Creating Balance in Dynamic Competitions

Derek J. Clark and Tore Nilssen*

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Abstract

Rivals often meet many times in competitions, sometimes gaining wins and sometimes suffering losses. We consider how the win-loss pattern affects effort incentives over time, and how a principal can divide up a prize fund in order to incite the rivals to exert effort. The stage game is a two-player all-pay auction, and contestants are heterogeneous in the sense that one of them has a bias in the first contest, meaning that his efforts are more productive than those of the rival. The winner of a contest gains an advantage in future ones, making his efforts more productive; the size of this win advantage also varies between players. When the initially disadvantaged player is sufficiently able to catch up with the leader, we show how the principal can balance the competition despite the two dimensions of heterogeneity, leading to efforts which exactly match the prize fund in expectation. The optimal reward scheme involves selecting the number of contests to be used, and the prize division between them.

Keywords: balanced competition; all-pay auction; bias; catching up.

JEL codes: D74, D72

*Clark: School of Business and Economics, UiT - The Arctic University of Norway, NO-9037 Tromsø, Norway; derek.clark@uit.no. Nilssen: Department of Economics, University of Oslo, P.O. Box 1095 Blindern, NO-0317 Oslo, Norway; tore.nilssen@econ.uio.no.

1 Introduction

For organizers of dynamic competitions, where players meet again and again, it is a challenge to keep up the efforts of the players, as an early loss often means a player becomes discouraged from putting in effort later in the competition. In this paper, we explore how to best meet this challenge in the case of a sequence of all-pay auctions with two players, where one of them, at the outset, has an advantageous bias in the sense of more productive efforts than the other player. Winning a contest creates a boost before the next contest, increasing the winner's productivity before later contests; and also this win advantage differs among the players. The principal, who organizes the competition, has available a fixed prize fund to distribute across the sequence of contests with the aim of maximizing total overall expected efforts.

The key question for the principal is whether there is a way to distribute the prize fund over time that overcomes the ex-ante heterogeneity of the players. As our analysis shows, this is in fact possibl. When the ex-ante heterogeneity is small enough and the ex-ante disadvantaged player is sufficiently better at gaining a boost from winning, the problem can be fully overcome and there is a way for the principal to distribute the prize fund such that there is full rent dissipation, with total expected efforts among the players equal to the prize fund.

We find that the extent to which full rent dissipation is achievable is increased when it is possible to organize the competition over a longer sequence of contests. In addition, for the two-contest case, we offer a complete characterization of the optimal organization of the contest and find that the principal, for an intermediate range of the ex-ante heterogeneity, should put all the prize fund in the second contest, while for large such heterogeneity, the best is to have all the prize fund in the first contest, essentially closing down the second one.

The central notion for the possibility of full rent dissipation is that of a balanced contest. Of course, a symmetric contest is balanced. In order to obtain balance in the first contest when there are ex-ante asymmetries, the value of winning that contest must be greater for the underdog, which translates into the underdog having the greater gain from winning a contest. When this is the case, there is a scope for splitting the prize fund across the contests such that a balanced first contest is created.

The paper closest to ours is Clark and Nilssen (2018a) who analyze a similar setting to the present one, except that the heterogeneities there are with respect to head starts rather than productivity biases. In particular, one player has an ex-ante head start, while the players also differ in their gains from an early win. Also there, rent dissipation can be obtained through an optimum distribution of the prize budget if the ex-ante heterogeneity is small enough. Although results are similar for the two-contest case, the mechanisms differ. With head starts, cases may occur where the leading player has such a big lead that he can win without exerting effort, simply trusting his head start, and the principal must take such cases, that would entail low total expected efforts, into account when designing the competition. This issue does not show up in the present case of productivity biases. One effect of this difference is that we here are able to get out some results for longer competitions, that is, sequences of three or more contests, while the other paper is limited to the analysis of the two-contest case.

A precursor to both these papers is Clark and Nilssen (2018b), which analyzes a sequence of all-pay auctions where prizes are constant across time, and players are identical. In particular, players obtain the same head start and/or productivity boost from winning a contest. In that paper, the design issue that we focus on presently does not appear, since identical players imply that the first contest is always balanced and therefore that there is always full rent dissipation in equilibrium. Instead, the focus is on the extent to which an initial laggard will stay in the game and eventually have higher expected efforts than the leader.[1] Clark and Nilssen (2018b) find that heterogeneity between the rivals occurring throughout the competition as wins and losses are recorded, can reduce their efforts since the weaker player reduces efforts due to a perceived increase in the probability of losing, and the stronger player reduces effort as a response.[2] Counteracting this mechanism through the contest design is a key issue in the current paper.

Other papers that discuss design in heterogeneous all-pay auctions have focused on how the principal can optimally set, or reset, biases in order to maximize total expected efforts; see, e.g., Epstein, et al. (2011), Li and Yu (2012), and Franke, et al. (2018).[3] This approach differs from ours, in that we take the view that biases are fixed and not possible to adjust directly and rather explore how, in dynamic competitions, distributing the prize fund across time can affect total expected efforts. A complicating – but realistic – factor in our analysis is that rivals have different productivities of effort initially, and that these evolve at different rates as the series of contests progresses. Contestants improve their effort productivity over time in relation to the pattern of losses and wins, and this opens up for the possibility that an productivity advantage may be enhanced, neutralized or overturned in the course of the play. As Rigney (2010; p. 1) puts it, "[i]nitial advantage does not always lead to further advantage, and initial disadvantage does not always lead to further disadvantage".

Rivals often face each other several times in competitions, and the sequence of winning and losing can create a synergy between the contests.[4] A successful contestant may gain a psychological boost (Krumer, 2013). The "hot-hand" phenomenon in basketball has been widely debated since the seminal paper by Gilovich et al. (1985). Previous success is purported – by many sports commentators at least – to increase the success rate of a shooter, rendering him "hot" or "on fire". On the whole, the literature has not found evidence to support this, although recently, Miller and Sanjurjo (2017) have pointed out a bias in earlier work; correcting for the bias, they reverse the conclusions of prominent studies

[1] Clark, et al. (2018) discuss a sequence of two Tullock contests in which the winner of the first has a lower cost of exerting effort in the second compared to the rival. In that model, players are ex-ante symmetric, and the optimum for the principal is to put the prize fund into the second contest.

[2] A similar mechanism is noted in the innovation tournament of Terwiesch and Xu (2008).

[3] See also the survey by Mealem and Nitzan (2016).

[4] Sela (2017) discusses different grounds that may lead to synergies between contests. See also Kanter's (2004) account of winning streaks and losing streaks in sports and business.

that label the hot-hand a fallacy. Gauriot and Page (2018) analyze a game of tennis as a dynamic contest, looking at the effect of being ahead or being behind on the probability of winning the next point. They find a significant momentum effect for professional male tennis players in which previous win-loss patterns affect current behavior.

In some competitions, successful players may gain access to material goods that make competing easier. In sales force management, it is not uncommon for the more successful agents to be given less administrative duties, better access to back-office resources, more training than the less successful, and better territories; see, *e.g.*, Skiera and Albers (1998), Farrell and Hakstian (2001), and Krishnamoorthy, *et al.* (2005). This is a mechanism in which winning creates winners. This latter phenomenon may also be recognized by scientists who compete for research grants. As noted by Gallini and Scotchmer (2002; p. 54): "[F]uture grants are contingent upon previous success. The linkage between previous success and future funding seems even more specific in the case of the National Science Council". Those who succeed in obtaining research grants may experience an increase in status, winning a grant to fund current work which again improves their chance of winning further grants. Losing teams must use time and resources in seeking presumably inferior forms of funding; in sum, this gives an advantage in future rounds of competition for scarce research funding. In the case of innovation contests, Davis and Davis (2004; p. 20) discuss reputational gains associated with winning a contest, stating: "[T]he impact of prizes on reputation, we feel, is a much overlooked phenomenon. They alone can provide the economic justification for a sponsor to design the contest and for contestants to enter." The experience of winning creates, in other words, momentum for future contests.

The impact of losing may well lead to a laggard being discouraged and giving up (Konrad, 2012). Perc (2014; p. 1) suggests the wider implications of situations in which an initial advantage may be self-amplifying as being prevalent whether we consider "economic wealth, political power, prestige, knowledge, or in fact any other scarce or valued resource". In our model, the effort-maximizing principal seeks to design a series of contests that balances the boost from winning and the negative effect of losing; the instruments at her disposal are the number of contests used and the division of the prize mass between them. One interesting application of our model is to franchising, which is suggested by Gillis, et al. (2011) to resemble a dynamic tournament setting. Franchisees compete with each other in order to gain more units in the franchise, as a prize for good relative performance over time. Initially, one franchisee may have an advantage over rivals due to location or other factors, and the franchisor can reward the high performer with another franchise unit. This allows the winner to build a business and enjoy economies of scale, scope and/or management in future contests. The franchisor must determine when, and how large a franchise to award. In our analysis, we show that several reward schemes can induce maximal expected effort; some of these involve periods of effort in which no direct prize (franchise) is awarded, but where rivals seek to build their claims for the next prize.

