Sweetening the Pill: The Four-Firm Case

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Abstract

This note discusses the four-firm version of the sequential merger game introduced in Fumagalli and Nilssen (2018). We delineate four motives for a firm not to merge at the first opportunity: the pill-sweetening motive; the external-effect motive; the bargaining-power motive; and the contraction motive. The last one does not arise in the three-firm case presented in detail in the other paper. Otherwise, the present discussion shows that the results of the three-firm model carry over to the four-firm case and that the prevalence of a firm passing up on opportunities to merge is, if anything, increasing as the number of firms in the industry increases.

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1 A four-firm model

As in Fumagalli and Nilssen (2018) [FN for short], we model a game consisting of two parts. The first part is a merger game, while the second part is a product-market competition game among the entities that are present after the merger game. There are some differences from the three-firm case dealt with in FN. The set of firms at the outset is $S := \{1, 2, 3, 4\}$. Firm 1 is of a different size than the other three, who are identical: $k_1 = k \in (0, 1)$, and $k_2 = k_3 = k_4 = \frac{1-k}{3}$. The symmetric case is at $k = \frac{1}{4}$.

In the four-firm case, there are seven principally different outcomes of the merger game: **SQ** - Status Quo, with no merger and the configuration {1,2,3,4}; **PO** - Partial Out merger, with a merger between two small firms and a configuration such as {1,23,4}; **PI** - Partial In merger, with a merger between firm 1 and one small firm and a configuration such as {12,3,4}; **FO** - Full Out merger, with a merger between all three small firms and the configuration {1,234}; **FI** - Full In merger, with a merger between firm 1 and two of the small firms and a configuration such as {123,4}; **AD** - Asymmetric Duopoly, with two pairwise mergers, one involving firm 1, and a configuration such as {12,34}; and **CM** - Complete Monopoly, with a merger between all four firms and the configuration {1234}. The new outcomes, compared to the three-firm case, are FO, FI, and AD, all three featuring two mergers in sequence, which now are one merger short of complete monopoly. Note that PO and PI, as in the three-firm case, are outcomes following from a single merger, but they now have three firms each. The set of possible outcomes is now $\Xi := \{SQ, PO, PI, FO, FI, AD, CM\}$, which in short-hand becomes $\Xi = \{Q, O, I, F, L, D, C\}$; note that we here, running out of characters, use L as shorthand for FI.

The merger game in the four-firm case is depicted in Figure 1. It consists of 22 decision nodes, out of which 10 are merger nodes and 12 are AA nodes; each of the two merger nodes 2 and 6 is followed by two AA nodes, node 2 by nodes $2A$ and $2B$ and node 6 by nodes $6A$ and $6B$, since there at those merger nodes are more than one merger to choose from. The set of decision nodes is thus $N := \{1, 1A, ..., 10, 10A\}$. The merger game starts out with firm 1 deciding whether or not to merge; this is node 1 in Figure 1. Since the other firms are of equal size, we randomly assign firm 2 the role of firm 1's merging mate. A merger is proposed if the joint profit of the merging firms is higher following a merger proposal than following a decision not to merge.[1]

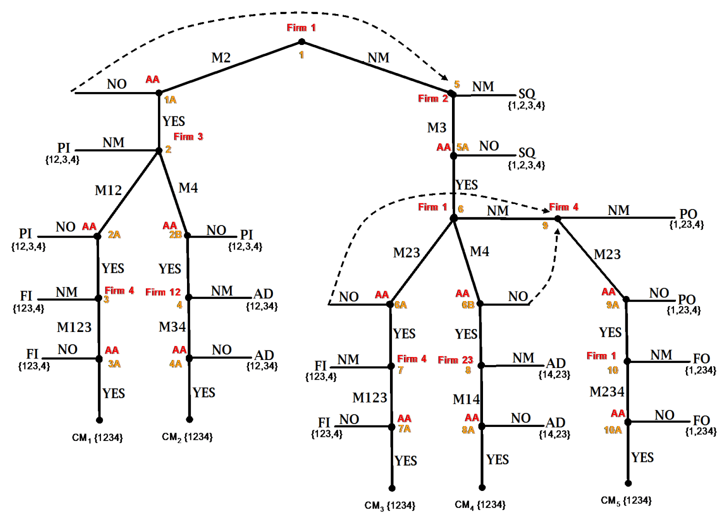

Figure 1. The merger game.

If firm 1 proposes a merger, then the antitrust authority (AA) makes a decision whether to approve the merger or not; this is node $1A$ in Figure 1. If AA says No, then we are at node 5, in the same situation as if firm 1 had decided not to merge; see below. If AA says Yes, then one of the remaining other firms, say firm 3, is given the choice.

[1] There is an exception to this statement that will be pointed out later on. This exception does not occur in the three-firm case.

Firm 3, at node 2 in Figure 1, has three alternatives, since the other two firms in the industry at this stage are of unequal size: firm 4 and the newly merged unit 12. So firm 3 chooses between no merger, a merger with 12, and a merger with 4. If firm 3 decides not to merge, then the process stops, and we end in a PI situation, with the firms in {12,3,4} playing a Cournot game. If firm 3 decides to merge with 12, then AA makes a decision whether to approve or not; this is node $2A$ in Figure 1. If AA says No, then the merger game ends again in a PI situation. If AA says Yes, then there is a final merger decision to be made, whether firm 4 should join with 123 or not; this is node 3 in Figure 1. If firm 4 decides not to merge, then the game ends in an FI situation, with firms 123 and 4 playing a Cournot game. If the merger between 4 and 123 is proposed, then we are in node $3A$ of Figure 1, where AA decides whether to approve or not. A No means we end in an FI situation again, while a Yes leads to CM with 1234 a monopolist in the industry.

If, at node 2, firm 3 decides to merge with 4, then AA makes a decision whether to approve or not; this is node $2B$ in Figure 1. If AA does not approve, then the merger game ends in a PI situation, and the firms in {12,3,4} play a Cournot game. If AA approves, then the industry consists of two firms, 12 and 34, which may want to merge. At node 4 in Figure 1, therefore, firm 12 decides whether or not to merge with 34. If no merger is decided, then the merger game ends in an AD situation, with 12 and 34 playing a Cournot game. If a merger is proposed, then AA decides whether to approve or not; this is node $4A$ in Figure 1. A No to merger means the merger ends in an AD situation again. A Yes to merger leads to a CM situation and 1234 being monopolist.

If, at node 1, firm 1 decides not to merge (or if, at node $1A$, AA says No), then there is still a possibility that a merger between two of the other firms would be viable, and so firm 2 makes a decision whether or not to merge with firm 3; this is node 5 in Figure 1. If again no merger is proposed, then the merger game ends with an SQ outcome, and the four firms play a Cournot game. In case of a merger proposal, the antitrust authority decides whether or not to approve; this is node $5A$ in Figure 1.

If the merger is approved at node $5A$, then a new situation has arisen, so that it is natural to give firm 1 a new chance to consider a merger. But now the two potential partners of firm 1 are of unequal size. So at node 6, firm 1 has three alternatives: merge with 23, merge with 4, or not merge at all. If firm 1 proposes a merger with 23, then AA decides whether or not to approve; this is node $6A$ in Figure 1. If AA says No, then we are at node 9 and the situation is the same as if firm 1 had decided not to merge; see below. If AA approves, then the situation is a duopoly with firms 123 and 4. At node 7, firm 4 decides whether or not to join 123. If there is no merger, then the merger game ends in an FI situation with 123 and 4 playing a Cournot game. If a merger between 4 and 123 is proposed, then it is for AA to decide whether to approve or not; this is node $7A$ in Figure 1. If AA says No, then the merger ends in an FI situation again. If AA says Yes, then we are in a CM situation with 1234 as monopolist.

If, at node 6, firm 1 chooses to merge with firm 4, then AA decides whether to approve or not; this is node $6B$ in Figure 1. If AA says No, then again we enter node 9, as if firm 1 had decided not to merge; see below. If AA says Yes, then we are in a situation with two firms, 14 and 23. At node 8, firm 23 decides whether or not to merge with 14. If the decision is not to merge, then the merger game ends with an AD situation, with firms 14 and 23 playing a Cournot game. If a merger between 23 and 14 is proposed, then AA makes a decision whether to approve it or not; this is node $8A$ in Figure 1. If AA says No, then we are again in an AD situation. If AA says Yes, then the game ends in a CM situation with 1234 as monopolist.

If, at node 6, firm 1 also this time chooses not to merge (or if, at node $6A$ or node $6B$, AA says No), then there is still a possibility that the three other firms would want to merge, so at node 9 we give firm 4 a chance to merge with firm 23. If also this is a decision not to merge, then the merger game ends with a PO outcome and the firms in {1,23,4} play a Cournot game. If firm 4 proposes to merge with 23, then AA makes an approval decision at node $9A$. If they say no, then we are again in the PO situation. If, however, they say yes, then once more a new situation has arisen, and firm 1 is given still another chance to merge, now with 234; this is node 10 in Figure 1. If the decision by firm 1 is not to merge – for the third time! – then the merger game ends in an FO situation, with 1 and 234 playing a Cournot game. If firm 1 proposes a merger, then AA decides whether or not to approve; this is finally node $10A$ in Figure 1. If the merger is not approved, then the merger game again ends in an FO situation. If AA says Yes, then the merger game ends with CM, and 1234 is a monopolist. This completes the description of the merger game.

