain of natural homotopy equivalences

$$\chi \ltimes E \simeq \chi \ltimes \psi E$$

Proof. For a compact W, this follows from [13, Proposition VI 2.12]. The general case follows from the fact that $\chi \ltimes -$ can be written as the colimit over the restriction of χ to the compact subspaces of W. See the discussion following [13, Definition VI 2.13] for more details. □

We now need to make sense of the p-fold smash product of a spectrum, considered as a C_p-spectrum indexed on $\mathscr{U}$. Let I be Bökstedt's category of finite sets and injective functions. For an object $\vec{x} = (x_0, \ldots, x_{p-1}) \in I^p$, we let $S^{\vec{x}}$ denote the smash product of spheres $S^{x_0} \wedge \ldots \wedge S^{x_{p-1}}$.

Definition 5.5. *Let $B \in \mathscr{S}\mathfrak{A}$. We let $B^{\wedge p}_{pre}$ be the prespectrum given by*

$$B^{\wedge p}_{pre}(V) = \operatorname*{hocolim}_{\vec{x} \in I^p} Map(S^{\vec{x}}, B(\mathbb{R}^{x_0}) \wedge \ldots \wedge B(\mathbb{R}^{x_{p-1}}) \wedge S^V).$$

for $V \in \mathfrak{A}$, together with the obvious structure maps. The group C_p acts on the indexing category and on the mapping spaces by adjunction.

We define the p-fold smash product of p copies of B with the permutation action of C_p as the spectrum

$$B^{\wedge p} = L(B^{\wedge p}_{pre}) \in C_p\mathscr{S}\mathscr{U}.$$

Here $L(-)$ is the (spectrification) functor which is left adjoint to the forgetful functor $l : C_p\mathscr{S}\mathscr{U} \to C_p\mathscr{P}\mathscr{U}$ from spectra to prespectra.

The following definition will be justified by Theorem 5.10.

Definition 5.6. *For $B \in \mathscr{S}\mathbb{R}^\infty$, let the Singer construction on B be the spectrum*

$$R_+(B) = (B^{\wedge p})^{tC_p}.$$

In the proof of Lemma 5.9, we will need the following technical results.

Lemma 5.7. *Let A be a compact space and $E \in \mathscr{S}\mathfrak{A}$ be a spectrum. Consider $A^{\wedge p}$ to be a C_p-space by permutation of factors. Then there is a natural equivariant homotopy equivalence of function spectra*

$$\mathrm{F}(A^{\wedge p}, E^{\wedge p}) \simeq \mathrm{F}(A, E)^{\wedge p}$$

in $C_p\mathscr{S}\mathfrak{A}$.

Proof. Since A is compact, then so is $A^{\wedge p}$ and we have a chain of equivalences

$$\begin{aligned} \mathrm{F}(A^{\wedge p}, E^{\wedge p}) &\simeq \mathrm{F}(A^{\wedge p}, S) \wedge E^{\wedge p} \\ &= \mathrm{F}(A^{\wedge p}, S) \wedge LE^{\wedge p}_{pre} \\ &\simeq L(\mathrm{F}(A^{\wedge p}, S) \wedge E^{\wedge p}_{pre}) \\ &\simeq L(\mathrm{F}(A^{\wedge p}, E^{\wedge p}_{pre}) \end{aligned}$$

The first and the last equivalence is where we use that $A^{\wedge p}$ is dualizable, and the equivalence in the middle is coming from commuting smash products of spectra with the spectrification functor L.

We then have the following chain of equivalences of C_p-prespectra

$$
\begin{aligned}
&\mathrm{F}(A^{\wedge p}, E^{\wedge p}_{pre})(\mathbb{R}^m) \\
&\qquad \simeq \operatorname*{hocolim}_{\vec{x}\in I^p} Map(S^{\vec{x}} \wedge A^{\wedge p}, E(\mathbb{R}^{x_0}) \wedge \ldots \wedge E(\mathbb{R}^{x_{p-1}}) \wedge S^m) \\
&\qquad \simeq \operatorname*{hocolim}_{\vec{x}\in I^p} Map(S^{\vec{x}}, Map(A, E(\mathbb{R}^{x_0})) \wedge \ldots \wedge Map(A, E(\mathbb{R}^{x_{p-1}})) \wedge S^m) \\
&\qquad = F(A,E)^{\wedge p}_{pre}(\mathbb{R}^m)\,.
\end{aligned}
$$

The Lemma now follows by passing to spectra. □

Lemma 5.8. *For a bounded below spectrum B, there is a natural chain of homotopy equivalences of spectra*

$$i^*\psi B^{(p)} \simeq i^* B^{\wedge p}$$

in $C_p\mathscr{S}\mathfrak{A}$.