The paper is organized as follows. In the next section, we do a preliminary analysis of the stage game. In Section 3, we present our model of the two-contest competition and provide a complete solution for the principal's optimum distribution of the prize fund. Section 4 extends the analysis to more than two contests but limits the discussion to finding conditions such that full rent dissipation is feasible. Section 5 offers some concluding remarks, while proofs are relegated to an Appendix.

2 Preliminaries

There are two risk-neutral players, s and w, who compete for a prize that they value at $v_s > 0$ and $v_w > 0$, respectively, by making irreversible efforts $x_s \geq 0$ and $x_w \geq 0$. The probability that player s wins the prize is

$$p_s(x_s, x_w) = \begin{cases} 1 \text{ if } \alpha_s x_s > \alpha_w x_w; \\ \frac{1}{2} \text{ if } \alpha_s x_s = \alpha_w x_w; \\ 0 \text{ if } \alpha_s x_s < \alpha_w x_w; \end{cases}$$

where $\alpha_i > 0$ is a bias parameter in favour of player $i \in \{s, w\}$, and the probability that w wins is $p_w = 1 - p_s$. We assume that $\alpha_s v_s \geq \alpha_w v_w$, implying that player s is the *stronger* one.

Let $F_i(x)$ be the cumulative distribution function of player i's mixed strategy, $i \in \{s, w\}$. The expected payoffs of the two players are given by

$$\begin{aligned} \pi_s(x_s) &= F_w\left(\frac{\alpha_s}{\alpha_w} x_s\right) v_s - x_s; \\ \pi_w(x_w) &= F_s\left(\frac{\alpha_w}{\alpha_s} x_w\right) v_w - x_w. \end{aligned}$$

The equilibrium of this game is described in the following Proposition, which is a variation of well-known results and therefore is stated without proof.[5]

Proposition 1 *The game has a unique equilibrium given by the mixed strategies*

$$\begin{aligned} F_s(x_s) &= \frac{\alpha_s}{\alpha_w v_w} x_s, \quad x_s \in \left[0, \frac{\alpha_w v_w}{\alpha_s}\right]; \\ F_w(x_w) &= 1 - \frac{\alpha_s (v_w - x_w)}{\alpha_s v_s}, \quad x_w \in [0, v_w]. \end{aligned}$$

In this equilibrium, the expected efforts of the players are

$$x_s^* = \frac{\alpha_w v_w}{2\alpha_s}, \text{ and } x_w^* = \frac{\alpha_w v_w^2}{2\alpha_s v_s}; \tag{1}$$

expected payoffs are

$$\pi_s^* = v_s - \frac{\alpha_w}{\alpha_s} v_w, \text{ and } \pi_w^* = 0;$$

and probabilities of winning are

$$p_s^* = 1 - \frac{\alpha_w v_w}{2\alpha_s v_s}, \text{ and } p_w^* = \frac{\alpha_w v_w}{2\alpha_s v_s}. \tag{2}$$

[5]It follows, for example, from a result in the online appendix of Clark and Nilssen (2018b).

From (1), we have that the total expected effort is

$$x_s^* + x_w^* = \frac{\alpha_w v_w}{\alpha_s v_s} \frac{(v_s + v_w)}{2}. \tag{3}$$

We say the contest is *balanced* when $\alpha_s v_s = \alpha_w v_w$. It follows from the above that, in a balanced contest,

$$\begin{aligned} x_s^* &= \frac{v_s}{2};\ x_w^* = \frac{v_w}{2};\ x_s^* + x_w^* = \frac{v_s + v_w}{2}; \\ \pi_s^* &= \pi_w^* = 0;\ \text{and } p_s^* = p_w^* = \frac{1}{2}. \end{aligned} \tag{4}$$

As noted in the Introduction, this notion of a balanced contest is crucial for the principal's search for full rent dissipation. Note that, in the biased contest, total expected effort (3) is a fraction of the average valuation of the prize. Balancing the contest yields expected efforts equal to the players' average valuation. We will return to this below.

The above simple all-pay auction with biases comprises the stage game of our analysis in the next sections.

3 The two-contest model

We start our analysis by completely solving the model for the case of two contests. In Section 4, we go on to discuss, for the case of a general number of contests, the extent to which the principal can obtain the maximum level of total expected efforts.

There are two risk-neutral players, $i \in \{1, 2\}$, who compete in two successive contests, $t \in \{1, 2\}$, by making irreversible efforts, $x_{i,t}$. The two players differ in two respects: in the biases they have before the game starts, and in the bias they can obtain before contest two by winning contest one. A principal has a prize mass of size one to divide between the two contests, making $1 - v$ available in the first, and v in the second.

Only efforts in the current contest affect the probability of winning, but do so according to a biased version of the all-pay auction. One of the players – player 1, without loss of generality – has a positive bias in contest one, so that the contest success function is

$$\rho_{1,1}(x_{1,1}, x_{2,1}) = \begin{cases} 1 \text{ if } bx_{1,1} > x_{2,1}; \\ \frac{1}{2} \text{ if } bx_{1,1} = x_{2,1}; \\ 0 \text{ if } bx_{1,1} < x_{2,1}; \end{cases}$$

where $b > 1$ is the bias in favour of player 1 in contest one. In contest two, the bias develops according to who has won the first contest. Should the already advantaged player 1 win the first contest, then his bias parameter is increased by a factor of $a_1 > 1$ to $a_1 b$. Should the initial laggard, player 2, win the first contest, then he has a bias parameter of $a_2 > 1$ in contest two, and player 1 retains his bias of b. Hence the probability that player 1 wins the second contest, having won the first, is

$$\rho_{1,2}(x_{1,2}, x_{2,2}; 1) = \begin{cases} 1 \text{ if } a_1 b x_{1,1} > x_{2,1}; \\ \frac{1}{2} \text{ if } a_1 b x_{1,1} = x_{2,1}; \\ 0 \text{ if } a_1 b x_{1,1} < x_{2,1}; \end{cases}$$

whilst the probability that player 1 wins the second contest after the opponent has won the first is

$$\rho_{1,2}(x_{1,2}, x_{2,2}; 2) = \begin{cases} 1 \text{ if } b x_{1,1} > a_2 x_{2,1}; \\ \frac{1}{2} \text{ if } b x_{1,1} = a_2 x_{2,1}; \\ 0 \text{ if } b x_{1,1} < a_2 x_{2,1}. \end{cases}$$

We look first at the second contest, where the prize on offer is v and there is no future contest after this. The results from Section 2 can be used to calculate expected efforts and payoffs. Suppose first that player 1 has won the first contest, in which case the bias parameter of this player is $a_1 b$. In terms of Section 2, we have player 1 as the stronger one, with $\alpha_s = a_1 b$, $\alpha_w = 1$, and $v_s = v_w = v$. From Proposition 1, we have the following equilibrium values:

$$x^*_{1,2}(1) = \frac{v}{2a_1 b}; x^*_{2,2}(1) = \frac{v}{2a_1 b} \tag{5}$$

$$x^*_{1,2}(1) + x^*_{2,2}(1) = \frac{v}{a_1 b} \tag{6}$$

$$\pi^*_{1,2}(1) = \frac{(a_1 b - 1)\, v}{a_1 b} \tag{7}$$

$$\pi^*_{2,2}(1) = 0 \tag{8}$$

where the number in brackets identifies the winner of the first contest.

When player 2 wins the first contest, we have to consider two different cases, A and B, depending on who is the stronger player in contest two.[6]

A) $a_2 \geq b$. In this case, player 2 is the stronger player, and we use Proposition 1 to get

$$x^{A*}_{1,2}(2) = \frac{bv}{2a_2}; \quad x^{A*}_{2,2}(2) = \frac{bv}{2a_2} \tag{9}$$

$$x^{A*}_{1,2}(2) + x^{A*}_{2,2}(2) = \frac{bv}{a_2} \tag{10}$$

$$\pi^{A*}_{1,2}(2) = 0 \tag{11}$$

$$\pi^{A*}_{2,2}(2) = \frac{(a_2 - b)\, v}{a_2} \tag{12}$$

B) $b \geq a_2$. Here player 1 is still the stronger player, despite losing contest one, but with less of a bias in his favour than he had earlier. Equilibrium values are,

[6] Note that, when $a_2 = b$, the two cases coincide and lead to the same outcome, such that $\pi^*_{1,2} = \pi^*_{2,2} = 0$.