Like in FN, our aim is, for each combination $(k,a) \in Z$ of market size and firm asymmetry, to find the corresponding equilibrium outcome, where Z is the set of combinations (k,a) such that all firms have positive quantities in all outcomes; the exact delineation of Z is done in the next section. We first solve the product-market game in each of the seven situations; see the next Section. Thereafter, we proceed by looking at each node $n \in N$ to determine, for each $(k,a) \in Z$, what the eventual outcome of the merger game is; *i.e.*, we are looking for an *outcome partition* Ω^n of Z at each node, where $\Omega^n := \left\{ Z^n_\xi, Z^n_\iota, ... \right\}$, and Z^n_ξ consists of all $(k,a) \in Z$ such that the outcome of the merger subgame starting at node $n \in N$ is outcome $\xi \in \Xi$. The equilibrium outcome of the whole merger game then corresponds to Ω^1, the outcome partition at node 1.

$\Gamma^n \subseteq \Xi$ is the set of outcomes that can occur after node n. $V^n_{\xi\iota} \subset Z$ is the relevant region of the parameter space at node $n \in N$ for the comparison between outcomes ξ and ι. ${}_\xi Y^m_\iota \subset Z$ is the set of parameter combinations for which decision maker m prefers outcome ξ to outcome ι. The decision maker $m = M(n)$ is the entity making the decision at node n. If it is a firm, then $m \in 2^S$. If it is the antitrust authority, then $m = A$. Thus the precise statement of Z^n_ξ is: $Z^n_\xi := \cup_{\iota \in \Gamma^n, \iota \neq \xi} \left(V^n_{\xi\iota} \cap {}_\xi Y^{M(n)}_\iota \right)$, $n \in N$, $\xi \in \Gamma^n$. With $\widetilde{N}$ being the set of end notes of the merger game, the relevant region at a decision node is constructed recursively through the outcome partitions of the node's immediate successors: $V^n_{\xi\iota} := \left[\cup_{l \in I^n \cap \Phi_\xi} Z^l_\xi \right] \cap \left[\cup_{l \in I^n \cap \Phi_\iota} Z^h_\iota \right]$, where I^n is the set of immediate successor nodes of node n and $\Phi_\xi := \left\{ n \in N \cup \widetilde{N} \mid Z_\xi^n \neq \varnothing \right\}$ is the set of nodes from which outcome ξ is a possible outcome. At most decision nodes, I^n consists of two nodes, so that the expression simplifies to: $V_{\xi\iota}^n = Z_\xi^l \cap Z_\iota^h$, where $l, h \in I^n$, and $l \neq h$, such that $l \in \Phi_\xi$ and $h \in \Phi_\iota$.[2]

2 Product-market competition

Below, we go through the seven different situations that may occur in the four-firm case in order to characterize the product-market equilibrium in each of them.

SQ: In this situation, one firm of size k and three firms each of size $\frac{1-k}{3}$ compete. The first-order condition of firm 1 is: $a - X - x_1 - \frac{1}{k} = 0$, while the first-order condition of firm $s \in \{2, 3, 4\}$ is: $a - X - x_s - \frac{3}{1-k} = 0$. Imposing symmetry on the identical firms 2 through 4, we can write these conditions as: $2x_1 + 3x_s = a - \frac{1}{k}$, and $x_1 + 4x_s = a - \frac{3}{1-k}$. Solving this system, we have:

$$\begin{aligned} x_1^{SQ} &= \frac{1}{5}\left(a - \frac{4-13k}{k(1-k)}\right), \\ x_2^{SQ} &= x_3^{SQ} = x_4^{SQ} = \frac{1}{5}\left(a - \frac{7k-1}{k(1-k)}\right), \end{aligned}$$

so that $\underline{a}^{SQ}(k) := \max\left\{\frac{4-13k}{k(1-k)}, \frac{7k-1}{k(1-k)}\right\}$; recall from FN that we construct $\underline{a}^{\xi}(k)$ for each $\xi \in \Xi$ in order to delineate the set Z of combinations (k, a) that we are interested in. Total quantity is:

$$X^{SQ} = \frac{1}{5}\left(4a - \frac{8k+1}{k(1-k)}\right).$$

PO: We have three firms: firm 1 of size k, firm 23 of size $\frac{2(1-k)}{3}$, and firm 4 of size $\frac{1-k}{3}$. The first-order conditions of the three firms are: $a - X - x_1 - \frac{1}{k} = 0$; $a - X - x_{23} - \frac{3}{2(1-k)} = 0$; and $a - X - x_4 - \frac{3}{1-k} = 0$. Rewriting, we have: $2x_1 + x_{23} + x_4 = a - \frac{1}{k}$; $x_1 + 2x_{23} + x_4 = a - \frac{3}{2(1-k)}$; $x_1 + x_{23} + 2x_4 = a - \frac{3}{1-k}$. Solving this system, we obtain:

$$\begin{aligned} x_1^{PO} &= \frac{1}{4}\left(a - \frac{6-15k}{2k(1-k)}\right), \\ x_{23}^{PO} &= \frac{1}{4}\left(a - \frac{5k-2}{2k(1-k)}\right), \\ x_4^{PO} &= \frac{1}{4}\left(a - \frac{17k-2}{2k(1-k)}\right). \end{aligned}$$

[2] In the three-firm case discussed in FN, I^n consists of two nodes in all cases. In the present four-firm case, there are two exceptions, nodes 2 and 6. However, at node 2, one of the nodes in I^2 is an end node. So the simplification in the text holds for all nodes except node 6.

Thus, $\underline{a}^{PO}(k) := \max\left\{\frac{3(2-5k)}{2k(1-k)}, \frac{5k-2}{2k(1-k)}, \frac{17k-2}{2k(1-k)}\right\}$. Total quantity is:

$$X^{PO} = \frac{1}{4}\left(3a - \frac{7k+2}{2k(1-k)}\right).$$

PI: We have one firm, 12, of size $k + \frac{1-k}{3} = \frac{2k+1}{3}$ and two firms, 3 and 4, each of size $\frac{1-k}{3}$. The first-order condition of firm 12 is: $a - X - x_{12} - \frac{3}{2k+1} = 0$, while the first-order condition of firm $s \in \{3,4\}$ is: $a - X - x_s - \frac{3}{1-k} = 0$. Imposing symmetry on the two identical firms 3 and 4, we rewrite: $2x_{12} + 2x_s = a - \frac{3}{2k+1}$; $x_{12} + 3x_s = a - \frac{3}{1-k}$. Solving the system, we obtain:

$$\begin{aligned} x_{12}^{PI} &= \frac{1}{4}\left(a - \frac{3(1-7k)}{1+k-2k^2}\right), \\ x_3^{PI} &= x_4^{PI} = \frac{1}{4}\left(a - \frac{3(5k+1)}{1+k-2k^2}\right), \end{aligned}$$

so that non-negative quantities require $a \geq \underline{a}^{PI}(k) := \max\left\{\frac{3(1-7k)}{1+k-2k^2}, \frac{3(5k+1)}{1+k-2k^2}\right\}$. Total quantity is:

$$X^{PI} = \frac{1}{4}\left(3a - \frac{9(1+k)}{1+k-2k^2}\right).$$

FO: There are two firms: firm 1 of size k and firm 234 of size $1-k$. The first-order condition of firm 1 is: $a - X - x_1 - \frac{1}{k} = 0$, while that of firm 234 is: $a - X - x_{234} - \frac{1}{1-k} = 0$. Rewriting, we have: $2x_1 + x_{234} = a - \frac{1}{k}$; $x_1 + 2x_{234} = a - \frac{1}{1-k}$. Solving this system, we get:

$$\begin{aligned} x_1^{FI} &= \frac{1}{3}\left(a - \frac{3k-1}{k(1-k)}\right), \\ x_{234}^{FI} &= \frac{1}{3}\left(a - \frac{2-3k}{k(1-k)}\right), \end{aligned}$$

so that $\underline{a}^{FI}(k) := \max\left\{\frac{3k-1}{k(1-k)}, \frac{2-3k}{k(1-k)}\right\}$, and total quantity is:

$$X^{FI} = \frac{1}{3}\left(2a - \frac{1}{k(1-k)}\right).$$

FI: Again, there are two firms, firm 123 of size $k + \frac{2(1-k)}{3} = \frac{k+2}{3}$ and firm 4 of size $\frac{1-k}{3}$. The first-order condition of firm 123 is: $a - X - x_{123} - \frac{3}{k+2} = 0$, while that of firm 4 is $a - X - x_4 - \frac{3}{1-k} = 0$. Rewriting, we obtain: $2x_{123} + x_4 = a - \frac{3}{k+2}$; $x_{123} + 2x_4 = a - \frac{3}{1-k}$. Solving this system, we have:

$$\begin{aligned} x_{123}^{FI} &= \frac{a}{3} + \frac{3k}{2-k-k^2}, \\ x_4^{FI} &= \frac{a}{3} - \frac{3k+1}{2-k-k^2}, \end{aligned}$$

