Uncertainty 9: Monty Hall

Welcome to the Monty Hall show. Behind one of these three doors is a big cash prize. Behind the two others, nothing. You can chose one door. .... **that one** .... Right. Before we open that door, lets open one other door. It's empty. I'll now give you the chance to switch to the other unopened door. Will you switch or will you stay?

The object is to maximize the probability of getting the prize, and with only two doors to choose from, you want to choose the door that has more than 50% chance of containing the prize. Note that Monty Hall never had this setup in real life and has said that he never would. This is a thought experiment.

This example of probabilistic reasoning is meant to illustrate that this may not be as intuitive as you may think. However, as it's described here, the probabilities are not uniquely defined, but are rather a function of what you believe concerning the described problem.

A Bayesian way of solving this problem can be attempted, since it's clear that we want to update probabilities after getting some information. First off, I want to define things, so the math will go a little smoother.

The unobserved stuff that we're interested in is the door that contains the prize, so we could call the three different options here the models. However, these 'model's does not uniquely define the data probabilities, so we need to take a closer look at the data, as well as find models that do uniquely define the data probabilities.

Back to the door models. The prize can be behind the door you chose, let's call that model N. or it can be behind one of the two others. We can call that model M. And model M can be subdivided into model M1, the prize is behind the leftmost of the doors you didn't choose. Or it can be behind the rightmost, M2.
<pause>

The data we got, is that Monty Hall opened one unoccupied door. This can be divided into three components. 1: He opened a door. He might not do that always. 2: The prize was not behind that door. It could be that Monty Hall sometimes opens the door with the prize and informs the participant that he/she didn't chose the correct one 3. He could've opened the rightmost or leftmost of the doors you didn't choose. You could imagine that Monty Hall chose the rightmost if you've chosen the correct door and the leftmost if not and this door is unoccupied. In which case you may want to switch if he's chosen the leftmost and not if he's chosen the rightmost. But without any such info on how this strategy is, it would not be possible to handle this information. See Prepoceros' video for more about this type of Monty Hall strategy.
<point left>

But here, I will not go further in this direction and simply strike off data 3 as irrelevant.

So we've got two data to relate to.

D2, the door is empty, has the characteristics from what I described in clip 5. One sub-model of a model

M has been falsified. Thus the probability for M must decrease. Or, if the prior probability for opening an unoccupied door is zero, the probability will stay the same. Normally we would not contemplate such a possibility. Who would want to falsify an already impossible model? But in this case, it could be that Monty Hall sits with privileged information and uses it.

From symmetry, see clip 7, model N starts off with 1/3 while M=M1+M2 starts out with probability 2/3. However, we would stay if Pr(M) it drops below ½, which it doesn't necessarily do.

If we take a look on the number line representing the probability for M, that is: the prize is behind one of the other doors, we start off at 2/3. We switch if the posterior probability is greater than 50% and stay if not.

So we don't know if we should switch or not, just from this. And of course, we haven't handled data 1, yet. The problem only describes a one-off event. However, in order to update the probabilities we need to know how likely that one-off event is. Fortunately we need only look at the set of Monty Hall playing rules which has the one-off outcome as a possibility. But there can be a wide variety of behavior that **can** deliver the described results. Here I will not go into every possibility but rather try to span out the possibilities with some archetypal examples.

Now I assign a probability to Monty Hall's behavior. The first specification is the one that's "**meant".** It's a real pity that this normally isn't specified when the

Monty Hall problem is described.

This strategy is simply that Monte Hall will always produce the data described in the problem. He will always open a door you didn't chose and that door will always be empty.

So data 1 and data 2 are irrelevant and will not change the probability for you having chosen the right door. See clip 6 for more on irrelevant info.

The probability for N still stays the same. Irrelevant info may sound bad, but in this case it's good. We started off with probability 1/3 for choosing the right model and 2/3 for choosing the wrong one. Since these probabilities do not change, you will have a 1/3 probability of getting the prize if you stay and 2/3 probability if you switch. So you definitely want to switch.

You can also see this by this graphical representation of the three possibilities, before the data arrives. When you got the data, one unoccupied door is removed. And you can see that in two of the cases, you will chose the prize by switching and in only one of the cases you'll get it by staying.

The strategy can be relaxed a little. It doesn't have to be that Monte Hall always opens a door. As long as he decides on opening a door independently of whether you've chosen the right one (model N) or nor (model M), data 1 will stay irrelevant. However, that gives an infinite variety of strategies and I only want to describe archetypal strategies here.

Before going to the other strategies, let's describe the expected return for each of your choices, staying or switching. If you switch, the expected return, using the weighting analogy from clip 8, is simply the probability for you having chosen the wrong door, the probability of M. If you stay, the expected return will be the probability for N.

So let's make a table, showing this. Let's also show the probability of getting the data. In this case, we'll always get that Monty Hall opens and unoccupied door if he opens one. Let's call the probability of him opening one for 'p'. We can the calculate the expected return before the data for each of the strategies of the contestants, to see it more from Monte Halls perspective. The expected loss from Monte Hall's perspective will be between 2/3 and 1/3 for contestants that switches, depending on how often he opens the door.