by Proposition 1,

$$x_{1,2}^{B*}(2) = \frac{a_2 v}{2b}; \quad x_{2,2}^{B*}(2) = \frac{a_2 v}{2b} \tag{13}$$

$$x_{1,2}^{B*}(2) + x_{2,2}^{*}(2) = \frac{a_2 v}{b} \tag{14}$$

$$\pi_{1,2}^{B*}(2) = \frac{(b - a_2)\, v}{b} \tag{15}$$

$$\pi_{2,2}^{B*}(2) = 0 \tag{16}$$

Turning to contest one, we set up the expected payoff functions for the players, where $k \in \{A, B\}$ denotes the cases discussed above following a win for player 2 in contest one:

$$\begin{aligned}
\pi_{1,1}^{k} &= \rho_{1,1}\left(1 - v + \pi_{1,2}^{*}(1)\right) + \left(1 - \rho_{1,1}\right)\pi_{1,2}^{k*}(2) - x_{1,1} \\
&= \pi_{1,2}^{k*}(2) + \rho_{1,1}\left(1 - v + \pi_{1,2}^{*}(1) - \pi_{1,2}^{k*}(2)\right) - x_{1,1} \\
&= \pi_{1,2}^{k*}(2) + \rho_{1,1} V_{1,1}^{k} - x_{1,1}; \\
\pi_{2,1}^{k} &= \rho_{2,1}\left(1 - v + \pi_{2,2}^{k*}(2)\right) - x_{2,1} \\
&= \rho_{2,1} V_{2,1}^{k} - x_{2,1}.
\end{aligned}$$

Winning the first contest gives the stage prize of $1 - v$ and the continuation value after having won: $\pi_{1,2}^{k*}(1)$ for player 1 and $\pi_{2,2}^{k*}(2)$ for player 2, $k \in \{A, B\}$. Losing gives the promise of $\pi_{i,2}^{k*}(j)$ in the next contest. The contest success function is biased in favour of player 1, who has a bias of b. If he wins contest one, then player 2 expects a positive payoff in contest two in case A, and zero in case B; he expects zero if he loses the first contest. Player 1 expects a positive losing payoff only in case B.

Player i is guaranteed $\pi_{i,2}^{k*}(j) \geq 0$ in the first contest in case $k = \{A, B\}$; this is his expected payoff in contest two if he loses contest one. Player 2 has no positive guaranteed payoff. Each player can win a further value of $V_{i,1}^{k} = 1 - v + \pi_{i,2}^{k*}(i) - \pi_{i,2}^{k*}(j)$, which is the stage prize in contest one, $1 - v$, plus the difference in expected contest-two payoff between winning and losing contest one. This value of winning the first contest for each player for each case is

$$V_{1,1}^{A} = 1 - v + \frac{(a_1 b - 1)\, v}{a_1 b} = 1 - \frac{1}{a_1 b} v; \tag{17}$$

$$V_{2,1}^{A} = 1 - v + \frac{(a_2 - b)\, v}{a_2} = 1 - \frac{b}{a_2} v; \tag{18}$$

$$\begin{aligned}
V_{1,1}^{B} &= 1 - v + \frac{(a_1 b - 1)\, v}{a_1 b} - \frac{(b - a_2)\, v}{b} = 1 - \frac{a_1 a_2 - a_1 b - 1}{a_1 b} v; \\
V_{2,1}^{B} &= 1 - v.
\end{aligned}$$

Clearly, player 1 is the stronger of the two in case B, since $b V_{1,1}^{B} > V_{2,1}^{B}$.[7] We

[7] This can be verified since the condition amounts to $\frac{v(a_1 a_2 - 1)}{a_1} + (b - 1)(1 - v) > 0$, where each component here is at least zero, and one component is positive if the other is exactly zero.

have, from equations (2) and (3),

$$
\begin{aligned}
x_{1,1}^{B*}+x_{2,1}^{B*} &= \frac{1-v}{2b}\frac{v-2ba_1+2bva_1-va_1a_2}{v-ba_1+bva_1-va_1a_2} \\
\rho_{1,1}^{B*} &= 1-\frac{1}{2}\frac{a_1(1-v)}{ba_1-bva_1+va_1a_2-v} \\
\rho_{2,1}^{B*} &= \frac{1}{2}\frac{a_1(1-v)}{ba_1-bva_1+va_1a_2-v}
\end{aligned}
$$

Given this, we can calculate the total expected effort over the two contests for case B as

$$
\begin{aligned}
X^{B*} &= \frac{1-v}{2b}\frac{v-2ba_1+2bva_1-va_1a_2}{v-ba_1+bva_1-va_1a_2} \\
&\quad+\left(1-\frac{1}{2}\frac{a_1(1-v)}{ba_1-bva_1+va_1a_2-v}\right)\frac{v}{a_1b} \\
&\quad+\left(\frac{1}{2}\frac{a_1(1-v)}{ba_1-bva_1+va_1a_2-v}\right)\frac{a_2v}{b} \\
&= \frac{a_1-v(a_1-1)}{ba_1},
\end{aligned}
$$

which is decreasing in v, since $a_1>1$, so that $v=0$ is the optimal contest-two prize for a principal who wants to maximize total expected effort. This gives $X^{B*}(v=0)=\frac{1}{b}$. In effect, with $v=0$, a single contest is optimal.

In case B, the initial lead cannot be surpassed, and the principal can do no better than run a single contest. In case A, the initial laggard gets a bias that surpasses the leader if he wins contest one, and the following Proposition indicates that an effort-maximizing principal now can do better than running a single contest, so that the optimal prize division depends on the exact size of the parameters. The proof of the Proposition is in the Appendix.

Proposition 2 *In the two-contest model, the optimal setting of v, and the corresponding realized total expected efforts, are as follows:*

(i) If $\frac{a_2(1+a_1)}{a_1(1+a_2)}\geq b>1$, then $v^*=\frac{a_1a_2(b-1)}{a_2-ba_1}:=\widehat{v}$, with total expected efforts 1.

(ii) If $\frac{a_2}{a_1}>b>\frac{a_2(1+a_1)}{a_1(1+a_2)}$, then $v^*=1$, with total expected efforts $\frac{1}{b}+\frac{a_2-ba_1}{ba_1a_2}\in\left(\frac{1}{b},1\right)$.

(iii) If $b=\frac{a_2}{a_1}$, then $v^*\in[0,1]$, with total expected efforts $\frac{1}{b}$.

(iv) If $b>\frac{a_2}{a_1}$, then $v^*=0$, with total expected efforts $\frac{1}{b}$.

Figure 1 gives an illustration of the results, depending on the value of b. When the initial bias in favour of player 1 is sufficiently small, it is possible to use the division of the prize mass to ensure balance in the first contest so that the principal can appropriate the whole of the prize on offer. For intermediate values of b, this is not attainable, but saving the whole prize for contest two yields the most expected effort. When the lead of player 1 at the outset is too large, then the principal can do no better than to run a single (biased) contest.

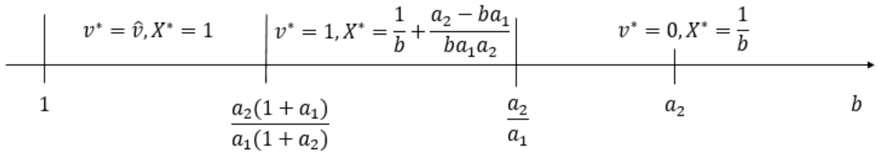

Figure 1: Illustration of Proposition 2

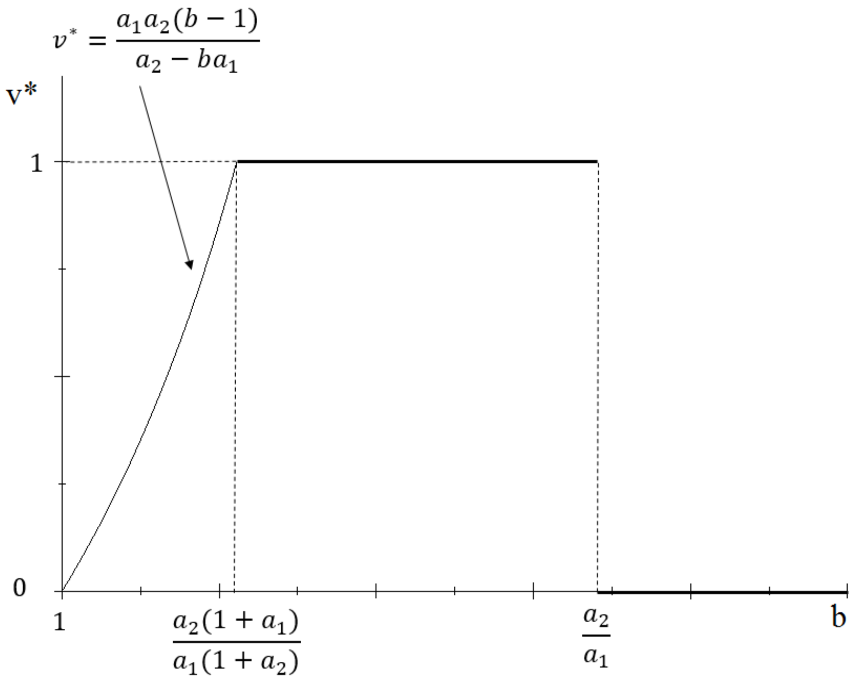

Figure 2: Optimal second contest prize

In the Appendix, we show that key to achieving full dissipation of the prize is making the first contest balanced. This requires dividing up the prize mass so that $bV_{1,1}^A = V_{2,1}^A$, giving $\widehat{v}$ as the second contest prize. The parameter restrictions in part (i) of Proposition 2 derive from the fact that, necessarily, $\widehat{v} \in [0,1]$. Figure 2 indicates how the second-contest prize depends on the initial bias in favour of player 1, increasing in this parameter until $v^* = 1$ is reached, and then falling to zero when the bias is too large.