Proof. Given indexing sets $\mathfrak{A}_1 \subset \mathfrak{A}_2$ and a prespectrum $D \in G\mathscr{P}\mathfrak{A}_1$. It follows from the definition that ψD is a spectrum, i.e. $\psi D \in G\mathscr{S}\mathfrak{A}_2$.

Assume that D is good prespectrum, in the sense of [10, Appendix A]. If it is not, we may replace D, in a natural way, by a fattened up version D^τ [10, Definition A1] such that there is a homotopy equivalence $D^\tau \to D$ of prespectra [10, Lemma A1]. For $U \in \mathfrak{A}_2$, it is easy to verify that

$$\psi L D(U) \cong \operatorname*{colim}_{W\in\mathfrak{A}_1} \Omega^{W-U} D(W) = \psi D(U)\,, \tag{5.8}$$

and so it follows that $\psi L D \cong L\psi D$ since the conunit $LlE \to E$ is always an isomorphism for any spectrum E by [13, Theorem I.2.2].

By the above, the lemma will follow from showing that there is a natural chain of homotopy equivalences of prespectra $\psi B^{(p)}_{pre} \simeq B^{\wedge p}_{pre}$.

Fix a natural number n. Then $\psi B^{(p)}_{pre}(\mathbb{R}^n)$ is equal to

$$
\begin{array}{c}
\operatorname*{colim}_{m} Map(S^{mp}, B(\mathbb{R}^m)^{\wedge p} \wedge S^n) \\
\simeq \big\uparrow (1) \\
\operatorname*{hocolim}_{m} Map(S^{mp}, B(\mathbb{R}^m)^{\wedge p} \wedge S^n) \\
\simeq \big\downarrow (2) \\
\operatorname*{hocolim}_{\vec{m}\in\mathbb{N}^p} Map(S^{m_0} \wedge \ldots \wedge S^{m_{p-1}}, B(\mathbb{R}^{m_0}) \wedge \ldots \wedge B(\mathbb{R}^{m_{p-q}}) \wedge S^n) \\
\simeq \big\downarrow (3) \\
\operatorname*{hocolim}_{\vec{x}\in I^p} Map(S^{x_0} \wedge \ldots \wedge S^{x_{p-1}}, B(\mathbb{R}^{x_0}) \wedge \ldots \wedge B(\mathbb{R}^{x_{p-q}}) \wedge S^n)
\end{array}
\tag{5.9}
$$

Here, the map (1) is an equivalence since B is a good prespectrum. The second map (2) is an equivalence since $\mathbb{N}^p$ is a filtering index category for the colimit. For (3) we need to apply Bökstedt's Approximation Lemma [4, 1.5]. To do this, we need the assumption that that B is bounded below. The last homotopy colimit is by definition equal to

$$\mathrm{sd}_p THH(B; S^n) = B^{\wedge p}_{pre}(\mathbb{R}^n)\,,$$

and the lemma follows. □

For any C_p-representation V we denote the unit sphere of V by $S(V)$ and the one-point compactification of V by S^V. We have a cofiber sequence of G-spaces

$$S(V)_+ \to S^0 \to S^V \tag{5.10}$$

where the first map collapses $S(V)$ to the non-basepoint of S^0.

Let $L = \mathbb{R}$ with the sign action of C_2. We choose $S(\infty L)$ as a model for EC_2. For $V = \infty L$, the cofiber sequence (5.10) is a model for (4.1) and we have an explicit filtration by skeleta $S(nL) = EC_2^{(n-1)} \subset EC_2$. We have then the following Greenlees filtration of $\widetilde{EC}_2$:

$$\widetilde{E}_n = \widetilde{EC}_2^{(n)} = S^{nL}$$

for all $n \in \mathbb{Z}$. When n is negative, this must interpreted as a spectrum.

For $p > 2$ we let $\mathbb{C}$ have the usual action of C_p and define the even filtrations as

$$\widetilde{E}_{2n} = \widetilde{EC}_p^{(2n)} = S^{n\mathbb{C}}\,.$$

We let $tF2n+1 = \widetilde{EC}_p^{(2n+1)}$ be obtained from $S^{n\mathbb{C}}$ by adjoining a C_p-free $(2n+1)$-cell, see [11]. The cellular chains on $\widetilde{EC}_p$ with mod p coefficients then give the standard C_p-free complete resolution of $\mathbb{F}_p$.