The second possible strategy that can cause the data you've got is that Monte Hall has no privileged information or does not use that privileged information. If you've chosen incorrectly, he'll have a 50% chance of opening the door with the prize. In this case, if he's opened an unoccupied door, the probability for you having chosen correctly is the same as the probability for you having chosen incorrectly, namely 50%. So staying or switching is all the same. This strategy may seem reasonable, as Monte Hall does not use privileged info. In strategy 1 he needs to know the whereabouts of the prize in order to avoid opening the door with the prize. Using that info in the strategy may for some feel like cheating, which may be part of the reason why so many answers that staying or switching doesn't matter. If first privileged info is used in the strategy, the worms are out of the can. How much cheating can be achieved when Monty Hall uses privileged info?

Going back to the table, the probability of getting the data is 2/3. the expected return conditioned on the data is ½ and unconditioned on the data it's 1/3 for both staying and switching.

That bring us to the sneaky third option. Monty Hall opens the door with the prize if given the chance, i.e. if you've chosen incorrectly. Only if you've chosen correctly will he open an unoccupied door and offer a switch. The probability of getting the prize by staying, conditioned on the data is then 100%. You'll only get the prize by staying! So this gives another result than the intended interpretation. Of course, you only get the data in 1/3 of the cases.

Going to the table, it's a 1/3 probability of getting the data and an unconditional expected return of 1/3 for staying, as usual.

Strategy four is more of the same. Now Monty Hall only opens an unoccupied door if you've chosen correctly. So data 1 is no longer irrelevant. The probability of getting D1 is now 1 if N is true and 0 if not. D2 is now irrelevant, though. The posterior probability for N and M is the same as for strategy three. And the probability for getting the data is also the same. So we can collapse the row for strategy three and four into one in the table.

The last strategy I'll describe is that Monty Hall only opens an unoccupied door if you've chosen incorrectly. Again D2 is irrelevant, while D1 is **highly** relevant. Of course, you'll always want to switch when given the chance in this scenario. So let's put that into the table,too.

There are infinitely many other strategies, as can be seen by simply combining the strategies described here. For instance, Monty Hall could flick a coin or throw a die and decide whether to adopt strategy 1 or 4. or he could in some cases stick you with your choice independently of what you chose. So there's all kinds of hybrid strategies, too. Websnarf described one of those in his video. The strategies I've described can be call archetypal strategies. Maybe there's even more of those.

The conclusion is that there's no fixed answer to the problem as it's usually stated. If it's stated so that it's clear that strategy 1 is used, then it's a small matter to find the probabilities, using the irrelevant data principle from clip 6. If this isn't specified, the question is what are your personal probabilities for the strategies of Monty Hall.

In normal circumstances, only strategy one or two will be the favored interpretation. However, a game theorist , or someone with a suspicious mind, may feel that strategy 3 or 4 is a good interpretation. Certainly, if you write the expected gain of the participant (and thus the expected loss of Monty Hall), you find that Monty Hall running strategy ¾ and you always staying is the Nash equilibrium. The table lists both expected gain given the data, the probability of the data and the unconditional probabilities. See clip 8 if you want to learn more about expectation. The table can be accessed in the video description (point left) and can be used for assessing prior probabilities of the different archetypal strategies.

In order to truly find out if you want to switch or not, you need to span out all possibilities that would result in the one-off data that you got and give a prior probability to all those. As this set of possibilities is almost hopelessly large and the analysis is probably not worth the effort, a look at the archetypes is better. However, even that may be going to far in a practical situation. Normally you would pick one interpretation and stick to it as long as the data wasn't too surprising.

Let's start off with taking the middle-road and assign prior probabilities to the strategies. First, let's assign 25% probability for each of the four archetypal strategies, before the data. After the data, the probabilities change, but not that much. Strategy 1 has become more probable but not exceeding 50%. Strategy two and five stays the same, while strategy 3+4 decreases in probability. You would want to switch in this case, as the probability for getting the prize if you stay is 37.5% and 62.5% is you switch.

If you start off with one interpretation, say strategy 2, but are suspicious enough to look at the other strategies too, you could assign 94% (for instance) to strategy 2 and 2% to the three other archetypes. After the data, the probabilities are 3%, 94%, 1% and 2%, respectively for strategy 1 to 4. the unconditional probability for winning will be 49% if you stay and 51% if you switch. So you want to switch, but the relative gain is very small indeed. Hardly worth the bother of going through the analysis.

But if you've got a suspicious mind, interpreted the problem as strategy ¾ being most likely, and assigned a prior probability of 94% for this, the data would in no way convince you otherwise. And you would still definitely want to switch,

This has been a much more involved presentation of the Monty Hall problem than what is usual. The object here was partially to say that if you want to present the problem, do it right or not at all, unless you want to delve into strategies and subjective probabilities. For the specially interested, an extended manuscript and slides are given. <point>