Composable Markov Processes
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COMPOSABLE MARKOV PROCESSES

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1. Introduction and summary

Many phenomena studied in the social sciences and elsewhere are complexes of more or less independent characteristics which develop simultaneously. Such phenomena may often be realistically described by time-continuous finite Markov processes. In order to define such a model which will take care of all the relevant a priori information, there ought to be a way of defining a Markov process as a vector of components representing the various characteristics constituting the phenomenon such that the dependences between the characteristics are represented by explicit requirements on the Markov process, preferably on its infinitesimal generator.

In this paper a stochastic process is defined to be composable if, from a probabilistic point of view, it may be regarded as a vector of distinct subprocesses. In a composable Markov process the concept of local independence between its components is defined by explicit restrictions on the infinitesimal generator. The latter concept formalizes the intuitive notion of direct but uncertain dependence between components.

The paper gives four theorems on the relation between stochastic and local independence and two examples which are intended to illustrate the practical usefulness of the concepts, which are both new.

2. Composable processes

2A. Let \( Y = Y(t) \) be a stochastic process with continuous time \( T \) and a finite state space \( E \). Assume that there are \( p > 2 \) spaces \( E_i ; i = 1, \cdots, p \); such that the number of elements of each space at least equals 2, and that there exists a one-to-one mapping \( f \) of \( E \) on to \( \times_{i=1}^p E_i \).

Definition. The process \( Y \) is a composable process with components \( Y_1, \cdots, Y_p \) given by \( f(Y(t)) = (Y_1(t), \cdots, Y_p(t)) \) if and only if for each \( A \subset \{1, 2, \cdots, p\} \) with at least 2 elements,

\[
\lim_{h \to 0} \frac{1}{h} \left\{ P \left( \bigcap_{i \in A} Y_i(t + h) \neq y_i \bigg| \bigcap_{i=1}^p Y_i(t) = y_i \right) \right\} = 0
\]

whenever \( y_i \in E_i ; i = 1, \cdots, p \); and \( t \in T \).

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In other words: \( Y \) is a composable process with components \( Y_i; \ i = 1, \ldots, p \); if the probability that more than one component changes value during a period of length \( h \), is of magnitude \( o(h) \). If this is the case, we write

\[ Y \sim (Y_1, \cdots, Y_p). \]

2B. The compositioning of \( Y \sim (Y_1, \cdots, Y_p) \) is not necessarily unique. If \( p > 2 \) let \( A_1, \cdots, A_r, 2 \leq r \leq p \), be a partitioning of \( \{1, \ldots, p\} \), i.e., if \( i \neq j \) then \( A_i \cap A_j = \emptyset \), \( A_i \neq \emptyset \) for \( i = 1, \ldots, r \), and \( \bigcup_{i=1}^{r} A_i = \{1, \ldots, p\} \). We can then define \( E'_j = \times \{A_i; E_i \} \) and \( f' \) as the one-to-one mapping of \( E \) on \( \times_{j=1}^{r} E'_j \) induced by \( f \). In this case we consequently have

\[ Y \sim (Y_1, \cdots, Y_p) \sim (Y'_1, \cdots, Y'_r) \]

where \((Y'_1(x), \ldots, Y'_r(x)) = f'(Y(x))\).

2C. If \( Y \sim (Y_1, \cdots, Y_p) \) is a composable Markov process such that all forces of transition \( \mu_x(y; y') \) exist, then \( \mu_x(y; y') \) equals zero if \( y \) and \( y' \) differ on more than one component; \( y \neq y' \in \times_{i=1}^{p} E_i \). This is an immediate consequence of the following definition of the forces of transition:

\[ \mu_x(y; y') = \lim_{h \downarrow 0} \frac{1}{h} P\{X + h = y' \mid X = y\}. \]

3. Local independence

3A. Let \( Y \) be a composable Markov process with finite state space. We shall call \( Y \) a CFMP (Composable Finite Markov Process) if for all \( y, y' \in E \) such that \( y \neq y' \), the force of transition \( \mu_x(y; y') \) exists and is a continuous and bounded function of \( x \) on any closed interval in \( T \).