Lemma 5.9. *Let $B \in \mathscr{S}\mathbb{R}^\infty$ and let $n \in \mathbb{Z}$. Then there is a chain of homotopy equivalences*

$$(B^{\wedge p})^{tC_p}[-n(p-1)+1] \simeq \Sigma^{1+n} D_{C_p}(\Sigma^{-n} B)\,,$$

natural in n and B.

I particular, we get that

$$(B^{\wedge p})^{tC_p} \simeq \operatorname{holim}_n \Sigma^{1+n} D_{C_p}(\Sigma^{-n} B)\,.$$

By naturality in n we mean naturality with respect to the maps in the inverse system (5.4).

Proof. Let $s \in \mathbb{Z}$. Since $\widetilde{EC_p}^{(s)}$ is dualizable, we have a natural homotopy equivalence of C_p-spectra

$$\Sigma\, \mathrm{F}(\widetilde{EC_p}^{(s)},\ EC_{p+} \wedge B^{\wedge p}) \simeq \Sigma EC_{p+} \wedge \mathrm{F}(\widetilde{EC_p}^{(s)}, B^{\wedge p})$$

This equivalence gives the first of the following equivalences, after applying C_p-fixed points

$$(B^{\wedge p})^{tC_p}[-n(p-1)+1] \simeq \Sigma[EC_{p+} \wedge \mathrm{F}(\widetilde{EC_p}^{(n(p-1))}, B^{\wedge p})]^{C_p} \tag{5.11}$$

$$\simeq \Sigma EC_{p+} \wedge_{C_p} i^* \,\mathrm{F}(\widetilde{EC_p}^{(n(p-1))}, B^{\wedge p})\,, \tag{5.12}$$

and the Adams transfer equivalence gives the second.

We now rewrite the term $\widetilde{EC_p}^{(n(p-1))}$, while at the same time keeping track of the naturality in n. When $p=2$, we have $\widetilde{EC}_2^{(n(p-1))} = \widetilde{EC}_2^{(n)} = S^{nL}$ and we have a commutative diagram of C_2-spaces

$$\begin{array}{ccc} \Sigma\Sigma^n S^{nL} & \longrightarrow & \Sigma\Sigma^n S^{(n+1)L} \\ \downarrow \cong & & \downarrow \cong \\ \Sigma S^n \wedge S^n & \longrightarrow & S^{n+1} \wedge S^{n+1} \end{array} \tag{5.13}$$

for every n. The lower row consist of C_2-spaces where C_2 acts by permutation of the factors. The upper horizontal map is the $(n+1)$th suspension of the skeleton inclusion and the lower horizontal map is induced by the diagonal map $S^1 \to S^1 \wedge S^1 \cong S^2$. The regular representation of C_2 has one-point compactification isomorphic to $S^1 \wedge S^1$ with the permutation action. The eigenvalues for the action of a generator of C_2 are ± 1, so the regular representation is isomorphic to $\mathbb{R} \oplus L$. The one-point compactification of the latter is ΣS^L. By desuspending (5.13) $(n+1)$ times, followed by taking Spanier-Whitehead duals, we then get a commutative diagram of spectra in $C_2\mathscr{S}\mathscr{U}$ as follows:

$$\begin{array}{ccc} D(\widetilde{EC}_2^{(n)}) & \longleftarrow & D(\widetilde{EC}_2^{(n+1)}) \\ \cong \uparrow & & \cong \uparrow \\ \Sigma^n D(S^n \wedge S^n) & \longleftarrow & \Sigma^{n+1} D(S^{n+1} \wedge S^{n+1}) \end{array} \tag{5.14}$$

Using this, we find that (5.12) is equivalent to

$$\Sigma^{1+n} EC_{2+} \wedge_{C_2} i^* \mathrm{F}((S^n)^{\wedge 2}, B^{\wedge 2})$$

which by Lemma 5.7 is equivalent to

$$\Sigma^{1+n} EC_{2+} \wedge_{C_2} i^* \mathrm{F}(S^n, B)^{\wedge 2} \tag{5.15}$$

$$\simeq \Sigma^{1+n} EC_{2+} \wedge_{C_2} i^* (\Sigma^{-n} B)^{\wedge 2} \,. \tag{5.16}$$

