Online appendix to "Keep On Fighting: The Dynamics of Head Starts in All-Pay Auctions" by Derek J. Clark and Tore Nilssen

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This appendix contains proofs and elaborations relating to Section 5 in our paper.

1 Head start vs handicap

The Lemma below extends Proposition 1 in the main text to allow for handicaps as well as head starts; by putting \( w = 1 \) in (B1), we are back to (1) in the paper.

**Lemma 1** Consider a single all-pay contest where players \( h \) and \( k \) compete. Let the contest success function be

\[
p_{h,t} (x_{h,t}, x_{k,t}; z, w) = \begin{cases} 
1 & \text{if } z + x_{h,t} > wx_{k,t} \\
\frac{1}{2} & \text{if } z + x_{h,t} = wx_{k,t} \\
0 & \text{if } z + x_{h,t} < wx_{k,t}
\end{cases}
\]

(B1)

where \( z \geq 0 \) and \( w \in (0,1] \). Let the values of the prize be \( v_h = v + a \) and \( v_k = v \) for players \( h \) and \( k \), respectively, where \( a \geq 0 \). The unique equilibrium is as follows:

(i) If \( v \leq \frac{z}{w} \), then \( x_h = x_k = 0 \).

(ii) If \( v > \frac{z}{w} \), then the equilibrium is given by

\[
F_h (x_h) = \frac{z + x_h}{vw}, \quad x_h \in [0, vw - z];
\]

\[
F_k (x_k) = \begin{cases} 
v(1-w)+a+z & \text{if } x_k \in [0, \frac{z}{w}) \\
v(1-w)+a+wx_k & \text{if } x_k \in [\frac{z}{w}, v] \\
\end{cases}
\]
Expected efforts are
\[ x_h^* (z, a, w) = \frac{(vw - z)^2}{2vw}, \text{ and } x_k^* (z, a, w) = \frac{v^2w^2 - z^2}{2w(v + a)}; \]
payoffs are
\[ \pi_h^* (z, a, w) = z + a + (1 - w) v, \text{ and } \pi_k^* (z, a, w) = 0; \quad (B2) \]
and probabilities of winning are
\[ 1 - \frac{v^2w^2 - z^2}{2vw(v + a)} \text{ for player } h, \]
\[ \frac{v^2w^2 - z^2}{2vw(v + a)} \text{ for player } k. \]

**Proof.** Player \( k \) will not spend more than \( v \), so that the maximum spent by player \( h \) is \( vw - z \). If player \( h \) sets \( x_h = 0 \), then he wins if \( z > wx_k \) so that player \( k \) will not choose positive effort below \( \frac{z}{w} \). Hence, \( x_h \in [0, vw - z] \), and \( x_k \in \{0\} \cup \left(\frac{z}{w}, v\right] \). By setting \( x_h = vw - z \), player \( h \) wins with probability 1 and secures a payoff of \( z + a + v(1 - w) \), whilst player \( k \) must get 0. The payoffs of the players are therefore
\[ \pi_h (X) = F_k (X) (v + a) - (wX - z) = z + a + (1 - w) v; \quad (B3) \]
\[ \pi_k (Y) = F_h (Y) v - \frac{Y + z}{w} = 0; \quad (B4) \]
where \( X = \frac{z + wx_k}{w}, Y = wx_k - z \), and \( F_i (x) \) is the mixed strategy of player \( i \in \{h, k\} \). The unique solution to this system of equations is
\[ F_k (X) = \frac{v(1 - w) + a + wX}{v + a}, \]
\[ F_h (Y) = \frac{Y + z}{vw}. \]
Player \( k \)'s probability of winning, \( \rho_k \), is found from the equation \( \rho_k v - x_k^* = 0 \), while that of player \( h \) is \( \rho_h = 1 - \rho_k \).

It is straightforward to verify that the disadvantaged player exerts the higher expected effort if
\[ a < \frac{2vz}{vw - z}. \quad (B5) \]
The right-hand side in (B5) decreases in \( w \). Thus, the laggard has more effort than his rival when the handicap is high, i.e., \( w \) is low.

In applying this result to the study of sequential contests, as we do in Section 5.1 in the paper with the contest success function in equation (25)
there, we need to make an assumption parallel to (7) in the paper, ensuring that no subgame can occur in which no effort is exerted. By (B2) above and (25) in the paper, the score improves by

\[ z + (1 - w) v = M_t s \left[ b + (1 - b) v \right] \]

for each win. We need that, even if player 1 were to win the first \((T - 1)\) contests, so that \(M_T = T - 1\), his score is still less than the stage prize \(v\); this is stated in (26) in the main text.