When the first contest is balanced, each player has an equal probability of being the victor there. In the second contest, each player has valuation v of winning, and from (1), one can see that each player has the same expected effort in that contest.[8] The amount of effort in the second contest depends on the identity of the winner of the first, being $\frac{v}{a_1 b}$ if player 1 wins, and $\frac{bv}{a_2}$ if 2 wins. In the Appendix, we show that expected effort in contest one is $1 - \frac{\widehat{v}}{2ba_1} - \frac{b\widehat{v}}{2a_2}$, which is decreasing in the second period balancing prize. However expected effort in contest two is $\frac{\widehat{v}}{2}\left(\frac{1}{ba_1} + \frac{b}{a_2}\right)$, which exactly neutralizes the effect that $\widehat{v}$ has on first contest effort,

[8]From (1), one can see that $\frac{x_s^*}{x_w^*} = \frac{v_s}{v_w}$.

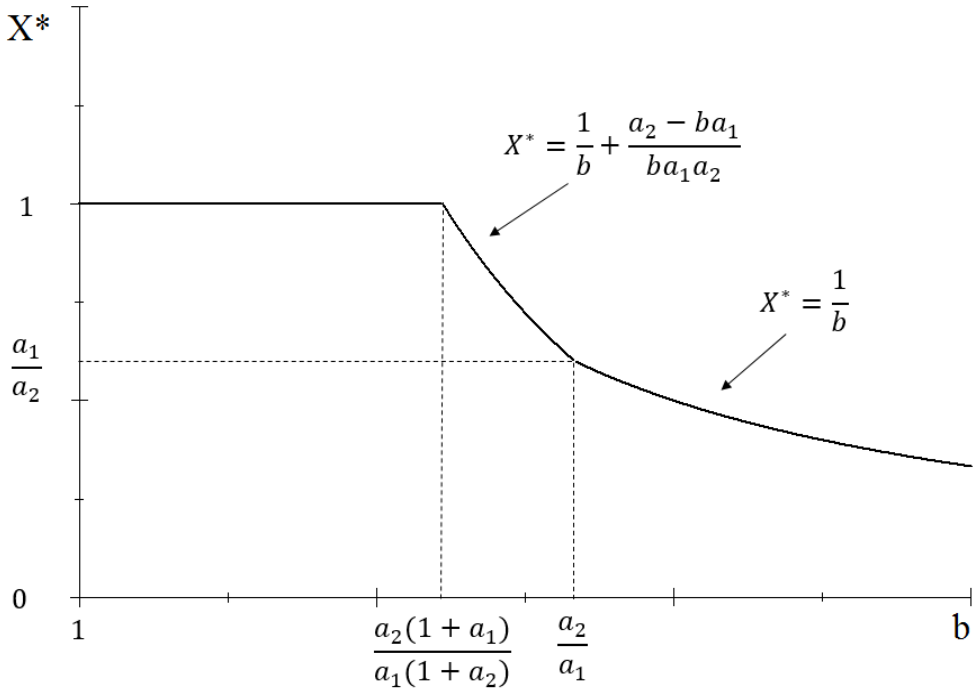

Figure 3: Maximum expected effort

leaving expected effort of 1.

With the first contest being balanced, the players compete away the full value of the total prize, each ending up with an overall payoff of zero. To see this, note that in the balanced contest for case A, $\rho_{i,1}^{A*} = \frac{1}{2}$ and $x_{i,1}^{A*} = \frac{V_{i,1}^{A}}{2}$, so that $\pi_{i,1}^{*} = \frac{V_{i,1}^{A}}{2} - \frac{V_{i,1}^{A}}{2} = 0$. This occurs for sufficiently small values of the ex-ante heterogeneity b. When $b > \frac{a_2(1+a_1)}{a_1(1+a_2)}$, it is no longer possible to balance the first contest, since doing so would require $\widehat{v} > 1$ which is not possible given the total value of the prize mass. It is well known that distributing the whole prize in contest one will give an expected effort of $\frac{1}{b}$. For intermediate values of b, part (ii) of Proposition 2 shows that it is possible to get some benefit from the possible catching up by player 2 by awarding the whole prize mass in contest two. This works as long as the catching-up parameter of player 2 is sufficiently large ($a_2 > ba_1$). We show in the Appendix that, when the first contest is not balanced, the total expected effort in the two contests is

$$\frac{v\,(a_2 - ba_1)}{ba_1a_2} + \frac{1}{b},$$

which is linear in v and depends on the sign of $a_2 - ba_1$. When $ba_1 > a_2$, the principal can do no better than setting $v = 0$, and getting the expected effort from a single contest, $\frac{1}{b}$. When $a_2 > ba_1$ then $v = 1$ gives more expected effort than $\frac{1}{b}$.

Figure 3 indicates the maximal amount of expected effort that can be achieved in total, being constant at $X^* = 1$ for sufficiently low values of b, and falling in b after this.

4 More than two contests

A complete solution for a game with more than two contests is complicated. Still, it is of interest to provide a partial characterization of the principal's optimum choice in order to see whether she, when having a longer sequence of contests available, can obtain full rent dissipation for a larger set of parameters than we have found above for the two-contest case. We answer this in the affirmative. In constructing this series of contests, we retain the assumption from the two-contest case that the initial laggard can catch up and surpass the leader. The key to full rent dissipation is again balancing the first contest. As in the two-contest case, when the first contest is balanced, the value of the game to each player is zero, since they compete away the full value of the prize mass in expectation. We present below a condition such that, in the general case, the first contest is balanced and there is full rent dissipation across the sequence of contests. The proof is in the Appendix.

Proposition 3 *Suppose that $b \leq \frac{a_2}{a_1}$, that the sequence consists of $T \geq 2$ contests, and that the principal allocates her total prize fund of 1 over the T contests such that the prize in contest $t \in \{1, ..., T\}$ is $v_t \geq 0$, and $\sum_{t=1}^{T} v_t = 1$. The first contest in the series is balanced, and the total expected effort equals the total prize fund, when*

$$v_1 (b-1) = \sum_{t=2}^{T} \phi_t v_t, \text{ where} \tag{19}$$

$$\phi_t = \frac{a_1^t - 1}{a_1^{t-1} (a_1 - 1)} - b \frac{(a_2^t - 1)}{a_2^{t-1} (a_2 - 1)}, t = 2, ..., T. \tag{20}$$

When $T = 2$, we see, from (20), that $\phi_2 > 0$ for $\frac{a_2(1+a_1)}{a_1(1+a_2)} \geq b$, consistent with part (i) of Proposition 2. Since the left-hand-side of (19) is at least zero, balance requires that the right-hand-side be non-negative. If $T = 2$, and $\frac{a_2(1+a_1)}{a_1(1+a_2)} < b$, then $\phi_2 < 0$, and a series of two contests cannot be balanced. Consider the case of three contests, $T = 3$, and suppose $b > \frac{a_2(a_1+1)}{a_1(a_2+1)}$. Now, $\phi_2 < 0$, which, from (19), goes against the aim of balancing the first contest, unless the principal annuls it by putting $v_2 = 0$. Instead, in order to obtain balance in contest 1, the principal must have a non-negative ϕ_3, which requires

$$b \leq \frac{1 + \frac{1}{a_1} + \frac{1}{a_1^2}}{1 + \frac{1}{a_2} + \frac{1}{a_2^2}} = \frac{a_2^2 (a_1 + a_1^2 + 1)}{a_1^2 (a_2 + a_2^2 + 1)}.$$

Thus, for

$$\frac{a_2 (a_1 + 1)}{a_1 (a_2 + 1)} < b \leq \frac{a_2^2 (a_1 + a_1^2 + 1)}{a_1^2 (a_2 + a_2^2 + 1)},$$