so that $\underline{a}^{FI}(k) := \frac{3k+1}{2-k-k^2}$ (since x_{123}^{FI} can never be negative), while total quantity is:

$$X^{FI} = \frac{2}{3}a - \frac{1}{2-k-k^2}.$$

AD: There are two firms, firm 12 of size $\frac{2k+1}{3}$ and firm 34 of size $\frac{2(1-k)}{3}$. The two firms' first-order conditions are $a - X - x_{12} - \frac{3}{2k+1} = 0$ and $a - X - x_{34} - \frac{3}{2(1-k)} = 0$. Rewriting, we obtain: $2x_{12}+x_{34} = a - \frac{3}{2k+1}$; $x_{12}+2x_{34} = a - \frac{3}{2(1-k)}$. Solving this system, we have:

$$\begin{aligned} x_{12}^{AD} &= \frac{a}{3} - \frac{3(1-2k)}{2(1+k-2k^2)}, \\ x_{34}^{AD} &= \frac{a}{3} - \frac{3k}{1+k-k^2}, \end{aligned}$$

so that $\underline{a}^{AD}(k) := \max\left\{\frac{3(1-2k)}{2(1+k-2k^2)}, \frac{3k}{1+k-k^2}\right\}$. Total quantity is:

$$X^{AD} = \frac{2}{3}a - \frac{3}{2(1+k-k^2)}.$$

CM: In complete monopoly, there is a single firm, 1234, whose first-order condition is: $a - 2x_{1234} - 1 = 0$. In other words:

$$X^{CM} = x_{1234}^{CM} = \frac{a-1}{2},$$

so that $\underline{a}^{CM}(k) := 1$.

Based on the above, we can now be specific about the function $\underline{a}(k)$, which restricts the set Z of combinations (k, a) of interest and is given by the following piecewise relationship:

$$\underline{a}(k) = \begin{cases} \underline{a}^{SQ}(k) = \frac{4-13k}{k(1-k)}, \text{ if } k \in \left(0, \frac{2}{11}\right); \\ \underline{a}^{PO}(k) = \frac{3(2-5k)}{2k(1-k)}, \text{ if } k \in \left[\frac{2}{11}, \frac{2}{9}\right); \\ \underline{a}^{FO}(k) = \frac{2-3k}{k(1-k)}, \text{ if } k \in \left[\frac{2}{9}, \frac{1}{4}\right); \\ \underline{a}^{FI}(k) = \frac{9(1+k)}{2-k-k^2}, \text{ if } k \in \left[\frac{1}{4}, 7-3\sqrt{5}\right); \\ \underline{a}^{PO}(k) = \frac{17k-2}{2k(1-k)}, \text{ if } k \in \left[7-3\sqrt{5}, 1\right). \end{cases}$$

3 The merger game

We suppose AA assesses mergers by the total-welfare (TW) standard and proceed by way of backward induction. Consider first node $10A$, where AA decides on whether to approve a merger between 1 and 234. If AA says no to the merger, then the merger game stops in the FO situation, whereas a yes leads to CM; in other words, $\Gamma^{10A} = \{FO, CM\}$. The two immediate successors to node $10A$ are both end nodes, implying that $V_{CF}^{10A} = Z$. In comparing FO and CM, AA compares TW in the two outcomes and approves the merger if and only if $(k,a) \in {}_{C}Y_{F}^{A} := \{(k,a) \in Z \mid a \leq a_{CF}^{A}(k)\}$, where

$$a_{CF}^{A}(k) := \frac{27}{5} + \frac{2}{5k(1-k)}\left[8 + 3\sqrt{1 - 4k + 28k^2 - 24k^3(2-k)}\right]. \qquad (1)$$

Intuitively, the merger is approved if the market is so small that there is no room for two firms in the market. Thus, the outcome partition at node $10A$ is $\Omega^{10A} = \{Z_{CM}^{10A}, Z_{FO}^{10A}\}$, where $Z_{CM}^{10A} = V_{CF}^{10A} \cap {}_{C}Y_{F}^{A} = {}_{C}Y_{F}^{A}$, and $Z_{FO}^{10A} = Z \backslash Z_{CM}^{10A}$.

At node 10, firm 1 decides whether or not to propose a merger with 234. Possible outcomes are $\Gamma^{10} = \Gamma^{10A}$. Throughout, our supposition is that a merger is never proposed if it will subsequently be turned down by the AA. In assessing whether a merger is profitable, here and throughout, a firm compares the profit it gains from this merger with the alternative(s), which normally is not to merge. (At two instances below, nodes 2 and 6, a firm has two different mergers to choose from, in addition to the no-merger alternative.) In many of these comparisons, how the extra profit obtained from merging is split between the merging parties is of no relevance for the assessment. However, firms are far-sighted and therefore compare profits obtained from the outcomes that eventually prevail after the various alternatives. This calls for a comparison of profits from merging that takes into account the way profits are split. We assume that the extra profit from merger, over and above what is obtained in total in the case of no merger, is split equally between the two merging parties.

The merger is profitable for firm 1 if the profit of the merged unit 1234 is greater than the joint pre-merger profit of 1 and 234. This is true for all $(k,a) \in Z$, so ${}_{C}Y_{F}^{1} = Z$. Thus, $\Omega^{10} = \Omega^{10A}$, and a merger is proposed at node 10 if and only if $a \leq a_{CF}^{A}(k)$; see (1).

At node $9A$, the AA is to decide whether or not to approve a merger between 23 and 4. Possible outcomes are $\Gamma^{9A} = \{CM, FO, PO\}$. In particular, if it says no, then the merger ends in a PO outcome; and if it says yes, then the game ends in CM if $a \leq a_{CF}^{A}(k)$, in FO otherwise. Consider first the comparison between CM and PO. The relevant region is $V_{CO}^{9A} = Z_{CM}^{10}$. The AA prefers CM to PO if $(k,a) \in {}_{C}Y_{O}^{A} := \{(k,a) \in Z \mid a \leq a_{CO}^{A}(k)\}$, where

$$a_{CO}^{A}(k) := \frac{24k^2 + 11k + 10 + \sqrt{60k^3(3k+1) - 659k^2 + 436k - 44}}{6k(1-k)}, \qquad (2)$$

and so $Z_{CM}^{9A} = V_{CO}^{9A} \cap {}_{C}Y_{O}^{A}$.

In the comparison between PO and FO, the relevant region is $V_{FO}^{9A} = Z_{FO}^{10}$. In this comparison, the AA finds that FO is preferred to PO if and only if $(k,a) \in {}_{F}Y_{O}^{A} := \{(k,a) \in Z \mid a \leq a_{FO}^{A}(k)\}$, where

$$a_{FO}^{A}(k) := \frac{45}{2(1-k)} - \frac{19 - 12\sqrt{77k^2 - 14k + 1}}{7k(1-k)}. \qquad (3)$$

The graph of $a_{FO}^{A}(k)$ is in Figure 2 below. Essentially, when k is large, it is better for the AA to have the big firm 1 of size k balanced by a single rival firm in an FO situation, than by two smaller firms in a PO situation.

This implies that $Z^{9A}_{FO} = V^{9A}_{FO} \cap {}_{F}Y^{A}_{O}$, $Z^{9A}_{PO} = Z \backslash \left(Z^{9A}_{CM} \cup Z^{9A}_{FO} \right)$, and $\Omega^{9A} = \left\{ Z^{9A}_{CM}, Z^{9A}_{FO}, Z^{9A}_{PO} \right\}$. In other words, if

$$a \leq \min \left\{ a^{A}_{CF}(k), a^{A}_{CO}(k) \right\},$$

where $a^{A}_{CF}(k)$ and $a^{A}_{CO}(k)$ are defined in (1) and (2), respectively, then the merger is accepted both at this stage and at the next stage, so that the game ends in CM; if

$$a^{A}_{CF}(k) < a \leq a^{A}_{FO}(k),$$

where $a^{A}_{FO}(k)$ is defined in (3), then the merger is accepted at this stage but not at the next, so that the game ends in FO; and if

$$a > \max \left\{ a^{A}_{FO}(k), a^{A}_{CO}(k) \right\},$$

then the merger at this stage is not approved and the game ends in PO.

At node 9, firm 4 decides whether or not to merge with firm 23. We have the same possible outcomes as at node $9A$: $\Gamma^{9} = \Gamma^{9A} = \{CM, FO, PO\}$. It turns out that a merger is profitable whenever it is approved, if proposed, at node $9A$. Therefore, the outcome partition at this node is identical to that of node $9A$: $\Omega^{9} = \Omega^{9A}$. In particular, the merger proposal is put forward if and only if

$$a \leq \max \left\{ a^{A}_{FO}(k), a^{A}_{CO}(k) \right\}.$$

At node $8A$, AA decides whether or not to approve a merger between 14 and 23. We have $\Gamma^{8A} = \{CM, AD\}$: If AA says no, then the merger game ends in the AD situation, whereas a yes means it ends in CM. Now, $V^{8A}_{CD} = Z$, and AA approves the merger if and only if $(k, a) \in {}_{C}Y^{A}_{D} := \left\{ (k, a) \in Z \mid a \leq a^{A}_{CD}(k) \right\}$, where

$$a^{A}_{CD}(k) := \frac{9\left(6k^{2} - 3k + 5\right) + 3\sqrt{105 + 120k - 144k^{2} - 384k^{3}(1 - k)}}{5(2k + 1)(1 - k)}. \quad (4)$$

Thus, $Z^{8A}_{CM} = {}_{C}Y^{TW}_{D}$, $Z^{8A}_{AD} = Z \backslash Z^{8A}_{CM}$, and $\Omega^{8A} = \left\{ Z^{8A}_{CM}, Z^{8A}_{AD} \right\}$.