As an illustration of the study of sequential contests with this contest success function, we put \(T = 3\). As in Section 3 in the main text, we let player 1 be the leader, without loss of generality. In contest 3, in case of symmetry, \(M_3 = 0\), each player’s expected effort is \(\frac{v}{2}\), and his expected net payoff is zero. In case of asymmetry in that contest, \(M_3 = 2\). By Lemma 1 above, the expected payoff to the leader is \(2s [b + v (1 - b)]\).

Consider next contest 2. Here, there is a leader for sure, and \(M_1 = 1\). The value of winning is

\[ a_2 = 2s [b + v (1 - b)] \]  \hspace{1cm} (B6)

The leader’s expected net surplus is

\[ z + a + v (1 - w) = 3s [b + v (1 - b)] \].

Thus, in contest 1, the value of winning is the above plus the prize in that contest, \(v\), that is,

\[ v + 3s [b + v (1 - b)] \].

Note that, at \(b = 1\), this becomes \(v + 3s\). Moreover, this value increases as \(b\) decreases, i.e., as more weight is put on handicapping relative to head start, if and only if \(v > 1\). Each player’s expected effort in contest 1 is

\[ \frac{1}{2} \{v + 3s [b + v (1 - b)]\} \].

Corollary 8 in the paper concerns relative efforts of the players in the second contest when \(T = 3\). Combining (B5) with the expression for the value of winning the second contest in (B6), we find that the laggard has the larger expected effort in the second contest if and only if

\[ s > \frac{v^2 (1 - b)}{[b + v (1 - b)]^2} \]  \hspace{1cm} (B7)

This puts a lower limit on the win advantage in order for the laggard to exert more effort than the leader in the second contest of a three-contest game. Combining this with the upper limit in (26) in the paper, we have that a value for the win advantage \(s\), satisfying both the constraints in (26) and (B7) when \(b < 1\), can only exist when \(v < \frac{b}{1 - b}\). In fact, when \(v > \frac{b}{1 - b}\), the opposite of Corollary 8 is true as noted in the text: the leader exerts the higher expected efforts at \(t = 2\) when \(T = 3\).
2 Discounting

Suppose players discount future payoffs with a discount factor \( \delta \in (0, 1] \). Discounting will affect the leader’s expected value of winning in a straightforward manner: equation (A1), in the proof of Proposition 2 in the paper, now becomes

\[
    u_{1,t}(M_t) = s \left( \sum_{i=0}^{T-t} \delta^i (M_t + i) \right). \tag{B8}
\]

Using (B8) together with (11) in the paper, we have, for \( \delta \in (0, 1) \),

\[
    a_t = u_{1,t+1}(M_{t+1}) - u_{1,t+1}(M_t - 1)
    = s \left[ \sum_{i=0}^{T-t-1} \delta^i (M_t + 1 + i) - \sum_{i=0}^{T-t-1} \delta^i (M_t - 1 + i) \right]
    = 2s \frac{1 - \delta^{T-t}}{1 - \delta}.
\]

Note that \( \lim_{\delta \to 1^-} \frac{1 - \delta^{T-t}}{1 - \delta} = T - t \), that \( \frac{da_t}{d\delta} > 0 \) – heavier discounting means a lower value of winning for the leader – and that, as before, \( \frac{da_t}{d(T-t)} > 0 \) – the more periods left, the higher is \( a_t \).

In Proposition 2 in the paper, the implication of discounting is that the laggard’s expected effort in contest \( t \), rather than (16) in the paper, becomes

\[
    x_{2,t}^*(M_t) = \frac{v^2 - (M_t s)^2}{2 \left[ v + 2s \frac{1 - \delta^{T-t}}{1 - \delta} \right]};
\]

thus, the more discounting, the higher is the laggard’s expected efforts for contests \( t \leq T - 2 \). The leader’s expected payoff in contest \( t \), in (19) in the paper, becomes, from (B8),

\[
    u_{1,t}^*(M_t) = \frac{s}{1 - \delta} \left\{ M_t (1 - \delta^{T-t}) + \delta \left[ \frac{1 - \delta^{T-t}}{1 - \delta} + \delta^{T-t} (T - t) \right] \right\}.
\]

Note that, as before, \( a_T = 0 \) and \( a_{T-1} = 2s \), so that Corollaries 2 and 8 in the paper still hold.

