Quantum Wires and Quantum Point Contacts
Quasi-1D systems with width $w$ comparable to the de Broglie wavelength

Classification of quasi-1D systems
Wave functions and eigenvalues

A strip of conducting material about 10 nanometers or less in width and thickness that displays quantum-mechanical effects.

Simple, but realistic model:

$$V(y) \propto y^2$$

- Split gate
- Interface
- Effective potential
- Electron density
Confinement (model) potential: \( V(y) = \frac{m^* \omega_0^2}{2} y^2 \)

Schrödinger equation:

\[
\left( -\frac{\hbar^2}{2m^*} \frac{\partial^2}{\partial x^2} - \frac{\hbar^2}{2m^*} \frac{\partial^2}{\partial y^2} + \frac{m^* \omega_0^2}{2} y^2 \right) \psi(x, y) = E \psi(x, y)
\]

Solution: \( \psi(x, y) = e^{i k x x} \varphi(y) \)

\[
\left( -\frac{\hbar^2}{2m^*} \frac{\partial^2}{\partial y^2} + \frac{m^* \omega_0^2}{2} y^2 \right) \varphi(y) = \left( E - \frac{\hbar^2 k_x^2}{2m^*} \right) \varphi(y)
\]

Harmonic oscillator!
Energy spectrum - set of 1D bands:

\[ E_j(k_x) = E_j + \frac{\hbar^2 k_x^2}{2m^*}, \quad E_j \equiv \hbar \omega_0 \left( j + \frac{1}{2} \right) \]

Density of states:

\[ k_x(E) = \pm \left( \frac{2m^*}{\hbar^2} \right)^{1/2} \sqrt{E - E_j} \]

\[ g_s \times \frac{d k_x(E)}{dE} \quad = \quad \left( \frac{2m^*}{\hbar^2} \right)^{1/2} \frac{1}{\sqrt{E - E_j}} \]

\[ D(E) = \frac{g_s \sqrt{2m^*}}{\pi \hbar} \sum_j \frac{\Theta(E - E_j)}{\sqrt{E - E_j}} \]

\[ g_s = 2 \]
Energy spectrum and density of states for parabolic confinement

\[ n = \frac{2g_s}{\pi \hbar} \sum_j \sqrt{2m^* (E - E_j)} \Theta(E - E_j) \]

\[ \propto \sum_j \frac{\Theta(E - E_j)}{\sqrt{E - E_j}} \]
Diffusive quantum wires

Top view of a diffusive quantum wire \((L = 40 \, \mu\text{m}, \text{geometric width } w_g = 150 \, \text{nm})\), patterned into a Ga[AI]As-HEMT by local oxidation.

The bright lines define the walls (a close-up is shown in the inset).

The electrostatic wire width \(w\) can be tuned by applying voltages to the two in-plane gates IPG 1 and IPG 2.
Parabolic confinement: \( V(y) = \frac{1}{2} m^* \omega_0^2 y^2 \)

Then we have the equation very similar to that leading to Landau quantization, however with replacements:

\[
\omega_c \rightarrow \omega(B) = \sqrt{\omega_c^2 + \omega_0^2}, \quad y_0 \rightarrow \bar{y} = y_0 \frac{\omega_c^2}{\omega^2(B)}
\]

The resulting equation is:

\[
H_{xy} = -\frac{\hbar^2}{2m^*} \frac{\partial^2}{\partial y^2} + \frac{m^* \omega^2}{2} (y - \bar{y})^2 + \frac{\hbar^2 k_x^2}{2m^*(B)}
\]

\[
m^*(B) = m^* \left( \frac{\omega(B)}{\omega_c} \right)^2 > m^*
\]
Magneto-electric modes

\[ E = \hbar \omega(B) \left( \frac{j}{2} + \frac{1}{2} \right) + \frac{\hbar^2 k_x^2}{2m^*(B)} \]

Magnetic field increases the confinement strength and can lead to depopulation of magneto-electric modes.

As a result, density of states oscillates as a function of magnetic field. However, the oscillations are not periodic in \(1/B\).

In very strong magnetic fields, \( \omega_0 \ll \omega_c \), we come back to the case of 2DEG.
The magnetoresistivity of the wire shown earlier.

The most prominent feature is the oscillation as a function of $B$, which detects the magnetic depopulation of the wire modes.

The positions of the oscillation minima are plotted vs. $1/ B$ as full circles in the inset. Here, the line is a least squares fit to theory, which gives an electronic wire width of 158 nm.
Boundary scattering in wires is important. If the roughness scale is less or of the order of the Fermi wave length, then the scattering is *diffusive*.

For smooth edges the scattering is *mirror*, it does not influence conductance very much.

Magneto-transport is specifically sensitive to the boundary scattering. Depending on the ratio between the wire width and cyclotron radius electron spend different time near the edges.