At node 8, firm 23 decides whether or not to merge with firm 14. We have $\Gamma^{8} = \Gamma^{8A} = \{CM, AD\}$, and $V^{8}_{CD} = Z^{8A}_{CM}$. Since ${}_{C}Y^{23}_{D} = Z$, *i.e.*, a merger is profitable for any $(k, a) \in Z$, we have $\Omega^{8} = \Omega^{8A}$: A merger is proposed at this stage if and only if it will be accepted, *i.e.*, if and only if $a \leq a^{A}_{CD}(k)$; see (4).

At node $6B$, AA decides whether or not to approve a merger between firms 1 and 4. If AA says no, then the game moves to node 9. If it says yes, then the game moves to node 8. All in all, there are four feasible outcomes that can be reached from this node: $\Gamma^{6B} = \{CM, FO, PO, AD\}$. Consider first the comparison between AD and PO. The relevant region is $V^{6B}_{DO} = Z^{8}_{AD} \cap Z^{9}_{PO}$, *i.e.*, combinations of a and k such that the merger game, if AA says yes, ends in AD, while if AA says no, it ends in PO. AA prefers AD to PS if and only if $(k, a) \in {}_{D}Y^{A}_{O} := \left\{ (k, a) \in Z \mid a \leq a^{A}_{DO}(k) \right\}$, where

$$a^{A}_{DO}(k) := \frac{18\left(70k^{2} - 9k + 10\right) + 24\sqrt{8(2 - 5k) + 1257k^{2} - 56k^{3}(41 - 32k)}}{28(2k + 1)(1 - k)}. \quad (5)$$

For the comparison between AD and FO, we have $V_{DF}^{6B} = Z_{AD}^{8} \cap Z_{FO}^{9}$. Moreover,

$$ {}_{D}Y_{F}^{A} := \left\{ (k,a) \in Z \mid k \leq \frac{2}{5} \right\}. \tag{6} $$

The intuition for this simple condition follows easily from the observation that, when $k = \frac{2}{5}$, the two situations AD and FO are identical, with one firm of size $\frac{2}{5}$ – firm 1 in the case of FO, and firm 23 in the case of AD – and one firm of size $\frac{3}{5}$ – firm 234 in the case of FO and firm 14 in the case of AD.

The comparison between CM and AD is straightforward: We know already that ${}_{D}Y_{F}^{A} = Z_{AD}^{8} \supset V_{CD}^{6B} = Z_{AD}^{8} \backslash \left(V_{DO}^{6B} \cup V_{DF}^{6B}\right)$. Therefore, combinations leading to AD at node $6B$ are in that part of Z_{AD}^{8} which is left after we have cut away combinations where either PO or FO is feasible and preferable: $Z_{AD}^{6B} = \left(V_{DO}^{6B} \cap {}_{D}Y_{O}^{A}\right) \cup \left(V_{DF}^{6B} \cap {}_{D}Y_{F}^{A}\right) \cup V_{CD}^{6B}$. The same is true for the comparisons between CM and PO, and between CM and FO. Clearly, ${}_{O}Y_{C}^{A} \supset V_{CO}^{6B} = Z_{CM}^{8} \cap Z_{PO}^{9}$. The implication is that combinations leading to PO are in that part of Z_{PO}^{9} which is left after we have cut away combinations where AD is feasible and preferable for the AA. That is, $Z_{PO}^{6B} = \left(V_{DO}^{6B} \cap {}_{O}Y_{D}^{A}\right) \cup V_{CO}^{6B}$. Similarly, ${}_{F}Y_{C}^{A} \supset V_{CF}^{6B} = Z_{CM}^{8} \cap Z_{FO}^{9}$, and $Z_{FO}^{6B} = \left(V_{DF}^{6B} \cap {}_{F}Y_{D}^{A}\right) \cup V_{CF}^{6B}$. Finally, therefore, we have $Z_{CM}^{6B} = Z \backslash \left(Z_{AD}^{6B} \cup Z_{FO}^{6B} \cup Z_{PO}^{6B}\right)$, and $\Omega^{6B} = \left\{Z_{CM}^{6B}, Z_{AD}^{6B}, Z_{FO}^{6B}, Z_{PO}^{6B}\right\}$. A picture of Ω^{6B} is provided in Figure 2.

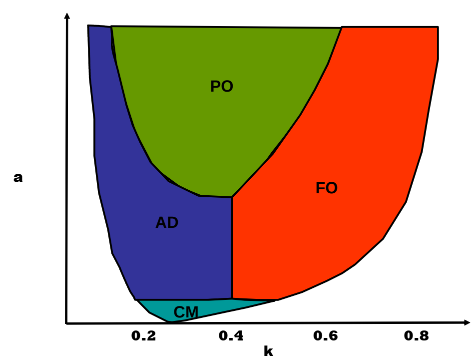

Figure 2. Outcomes at node $6B$.

In summary, the outcome of AA's decision at node $6B$ depends on a and k as follows: If

$$ a \leq \min\left\{a_{CF}^{A}(k), a_{CD}^{A}(k)\right\}, $$

where $a_{CF}^{A}(k)$ and $a_{CD}^{A}(k)$ are defined in (1) and (4), respectively, then the merger is approved, so is the subsequent merger proposal at node $8A$, and the merger game ends in CM. Note that, in cases where CM is the outcome whether AA says yes or no, we suppose here at node $6B$ and throughout that that outcome is reached in the most natural way, by AA saying yes right away. If

$$ k \leq \frac{2}{5}, \text{ and } a_{CD}^{A}(k) < a \leq a_{DO}^{A}(k), $$

where $a_{DO}^A(k)$ is defined in (5), then the merger is again approved, but a subsequent merger proposal would not have been approved at node $8A$, so that, at node 8, firms 23 and 14 decide not to merge, and the game ends in AD. If, on the other hand,

$$k > \frac{2}{5}, \text{ and } a_{CF}^A(k) < a \leq a_{FO}^A(k), \tag{7}$$

then the merger between firms 1 and 4 is not approved at node $6B$, the game moves to node 9, where a merger between 23 and 4 is proposed and subsequently accepted at node $9A$, and the game ends in FO. If

$$a > \max\left\{a_{DO}^A(k), a_{FO}^A(k)\right\},$$

then the merger between firms 1 and 4 is also not approved at node $6B$, but now, at node $9A$, a merger between 23 and 4 would not be approved either, so that, at node 9, the firms decide not to merge and the game ends in PO.

In short, if the market is small, then the AA prefers a complete monopoly. If the market is large, then the AA wants no more mergers, and PO is the outcome. For medium-sized markets, the AA allows one more merger. It goes for the FO outcome if k, the size of firm 1, is large, while it otherwise gets the best balance through the AD outcome.

At node $7A$, AA decides whether or not to approve a merger between firms 123 and 4. We have $\Gamma^{7A} = \{CM, FI\}$: If AA says no, then the game ends in FI; if it says yes, then the game ends in CM. Thus, $V_{CL}^{7A} = Z$. AA finds that CM is better, and thus approves the merger, if and only if $(k,a) \in {}_CY_L^A := \left\{(k,a) \in Z \mid a \leq a_{CL}^A(k)\right\}$, where

$$a_{CL}^A(k) := \frac{6\left(10 + 15k - k^3\right) + \sqrt{21(7+13k) - 6k^2(13+47k) - 3k^4(19+k)}}{2(3k+1)(k+2)(1-k)}. \tag{8}$$

Thus, $Z_{CM}^{7A} = {}_CY_L^A$, $Z_{FI}^{7A} = Z \backslash Z_{CM}^{7A}$, and $\Omega^{7A} = \left\{Z_{CM}^{7A}, Z_{FI}^{7A}\right\}$.

At node 7, firm 4 decides whether to merge with firm 123. Again, $\Gamma^7 = \{CM, FI\}$, and we have $V_{CL}^7 = Z_{CM}^{7A}$. This merger is always profitable: ${}_CY_L^4 = Z$. Thus, $Z_{CM}^7 = Z_{CM}^{7A}$, and $\Omega^7 = \Omega^{7A}$; the merger between 123 and 4 is proposed at node 7 whenever (8) holds.