A CFMP \( Y \) has a normal transition-probability \( P_{xt}(y, y') = P\{X + t = y' \mid X = y\} \), i.e., \( \lim_{t \downarrow 0} P_{xt}(y, y') \) equals 0 or 1 according as \( y \) and \( y' \) are different or equal. In this case the total force of transition

\[ \bar{\mu}_x(y) = \sum_{y' \neq y} \mu_x(y; y') = \lim_{t \downarrow 0} \frac{1}{t} (1 - P_{xt}(y, y)) \]

is a continuous and bounded function of \( t \).

3B. Let \( Y \sim (Y_1, \cdots, Y_p) \) be a CFMP. According to Section 2C only those \( \mu_x(y; y') \) differ from 0 for which \( y \) and \( y' \) are equal in all but one component, say the \( r \)th. In order to suppress superfluous arguments, we let

\[ \gamma'_r(y; y') = \mu_x(y; y') \]

where \( y' \) is the \( r \)th component of \( y' \).

Definition. The component \( Y_q \) is locally independent of the component \( Y_r \),
if and only if \( \gamma^q_i(x; y_q') \) is a constant function of the \( r \)th component \( y_r \) of \( y \) for all \( x \in T, y_q' \in E_q \) and \( y_{i'} \in E_i; i \neq r \).

The relation "locally independent of" is neither symmetric, reflexive, nor transitive.

\( Y_j \) will be said to be locally dependent on \( Y_q \) when it is not locally independent of \( Y_q \). When \( Y_j \) is locally dependent on exactly \( Y_{i_1}, \ldots, Y_{i_k} \) it is convenient to write

\[
\lambda^1_{i_1}(y_{i_1}, \ldots, y_{i_k}; y_j') = \gamma^1_j(y'; y_j').
\]

3C. We shall elucidate the relation between local independence and stochastic independence by proving some theorems.

**Theorem 1.** Let \( Y \sim (Y_1, Y_2) \) be a CFMP. \( Y_1 \) is locally independent of \( Y_2 \) if and only if for all \( t > 0 \) and \( x, x + t \in T, Y_1(x + t) \) is stochastically independent of \( Y_2(x) \) given \( Y_1(x) \).

**Proof.** Assume that \( Y_1 \) is locally independent of \( Y_2 \). For arbitrary \( x, t > 0, y_1, y \in E_1 \) and \( \gamma_{ii} \in E_2 \) we shall prove that the relation

1. \( P(Y_1(x + t) = y \mid Y_1(x) = y_1, Y_2(x) = y_2) = P(Y_1(x) = y \mid Y_1(x) = y_1) \)

holds. Denote by \( P_{x,x+t}(y_1, y_2; y) \) the left member of (1). By the composability and locally independence, we have for \( u \in E_1, v \in E_2 \):

\[
\lim_{h \downarrow 0} \frac{1}{h} P_{x,x+t,x+t+h}(u,v; y) = \lambda^1_{x+t}(u; y) \quad \text{for } y \neq u,
\]

and

\[
\lim_{h \downarrow 0} \frac{1}{h} (1 - P_{x,x+t,x+t+h}(y,v; y)) = \sum_{u \neq y} \lambda^1_{x+t}(y; u).
\]

Dividing by \( h \), letting \( h \downarrow 0 \) in the obvious relation

\[
P_{x,x+t,x+t+h}(y_1, y_2; y) = \sum_{u \in E_1, v \in E_2} P(Y_1(x + t) = u, Y_2(x + t) = v \mid Y_1(x) = y_1, Y_2(x) = y_2) P_{x+t,x+t+h}(u,v; y),
\]

utilizing (2) and (3), and rearranging we obtain

\[
\frac{d}{dt} P_{x,x+t}(y_1, y_2; y) = \sum_{u \neq y} \left[ P_{x,x+t}(y_1, y_2; u) \lambda^1_{x+t}(u; y) - P_{x,x+t}(y_1, y_2; y) \lambda^1_{x+t}(y; u) \right].
\]