3 Varying prizes

Here we present details of the analysis of the case when prizes vary over time, discussed in Section 5.3 in the paper. Again, without loss of generality, we put player 1 as the leader, if there is one. We start with considering the last contest, \( t = 3 \). There are two possibilities, either symmetry, with one win to each player in the previous rounds, or asymmetry, with one
player having won both previous rounds. In case of symmetry, $M_3 = 0$, each player’s expected effort is $v_3/2 = (1 - v_1 - v_2)/2$, and each player’s expected net payoff is zero.

In case of asymmetry, $M_3 = 2$. We need to distinguish between two cases. If $v_3 = 1 - v_1 - v_2 \leq 2s$, then, by part (i) of Proposition 1 in the paper, players have zero efforts in the last contest and the leader is certain to win, with net payoff $1 - v_1 - v_2$ to the leader and zero to the laggard. Otherwise, if $1 - v_1 - v_2 > 2s$, then, by (4) in the main text, the expected efforts of the leader and the laggard are

$$\frac{(1 - v_1 - v_2 - 2s)^2}{2(1 - v_1 - v_2)} \quad \text{and} \quad \frac{(1 - v_1 - v_2)^2 - 4s^2}{2(1 - v_1 - v_2)}, \quad (B9)$$

respectively, so that total expected efforts in contest 3 in this case, the sum of the two expressions above, is

$$1 - v_1 - v_2 - 2s.$$ 

The expected net payoffs are $2s$ to the leader and, again, zero to the laggard.

Consider next the next-to-last contest, that is, $t = 2$. In this case, there is surely asymmetry. With player 1 as the leader, we have $M_2 = 1$. Again, we need to consider two possibilities. If $v_2 \leq s$, then players have zero efforts and the leader wins contest 2. If, in addition, $1 - v_1 - v_2 \leq 2s$, then the leader wins also contest 3 with zero efforts. Thus, if $1 - v_1 - 2s \leq v_2 \leq s$, which can only happen if $v_1 \geq 1 - 3s$, then the winner of contest 1 wins the next two contests without spending further efforts and the expected value of winning contest 1 is 1; the variable restrictions in this case corresponds to area I in Figure 1 in the paper, where feasible combinations of $(v_1, v_2)$ are depicted. If $v_2 < 1 - v_1 - 2s$ at the same time as $v_2 \leq s$, however, then the winner of contest 1 wins again in contest 2 and has an expected net payoff of $2s$ in contest 3, with a total value of winning contest 1 of $v_1 + v_2 + 2s < 1$; this is area II in Figure 1 in the paper.

If $v_2 > s$, then, by (4) in the paper, the expected efforts of the leader and the laggard are

$$\frac{(v_2 - s)^2}{2v_2} \quad \text{and} \quad \frac{v_2^2 - s^2}{2(v_2 + a_2)},$$

respectively. To get any further, we need to find $a_2$. For this, we distinguish two subcases.

If $v_2 \geq 1 - v_1 - 2s$, as well as $v_2 > s$, then the leader, if he wins also here, will win again in contest 3 without efforts, so $a_2 = 1 - v_1 - v_2 \leq 2s$, the laggard’s expected effort is

$$\frac{v_2^2 - s^2}{2(1 - v_1)},$$
and total expected effort in contest 2 is
\[ \frac{v_2 - s}{2v_2 (1 - v_1)} \left[ v_2^2 + (1 - v_1 + s) v_2 - s (1 - v_1) \right]. \] (B10)

The expected payoff to the leader is \( z + a \), which here is \( 1 - v_1 - v_2 + s \).
Thus, the value of winning contest 1 is, in this case, \( v_1 + (1 - v_1 - v_2 + s) = 1 - (v_2 - s) < 1 \). The case corresponds to area III in Figure 1.

If, on the other hand, \( s < v_2 < 1 - v_1 - 2s \), which can only happen if \( v_1 < 1 - 3s \), then \( a_2 = 2s \) by equation (23) in the main text, the laggard’s expected effort is
\[ \frac{v_2^2 - s^2}{2 (v_2 + 2s)^2}. \]

and total expected effort in contest 2 is
\[ \frac{v_2 - s}{v_2 (v_2 + 2s)} \left( v_2^2 + sv_2 - s^2 \right). \] (B11)

The expected payoff to the leader is \( z + a = 3s \), and the value of winning contest 1 is \( v_1 + 3s < 1 \). This case corresponds to area IV in Figure 1 in the paper.