At node $6A$, the AA decides whether or not to approve a merger between firms 1 and 23. This node is like node $6B$: If the AA says no, then the game moves to node 9, where firm 4 is to decide whether to merge with firm 23. If it says yes, then the game moves to node 7. Thus, $\Gamma^{6A} = \{CM, FI, PO, FO\}$. This gives rise to two new comparisons for the AA not already discussed above, that between FI and PO, and that between FI and FO. In the comparison between FI and PO, $V_{LO}^{6A} = Z_{FI}^7 \cap Z_{PO}^9$, and the AA prefers FI to PO if and only if $(k,a) \in {}_LY_O^A := \left\{(k,a) \in Z \mid a \leq a_{LO}^A(k)\right\}$, where

$$a_{LO}^A(k) := \frac{9\left(35k^2 - 48k + 20\right) + 24\sqrt{16 - 11k(8-39k) - 7k^3(94-43k)}}{14(k+2)(1-k)}. \tag{9}$$

Consider the comparison between FI and FO. Here, $V_{LF}^{6A} = Z_{FI}^{7} \cap Z_{FO}^{9}$, while

$$_{L}Y_{F}^{A} := \left\{ (k, a) \in Z \mid k \leq \frac{1}{4} \right\}.$$

The intuition is similar to that of the comparison between AD and FS above: When $k = \frac{1}{4}$, all the firms in the *status quo* are of identical size, and so the two situations FI and FO are identical, with one firm – firm 4 in the FI situation and firm 1 in the FO situation – of size $\frac{1}{4}$ and one firm – firm 123 in the FI situation and firm 234 in the FO situation – of size $\frac{3}{4}$.

The other comparisons at node $6A$ are straightforward. For the comparison between CM and FI, we know that $_{L}Y_{C}^{A} = Z_{FI}^{7} \supset V_{CL}^{6A} = Z_{FI}^{7} \backslash \left(V_{LO}^{6A} \cup V_{LF}^{6A}\right)$. This implies that combinations leading to FI at node $6A$ are in that part of Z_{FI}^{7} which is left after we cut away combinations where PO or FO is both feasible and preferable: $Z_{FI}^{6A} = \left(V_{LO}^{6A} \cap {}_{L}Y_{O}^{A}\right) \cup \left(V_{LF}^{6A} \cap {}_{L}Y_{F}^{A}\right) \cup V_{CL}^{6A}$. For the comparison between CM and PO, we have $_{O}Y_{C}^{A} \supset V_{CL}^{6A} = Z_{CM}^{7} \cap Z_{PO}^{9}$, and so $Z_{PO}^{6A} = \left(V_{LO}^{6A} \cap {}_{O}Y_{L}^{A}\right) \cup V_{CO}^{6A}$. For the comparison between CM and FO, we have $_{F}Y_{C}^{A} \supset V_{CF}^{6A} = Z_{CM}^{7} \cap Z_{FO}^{9}$, so that $Z_{PO}^{6A} = \left(V_{LF}^{6A} \cap {}_{F}Y_{L}^{A}\right) \cup V_{CF}^{6A}$. Finally, $Z_{CM}^{6A} = Z \backslash \left(Z_{FI}^{6A} \cup Z_{FO}^{6A} \cup Z_{PO}^{6A}\right)$, and $\Omega^{6A} = \left\{Z_{CM}^{6A}, Z_{FI}^{6A}, Z_{FO}^{6A}, Z_{PO}^{6A}\right\}$. A picture of Ω^{6A} is provided in Figure 3.

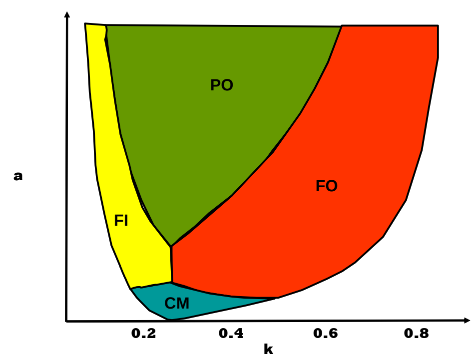

Figure 3. Outcomes at node $6A$.

In summary, we can describe the outcome of the AA's decision at node $6A$ as follows. If

$$a \leq \min\left\{a_{CF}^{A}(k), a_{CL}^{A}(k)\right\},$$

where $a_{CF}^{A}(k)$ and $a_{CL}^{A}(k)$ are defined in (1) and (8), respectively, then the merger is approved, and so is the subsequent merger at node $7A$, so that the merger game ends in CM. If

$$k \leq \frac{1}{4}, \text{ and } a_{CL}^{A}(k) < a \leq a_{LO}^{A}(k),$$

where $a_{LO}^{A}(k)$ is defined in (9), then again the merger is approved at node $6A$, but a merger would not be approved at node $7A$, so that the merger ends at node 7 with firms 123 and 4 deciding not to merge, leading to FI. If

$$k > \frac{1}{4}, \text{ and } a^A_{CF}(k) < a \leq a^A_{FO}(k),$$

where $a^A_{FO}(k)$ is defined in (3), then the merger between firms 1 and 23 is not approved, the game moves to node 9 where a merger between firms 4 and 23 is proposed and subsequently approved, and the game ends in FO. Finally, if

$$a > \max\left\{a^A_{LO}(k), a^A_{FO}(k)\right\},$$

then the merger is again not approved at node $6A$, and a subsequent merger proposal would have been stopped at node $9A$, so that the merger game ends with firms 4 and 23 deciding at node 9 not to merge, leading to a PO outcome.

In short, the story is much the same at node $6A$ as at node $6B$ discussed above: If the market is small, then the AA prefers a complete monopoly. If the market is large, then the AA wants no more mergers, and PO is the outcome. For medium-sized markets, the AA allows one more merger. It goes for the FO outcome if k is large, while it otherwise gets the best balance through the FI outcome. There is a difference from node $6B$ in the increased prevalence of the FO outcome at node $6A$: With the rather balanced outcome AD not available, the AA has only FI as an alternative to FO for medium-sized markets.

At node 6, firm 1 has three alternatives: It can merge with firm 23, leading to node $6A$; it can merge with firm 4, leading to node $6B$; or it can choose not to merge, leading to node 9. This case is slightly more difficult to analyze than those discussed so far, since there now are three alternatives for the decision maker. As noted before, a merger is not proposed unless it is going to be accepted. We have $\Gamma^6 = \{CM, AD, FI, FO, PO\}$.

When comparing the three alternatives, we see that, for a sufficiently small, CM is anyway the outcome. However, firm 1's gain from complete monopoly depends on the sequence of firms into that merger. If firm 1 chooses to merge with 23 and ends up with CM, the sequence is 23-1-4; if it merges with 4 and ends up with CM, the sequence is 1-4-23; and if it does not merge at node 6 but still ends up with CM, then the sequence is 4-23-1. We find that firm 1 always prefers CM after merging with 23 to CM after merging with 4, whereas CM after merging with 23 is better than CM after no merger if and only if $(k,a) \in Y^1_{CC1} := \left\{(k,a) \in Z \mid a \leq a^1_{CC1}(k)\right\}$, where

$$a^1_{CC1} := \frac{4 + 18k + 20k^2 - 3k^3 + \sqrt{64 - 336k + 508k^2 + 852k^3 + 544k^4 - 138k^5}}{3k(k+2)(1-k)}.$$

When comparing, in the eyes of firm 1, the outcomes AD and FI, we have that the relevant region is $V^6_{DL} = Z^{6A}_{FI} \cap Z^{6B}_{AD}$. To be precise, we can write

$$V^6_{DL} = \left\{ \begin{array}{c} (k,a) \in Z \mid k \leq \frac{1}{4}, \text{ and } \max\left\{a^A_{CF}(k), a^A_{CD}(k)\right\} < \\ a \leq \min\left\{a^A_{LO}(k), \max\left\{a^A_{DO}(k), a^A_{FO}(k)\right\}\right\} \end{array} \right\}.$$

In all of this relevant region, firm 1 prefers AD to FI: ${}_DY^1_L \supset V^6_{DL}$.

Consider next the comparison between AD and PO. The relevant region is $V^6_{DO} = Z^{6B}_{AD} \cap Z^9_{PO}$; recall that we disregard the possibility of getting to PO through a No from the AA at node $6A$. Also here, ${}_DY^1_O \supset V^6_{DO}$, meaning firm 1 prefers AD to PO whenever this comparison is relevant at node 6.

In the comparison between AD and FO, the relevant region is $V^6_{DF} = Z^{6B}_{AD} \cap Z^9_{FO}$. For almost all combinations in V^6_{DF}, firm 1 prefers FO to AD. In particular, FO is preferred whenever $(k, a) \in {}_FY^1_D := \left\{(k, a) \in Z \mid a \geq a^1_{FD}(k)\right\}$, where

$$a^1_{FD}(k) := \frac{60k^2 + 8k + \sqrt{3168k^4 - 1536k^3 + 1100k^2 + 20k - 7}}{4k(2k + 1)(1 - k)}.$$

The comparison between FI and PO is redundant, since AD dominates both FI and PO in the respective relevant regions. Also the comparison between FI and FO is redundant, but now because the relevant region is empty: $V^6_{LF} = Z^{6A}_{FI} \cap Z^9_{FO} = \varnothing$; *i.e.*, there does not exist a combination (k, a) such that firm 1 at node 6 has a choice to make between the outcomes FI and FO.