For fixed \( y_1, y \in E_1 \) and \( x \) this is a finite set of linear differential equations which together with the initial requirement

\[
P_{x,x}(y_1, y_2; y) = \begin{cases} 1 & \text{when } y = y_1 \\ 0 & \text{when } y \neq y_1 \end{cases}
\]

utilizing (2) and (3), and rearranging we obtain

\[
\frac{d}{dt} P_{x,x+t}(y_1, y_2; y) = \sum_{u \neq y} \left[ P_{x,x+t}(y_1, y_2; u) \lambda^1_{x+t}(u; y) - P_{x,x+t}(y_1, y_2; y) \lambda^1_{x+t}(y; u) \right].
\]
Composable Markov processes uniquely determines $P_{x,x+t}(y_1, y_2; y)$. But the coefficients in the differential equations and also the initial requirements are independent of $y_2$, and hence the solution must also have this property.

Assume conversely that (1) is true. For $y_1 \neq y$ we have by definition

$$\gamma^1_2(y_1, y_2; y) = \lim_{t \downarrow 0} \frac{1}{t} P_{x,x+t}(y_1, y_2; y) = \lim_{t \downarrow 0} \frac{1}{t} P(Y_1(x+t) = y_1, Y_2(x+t) \neq y_2 | Y_1(x) = y_1, Y_2(x) = y_2).$$

Since the last term equals 0, $\gamma^1_2(y_1, y_2; y)$ must then, because of (1), be a constant function of $y_2$.

**Theorem 2.** Let $Y \sim (Y_1, Y_2)$ be a CFMP. If $Y_1$ is locally independent of $Y_2$ then $Y_1$ is a Markov process with forces of transition $\lambda^1_2(y_1; y)$.

**Proof.** Let $t_1 < t_2 < \cdots < t_n < t$.

$$P(Y_1(t) = y_1 | Y_1(t_1), \ldots, Y_1(t_n)) = E[P(Y_1(t) = y_1 | Y_1(t_1), Y_2(t_1), \ldots, Y_1(t_n), Y_2(t_n)) | Y_1(t_1), \ldots, Y_1(t_n)]$$

By Theorem 1 this equals

$$E[P(Y_1(t) = y_1 | Y_1(t_n)) | Y_1(t_1), \ldots, Y_1(t_n)] = P(Y_1(t) = y_1 | Y_1(t_n))$$

and $Y_1$ is hence a Markov process. The forces of transition are obviously $\lambda^1_2(y_1; y)$.

The converse of Theorem 2 is, however, not true. If for example $Y \sim (Y_1, Y_2)$ is a CFMP such that $E_1 = \{0,1\}$ and 1 is an absorbing state for $Y_1$, then $Y_1$ is a Markov process. This is seen by noting that for epochs $t_1 < t_2 < \cdots < t_n < t$ $Y_1(t_n) = 0 \Rightarrow Y_1(t_1) = 0, \ldots, Y_1(t_{n-1}) = 0$ and hence

$$P(Y_1(t) = i | Y_1(t_1) = 0, \ldots, Y_1(t_n) = 0) = P(Y_1(t) = i | Y_1(t_n) = 0)$$

for $i = 0, 1$. On the other hand

$$P(Y_1(t) = i | Y_1(t_1) = i_1, \ldots, Y_1(t_{n-1}) = i_{n-1}, Y_1(t_n) = 1)$$

$$= P(Y_1(t) = i | Y_1(t_n) = 1) = i; \quad i = 0, 1$$

for all possible values of $i_1, \ldots, i_{n-1}$. $Y_1$ may of course be locally dependent on $Y_2$.

It seems, however, that $Y_1$ being a Markov process and locally dependent on $Y_2$ is a rather special case. Probably $Y_1$ will generally not be a Markov process when it is locally dependent on $Y_2$. See Example 1 of Section 5.

If $Y \sim (Y_1, \ldots, Y_p)$ is a CFMP and $A$ is a subset of $\{1, \ldots, p\}$ such that all the elements in the set $\{Y_j | j \in A\}$ are locally independent of the rest of the components,
then the vector $Y'_1$ containing the components $\{Y_j \mid j \in A\}$ is locally independent of the vector $Y'_2$ containing the rest of the components. Hence $Y'_1$ is a Markov process.

We shall now show that complete mutual local independence of all components is equivalent to their stochastic independence.