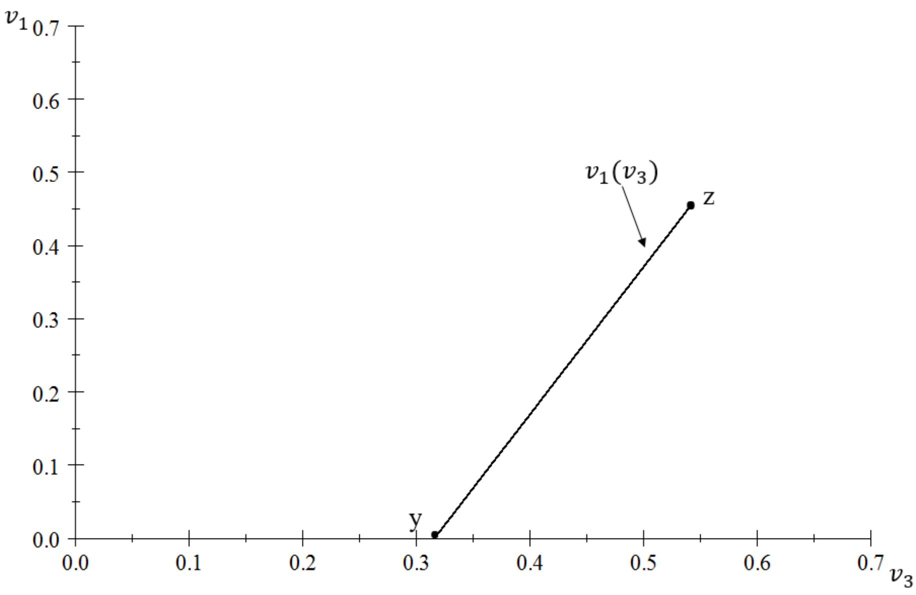

Figure 4: Reward schemes for $T = 3, b = 1.3, a_1 = 1.2, a_2 = 2$

the principal can obtain a balanced first contest by putting $v_2 = 0$ and some $1 \geq v_3 \geq 0$. The solution to (19) in this case is

$$\begin{aligned} v_3 &= \frac{a_1^2 a_2^2 (b-1)}{a_2^2 (1+a_1) - b a_1^2 (1+a_2)}, \\ v_1 &= 1 - v_3, \\ v_2 &= 0. \end{aligned}$$

There is, of course, no unique reward scheme for achieving full rent dissipation here, since any combination of non-negative prizes v_1, v_2, and v_3 that sum to 1 and solve (19) will work. As an example, consider $a_1 = 1.2, a_2 = 2, b = 1.3$; this example is pursued later. Supposing that $T = 3$, one can calculate $\phi_2 = -0.11667$, and $\phi_3 = 0.25278$, so that balance in the first contest requires

$$0.3 v_1 = -0.11667 v_2 + 0.25278 v_3; \text{ and} \tag{21}$$

$$v_1 + v_2 + v_3 = 1. \tag{22}$$

Figure 4 depicts combinations of first-contest and third-contest prizes in (v_3, v_1) space. The $v_1(v_3)$ line in the figure is the locus of combinations of the two prizes that satisfy the two equations (21) and (22) in addition to the non-negativity constraints $v_t \geq 0$, $t \in \{1, 2, 3\}$. At point y in the figure, the first-contest prize is at zero and we have the prize vector $(v_1 = 0, v_2 = 0.68, v_3 = 0.32)$. At point z, it is the second-contest prize that is at zero, and the prize vector is $(v_1 = 0.46, v_2 = 0, v_3 = 0.54)$. In between the two extremes, the third-contest prize moves in the range $[0.32, 0.54]$.

When $b > \frac{a_2^2\left(a_1+a_1^2+1\right)}{a_1^2\left(a_2+a_2^2+1\right)}$, three contests will not achieve balance, since $\phi_2 < 0$ and $\phi_3 < 0$, and the principal can consider adding a fourth contest. One can calculate that $\phi_4 > 0$ for

$$b \leq \frac{1+\frac{1}{a_1}+\frac{1}{a_1^2}+\frac{1}{a_1^3}}{1+\frac{1}{a_2}+\frac{1}{a_2^2}+\frac{1}{a_2^3}} = \frac{a_2^3\left(a_1^3+a_1^2+a_1+1\right)}{a_1^3\left(a_2^3+a_2^2+a_2+1\right)} = \frac{a_2^3\left(a_1+1\right)\left(a_1^2+1\right)}{a_1^3\left(a_2+1\right)\left(a_2^2+1\right)}.$$

This restriction does not necessarily obey the imposed $b < \frac{a_2}{a_1}$. We can, however, state that, for $T = 4$, the principal can get full rent dissipation for any

$$b \leq \min\left\{\frac{a_2}{a_1}, \frac{a_2^3\left(a_1+1\right)\left(a_1^2+1\right)}{a_1^3\left(a_2+1\right)\left(a_2^2+1\right)}\right\}$$

by putting $v_2 = v_3 = 0$ and putting some $v_4 > 0$, following the previous logic.

Note that the critical value of b making ϕ_t positive increases in t, and that it can be expressed as the ratio between two geometrical series. We see that, as we increase the number of contests, the restriction gets weaker and weaker, modulo the other restriction of $b < \frac{a_2}{a_1}$. We have that $\phi_t > 0$ for

$$b < \frac{1+\frac{1}{a_1}+...+\frac{1}{a_1^{t-1}}}{1+\frac{1}{a_2}+...+\frac{1}{a_2^{t-1}}} = \frac{\frac{1-\frac{1}{a_1^t}}{1-\frac{1}{a_1}}}{\frac{1-\frac{1}{a_2^t}}{1-\frac{1}{a_2}}} = \frac{\left(a_1-\frac{1}{a_1^{t-1}}\right)\left(a_2-1\right)}{\left(a_2-\frac{1}{a_2^{t-1}}\right)\left(a_1-1\right)} := \theta_t$$

Hence, the series of critical values for the initial bias parameter θ_t that makes ϕ_t positive increases as each new contest is added.[9] In principle, this means that, for a value of b, the principal can add contests to the series until $\phi_T > 0$ to balance the first contest – up to the point where ϕ_T surpasses $\frac{a_2}{a_1}$, which is the upper restriction on b. A simple reward scheme for achieving balance is then to only place prize weight on the first and last prize, stated formally as the following Corollary:

Corollary 1 *With a series of T contests, the principal obtains full rent dissipation by distributing the prize fund strictly between the two contests 1 and T, keeping v_2 through v_{T-1} at zero, as long as*

$$b \leq \min\left\{\frac{a_2}{a_1}, \frac{\left(a_1-\frac{1}{a_1^{T-1}}\right)\left(a_2-1\right)}{\left(a_2-\frac{1}{a_2^{T-1}}\right)\left(a_1-1\right)}\right\}. \tag{23}$$

Setting $v_2, ...,v_{T-1} = 0$, $v_1 \geq 0$, $v_T \geq 1$, and $v_1 + v_T = 1$ into (19) gives a reward scheme that secures full expected prize dissipation:

$$\begin{aligned} v_T &= \frac{b-1}{b-1+\phi_T}; \\ v_1 &= \frac{\phi_T}{b-1+\phi_T}; \\ v_2, ..., v_{T-1} &= 0; \end{aligned} \tag{24}$$

[9] Also $\frac{\partial \theta_t}{\partial a_1} < 0$ and $\frac{\partial \theta_t}{\partial a_2} > 0$.

where ϕ_T is defined in (20).

Note, from (20), that $\frac{\partial \phi_t}{\partial t} > 0$. By inserting from (20) in (24), we can derive the following comparative-statics properties of the final-contest prize:

$$\frac{\partial v_T}{\partial b} > 0; \; \frac{\partial v_T}{\partial a_1} > 0; \; \frac{\partial v_T}{\partial a_2} < 0; \; \frac{\partial v_T}{\partial T} < 0.$$

If the initial bias in favor of player 1 increases, and/or if the gain to this player from winning increases, then the final contest prize should be increased, as long as (23) is still satisfied. Saving the prize mass until later encourages the laggard to stay in the game, fighting for the possibility of winning a large reward at the end of the series. The more the disadvantaged player gains from winning a contest, the more of the prize mass it is optimal to have early. This gives the initial laggard a large incentive to win early, catching up and surpassing the initial leader. As the number of contests in the series becomes larger, the principal should give a larger share of the spoils early to balance the contest. This is easily seen from (19), since ϕ_T is increasing in T and all prizes from v_2 through v_{T-1} are zero in this particular reward scheme. The principal induces most effort in the first contest, since many of the following contests are simply for position, with a modest prize in the end. Note, however, that the final prize is always positive.

There is an interesting interplay between the comparative-statics effects noted above. Ceteris paribus, increasing the initial bias b makes it optimal to shift prize mass to late in the series. However, this also increases the number of contests that must be used in order to achieve full rent dissipation, which lowers the optimal final prize. This can be illustrated by a numerical example in which $a_1 = 1.2$, and $a_2 = 2$, and where we vary the initial bias b and the number of contests T; at $b = 1.3$ and $T = 3$, this example is identical to the one used in conjunction with Figure 4 above. In the following table, for each row, a "+" indicates the contest in which ϕ_t turns positive.

b	ϕ_2	ϕ_3	ϕ_4	ϕ_5	v_T
1.1	+				0.35
1.3	−	+			0.54
1.5	−	−	+		0.63
1.6	−	−	+		0.85
1.66	−	−	−	+	0.64

Two contests can be used for a low value of the initial bias ($b = 1.1$), with most of the prize mass given in the first contest. With a bias of $b = 1.3$, three contests are utilized and more than half of the prize is distributed in the final contest. Four contests are used for biases of 1.5 and 1.6, with the final prize increasing in b. When the initial bias is at 1.66, which is close to its top level at $\frac{a_2}{a_1} = \frac{2}{1.2} = \frac{5}{3}$, five contests are necessary, and here we see that the amount of prize that is awarded late falls from 0.85 with $b = 1.6$ to 0.64 with the higher bias parameter. As discussed above, the increase in the bias parameter tends to increase the late prize, whereas the fact that it is awarded one contest later reduces the late prize.