The dominance of AD over PO implies that firm 1 goes for PO only when alternatives do not exist: $Z^6_{PO} = Z^{6B}_{PO}$, since $Z^{6A}_{PO}, Z^9_{PO} \subset Z^{6B}_{PO}$. Also, CM dominates all alternatives, so that firm 1 goes for CM whenever CM is feasible: $Z^6_{CM} = Z^{6A}_{CM} \cup Z^{6B}_{CM} \cup Z^9_{CM}$. This expression, however, hides the fact that firm 1 has two different ways of obtaining CM at node 6. If $(k, a) \in {}_LY^A_C \cap Z^6_{CM}$, then CM is obtained by firm 1 choosing no merger at node 6, leading the game to node 9, and subsequently to CM. If, on the other hand, $(k, a) \in {}_CY^A_L \cap Z^6_{CM}$, then CM is obtained by firm 1 choosing to merge with 23 at node 6.

Note that $Z^{6A}_{FO}, Z^{6B}_{FO} \subset Z^9_{FO}$. Some combinations in Z^9_{FO} are dominated by AD in the relevant region, and so firm 1 merges with 4 to obtain AD. For other combinations in Z^9_{FO}, firm 1's best plan is not to merge in order to obtain FO. In particular, for $k \geq \frac{2}{5}$, firm 1 choosing not to merge and ending up as the one firm outside all the other firms merged into one is a prevalent outcome at node 6.

We also note that $Z^6_{FI} = \varnothing$; for all combinations for which FI is a feasible outcome for firm 1 at node 6, *i.e.*, for all $(k, a) \in Z^{6A}_{FI}$, firm 1 prefers merging with firm 4 to obtain AD to merging with firm 23 to obtain FI. Thus, Z^6_{AD} is the residual: $Z^6_{AD} = Z \backslash \left(Z^6_{CM} \cup Z^6_{FO} \cup Z^6_{PO}\right)$, and $\Omega^6 = \left\{Z^6_{CM}, Z^6_{AD}, Z^6_{FO}, Z^6_{PO}\right\}$; the latter is pictured in Figure 4.

We see from Figure 4 that, when firm 1 is big, *i.e.*, k is large, the firm chooses to abstain from merger, because the need to get approval from the AA makes it impossible to get a merger through anyway. The exception is when the market is very small, in particular when, in addition to $k > \frac{2}{5}$, we also have $a \leq a^A_{CF}(k)$, in which case the AA accepts mergers to complete monopoly. We also see that firm 1 chooses not to merge at node 6 also for some cases where $k < \frac{2}{5}$: When $\max\left\{a^A_{CF}(k), a^1_{FD}(k)\right\} < a < a^A_{FO}(k)$, firm 1 again prefers not to merge, since its best option is a life on its own outside the merged entity 234 in the FO outcome, despite its not being as large as in the $k > \frac{2}{5}$ case. And a particularly interesting case occurs for $a^A_{CL}(k) < a < \min\left\{a^A_{CO}(k), a^A_{CD}(k)\right\}$:

A merger to complete monopoly would not have been accepted by the AA if firm 1 were to merge with 23. Instead, firm 1 says no to merger (for the second time), so that all the other firms – 2, 3, and 4 – are merged into one firm before the AA is asked to approve of a complete-monopoly outcome.

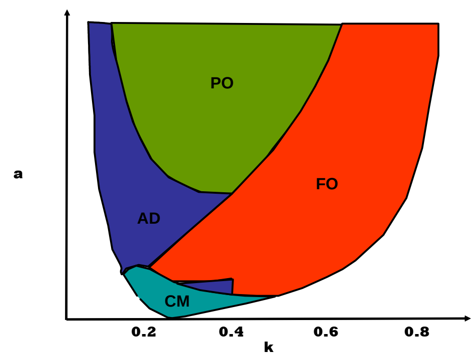

Figure 4. Outcomes at node 6.

At node $5A$, the AA decides whether or not to approve the merger between firms 2 and 3. If it says no, then the process stops and the merger ends in the initial situation SQ. If it says yes, then the game proceeds to node 6, described above. Thus, $\Gamma^{5A} = \{CM, AD, FO, PO, SQ\}$; as already noted above, there cannot be any FI outcome in equilibrium following node 6, and therefore neither following node $5A$. We need to compare SQ with each of the other outcomes in terms of total welfare.

In the comparison between SQ and PO, we have $V^{5A}_{QO} = Z^{6}_{PO}$ $\left(= Z^{6A}_{PO}\right)$. The AA prefers PO to SQ if and only if $(k,a) \in {}_{O}Y^{A}_{Q} := \left\{(k,a) \in Z \mid a \leq a^{A}_{OQ}(k)\right\}$, where

$$a^{A}_{OQ}(k) := \frac{661k - 58 + 40\sqrt{181k^2 - 20k + 1}}{18k(1-k)}. \tag{10}$$

From node $5A$, we end up in outcome PO if $(k,a) \in Z^{5A}_{PO} = V^{5A}_{QO} \cap {}_{O}Y^{A}_{Q}$.

In the comparison between SQ and AD, $V^{5A}_{QD} = Z^{6}_{AD}$. The AA prefers AD to SQ if and only if $(k,a) \in {}_{D}Y^{A}_{Q} := \left\{(k,a) \in Z \mid a \leq a^{A}_{DQ}(k)\right\}$, where

$$a^{A}_{DQ}(k) := \frac{432k^2 + 45k + 15\sqrt{528k^4 - 168k^3 + 180k^2 + 30k + 3}}{8k(2k+1)(1-k)}. \tag{11}$$

Thus, $Z^{5A}_{AD} = V^{5A}_{QD} \cap {}_{D}Y^{A}_{Q}$.

In the comparison between SQ and FO, we have $V^{5A}_{QF} = Z^{6}_{FO}$ and find that ${}_{F}Y^{A}_{Q} \supset V^{5A}_{QF}$, so that FO is preferred to SQ by the AA in all of the relevant region. This implies that $Z^{5A}_{FO} = Z^{6}_{FO}$. The same holds for the comparison between SQ and CM: $V^{5A}_{QC} = Z^{6}_{CM}$, and ${}_{C}Y^{A}_{Q} \supset V^{5A}_{QC}$, so that $Z^{5A}_{CM} = Z^{6}_{CM}$. Thus, the AA says no to the merger between firms 2 and 3 at node $5A$, and the merger game ends with status quo (SQ), if $(k,a) \in Z^{5A}_{SQ} = Z \backslash \left(Z^{5A}_{CM} \cup Z^{5A}_{AD} \cup Z^{5A}_{FO} \cup Z^{5A}_{PO}\right)$, which happens when a is sufficiently large. Otherwise, *i.e.*, if $(k,a) \in Z \backslash Z^{5A}_{SQ}$, then the AA approves the merger. We have $\Omega^{5A} = \left\{ Z^{5A}_{CM}, Z^{5A}_{AD}, Z^{5A}_{FO}, Z^{5A}_{PO}, Z^{5A}_{SQ} \right\}$.

At node 5, no merger has taken place so far in the game when firm 2 considers whether or not to merge with firm 3. We have $\Gamma^5 = \{CM, AD, FO, PO, SQ\}$; again, since FI cannot occur in equilibrium following node 5, it is excluded from the list. In parallel to node $5A$ discussed above, we need to compare SQ with the outcomes CM, AD, FO, and PO, but this time from the perspective of firm 2 rather than that of the AA. Three of the four comparisons are straightforward, with firm 2 always preferring merger to no merger in all the relevant region; these are the comparisons between SQ and CM, AD, and FO, respectively. Formally, $V^5_{QC} = Z^{5A}_{CM}$, and ${}_CY^2_Q \supset V^5_{QC}$, implying $Z^5_{CM} = Z^{5A}_{CM}$; $V^5_{QD} = Z^{5A}_{AD}$, and ${}_DY^2_Q \supset V^5_{QD}$, implying $Z^5_{AD} = Z^{5A}_{AD}$; and $V^5_{QF} = Z^{5A}_{FO}$, and ${}_FY^2_Q \supset V^5_{QF}$, implying $Z^5_{FO} = Z^{5A}_{FO}$. But the comparison between SQ and PO is interesting: the two firms may choose not to merge even in cases where a merger would have been approved. We have $V^5_{QS} = Z^{5A}_{PS}$. Firm 2 prefers PO to SQ, meaning that the merger with firm 3 is profitable for firm 2, if and only if $(k,a) \in {}_OY^2_Q := \left\{ (k,a) \in Z \mid a \leq a^A_{OQ}(k) \right\}$, where

$$a^A_{OQ}(k) := \frac{\left(323 + 180\sqrt{2}\right)k - 14}{14k(1-k)}.$$

Now, $Z^5_{PO} = Z^{5A}_{PO} \cap_O Y^2_Q$, $Z^5_{SQ} = Z \backslash \left(Z^5_{CM} \cup Z^5_{AD} \cup Z^5_{FO} \cup Z^5_{PO} \right)$, and $\Omega^5 = \left\{ Z^5_{CM}, Z^5_{AD}, Z^5_{FO}, Z^5_{PO}, Z^5_{SQ} \right\}$, depicted in Figure 5.

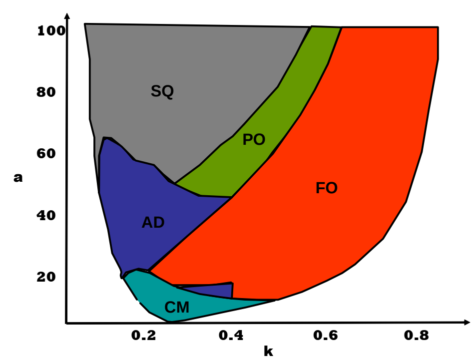

Figure 5. Outcomes at node 5.