Lemma 5.8 implies that (5.16) is equivalent to

$$\Sigma^{1+n} EC_{2+} \wedge_{C_2} i^* \psi (\Sigma^{-n} B)^{(2)} \,. \tag{5.17}$$

Let $\chi : EC_p \to \mathscr{J}(\mathscr{U}, \mathbb{R}^\infty)$ by any C_p-equivariant map and let $E \in C_p\mathscr{S}\mathscr{U}$. Then by [13, Theorem VI.1.17] there is a natural homotopy equivalence

$$\chi \ltimes E \simeq EC_{p+} \wedge i^* E$$

of spectra in $C_p\mathscr{S}\mathbb{R}^\infty$. We apply this fact to see that (5.17) is equivalent to

$$\begin{aligned} &\Sigma^{1+n} \chi \ltimes_{C_2} \psi (\Sigma^{-n} B)^{(2)} \\ &\simeq \Sigma^{1+n} \chi \ltimes_{C_2} (\Sigma^{-n} B)^{(2)} \end{aligned}$$

which is our spectrum level model for $\Sigma^{1+n} D_{C_2}(\Sigma^{-n} B)$, so the lemma follows for $p = 2$.

Likewise, for $p > 2$, we have that $(S^1)^{\wedge p} \cong \Sigma(S^{((p-1)/2)\mathbb{C}})$ as C_p-spaces. Thus we get a diagram

$$\begin{array}{ccc} \Sigma\Sigma^n (S^{(n(p-1)/2)\mathbb{C}}) & \longrightarrow & \Sigma\Sigma^n (S^{((n+1)(p-1)/2)\mathbb{C}}) \\ \downarrow \cong & & \downarrow \cong \\ \Sigma (S^n)^{\wedge p} & \longrightarrow & (S^{n+1})^{\wedge p} \end{array}$$

analogous to (5.13). Recalling that $\widetilde{EC_p}^{2n} = S^{n\mathbb{C}}$, we apply the same calculation as for the case $p=2$, and the lemma follows for $p>2$.

□

Theorem 5.10. *Let $B \in \mathscr{S}\mathbb{R}^\infty$ be a spectrum. Then there are natural isomorphisms*

$$H_c^*((B^{\wedge p})^{tC_p}) \cong R_+(H^*(B))$$
$$H_*^c((B^{\wedge p})^{tC_p}) \cong R_+(H_*(B))$$

Proof. By Lemma 5.9, we have a natural isomorphism

$$H_c^*((B^{\wedge p})^{tC_p}) \cong \operatorname*{colim}_{n\to\infty} H^*(\Sigma\Sigma^n D_{C_p}(\Sigma^{-n}B))\,.$$

The colimit on the right is by Theorem 5.3 isomorphic to

$$R_+(H^*(B))\,.$$

The $\hom_{\mathbb{F}_p}$ functional dual of the two isomorphism above yields the natural isomorphism

$$H_*^c((B^{\wedge p})^{tC_p}) \cong R_+(H_*(B))\,.$$

□

5.3. **The topological Singer ϵ-map and the fixed points of $B^{\wedge p}$.** Let $B \in \mathscr{S}\mathbb{R}^\infty$ be bounded below and of finite type. Then set $X = B^{\wedge p}$. Diagram (4.7) now reads

(5.18)
$$\begin{array}{ccccc}
(B^{\wedge p})_{hC_p} & \longrightarrow & (B^{\wedge p})^{C_p} & \longrightarrow & [\widetilde{EC_p} \wedge B^{\wedge p}]^{C_p} \\
\downarrow{=} & & \downarrow{\Gamma} & & \downarrow{\gamma} \\
(B^{\wedge p})_{hC_p} & \longrightarrow & (B^{\wedge p})^{hC_p} & \longrightarrow & R_+(B)\,.
\end{array}$$

For a general Lie group G and any $X \in G\mathscr{S}\mathscr{U}$, the construction $[\widetilde{EG} \wedge X]^G$ is a model for the so-called geometric fixed points of X, denoted $\Phi^G(X)$. We are not interested in its general properties here, but we introduce the notation to make the exposition easier. In [10, Propsition 2.5], the authors show the following: There is a natural homotopy equivalence