Note that, as the number T of contests goes to infinity, the condition in (23) becomes

$$b \le \min\left\{\frac{a_2}{a_1}, \frac{a_1(a_2-1)}{a_2(a_1-1)}\right\}.$$

When $\frac{a_2}{a_1} > \frac{a_1(a_2-1)}{a_2(a_1-1)}$, i.e. $a_2 > \frac{a_1}{a_1-1}$, the critical value θ_T never quite reaches the upper bound for b of $\frac{a_2}{a_1}$ as T goes to infinity, and the reward scheme in Corollary 1 will not work for $b > \frac{a_1(a_2-1)}{a_2(a_1-1)}$. In other cases, $b = \frac{a_2}{a_1}$ will be the upper limit. For the lowest value of T such that $\theta_T > \frac{a_2}{a_1}$, it will not be possible to add more contests to induce full prize dissipation.

As noted, the simple reward scheme in Corollary 1 is not the unique one that balances the first contest. But there is no other reward scheme that achieves this goal by using fewer contests in the series. To see this, consider (19). If the left-hand side is positive (i.e. $v_1 > 0$), then the right-hand side must also be. Hence contests must be added until we find T such that $\phi_T > 0$, just as in the simple scheme above. If $v_1 = 0$, then the right-hand-side must sum to zero, so that the early negative values of ϕ_t must be canceled out by the first positive one, just as above. Hence, we can state the following:

Corollary 2 *The reward scheme in which* $v_1 > 0$*,* $v_T > 0$*,* $v_1 + v_T = 1$*, and* $v_2, ..., v_{T-1} = 0$*, achieves balance in contest* 1 *with the fewest number of contests possible.*

It is not difficult to construct examples in which a reward scheme uses more contests than that in Corollary 2. Suppose for instance that the principal wants to divide the prize mass as equally as possible across contests, whilst still maintaining balance, so that all contest prizes after the first are of equal value v, whilst the prize in contest one is $v_1 > 0$, and $v_1 + (T-1)v = 1$. The condition for balance in this case is

$$v_1(b-1) = v\sum_{t=2}^{T}\phi_t,$$

requiring that $\sum_{t=2}^{T}\phi_t > 0$. As an illustration, return to the numerical example above, and put $b = 1.5$. The condition $\sum_{t=2}^{T}\phi_t > 0$ is not fulfilled for four contests, so the principal must use five in this instance, one more than discussed above. Contest 1 is balanced in this case for $T = 5$, $v_1 = 0.188$, and $v_2 = v_3 = v_4 = v_5 = 0.203$. A more even distribution of the prize mass thus requires more contests in order to preserve balance.

5 Conclusion

Rivals often face each other in competition repeatedly, and the experience of winning or losing is suggested to affect future competitions. A winner may gain some psychological momentum, or some material goods that makes competing relatively easier. This belies the saying that "success breeds success", or "winning makes winners". In such a situation, it is easy to think that a laggard may simply give up, rendering the competition as a futile method for inducing effort.

We have looked at conditions under which a principal may divide a prize mass among several contests in order to induce effort, when strength or ability evolves according to previous wins and losses. The synergy created between the contests in the series can be exploited in order to achieve full rent dissipation even if the rivals are not equally strong at the outset.

In order to derive this result we have shown that the initial laggard must have the possibility of catching up and surpassing his rival's strength in the course of the play. We have characterized the result completely for the case of two interlinked contests. We have also presented conditions under which the principal can expect the whole value of the prize to be dissipated in longer series. To achieve this, the principal must set the optimal number of contests, and set an appropriate prize division across these contests. We have shown that a particularly simple scheme can be used, and that this minimizes the number of contests needed in the series. The principal should just place weight on the first and the last prize in the series, rendering intermediate rounds as contests for position to win the final prize.

In some applications, such awarding of prizes may not be possible; researchers compete yearly for grants, for example, and not one every few years. If the principal has access to a more or less even stream of financing for the prize (a research council with a yearly budget for example), we have demonstrated that balance can still be achieved – with ensuing full prize dissipation in expectation – albeit at the cost of having to use more contests.

A Appendix

Proof of Proposition 2

Consider case A in the text, and part (i) of the Proposition. When

$$bV_{1,1}^{A} = V_{2,1}^{A}, \tag{A1}$$

contest 1 is balanced, and each player has an equal chance of winning. From equations (17), (18), and (A1), we see that we obtain balance in contest one by putting

$$v = \widehat{v} := \frac{a_1 a_2 (b-1)}{a_2 - b a_1}.$$

Because of the restriction $v \in [0,1]$, we can only have balance in contest one if $\widehat{v} \in [0,1]$, i.e., if $\frac{a_2(1+a_1)}{a_1(1+a_2)} > b > 1$.

With balance in contest 1, total expected effort in that contest, by equations (4), (17), and (18), is

$$\frac{V_{1,1}^{A} + V_{2,1}^{A}}{2} = \frac{1}{2}\left(\frac{ba_1 - v}{ba_1} + \frac{a_2 - bv}{a_2}\right),$$

and win probabilities are $\frac{1}{2}$ for each player. If player 1 wins contest 1, then total expected effort in contest 2 is $\frac{v}{a_1 b}$, by equation (6). If player 2 wins contest 1, then total expected effort in contest 2 is $\frac{bv}{a_2}$, by equation (10). Thus, total expected efforts across the two contests, when contest 1 is balanced, is

$$\frac{1}{2}\left(\frac{ba_1 - v}{ba_1} + \frac{a_2 - bv}{a_2} + \frac{v}{a_1 b} + \frac{bv}{a_2}\right) = 1.$$

This is obviously the best the principal can obtain, so the principal's choice is to get contest one balanced by putting $v^* = \widehat{v} := \frac{a_1 a_2 (b-1)}{a_2 - ba_1}$.

Consider next part (ii) of the Proposition. Now, $\widehat{v} > 1$. This means that it can never be the case that player 2 is the strong player in contest 1, since this would require $v \geq \widehat{v}$. Thus, in the present case, we have $0 \leq v < \widehat{v}$, so that $bV_{1,1}^A > V_{2,1}^A$, with player 1 being the strong player in contest 1. By equations (3), (17), and (18), total expected effort in contest 1 is now

$$\frac{V_{2,1}^A \left(V_{1,1}^A + V_{2,1}^A\right)}{2bV_{1,1}^A} = \frac{1}{2}\left(1 - \frac{bv}{a_2}\right)\left(\frac{1}{b} + \frac{a_1 (a_2 - bv)}{a_2 (ba_1 - v)}\right),$$

while the win probability of the weak player 2 in contest 1, by equation (2), is

$$\frac{V_{2,1}^A}{2bV_{1,1}^A} = \frac{a_1 (a_2 - bv)}{2a_2 (ba_1 - v)}.$$

By again using equations (6) and (10), we have that total expected effort over the two contests is

$$\begin{aligned} & \frac{1}{2}\left(1 - \frac{bv}{a_2}\right)\left(\frac{1}{b} + \frac{a_1 (a_2 - bv)}{a_2 (ba_1 - v)}\right) \\ & + \left(1 - \frac{a_1 (a_2 - bv)}{2a_2 (ba_1 - v)}\right)\frac{v}{ba_1} + \frac{a_1 (a_2 - bv)}{2a_2 (ba_1 - v)}\frac{bv}{a_2} \\ = & \frac{v(a_2 - ba_1) + a_1 a_2}{ba_1 a_2}. \end{aligned}$$

This is linear in v, and depends upon the sign of $a_2 - ba_1$. When $b > \frac{a_2}{a_1}$, we are in part (iv) of the Proposition and it is optimal to set $v = 0$, with a total expected effort of $\frac{1}{b}$. Here in part (ii), $b < \frac{a_2}{a_1}$, and the expression is increasing in v, which should be set at the top of its range, i.e., at $v^* = 1$, for a total expected effort of $\frac{1}{b} + \frac{a_2 - ba_1}{ba_1 a_2} > \frac{1}{b}$. In the knife-edge case of $b = \frac{a_2}{a_1}$, the principal is indifferent, since any $v \in [0, 1]$ makes contest 1 balanced. ■

Proof of Proposition 3

Consider a sequence of T contests with prizes $(v_1, ..., v_T)$. Let the history of the game up until contest $t \in \{2, ..., T\}$ be summarized by (m_t, n_t), where m_t (n_t) is the number of wins so far by player 1 (2), and $m_t + n_t = t - 1$. We extend the notation from Section 3 to let the payoff to player i in contest t be given by $\pi_{i,t}^* (m_t, n_t)$.