Moving to the left-hand side of Figure 1, we have that node $4A$ is identical to node $8A$, and so is also always the outcome: $\Omega^{4A} = \Omega^{8A}$. Similarly, node 4 is identical to node 8, so that $\Omega^4 = \Omega^8 \left(= \Omega^{8A}\right)$.

Node $2B$ is not identical to node $6B$, though. Here, the AA not approving the merger between firms 3 and 4 leads to the merger game ending in the PI outcome, whereas at node $6B$, a No from the AA implies a move to node 9 and a decision by firm 4 on whether or not to merge with 23. We have $\Gamma^{2B} = \{CM, AD, PI\}$.

In the comparison between AD and PI, we have that $V_{DI}^{2B} = Z_{AD}^{4} \left(= Z_{AD}^{8A}\right)$. The AA prefers AD to PI if and only if $(k,a) \in {}_{D}Y_{I}^{A} := \left\{(k,a) \in Z \mid a \leq a_{DI}^{A}(k)\right\}$, where

$$a_{DI}^{A}(k) := \frac{9}{7} \frac{45k + 13 + 2\sqrt{256k^2 + 184k + 37}}{(2k+1)(1-k)}. \tag{12}$$

Thus, $Z_{AD}^{2B} = V_{DI}^{2B} \cap {}_{D}Y_{I}^{A}$.

In the comparison between CM and PI, the AA prefers CM to PI in all of the relevant region: ${}_{C}Y_{I}^{A} \supset V_{QI}^{2B} = Z_{CM}^{4} \left(= Z_{CM}^{8A}\right)$, so that $Z_{CM}^{2B} = Z_{CM}^{4}$. This implies that $Z_{PI}^{2B} = Z_{AD}^{4} \backslash {}_{D}Y_{I}^{A}$, and $\Omega^{2B} = \left\{Z_{CM}^{2B}, Z_{AD}^{2B}, Z_{PI}^{2B}\right\}$, which is depicted in Figure 6. We see that CM is preferred by the AA only when a is very small. PI is preferred when a is large and k is small, while AD is preferred for large k. Intuitively, when k is large and thus firm 1 is big, the AA would like to have another merger between two small firms in order to counterbalance the large merger already in place between firm 1 and one of the small ones.

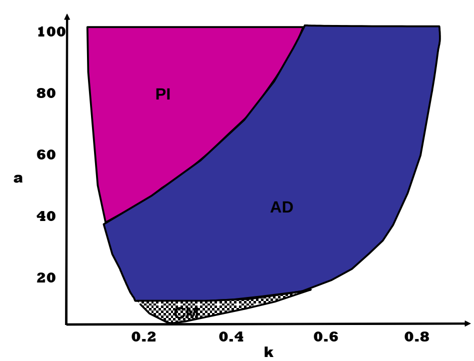

Figure 6. Outcomes at node $2B$.

Node $3A$ is identical to node $7A$, and node 3 is identical to node 7; thus, $\Omega^{3A} = \Omega^{7A}$, and $\Omega^{3} = \Omega^{7} \left(= \Omega^{7A}\right)$.

At node $2A$, the AA decides whether or not to approve a merger between firms 12 and 3. If it says no, then the merger game ends in the PI outcome. If it says yes, then the game moves to node 3 and we get, like at node 7, either FI or CM. We have: $\Gamma^{2A} = \{CM, FI, PI\}$. In the comparison between PI and FI, we have $V_{IL}^{2A} = Z_{FI}^{3} \left(= Z_{FI}^{7A}\right)$. The AA prefers FI to PI if and only if $(k,a) \in {}_{L}Y_{I}^{A} := \left\{(k,a) \in Z \mid a \leq a_{LI}^{A}(k)\right\}$, where

$$a_{LI}^{A}(k) := \frac{9}{7} \frac{45k^2 + k + 26 + 4\sqrt{148k^4 - 112k^3 - 48k^2 + 56k + 37}}{(2k+1)(k+2)(1-k)}. \tag{13}$$

Thus, $Z_{FI}^{2A} = V_{IL}^{2A} \cap {}_{L}Y_{I}^{A}$. In the comparison between PI and CM, CM is preferred by the AA in all of the relevant region: ${}_{C}Y_{I}^{A} \supset V_{CI}^{2A} = Z_{CM}^{3} \left(= Z_{CM}^{7A}\right)$. Thus, $Z_{CM}^{2A} = Z_{CM}^{3}$, and therefore $Z_{PI}^{2A} = V_{IL}^{2A} \backslash {}_{L}Y_{I}^{A}$ and $\Omega^{2A} = \left\{Z_{CM}^{2A}, Z_{FI}^{2A}, Z_{PI}^{2A}\right\}$. A picture of Ω^{2A} is in Figure 7. Compared to node $2B$, discussed above, there is no AD alternative available here. So the AA will have to settle for the CM

alternative not only when a is small, but also in cases where k is very large: At node $3A$, and again at node $7A$, the AA prefers to allow 123 to merge with 4 to create a complete monopoly when k is so large that the cost savings from this merger make CM preferable to FI.

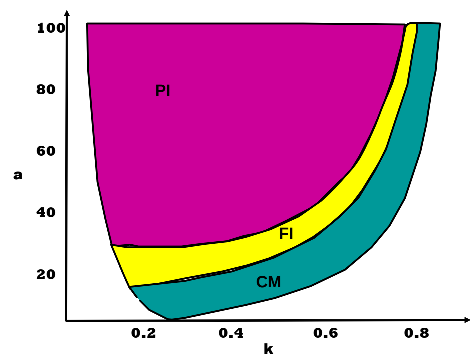

Figure 7. Outcomes at node $2A$.

At node 2, firm 3 has three alternatives to choose from: It can merge with firm 12, leading to node $2A$; it can merge with firm 4, leading to node $2B$; and it can choose not to merge, ending the merger game with a PI outcome. This is the only decision node, in addition to node 6, where there are three alternatives. However, the analysis of node 2 is a lot simpler than that of node 6, since one of the alternatives of firm 3 at node 2 leads to an end node of the merger game. We have $\Gamma^2 = \{CM, AD, FI, PI\}$.

In the comparison between AD and PI, we have that firm 3 prefers AD in all of the relevant region: ${}_DY_I^3 \supset V_{DI}^2 = Z_{PI}^{2A} \cap Z_{AD}^{2B}$. Thus, PI is chosen at node 2 only when there are no alternatives: $Z_{PI}^2 = Z_{PI}^{2B}$. The other comparisons at node 2 turn out to be equally straightforward. In the comparison between AD and FI, AD is again preferred in all of the relevant region: ${}_DY_L^3 \supset V_{DL}^2 = Z_{FI}^{2A} \cap Z_{AD}^{2B}$. In the comparison between AD and CM, it is AD that is dominated: CM is preferred in all the relevant region: ${}_CY_D^3 \supset V_{CD}^2 = Z_{CM}^{2A} \cap Z_{AD}^{2B}$. The same happens in the comparison between CM and PI: ${}_CY_I^3 \supset V_{CI}^2 = Z_{CM}^{2A}$. This implies that $Z_{CM}^2 = Z_{CM}^{2A}$, and $Z_{AD}^2 = Z_{AD}^{2B} \backslash Z_{CM}^{2A}$. We see that, like at node 6 discussed above, $Z_{FI}^2 = \varnothing$; whenever the FI outcome is feasible, the firm perfers an alternative. The upshot is that FI, with a market structure like {123,4}, never occurs in equilibrium in this merger game. We have $\Omega^2 = \left\{Z_{CM}^2, Z_{AD}^2, Z_{PI}^2\right\}$. A picture of Ω^2 is provided in Figure 8.

We see that firm 3 is able to obtain CM for k sufficiently large; interestingly, though, firm 3 does this by merging with firm 4, since this gives firm 3 more out of the merger process than it obtains from first merging with firm 12. In a medium range, the outcome is AD through a merger between firms 3 and 4, while for low k and large a, firm 3 will have to settle for the no-merger alternative and the PI outcome.

At node $1A$, the AA decides whether or not to approve the merger between firms 1 and 2. If it says No, then the game moves to node 5, with firm 2 deciding whether or not to leave the SQ situation by merging with another small firm. If it says Yes, then the game moves to node 2, with firm 3 making a merger decision. So a negative decision by the AA turns the game over to the right-hand side of Figure 1. This means that all outcomes, except FI, are feasible at this node: $\Gamma^{1A} = \{CM, AD, FO, PO, PI, SQ\}$. It also means that a lot of the comparisons, although made before, now have quite different relevant regions than above.

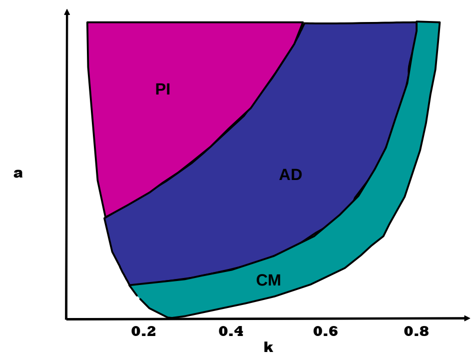

Figure 8. Outcomes at node 2.