(5.19)
$$\Phi^{C_p} B^{\wedge p} \xrightarrow{\simeq} B\,.$$

In fact, if B is symmetric ring spectrum, we think of $B \in \mathscr{S}\mathbb{R}^\infty$ as the spectrification of the pre-spectrum indexed on the subspaces $\mathbb{R}^n \subset \mathbb{R}^\infty$ given by

$$B(\mathbb{R}^n) = B_n\,.$$

Then our definition of $B^{\wedge p}$ coincides with the zero simplices of their model of the p-fold edgewise subdivision of the simplicial spectrum $THH(B)$. In the notation of [10],

$$B^{\wedge p} = \mathrm{sd}_p\, THH(B)_0\,.$$

In [10, Propsition 2.5], the authors show that there is a simplicial homotopy equivalence

(5.20)
$$\Phi^{C_p}\, \mathrm{sd}_p\, THH(B)_r \xrightarrow{\simeq} THH(B)_r\,.$$

Restriction this equivalence to simplicial degree $r=0$, gives (5.19). We will return to this fact in more detail in [15].

Definition 5.11. *Let $\epsilon_B : B \to R_+(B)$ be the the stable map given by the composite*

$$B \simeq \Phi^{C_p} B^{\wedge p} \xrightarrow{\gamma} R_+(B)\,.$$

When B is an E_∞ ring spectrum, both $THH(B)$ and its zero simplices $B^{\wedge p}$ get induced structures as E_∞ ring spectra. As noted in §4.6, the right hand square of diagram 5.18 then consists of commutative ring- spectra and maps. Morover, the map (5.20) is a map of commutative ring spectra [3, ???] and so the stable map

$$\epsilon_B : B \to R_+(B)$$

is a map of commutative ring spectra.

Lemma 5.12. *The map $\epsilon_B : B \to R_+(B)$ is natural in B and commutes with suspension.*

On continuous cohomology have

$$\epsilon^*_{\Sigma B}(\Sigma u^i t^n \otimes x) = \Sigma \epsilon^*_B(u^i t^n \otimes x).$$

Proof. Naturality follows immediately from the definition, using naturality of $\Phi^C(-)$ and $R_+(-)$.

For the second statement, consider the diagram

(5.21)
$$\begin{array}{ccc}
\Sigma B & = & \Sigma B \\
\simeq\uparrow & & \uparrow\simeq \\
\Sigma\Phi^{C_p}(B^{\wedge p}) & \xrightarrow{\Delta} & \Phi^{C_p}((\Sigma B)^{\wedge p}) \\
\downarrow\Sigma\gamma & & \downarrow\gamma \\
\Sigma R_+(B) & \xrightarrow{\Delta} & R_+(\Sigma B)\,.
\end{array}$$

The lower two horizontal maps are induced by the diagonal inclusion $S^1 \hookrightarrow S^p$, which in the case of the Tate construction is the same map used in the definition of $R_+(B)$ as the inverse system of suspended extended power constructions. □

Lemma 5.13. *The map $\epsilon_S : S \to R_+(S)$ of ring spectra induces an isomorphism of continuous homology groups in dimension* 0*:*

$$\epsilon_{S*} : H_0(S) \to H^c_0(R_+(S))\,.$$

Proof. We let $B = S$ and consider $S^{\wedge p}$ to be a model for the equivariant sphere spectrum $S_{C_p} \in C_p\mathscr{S}\mathscr{U}$. We use that the S_{C_p} is split (see [9, §1]). This means that there is a map $s : S_{C_p} \to S^{C_p}_{C_p}$ such that s followed by the inclusion of the fixed points by the Frobenius map $F : S^{C_p}_{C_p} \to S_{C_p}$ is a non-equivariant equivalence.

The map F can be factored as the composition $S^{C_p}_{C_p} \xrightarrow{\Gamma} S^{hC_p}_{C_p} \to S_{C_p}$ where $S^{hC_p}_{C_p} \to S_{C_p}$ is the map forgetting equivariance induced by the restriction map

$C_p \hookrightarrow EC_p$. We have a commutative diagram

$$\begin{array}{ccccccc}
 & & S_{C_p} & & & & \\
 & & \downarrow{\scriptstyle s} & & & & \\
(S_{C_p})_{hC_p} & \xrightarrow{R} & S_{C_p}^{C_p} & \longrightarrow & S_{C_p} & & \\
\| & & \downarrow{\scriptstyle \Gamma} & & \downarrow{\scriptstyle \gamma} & & \\
(S_{C_p})_{hC_p} & \xrightarrow{N^h} & S_{C_p}^{hC_p} & \longrightarrow & S_{C_p}^{tC_p} & & \\
\| & & \downarrow & & \downarrow & & \\
(S_{C_p})_{hC_p} & \xrightarrow{trf} & S_{C_p} & \xrightarrow{g} & S_{C_p}^{tC_p}[0] & \longrightarrow & \Sigma(S_{C_p})_{hC_p}
\end{array}$$

where the middle vertical composite is an equivalence. We know from the homological Tate spectral sequence that $H_0(S_{C_p}^{tC_p}[0]) \cong H_0^c(S_{C_p}^{tC_p}) \cong \mathbb{F}_p$. To show that the map γ_* is non-trivial on continuous homology groups, it suffices to show that the map g_* is surjective in degree 0. This follows from the long exact sequence in homology since $(S_{C_p})_{hC}$ is 0-connected. □