**Theorem 3.** Let $Y \sim (Y_1, \ldots, Y_p)$ be a CFMP. Then $Y_1, \ldots, Y_p$ are stochastically independent Markov processes if and only if each component is locally independent of all the others.

**Proof.** Assume first that $Y_1, \ldots, Y_p$ are stochastically independent Markov processes. Let $P_{st}^j(y_j, y_j')$ be the transition probabilities of $Y_j$, and let $y_i, y_i' \in E_i$; $i = 1, \ldots, p$ be such that $y_q \neq y'_q$ and $y_j = y'_j; j \neq q$. Finally, let $y = (y_1, \ldots, y_p)$ and $y' = (y'_1, \ldots, y'_p)$. By the assumption,

$$P_t^q(y_i y_i'; y_q) = \lim_{t \to 0} \prod_{j=1}^p P_{st}^j(y_j, y_j')$$

which is independent of $y_j$ for all $j \neq q$ since $P_{st}^q(y_j, y_j')$ tends to 1 as $t$ tends to 0.

Assume conversely complete mutual local independence of the components $Y_1, \ldots, Y_p$. By Theorem 2 all $Y_q$ are Markov processes with forces of transition $P_{st}^q(y_q; y_q')$ and corresponding transition probabilities $P_t^q(y_q, y_q')$. These transition probabilities determine a new set of transition probabilities

$$P'_t(y, y') = \prod_{j=1}^p P_{st}^j(y_j, y_j')$$

which belong to a CFMP $Y'$ with stochastically independent components. Since however, the two CFMP's $Y'$ and $Y$ have common forces of transition, they must have identical transition probabilities and hence the Markov processes $Y_1, \ldots, Y_p$ must be stochastically independent.

The following extension of the theorem is obvious and needs no special proof.

**Corollary.** Let $Y \sim (Y_1, \ldots, Y_p)$ be a CFMP and let $A_1, \ldots, A_r$ be a partitioning of $\{1, \ldots, p\}$. Then the vectors $Y'_j$ consisting of $\{Y_i \mid i \in A_j\}$; $j = 1, \ldots, r$, are stochastically independent Markov processes if and only if $Y_j$ and $Y_k$ are mutually locally independent whenever $i$ and $k$ belong to different $A$'s.

**Complete** mutual local independence is not necessary however, for some components to be stochastically independent.

**Theorem 4.** If $Y \sim (Y_1, Y_2, Y_3)$ is CFMP such that both $Y_1$ and $Y_2$ are locally independent of $Y_3$, then $Y_1$ and $Y_2$ are stochastically independent Markov processes if and only if they are mutually locally independent.
Proof. Let $Y' = (Y_1, Y_2)$. Then $Y \sim (Y', Y_3)$ and $Y'$ is locally independent of $Y_3$. By Theorem 2, $Y' = (Y_1, Y_2)$ is a CFMP with forces of transition

$$\gamma_1^2(y_1, y_2; y_1') = \mu_4(y; y'), \text{ where } y' = (y_1', y_2, y_3),$$

and

$$\gamma_2^2(y_1, y_2; y_2^0) = \mu_4(y; y^0), \text{ where } y^0 = (y_1, y_2^0, y_3).$$

The equivalence then follows from Theorem 3.

If $Y_i$ is locally dependent on $Y_j$ and $Y_j$ is locally dependent on $Y_k$ then $Y_i$ is, in a sense, dependent on $Y_k$ through $Y_j$. This may be an important dependence even if $Y_i$ is locally independent of $Y_k$. In order to formalize this concept we introduce the relation $<$ with which we can describe the dependence structure of the process.

Definition. The binary relation $<$ between components is defined as follows.

(i) If $Y_j$ is locally dependent on $Y_i$, then $Y_i < Y_j$.

(ii) $<$ is transitive and reflexive.

Note that the relation "$Y_j$ is locally dependent on $Y_i$" is neither transitive nor reflexive, and hence line (ii) of the definition is essential.

We shall say that $Y_k$ is a predecessor of $Y_j$ whenever $Y_k < Y_j$.

By this concept we get the following extension of Theorem 4.

Corollary to Theorem 4. Let $Y \sim (Y_1, \ldots, Y_p)$ be CFMP. If the components $Y_r$ and $Y_s$ have no common predecessors, then $Y_r$ and $Y_s$ are stochastically independent random processes.