In the comparison between CM and AD, the relevant region is $V_{CD}^{1A} = Z_{AD}^{2} \cap Z_{CM}^{5}$, and ${}_{D}Y_{C}^{A} \supset V_{CD}^{1A}$. Thus, the AA can avoid the CM outcome for some combinations (k, a), namely those in V_{CD}^{1A}, by approving the merger at node $1A$. In the comparison between CM and FO, recall that, for large k, approval of the merger will lead to CM (see Figure 8): $V_{CF}^{1A} = Z_{CM}^{2} \cap Z_{FS}^{5}$. The AA prefers FO to CM in all of the relevant region, however: ${}_{F}Y_{C}^{A} \supset V_{CF}^{1A}$. Taken together, these two comparisons imply that CM is the outcome at node $1A$ only when it is the only alternative: $Z_{CM}^{1A} = Z_{CM}^{2} \cap Z_{CM}^{5}$.

In the comparison between outcomes AD and FO at node $1A$, $V_{DF}^{1A} = Z_{AD}^{2} \cap Z_{FS}^{5}$. As noted in eq. (6) above, ${}_{D}Y_{F}^{A} := \left\{(k, a) \in Z \mid k \leq \frac{2}{5}\right\}$. Thus, the AA says No to a merger for any combination in V_{DF}^{1A} for which $k > \frac{2}{5}$. Combinations leading to FO at node $1A$ are those leading to FO at node 5, except those with $k \leq \frac{2}{5}$ for which AD is the alternative: $Z_{FO}^{1A} = Z_{FO}^{5} \backslash \left(V_{DF}^{1A} \cap {}_{D}Y_{F}^{A}\right)$.

In the comparison between AD and PO, $V_{DO}^{1A} = Z_{AD}^{2} \cap Z_{PO}^{5}$, and here again AD is chosen away: ${}_{O}Y_{D}^{A} \supset V_{DO}^{1A}$. Thus, the AA chooses no merger whenever the continuation at node 5 leads to the PO outcome: $Z_{PO}^{1A} = Z_{PO}^{5}$.

In the comparison between AD and SQ, $V_{DQ}^{1A} = Z_{AD}^{2} \cap Z_{SQ}^{5}$, a very small region. The AA prefers AD to SQ if and only if $(k, a) \in {}_{D}Y_{Q}^{A} := \left\{(k, a) \in Z \mid a \leq a_{DQ}^{A}(k)\right\}$, where $a_{DQ}^{A}(k)$ is defined in (11).

In the comparison between PI and AD, the AA prefers PI in all of the relevant region: ${}_{I}Y_{D}^{A} \supset V_{ID}^{1A} = Z_{PI}^{2} \cap Z_{AD}^{5}$. In the comparison between PI and SQ, $V_{IQ}^{1A} = Z_{PI}^{2} \cap Z_{SQ}^{5}$, and the AA prefers PI to SQ if and only if $(k, a) \in {}_{I}Y_{Q}^{A} := \left\{(k, a) \in Z \mid a \leq a_{IQ}^{A}(k)\right\}$, where

$$a_{IQ}^{A}(k) := \frac{137k^2 - 55k + 32 + 20\sqrt{52k^4 - 88k^3 + 48k^2 - 4k + 1}}{3k(2k+1)(1-k)}. \tag{14}$$

This implies that $Z_{SQ}^{1A} = \left(V_{DQ}^{1A} \setminus {}_DY_Q^A\right) \cup \left(V_{IQ}^{1A} \setminus {}_IY_Q^A\right)$, and $Z_{PI}^{1A} = Z_{PI}^2 \setminus \left(V_{IQ}^{1A} \setminus {}_IY_Q^A\right)$. Thus, $Z_{AD}^{1A} = Z \setminus \left(Z_{CM}^{1A} \cup Z_{FO}^{1A} \cup Z_{PO}^{1A} \cup Z_{SQ}^{1A} \cup Z_{PI}^{1A}\right)$, and $\Omega^{1A} = \left\{Z_{CM}^{1A}, Z_{AD}^{1A}, Z_{FO}^{1A}, Z_{PO}^{1A}, Z_{SQ}^{1A}, Z_{PI}^{1A}\right\}$.

Finally, we have reached node 1, where firm 1 decides whether or not to merge with one of the small firms. If firm 1 decides to merge, then the game moves to node $1A$, where the AA decides whether or not to accept that merger. If firm 1 decides not to merge, then the game moves to node 5, giving instead two of the small firms a chance to merge. We have $\Gamma^1 = \{CM, AD, FO, PO, PI, SQ\}$. We note again that FI cannot occur in equilibrium. We also note that $Z_{PO}^{1A} = Z_{PO}^5$. Thus, we do not need to compare at this node PO with any other outcome and have $Z_{PO}^1 = Z_{PO}^5 \ \left(= Z_{PO}^{1A}\right)$.

In the comparison between CM and AD, we have that ${}_CY_D^1 \supset V_{CD}^1 = Z_{AD}^{1A} \cap Z_{CM}^5$. This implies that $Z_{CM}^1 = Z_{CM}^5$; however, Z_{CM}^1 is again further split into smaller regions according to actions taken at node 1. In the comparison between AD and FO, we have ${}_FY_D^1 \supset V_{DF}^1 = Z_{AD}^{1A} \cap Z_{FO}^5$. Thus, $Z_{FO}^1 = Z_{FO}^5$.

In the comparison between SQ and PI, we have $V_{QI}^1 = Z_{PI}^{1A} \cap Z_{SQ}^5$. The outcome PI is preferred by firm 1 to SQ if and only if $(k,a) \in {}_IY_Q^1 := \left\{(k,a) \in Z \mid a \leq a_{IQ}^1(k)\right\}$, where

$$a_{IQ}^1 := \frac{333k^2 - 75k + 48 + 20\sqrt{226k^4 - 292k^3 + 150k^2 - 4k + 1}}{7k\,(2k+1)\,(1-k)}.$$

In the comparison between SQ and AD, AD is preferred by firm 1 in all of the relevant region: ${}_DY_Q^1 \supset V_{DQ}^1 = Z_{AD}^{1A} \cap Z_{SQ}^5$. This implies that we end up with SQ in cases where the AA wouldn't allow anything else (Z_{SQ}^{1A}) plus some cases where the AA would allow a merger between firms 1 and 2 but nothing more and where firm 1 then prefers going alone ($V_{QI}^1 \cap {}_IY_Q^1$); *i.e.*, $Z_{SQ}^1 = Z_{SQ}^{1A} \cup \left(V_{QI}^1 \cap {}_IY_Q^1\right)$. Finally, in the comparison between PI and AD, AD is also here preferred in all of the relevant region: ${}_DY_I^1 \supset V_{ID}^1 = Z_{PI}^{1A} \cap Z_{AD}^5$. This means that $Z_{AD}^1 = Z_{SQ}^5 \cup V_{DQ}^1$, and that $Z_{PI}^1 = Z \setminus \left(Z_{CM}^1 \cup Z_{AD}^1 \cup Z_{FO}^1 \cup Z_{PO}^1 \cup Z_{SQ}^1\right)$. We can therefore come with the conclusion to our analysis. The equilibrium of the merger game under study here is $\Omega^1 = \left\{Z_{CM}^1, Z_{AD}^1, Z_{FO}^1, Z_{PO}^1, Z_{SQ}^1, Z_{PI}^1\right\}$, and it is depicted in Figure 9.

The picture in Figure 9 shows a great variety in possible outcomes. It also shows that firms' decisions to merge are heavily influenced by what will get through the antitrust authority. When k is large, so that the industry consists of one big firm and three small ones, firm 1 realizes that its best shot is sitting outside the merger process and letting the small firms merge, either all three or at least two of them. When k is small, so that the industry consists of one small firm and three larger ones, then there is a greater potential for at least one of the larger firms to get involved in a merger. If a is not very large, then it is actually possible for all the large firms to get involved in a merger, as in the AD outcome, with two of the larger firms merging with each other and one of them merging with the small firm. When a is very small, then there is not room for more than one firm in the industry, also by the AA's standard, and so we end up with CM. In the opposite end, when a is very large, there is no scope for a merger seen from the AA's point of view, and SQ, the situation we start out with, is also the final outcome. We refer to FN for further discussion.

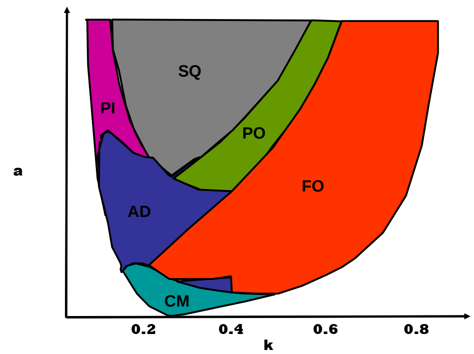

Figure 9. Equilibrium outcomes.

References

[1] Fumagalli, E. and T. Nilssen (2018), "Sweetening the Pill: A Theory of Waiting To Merge", unpublished manuscript.