The authors would like to thank M. Bökstedt for a helpful discussion on the following result.

Proposition 5.14. *Let B be a bounded below spectrum of finite type and let $M = H^*(B)$. Then $\epsilon_B^* = \epsilon$, the latter being the map described in §3.1.1.*

Proof. First, consider the free case $B = H\mathbb{F}_p$. Then $H^*(H\mathbb{F}_p) = \mathscr{A}$ is degreewise finite dimensional and bounded below and thus by Corollary 3.5 there is a unique self map $f_{H\mathbb{F}_p} : \mathscr{A} \to \mathscr{A}$ of $\mathscr{A}$-modules such that the diagram

$$\begin{array}{ccc}
R_+(\mathscr{A}) & \xrightarrow{\epsilon^*_{H\mathbb{F}_p}} & \mathscr{A} \\
\downarrow{\scriptstyle \epsilon} & \nearrow{\scriptstyle f_{H\mathbb{F}_p}} & \\
\mathscr{A} & &
\end{array} \tag{5.22}$$

of $\mathscr{A}$-modules commutes. The claim is that $f_{H\mathbb{F}_p}$ in (5.22) is the identity on $\mathscr{A}$. In fact, since $f_{H\mathbb{F}_p}$ is $\mathscr{A}$-linear, it is determined by $\epsilon_B^*(1)$ in the case $B = H\mathbb{F}_p$, and we know that dually, in continuous homology, ϵ_{B*} sends the unit to the unit. The latter claim follows from Lemma 5.13 by naturality with respect to the unit $S \to H\mathbb{F}_p$ and the fact that $\epsilon_{H\mathbb{F}_p}$ is a map of ring spectra.

Consider now the general case. Choose a map $x : B \to \Sigma^n H\mathbb{F}_p$ representing a class in cohomology $x^*(\iota_n) \in H^n(B)$, where $\iota_n \in H^n(H\mathbb{F}_p)$ is the fundamental class. The Proposition follows by naturality using the following commutative diagram of $\mathscr{A}$-modules:

$$\begin{array}{ccccc}
\Sigma^n H^*(H\mathbb{F}_p) & \xrightarrow{\cong} & H^*(\Sigma^n H\mathbb{F}_p) & \xrightarrow{x^*} & H^*(B) \\
\uparrow{\scriptstyle \Sigma^n \epsilon_{H\mathbb{F}_p}} & & \uparrow{\scriptstyle \epsilon^*_{\Sigma^n H\mathbb{F}_p}} & & \uparrow{\scriptstyle \epsilon^*_B} \\
\Sigma^n R_+(H^*(H\mathbb{F}_p)) & \xrightarrow{\cong} & R_+(H^*(\Sigma^n H\mathbb{F}_p)) & \xrightarrow{R_+(x)^*} & R_+(H^*(B))\,.
\end{array} \tag{5.23}$$

Lemma 5.12 ensures that both squares commute. □

The following corollary generalizes the Segal conjecture for C_p.

Corollary 5.15. *Let B be a bounded below spectrum of finite type over $\mathbb{F}_p$. Then the canonical map*

$$\Gamma : (B^{\wedge p})^{C_p} \to (B^{\wedge p})^{hC_p}$$

is a p-adic equivalence of spectra.

Proof. By Proposition 5.14, we know that γ of diagram (5.18) induces an Ext-isomorphism of continuous cohomology groups and thus an isomorphism of the E_2-terms, and thus of the E_∞-terms, of the inverse limit Adams spectral sequence associated to the Tate constuction $R_+(B)$. Hence it is a p-adic homotopy equivalence of spectra.