Proof. Define $A_k = \{i \mid Y_i < Y_k\}$. Since $<$ is reflexive, $k \in A_k$. The antecedent in the corollary is equivalent to $A_r \cap A_s = \emptyset$. Let now $Y'$ be the vector consisting of the components $\{Y_i \mid i \in A_r\}$, and let $Y''$ be the vector consisting of $\{Y_i \mid i \in A_s\}$. If $A_r \cup A_s = \{1, \ldots, p\}$ then $Y \sim (Y', Y'')$ where $Y'$ and $Y''$ are mutually locally independent by construction. Consequently $Y_r$ and $Y_s$ are stochastically independent because $Y'$ and $Y''$ are. If, however, $A = \{1, \ldots, p\} - (A_r \cup A_s) \neq \emptyset$ then define $Y''$ to be the vector consisting of the components $\{Y_i \mid i \in A\}$. By our construction, we have $Y \sim (Y', Y'', Y''')$ where $Y'$ and $Y''$ are both locally independent of $Y'''$. The corollary is now obtained by Theorem 4.

Note that $Y_r$ and $Y_s$ need not be Markov processes, even if they have no common predecessors. (See Example 2 of Section 5 below.)

4. Conditional Markov processes

4A. Starting with a CFMP it is sometimes possible to construct new Markov processes by conditioning. Assume for example that $Y \sim (Y_1, Y_2)$ is a CFMP with time space $T = [0, \infty)$ and with the property that there exists a state 1, say, in $E_2$ such that $\Pr(Y_2(0) = 1) = 1$. Let $(\Omega, \mathcal{B}, P)$ be the canonical probability
space defining $Y$, (Dynkin (1965), page 85), i.e., every sample point $\omega$ of $\Omega$ represents a unique sample path $y(t, \omega) = (y_1(t, \omega), y_2(t, \omega))$ with $y_2(0, \omega) = 1$. Connect to each $\omega$ in $\Omega$ a $\omega^* = g(\omega)$ which is the sample point representing the terminating sample path $y_1(t, \omega) = y^*(t, \omega^*)$; $t \in [0, D(\omega^*)]$, where $D(\omega^*)$ is the time of first departure from state 1 for $y_2(t, \omega)$. $\Omega^* = g(\Omega)$ is then a sample space to which there corresponds a probability space $(\Omega^*, \mathcal{G}^*, P^*)$ where $\mathcal{G}^*$ may be taken as the largest $\sigma$-algebra such that $g$ is measurable, and $P^* = Pg^{-1}$. This probability space determines a Markov process $Y^*$ with state space $E_1$, time space $[0, \infty]$, terminal time $D$, forces of transition $\gamma^*_2(y_1, y'_1) = \gamma^*_1(y_1, 1; y'_1)$, and transition probabilities

$$P^*_x(y_1, y'_1) = P(Y_1(x + t) = y_1', Y_2(\xi) = 1 \mid x < \xi \leq x + t \mid Y_1(x) = y_1, Y_2(x) = 1).$$

The truth of this is seen by elementary conditional probability.

4B. Consider the probability $P(Y^*(x) = y \mid \cap_{i=1}^n Y^*(x_i) = y_i \cap D > \tau)$ for $0 < x_1 < \cdots < x_n < x < \tau$, and $y, y_i \in E_1$. This probability has the Markov property in the sense that for all $0 \leq x_1 < \cdots < x_n < x < \tau$, $P(Y^*(x) = y \mid \cap_{i=1}^n Y^*(x_i) = y_i \cap D > \tau) = P(Y^*(x) = y \mid Y^*(x_n) = y_n \cap D > \tau)$. These conditional probabilities therefore are transition probabilities for a Markov process with time space $[0, \tau]$ and with forces of transition $\gamma^*_x(y_1, y'_1)$. We shall denote this process the conditional Markov process, given $Y_2 = 1$. As will be seen in the examples below, this conditional process may have a structure which is much simpler and more informative than the structure of the process from which it is determined.