With this notation, we can express the requirement of a balanced first contest. From Section 2, we have that a contest is balanced when $\alpha_s v_s = \alpha_w v_w$. Player 1 is the strong player in contest 1, since he has an ex-ante bias. We have $\alpha_s = b$, $\alpha_w = 1$, $v_s = v_1 + \pi^*_{1,2}(1,0)$, and $v_w = v_1 + \pi^*_{2,2}(0,1)$. Thus, contest 1 is balanced when $b\left(v_1 + \pi^*_{1,2}(1,0)\right) = v_1 + \pi^*_{2,2}(0,1)$, or:

$$v_1(b-1) = \pi^*_{2,2}(0,1) - b\pi^*_{1,2}(1,0). \tag{A2}$$

Consider a contest $t \in \{2, ..., T\}$ where player 1 is strong whenever $m_t > n_t$, whilst player 2 is strong whenever $m_t \leq n_t$.[10] If $m_t > n_t + 1$, then player 2 only fights for the stage prize at t, and not for the continuation value of being the leader after contest t, since player 1 will be strong in contest $t+1$ whatever the outcome in contest t. Hence, following Proposition 1, we have

$$\begin{aligned}\pi^*_{1,t}(m_t, n_t)|_{m_t > n_t+1} &= v_t + \pi^*_{1,t+1}(m_t+1, n_t)|_{m_t>n_t+1} - \frac{a_2^{n_t}}{a_1^{m_t} b} v_t \\ &= v_t\left(\frac{a_1^{m_t} b - a_2^{n_t}}{a_1^{m_t} b}\right) + \pi^*_{1,t+1}(m_t+1, n_t)|_{m_t>n_t+1}.\end{aligned}$$

In turn, working out $\pi^*_{i,t+1}(m_t+1, n_t)|_{m_t>n_t+1}$ reveals that player 1 is strong at each continuation, and we can by backward recursion find

$$\pi^*_{1,t}(m_t, n_t)|_{m_t>n_t+1} = \sum_{k=t}^{T} v_k \left(\frac{a_1^{m_t+k-t} b - a_2^{n_t}}{a_1^{m_t+k-t} b}\right). \tag{A3}$$

Similarly, if $m_t < n_t$, then player 2 is strong in contest $t+1$, whatever the outcome of contest t. By the same reasoning as above,

$$\pi^*_{2,t}(m_t, n_t)|_{m_t<n_t} = \sum_{k=t}^{T} v_k \left(\frac{a_2^{n_t+k-t} - a_1^{m_t} b}{a_2^{n_t+k-t}}\right). \tag{A4}$$

Hence, we can write the value for player 2 of winning contest 1, with $t = 2$, $n_2 = 1$, and $m_2 = 0$, as

$$\pi^*_{2,2}(0,1) = \sum_{k=2}^{T} v_k \left(\frac{a_2^{k-1} - b}{a_2^{k-1}}\right). \tag{A5}$$

Consider next the value for player 1 of winning contest 1, $\pi^*_{1,2}(1,0)$, the determination of which follows

$$\pi^*_{1,2}(1,0) = v_2 + \pi^*_{1,3}(2,0) - \frac{1}{a_1 b}\left(v_2 + \pi^*_{2,3}(1,1)\right),$$

[10] Analysis of the two-contest case has shown that full dissipation is achievable only if the initial laggard can catch up to a sufficient degree. The assumption made here ensures this. Restricting attention to this case is not limiting. To see this, note that we could have considered the case where player 1 is strong in contest t if and only if $m_t > n_t + 1$, the case where player 1 is strong in contest t if and only if $m_t > n_t + 2$, and so on, as far as it is appropriate. All these cases would involve the calculations we present below. The important point for the result is that player 2 is strong if each have won equally many times; such catching up is necessary for the principal to be able to balance the first contest.

where, by (A3), we can write

$$\pi_{1,3}^{*}(2,0)=\sum_{k=3}^{T}v_{k}\left(\frac{a_{1}^{k-1}b-1}{a_{1}^{k-1}b}\right).$$

Thus,

$$\pi_{1,2}^{*}(1,0)=\sum_{k=2}^{T}v_{k}\left(\frac{a_{1}^{k-1}b-1}{a_{1}^{k-1}b}\right)-\frac{1}{a_{1}b}\pi_{2,3}^{*}(1,1). \tag{A6}$$

The challenge is now to find an expression for $\pi_{2,3}^{*}(1,1)$, since the continuation value here is positive for the player who wins contest 2. We write

$$\pi_{2,3}^{*}(1,1)=v_{3}+\pi_{2,4}^{*}(1,2)-\frac{a_{1}b}{a_{2}}\left(v_{3}+\pi_{1,4}^{*}(2,1)\right),$$

where, by (A4),

$$\pi_{2,4}^{*}(1,2)=\sum_{k=4}^{T}v_{k}\left(\frac{a_{2}^{k-2}-a_{1}b}{a_{2}^{k-2}}\right).$$

Hence,

$$\pi_{2,3}^{*}(1,1)=\sum_{k=3}^{T}v_{k}\left(\frac{a_{2}^{k-2}-a_{1}b}{a_{2}^{k-2}}\right)-\frac{a_{1}b}{a_{2}}\pi_{1,4}^{*}(2,1).$$

Continuing, we have

$$\begin{aligned}\pi_{1,4}^{*}(2,1) &= v_{4}+\pi_{1,5}^{*}(3,1)-\frac{a_{2}}{a_{1}^{2}b}\left(v_{4}+\pi_{2,5}^{*}(2,2)\right)\\ &= \sum_{k=4}^{T}v_{k}\left(\frac{a_{1}^{k-2}b-a_{2}}{a_{1}^{k-2}b}\right)-\frac{a_{2}}{a_{1}^{2}b}\pi_{2,5}^{*}(2,2),\end{aligned}$$

where

$$\begin{aligned}\pi_{2,5}^{*}(2,2) &= v_{5}+\pi_{2,6}^{*}(2,3)-\frac{a_{1}^{2}b}{a_{2}^{2}}\left(v_{5}+\pi_{1,6}^{*}(3,2)\right)\\ &= \sum_{k=5}^{T}v_{k}\left(\frac{a_{2}^{k-3}-a_{1}^{2}b}{a_{2}^{k-3}}\right)-\frac{a_{1}^{2}b}{a_{2}^{2}}\pi_{1,6}^{*}(3,2),\end{aligned}$$

$$\begin{aligned}\pi_{1,6}^{*}(3,2) &= v_{6}+\pi_{1,7}^{*}(4,2)-\frac{a_{2}^{2}}{a_{1}^{3}b}\left(v_{6}+\pi_{2,7}^{*}(3,3)\right)\\ &= \sum_{k=6}^{T}v_{k}\left(\frac{a_{1}^{k-3}b-a_{2}^{2}}{a_{1}^{k-3}b}\right)-\frac{a_{2}^{2}}{a_{1}^{3}b}\pi_{2,7}^{*}(3,3),\end{aligned}$$

and

$$\begin{aligned}\pi_{2,7}^{*}(3,3) &= v_{7}+\pi_{2,8}^{*}(3,4)-\frac{a_{1}^{3}b}{a_{2}^{3}}\left(v_{7}+\pi_{1,8}^{*}(4,3)\right)\\ &= \sum_{k=7}^{T}v_{k}\left(\frac{a_{2}^{k-4}-a_{1}^{3}b}{a_{2}^{k-4}}\right)-\frac{a_{1}^{3}b}{a_{2}^{3}}\pi_{1,8}^{*}(4,3).\end{aligned}$$