Since the right hand square of the diagram is the homotopy Cartesian, the Corollary follows. □

5.4. **The Tate spectral sequence for the topological Singer construction.** We have seen how the Singer construction can be realized as the continuous cohomology of the C_p-Tate construction. For our limited applications so far, we have not been forced to do explicit calculations involving the Tate spectral sequence. For future reference we now proceed by relating the definition of the homological, algebraic Singer construction to the homological Tate spectral sequence associated with the topological Singer construction $R_+(B)$.

By Lemma 5.9, we have an equivalence of towers of spectra

$$(5.24)\qquad \begin{array}{ccccc} \cdots \longrightarrow & (B^{\wedge p})^{tC_p}[-(n+1)(p-1)+1] & \longrightarrow & (B^{\wedge p})^{tC_p}[-n(p-1)+1] \\ & \downarrow \simeq & & \downarrow \simeq \\ \cdots \longrightarrow & \Sigma\Sigma^{n+1} D_{C_p}(\Sigma^{-n-1}B) & \longrightarrow & \Sigma\Sigma^{n} D_{C_p}(\Sigma^{-n}B) \end{array}$$

where the homotopy inverse limit of both towers are equivalent to $R_+(B) = (B^{\wedge p})^{tC_p}$.

When $p = 2$, the upper row of the above diagram is equal to the tower of spectra defining the Tate filtration (4.10) and the exact couple giving, us the Tate spectral sequences Proposition 4.10 and 4.11.

Proposition 5.16. *Let B be a bounded below spectrum of finite type. The homological Tate spectral sequence $\hat{E}^r(R_+(B))$ converging to $H^c_*(R_+(B)) \cong R_+(H_*(B))$ collapses at the $\hat{E}^2$-term.*

For $p > 2$ we have

$$\hat{E}^\infty_{*,*} \cong \hat{E}^2_{*,*} = \widehat{H}^{-*}(C_p; H_*(B)^{\otimes p}) \cong E(u) \otimes P(t, t^{-1}) \otimes \mathbb{F}_p\{\alpha^{\otimes p}\}_{\alpha \in H_*(B)}$$

where α runs through an $\mathbb{F}_p$-basis of $H_*(B)$.

For $\alpha \in H_q(B)$, the element $u^\epsilon t^i \otimes \alpha \in R_+(H_*(B))$ is represented in the Tate spectral sequence by the class $u^\epsilon t^{i+q(p-1)/2} \otimes \alpha^{\otimes p}$ in bidegree $(-\epsilon - 2i - q(p-1), pq)$, modulo multiplication by non-zero scalars in $\mathbb{F}_p$.

For $p = 2$ the corresponding statement is that

$$\hat{E}^\infty_{*,*} \cong \hat{E}^2_{*,*} = \widehat{H}^{-*}(C_2; H_*(B)^{\otimes 2}) \cong P(u, u^{-1}) \otimes \mathbb{F}_2\{\alpha^{\otimes 2}\}_{\alpha \in H_*(B)}$$

where α runs through an $\mathbb{F}_2$-basis of $H_*(B)$.

For $\alpha \in H_q(B)$, the element $u^i \otimes \alpha \in R_+(H_*(B))$ is represented in the Tate spectral sequence by the class $u^{i+q} \otimes \alpha^{\otimes 2}$ in bidegree $(-i - q, 2q)$.

Proof. Consider first the case $B = S^q$. Then the result is trivial for dimensional reasons: the spectral sequence is trivial except in bidegrees $(*, pq)$, so there is no room for differentials after the $\hat{E}^2$-term. Since each total degree is of rank 1 over $\mathbb{F}_p$, there are also no extension problems to be solved.

Let H be the $\mathbb{F}_p$ Eilenberg-Mac Lane spectrum. We model H by a commutative symmetric ring spectrum and form the spectrum $H^{\wedge p} \in C_p\mathscr{S}\mathscr{U}$ according to definition 5.5. The iterated multiplication on H then induces a C_p-equivariant map

$$H^{\wedge p} \to H$$

In general, let $\alpha : S^q \to H \wedge B$ represent a class $[\alpha]$ in $H_q(B)$. We then have a C_p-equivariant composite map

$$\alpha^{\otimes p} : H \wedge S^{pq} \to H \wedge (H \wedge B)^{\wedge p} \cong H \wedge H^{\wedge p} \wedge B^{\wedge p} \to H \wedge B^{\wedge p}$$

which on passage to homotopy groups yields a map $\Sigma^{pq}\mathbb{F}_p \to H_*(B)^{\otimes p}$ with image generated by the class $[\alpha]^{\otimes p}$.