5. Examples

Example 1. Let us consider a queueing model described by Khintchine ((1960), page 82). Calls arrive at a telephone exchange with $R$ lines, $L_1, \ldots, L_R$, according to a Poisson process with parameter $\lambda$. The service pattern is as follows. If at time $x$ a call arrives and the lines $L_1, L_2, \ldots, L_{k-1}$ are busy while $L_k$ is free ($1 < k < R$), this call is transferred via $L_k$. If all $R$ lines are busy, the call is lost.

Assume that the conversation periods are stochastically independent with a common exponential distribution with parameter $1$, and that they are stochastically independent of the incoming stream of calls.

Define the random variables

$$Y_i(x) = \begin{cases} 0 & \text{if } L_i \text{ is free at time } x, \\ 1 & \text{if } L_i \text{ is busy at time } x, \end{cases} \quad i = 1, \ldots, R.$$

Obviously the stochastic process $Y(x) = (Y_1(x), \ldots, Y_R(x))$ is a composable finite Markov process with forces of transition given by
Composable Markov processes

Consequently $Y_k$ is locally dependent on $Y_1, \ldots, Y_k$, and locally independent of $Y_{k+1}, \ldots, Y_R$. When $R = 4$ we can draw a picture of this structure as in Figure 1 where an arrow from $Y_j$ to $Y_k$ indicates that $Y_k$ is locally dependent of $Y_j$.

![Figure 1](image)

For $K < R$ we may define $Y' = (Y_1, \ldots, Y_K)$ and $Y'' = (Y_{K+1}, \ldots, Y_R)$. Thus $Y \sim (Y', Y'')$ where $Y'$ is locally independent of $Y''$. Consequently $Y'$ is a Markov process by Theorem 1—something which is also self-evident.

Khintchine ((1960), page 83) has shown that $Y_2$ is not a Markov process despite the fact that both $Y' = (Y_1, Y_2)$ and $Y_1$ are.

**Example 2.** Suppose that one wishes to investigate the simultaneous influence on mortality of the five diseases

- $Y_1$: chill
- $Y_2$: pneumonia
- $Y_3$: bronchitis
- $Y_4$: hypertoni
- $Y_5$: angina pectoris

A person may have or be free from each of these diseases. A live person of age $x$ is characterized by the vector $(Y_1(x), \ldots, Y_5(x))$, where $Y_i(x)$ equals 1 or 0 according as he has or does not have the $i$th disease. If the person dies at age $\tau$, we shall say that at age $x > \tau$ he is characterized by the vector

$$(Y_1(x), \ldots, Y_5(x)) = (Y_1(\tau), \ldots, Y_5(\tau)),$$

which in fact gives his status at death.
By introducing the component \( Y_n(x) \) which equals 1 or 0 according as he is alive at age \( x \) or he has died at an age \( \tau \leq x \), we may give a complete characterization of him by the vector \( (Y_1(x), \ldots, Y_6(x)) \).

A person cannot recover from any disease nor get a new one at death. It is further natural to assume that a person cannot simultaneously get or recover from two diseases, nor can he recover from one disease in the same instant as he gets another.

\( Y = (Y_1, \ldots, Y_6) \) is then a composable stochastic process with the finite state space \( \mathcal{E} = \times_{i=1}^6 \{0,1\} \).

We shall assume, possibly with some lack of realism, that \( Y \) is a Markov process.

From the moment when \( Y_6 \) first equals 0, no more transfers are possible. Consequently \( Y_1, \ldots, Y_5 \) are locally dependent on \( Y_6 \). Conversely, mortality depends on the state of health, so \( Y_6 \) is locally dependent on \( Y_1, \ldots, Y_5 \). Although we will not give any guarantee of medical realism, it is probably reasonable to assume that \( Y_1 \) is locally independent of \( Y_2, \ldots, Y_5 \); \( Y_2 \) is locally dependent on \( Y_1 \) and \( Y_3 \); \( Y_3 \) is locally independent of \( Y_4, Y_5 \); \( Y_4 \) is locally dependent on \( Y_1 \) and \( Y_3 \); \( Y_5 \) is locally independent of \( Y_1, Y_2, Y_3, \) and \( Y_5 \); and \( Y_6 \) is locally dependent on \( Y_4 \) and locally independent of \( Y_1, Y_2, Y_3 \).