Finally, consider the full game, noting that, at contest 1, there is symmetry and \( M_1 = 0 \). We can now specify the equilibrium play in each of the four cases introduced above. If \( 1 - v_1 - 2s \leq v_2 \leq s \), then the value of winning contest 1 is 1, and each player exerts expected effort in that contest equal to \( \frac{1}{2} \). No efforts are exerted in contests 2 and 3, so that total expected effort in the game is 1. This is area I in Figure 1 in the paper.

If \( v_2 < 1 - v_1 - 2s \) at the same time as \( v_2 \leq s \), then the value of winning contest 1 is \( v_1 + v_2 + 2s \); each player’s expected effort in contest 1 is \( (v_1 + v_2 + 2s) / 2 \), and total expected effort in contest 1 is \( v_1 + v_2 + 2s \). In contest 2, no player exerts effort and the leader wins that contest for certain. In contest 3, the expected efforts of leader and laggard are given in (B9), and total expected effort is \( 1 - v_1 - v_2 - 2s \). Thus, total expected effort across the three contests is 1. This is area II in Figure 1.

If \( v_2 \geq 1 - v_1 - 2s \), as well as \( v_2 > s \), then the value of winning contest 1 is \( 1 - (v_2 - s) \). Each player exerts in expectation \( [1 - (v_2 - s)] / 2 \) in contest 1, and total expected effort in that contest is \( 1 - (v_2 - s) \). In contest 2, the expected efforts of leader and laggard are
\[ \frac{(v_2 - s)^2}{2v_2} \text{ and } \frac{v_2^2 - s^2}{2 (1 - v_1)}, \]
respectively, with total expected effort given by (B10). The laggard wins with probability
\[ \frac{v_2^2 - s^2}{2v_2 (1 - v_1)}. \]
in which case the game moves to symmetry in contest 3 where each player’s expected effort is \((1 - v_1 - v_2) / 2\), with total expected efforts in contest 3 equal to \(1 - v_1 - v_2\). With probability

\[
1 - \frac{v_2^2 - s^2}{2v_2 (1 - v_1)},
\]

the leader wins contest 2, in which case no effort is exerted in contest 3 and the leader wins for sure. The total expected effort across all contests is

\[
1 - (v_2 - s) + \frac{v_2 - s}{2v_2 (1 - v_1)} \left[ v_2^2 + (1 - v_1 + s) v_2 - s (1 - v_1) \right] + \frac{v_2^2 - s^2}{2v_2 (1 - v_1)} (1 - v_1 - v_2) = 1.
\]

This is area III in Figure 1.

Finally, consider the case \(s < v_2 < 1 - v_1 - 2s\), which covers the special case of \(v_1 = v_2 = \frac{1}{3}\) discussed in Section 3 in the paper. The value of winning contest 1 is \(v_1 + 3s\), and so each player’s expected effort in contest 1 is \((v_1 + 3s) / 2\) with a total expected effort in contest 1 of \(v_1 + 3s\). In contest 2, expected efforts of the leader and the laggard are

\[
\frac{(v_2 - s)^2}{2v_2} \text{ and } \frac{v_2^2 - s^2}{2(v_2 + 2s)},
\]

respectively, with total expected effort given in (B11). The laggard wins with probability

\[
\frac{v_2^2 - s^2}{2v_2 (v_2 + 2s)},
\]

in which case there is symmetry in contest 3 and total expected effort in that contest equal to \(1 - v_1 - v_2\). The leader wins contest 2 with probability

\[
1 - \frac{v_2^2 - s^2}{2v_2 (v_2 + 2s)},
\]

and the game moves to an instance of asymmetry in contest 3 with the players’ expected efforts in that contest given in (B9) and total expected efforts equal to \(1 - v_1 - v_2 - 2s\). The total expected effort across all three contests is

\[
v_1 + 3s + \frac{v_2 - s}{v_2 (v_2 + 2s)} \left( v_2^2 + sv_2 - s^2 \right) + \frac{v_2^2 - s^2}{2v_2 (v_2 + 2s)} (1 - v_1 - v_2) + \left[ 1 - \frac{v_2^2 - s^2}{2v_2 (v_2 + 2s)} \right] (1 - v_1 - v_2 - 2s) = 1.
\]

This is area IV in Figure 1 in the paper.