Substituting in these expressions, we have

$$\begin{aligned}
\pi^*_{2,3}(1,1) &= \sum_{k=3}^{T} v_k \left(\frac{a_2^{k-2}-a_1 b}{a_2^{k-2}}\right) - \frac{a_1 b}{a_2}\pi^*_{1,4}(2,1) \\
&= \sum_{k=3}^{T} v_k \left(\frac{a_2^{k-2}-a_1 b}{a_2^{k-2}}\right) - \frac{a_1 b}{a_2}\left(\sum_{k=4}^{T} v_k \left(\frac{a_1^{k-2}b-a_2}{a_1^{k-2}b}\right) - \frac{a_2}{a_1^2 b}\pi^*_{2,5}(2,2)\right) \\
&= \sum_{k=3}^{T} v_k \left(\frac{a_2^{k-2}-a_1 b}{a_2^{k-2}}\right) - \frac{a_1 b}{a_2}\sum_{k=4}^{T} v_k \left(\frac{a_1^{k-2}b-a_2}{a_1^{k-2}b}\right) + \frac{a_1 b}{a_2}\frac{a_2}{a_1^2 b}\pi^*_{2,5}(2,2) \\
&= \sum_{k=3}^{T} v_k \left(\frac{a_2^{k-2}-a_1 b}{a_2^{k-2}}\right) - \frac{a_1 b}{a_2}\sum_{k=4}^{T} v_k \left(\frac{a_1^{k-2}b-a_2}{a_1^{k-2}b}\right) \\
&\quad + \frac{a_1 b}{a_2}\frac{a_2}{a_1^2 b}\left(\sum_{k=5}^{T} v_k \left(\frac{a_2^{k-3}-a_1^2 b}{a_2^{k-3}}\right) - \frac{a_1^2 b}{a_2^2}\pi^*_{1,6}(3,2)\right) \\
&= \sum_{k=3}^{T} v_k \left(\frac{a_2^{k-2}-a_1 b}{a_2^{k-2}}\right) - \frac{a_1 b}{a_2}\sum_{k=4}^{T} v_k \left(\frac{a_1^{k-2}b-a_2}{a_1^{k-2}b}\right) \\
&\quad + \frac{a_1 b}{a_2}\frac{a_2}{a_1^2 b}\sum_{k=5}^{T} v_k \left(\frac{a_2^{k-3}-a_1^2 b}{a_2^{k-3}}\right) - \frac{a_1 b}{a_2}\frac{a_2}{a_1^2 b}\frac{a_1^2 b}{a_2^2}\pi^*_{1,6}(3,2) \\
&= \sum_{k=3}^{T} v_k \left(\frac{a_2^{k-2}-a_1 b}{a_2^{k-2}}\right) - \frac{a_1 b}{a_2}\sum_{k=4}^{T} v_k \left(\frac{a_1^{k-2}b-a_2}{a_1^{k-2}b}\right) \\
&\quad + \frac{a_1 b}{a_2}\frac{a_2}{a_1^2 b}\sum_{k=5}^{T} v_k \left(\frac{a_2^{k-3}-a_1^2 b}{a_2^{k-3}}\right) - \frac{a_1 b}{a_2}\frac{a_2}{a_1^2 b}\frac{a_1^2 b}{a_2^2}\sum_{k=6}^{T} v_k \left(\frac{a_1^{k-3}b-a_2^2}{a_1^{k-3}b}\right) \\
&\quad + \frac{a_1 b}{a_2}\frac{a_2}{a_1^2 b}\frac{a_1^2 b}{a_2^2}\frac{a_2^2}{a_1^3 b}\pi^*_{2,7}(3,3) \\
&= \sum_{k=3}^{T} v_k \left(\frac{a_2^{k-2}-a_1 b}{a_2^{k-2}}\right) - \frac{a_1 b}{a_2}\sum_{k=4}^{T} v_k \left(\frac{a_1^{k-2}b-a_2}{a_1^{k-2}b}\right) \\
&\quad + \frac{1}{a_1}\sum_{k=5}^{T} v_k \left(\frac{a_2^{k-3}-a_1^2 b}{a_2^{k-3}}\right) - \frac{a_1 b}{a_2^2}\sum_{k=6}^{T} v_k \left(\frac{a_1^{k-3}b-a_2^2}{a_1^{k-3}b}\right) \\
&\quad + \frac{1}{a_1^2}\left(\sum_{k=7}^{T} v_k \left(\frac{a_2^{k-4}-a_1^3 b}{a_2^{k-4}}\right) - \frac{a_1^3 b}{a_2^3}\pi^*_{1,8}(4,3)\right),
\end{aligned}$$

and, eventually,

$$\begin{aligned}
\pi^*_{2,3}(1,1) &= \sum_{k=3}^{T} v_k \left(\frac{a_2^{k-2}-a_1 b}{a_2^{k-2}}\right) - \frac{a_1 b}{a_2}\sum_{k=4}^{T} v_k \left(\frac{a_1^{k-2}b-a_2}{a_1^{k-2}b}\right) + \frac{1}{a_1}\sum_{k=5}^{T} v_k \left(\frac{a_2^{k-3}-a_1^2 b}{a_2^{k-3}}\right) \\
&\quad - \frac{a_1 b}{a_2^2}\sum_{k=6}^{T} v_k \left(\frac{a_1^{k-3}b-a_2^2}{a_1^{k-3}b}\right) + \frac{1}{a_1^2}\sum_{k=7}^{T} v_k \left(\frac{a_2^{k-4}-a_1^3 b}{a_2^{k-4}}\right) - \frac{a_1 b}{a_2^3}\pi^*_{1,8}(4,3). \quad \text{(A7)}
\end{aligned}$$

The process continues until we reach T.

Inserting from equations (A5), (A6), and (A7), we can write the right-hand side of (A2) as

$$\pi^*_{2,2}(0,1) - b\pi^*_{1,2}(1,0) = \sum_{k=2}^{T} v_k \left(\frac{a_2^{k-1} - b}{a_2^{k-1}}\right)$$

$$- b\left(\sum_{k=2}^{T} v_k \left(\frac{a_1^{k-1}b - 1}{a_1^{k-1}b}\right) - \frac{1}{a_1 b}\sum_{k=3}^{T} v_k \left(\frac{a_2^{k-2} - a_1 b}{a_2^{k-2}}\right) + \frac{1}{a_2}\sum_{k=4}^{T} v_k \left(\frac{a_1^{k-2}b - a_2}{a_1^{k-2}b}\right)\right.$$

$$\left. - \frac{1}{a_1^2 b}\sum_{k=5}^{T} v_k \left(\frac{a_2^{k-3} - a_1^2 b}{a_2^{k-3}}\right) + \frac{1}{a_2^2}\sum_{k=6}^{T} v_k \left(\frac{a_1^{k-3}b - a_2^2}{a_1^{k-3}b}\right) - \frac{1}{a_1^3 b}\sum_{k=7}^{T} v_k \left(\frac{a_2^{k-4} - a_1^3 b}{a_2^{k-4}}\right) + ...\right),$$

or, collecting terms,

$$\pi^*_{2,2}(0,1) - b\pi^*_{1,2}(1,0) =$$
$$v_2\left(\frac{a_2 - b}{a_2} - \frac{a_1 b - 1}{a_1}\right) + v_3\left(\frac{a_2^2 - b}{a_2^2} - \frac{a_1^2 b - 1}{a_1^2} + \frac{1}{a_1}\left(\frac{a_2 - a_1 b}{a_2}\right)\right)$$
$$+ v_4\left(\frac{a_2^3 - b}{a_2^3} - \frac{a_1^3 b - 1}{a_1^3} + \frac{1}{a_1}\left(\frac{a_2^2 - a_1 b}{a_2^2}\right) - \frac{1}{a_2}\frac{a_1^2 b - a_2}{a_1^2}\right) +$$
$$v_5\left(\frac{a_2^4 - b}{a_2^4} - \frac{a_1^4 b - 1}{a_1^4} + \frac{1}{a_1}\left(\frac{a_2^3 - a_1 b}{a_2^3}\right) - \frac{1}{a_2}\frac{a_1^3 b - a_2}{a_1^3} + \frac{1}{a_1^2}\left(\frac{a_2^2 - a_1^2 b}{a_2^2}\right)\right) + ...$$

This expression can be further simplified so that it becomes

$$\pi^*_{2,2}(0,1) - b\pi^*_{1,2}(1,0) = \sum_{t=2}^{T} \phi_t v_t, \text{ where}$$

$$\phi_t = \sum_{k=1}^{t}\left(\frac{1}{a_1^{k-1}} - \frac{b}{a_2^{k-1}}\right) = \frac{a_1^t - 1}{a_1^{t-1}(a_1 - 1)} - b\frac{(a_2^t - 1)}{a_2^{t-1}(a_2 - 1)}, t = 2, ..., T$$

Combining this with (A2) gives the result.

To show that a balanced contest yields full rent dissipation involves computing the expected efforts at each node in the game tree, and multiplying by the probability that the node is reached. Denote by $X_2(1,0)$ and $X_2(0,1)$ the total expected efforts from nodes $(1,0)$ and $(0,1)$. Rounds of recursion from the final contest and backwards yields the following pattern:

$$\pi^*_{1,2}(1,0) = \sum_{t=2}^{T} v_t - X_2(1,0)$$
$$\pi^*_{2,2}(0,1) = \sum_{t=2}^{T} v_t - X_2(0,1)$$

The value for each player of winning the first contest is $V_{1,1}(0,0) = v_1 + \pi^*_{1,2}(1,0)$ and $V_{2,1}(0,0) = v_1 + \pi^*_{2,2}(0,1)$. From (3) this gives a total expected effort in contest one of

$$X_1 = \frac{v_1 + \pi^*_{2,2}(0,1)}{b\left(v_1 + \pi^*_{1,2}(1,0)\right)}\left(\frac{2v_1 + \pi^*_{1,2}(1,0) + \pi^*_{2,2}(0,1)}{2}\right)$$

When the first contest is balanced, $v_1 + \pi^*_{2,2}(0,1) = b\left(v_1 + \pi^*_{1,2}(1,0)\right)$, and the probability of each player winning is $\frac{1}{2}$ (i.e. nodes $(1,0)$ and $(0,1)$ are reached with equal probability). Using balance, the total expected effort X is

$$\begin{aligned}
X &= X_1 + \frac{1}{2}\left(X_2(1,0) + X_2(0,1)\right) \\
&= \left(\frac{2v_1 + \pi^*_{1,2}(1,0) + \pi^*_{2,2}(0,1)}{2}\right) + \frac{1}{2}\left(\sum_{t=2}^{T} v_t - \pi^*_{1,2}(1,0) + \sum_{t=2}^{T} v_t - \pi^*_{2,2}(0,1)\right) \\
&= \sum_{t=1}^{T} v_t = 1.
\end{aligned}$$

■

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