Figure 2 gives a picture of this structure.

![Figure 2](image)

By looking at Figure 2 we immediately see that for all \( i, j \) we have \( Y_i < Y_j \). Thus all components are stochastically dependent.

If we proceed as in Section 4, however, and construct the conditional Markov process \( \mathcal{P} \), given that \( Y_6 = 1 \), \( \mathcal{P} = (\mathcal{P}_1, \ldots, \mathcal{P}_5) \) is a CFMP with local dependence structure as shown in Figure 3.
We see that the components $Y' = (Y_1, Y_2, Y_3)$ and $Y'' = (Y_4, Y_5)$ are mutually locally independent, and consequently stochastically independent Markov processes.

This illustrates a feature common to many situations where CFMP models are useful. The CFMP model describes the evolution for instance of a person, an animal, a machine, or another individual or unit which may die or stop functioning. One of the components of the CFMP indicates whether the individual (unit) is alive (functioning) or dead (out of function). In this way the rest of the components are locally dependent on this particular one, and if the latter is locally dependent on the others (which often is the case), then all components are stochastically dependent. If, however, we construct the conditional process described, a more interesting local dependence structure may be obtained. This structure seems to correspond to our intuitive understanding of the relations between the phenomena under consideration. In fact, we probably take into account only what happens to the individual (or unit) up to its death (or so long as it functions).

Returning to our example, let us recompose $\bar{Y}$ by letting $Y' = (Y_1, Y_4, Y_5)$, $Y'' = (Y_2, Y_3)$, and $\bar{Y} \sim (Y', Y'')$. Because $Y'$ is locally independent of $Y''$, $Y'$ is a (conditional) CFMP and since $\bar{Y}_1$ is locally independent of $\bar{Y}_4$ and $\bar{Y}_5$, $\bar{Y}_1$ is a Markov process. $Y''$ need not, however, be a Markov process. If, on the other hand, we had recomposed $\bar{Y}$ into $\bar{Y} \sim (Y_1^0, Y_2^0)$ where $Y_1^0 = (Y_1, Y_3)$ and $Y_2^0 = (Y_2, Y_3, Y_4)$, then neither $Y_1^0$ nor $Y_2^0$ need be Markov processes. The reason is that neither $Y_1^0$ nor $Y_2^0$ consists of components from the "top of the local dependence tree" in Figure 3.

This example throws some further light upon Markov process models in general. Let, in fact, a complicated phenomenon be described by a CFMP. This CFMP may be difficult to handle if it has too many components. The following question then arises. Is it possible to take under investigation only some part of the phenomenon which possesses its main features? Restating this question in terms of the components of the CFMP $Y \sim (Y_1, \ldots, Y_p)$, we may ask whether it is possible to recompose $Y$ into $(Y', Y'')$, where $Y' = (Y_1, \ldots, Y_a)$ describes these main features and where $Y'$ is not too complicated for investigation? If investigation means estimation of the probability structure of the random process $Y'$, this may be difficult unless $Y'$ is a Markov process. A reasonable requirement
for the decomposition of \( Y \) is therefore that \( Y' \) be such a process. If we know the local dependence structure of the process \( Y \), we may draw a (mental or actual) picture of the “local dependence tree” as we have done in Figures 1 to 3. From Theorem 1 we then know that a set of components \( Y_{i_1}, \ldots, Y_i \) forming a “top” of this tree, if any, constitute a component \( Y' = (Y_{i_1}, \ldots, Y_i) \) which is a Markov process. We shall call such a component Markovian. A Markovian component of a CFMP \( Y \sim (Y_1, \ldots, Y_p) \) is then by definition a component \( Y' = (Y_{i_1}, \ldots, Y_i) \) such that for all \( k: q < k \leq p \), \( Y_{i_k} \) is not a predecessor of any of the components \( Y_{i_1}, \ldots, Y_i \). Alternatively, if \( Y \sim (Y', Y'') \) is a CFMP, then \( Y' \) is a Markovian component if \( Y' \) is locally independent of \( Y'' \).

The question asked above may then be answered by looking through the possible Markovian components of \( Y \) and judging them with respect to complexity and adequacy.

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**References**