By applying the Tate construction to this map, we get a map of spectra

$$\bar{\alpha} : (H \wedge S^{pq})^{tC_p} \to (H \wedge B^{\wedge p})^{tC_p}$$

which preserves the Tate filtration.

According to Proposition 4.12, we get a map of continuous homology groups by taking homotopy on both sides :

$$H^c_*(R_+(S^q)) \to H^c_*(R_+(B)) \, . \tag{5.25}$$

This map is injective for $[\alpha] \neq 0$ and $\mathbb{F}_p$-linear. For p odd, the map is given on the level of spectral sequences as the map sending $u^\epsilon t^i \otimes \iota_q^{\otimes p} \mapsto u^\epsilon t^i \otimes \alpha^{\otimes p}$. Similarly for $p = 2$.

The statement of the proposition now follows by naturality. □

Note that the $\mathbb{F}_p$-linear map (5.25) is not a homomorphism of $\mathscr{A}_*$-comodules because it was formed using the iterated product on H.

References

[1] J. F. Adams, J. H. Gunawardena, and H. Miller, *The Segal conjecture for elementary abelian p-groups*, Topology **24** (1985), no. 4, 435–460. MR816524 (87m:55026)

[2] J. Michael Boardman, *Conditionally convergent spectral sequences*, Homotopy invariant algebraic structures (baltimore, md, 1998), 1999, pp. 49–84. MR2000m:55024

[3] M. Bökstedt and I. Madsen, *Topological cyclic homology of the integers*, Astérisque **226** (1994), 7–8, 57–143. K-theory (Strasbourg, 1992). MR96d:19004

[4] Marcel Bökstedt, *Topological Hochschild homology*, Preprint.

[5] R. R. Bruner, J. P. May, J. E. McClure, and M. Steinberger, *H_∞ ring spectra and their applications*, Lecture Notes in Mathematics, vol. 1176, Springer-Verlag, Berlin, 1986. MR88e:55001

[6] Henri Cartan and Samuel Eilenberg, *Homological algebra*, Princeton University Press, Princeton, N. J., 1956. MR17,1040e

[7] J. Caruso, J. P. May, and S. B. Priddy, *The Segal conjecture for elementary abelian p-groups. II. p-adic completion in equivariant cohomology*, Topology **26** (1987), no. 4, 413–433. MR919728 (90c:55005)

[8] J. P. C. Greenlees, *Representing Tate cohomology of G-spaces*, Proc. Edinburgh Math. Soc. (2) **30** (1987), no. 3, 435–443. MR88m:57055

[9] J. P. C. Greenlees and J. P. May, *Generalized Tate cohomology*, Mem. Amer. Math. Soc. **113** (1995), no. 543, viii+178. MR1230773 (96e:55006)

[10] Lars Hesselholt and Ib Madsen, *On the K-theory of finite algebras over Witt vectors of perfect fields*, Topology **36** (1997), no. 1, 29–101. MR1410465 (97i:19002)

[11] ———, *On the K-theory of local fields*, Ann. of Math. (2) **158** (2003), no. 1, 1–113. MR1998478 (2004k:19003)

[12] John D.S. Jones, *Root invariants, cup-r-products and the Kahn-Priddy theorem*, Bull. London Math. Soc. **17** (1985), no. 5, 479–483.

[13] L. G. Lewis Jr., J. P. May, M. Steinberger, and J. E. McClure, *Equivariant stable homotopy theory*, Lecture Notes in Mathematics, vol. 1213, Springer-Verlag, Berlin, 1986. With contributions by J. E. McClure. MR88e:55002

[14] W. H. Lin, D. M. Davis, M. E. Mahowald, and J. F. Adams, *Calculation of Lin's Ext groups*, Math. Proc. Cambridge Philos. Soc. **87** (1980), no. 3, 459–469. MR81e:55025

[15] Sverre Lunøe-Nielsen and John Rognes, *The Segal conjecture for topological Hochschild homology of complex cobordism*, In preparation.

[16] William M. Singer, *On the localization of modules over the Steenrod algebra*, J. Pure Appl. Algebra **16** (1980), no. 1, 75–84. MR549705 (81b:55039)

[17] ———, *A new chain complex for the homology of the Steenrod algebra*, Math. Proc. Cambridge Philos. Soc. **90** (1981), no. 2, 279–292. MR620738 (82k:55018)

[18] Clarence Wilkerson, *Classifying spaces, Steenrod operations and algebraic closure*, Topology **16** (1977), no. 3, 227–237. MR0442932 (56 #1307)