It influences the conductance, see next slide.
Quantum wires and point contacts

Wire peaks in QWs defined by top gates of similar widths (shown in the inset) are much weaker (the reflection is more specular). The length of the wires was $L = 12 \mu m$.

Diffusive scattering at the wire boundaries causes a magnetoresistivity peak at $\omega = 0.55r_c$.

The main figure shows these peaks for wires of different widths, studied at QW made by ion beam implantation, which generates particularly rough, diffusive walls.
Ballistic quantum wires

First system that have been studied - ballistic point contact

GaAlAs-HEMT

Two traces correspond to different concentrations tuned by illumination.

T = 60 mK

Conductance is quantized in units of $j \cdot \frac{2e^2}{h}$
The quantization seems to be similar to the quantum Hall effect.

However:

- No impurities in the contact region
- The typical accuracy is \( \delta R/(2e^2/h) = 10^{-2} \)
- Conductance quantization usually disappears at \( L > 2 \mu m \)

Principal questions:

- How finite conductance can exist without scattering?
- Where the heat is released?
- What is the reason for quantization?
Simple model:

A barrier with tunneling probability $T$, connecting 1D wires

Voltage $V = (\mu_S - \mu_D)/e \rightarrow$ current $I = n_1 e \nu$.

Let us calculate the current in a symmetric system:

$$I(E) = eT \left[ \overrightarrow{n}_S(E)v_S(E) - \overleftarrow{n}_D(E)v_D(E) \right]$$

$$= eT \left[ \overrightarrow{D}_S(E)v_S(E)f(E - \mu_S) - \overleftarrow{D}_D(E)v_D(E)f(E - \mu_D) \right]$$
Quantum wires and point contacts

For a 1D system the density of states is inversely proportional to the velocity!

\[ I(E) = \frac{eT}{2\epsilon_2TV} \int_{-\Theta - \theta}^{\Theta + \theta} \left[ f(E - \mu) - f(E - \mu_D) \right] dE \]

\[ \frac{D_j(E)}{D_i(E)} = \frac{1}{2} \frac{1}{\pi \hbar v_j(E)} \]
At $T=1$, $G = \frac{2e^2}{h}$

Here we have assumed that the distributions in the leads are described by the Fermi functions, i.e., the leads are in the equilibrium.

Such an equilibrium is achieved only at the distances larger than the electron mean free path. Thus, the dissipation takes place in the leads close to the contacts.

What we have calculated is the total resistance of the device, which can be understood as an intrinsic contact resistance.
What happens if the wire is quasi-1D, i.e., has several modes?

\[
    G = \frac{2e^2}{h} \sum_{\alpha, \beta} |t_{\alpha\beta}|^2
\]

\( t_{\alpha\beta} \) - transmission amplitudes

If there are no inter-mode scatterings, then the transport is called **adiabatic**.

This is the case for the wires with smooth modulation of the width.

In case of pronounced backscattering the shape of the conductance steps is modified.
A model for smooth point contacts leads to the following criterion for the transition regions to be narrow:

$$\pi^2 \sqrt{2\rho/d_0} \gg 1$$
The Landauer formula is applicable only if nothing happens inside the wire – no interactions, inelastic processes, etc.

Example: puzzle of 0.7 plateaus

Conductance of a QPC as a function of the spit-gate voltage.

(a) The "0.7 feature" becomes sharper with temperature increase.

(b) In strong parallel magnetic fields, the spin degeneracy is removed and additional plateaus are visible at odd integers of $\frac{e^2}{h}$. As $B$ is reduced, the spin-split plateau at $G = 0.5\frac{e^2}{h}$ evolves into the 0.7 feature.
How one can avoid contact resistance?

Use four-probe measurements!

Four-terminal resistance measurements on a ballistic quantum wire.
A quantum well is grown in [100] direction, and three tungsten gate stripes of are evaporated on top. The wafer is then cleaved, and a modulation-doped layer of Al$_{0.3}$Ga$_{0.7}$As is grown along the [110] direction. The wire extends along the cleave.

4-terminal experiments beautifully confirm the theory of 1D ballistic transport
The two-terminal resistance of the wire (A) along gate 2 and its four-terminal resistance (B) are compared.

Four-terminal resistance vanishes!
What is the current distribution in a QPC?

A way to measure – Scanning Gate Microscopy (SGM)

The tip introduces a movable depletion region which scatters electron waves flowing from the quantum point contact (QPC).

An image of electron flow is obtained by measuring the effect the tip has on QPC conductance as a function of tip position.

Explanation: the branching of current flux is due to focusing of the electron paths by ripples in the background potential.

More sophisticated SGM technique allows analyzing e-e interaction.

The experimental current distribution can be explained by account of non-equilibrium effects in QPC.

Topinka et al., Nature 2001

Jura et al., preprint 2010
Landauer formula: \[ G = \frac{2e^2}{h} \sum_{\alpha, \beta} |t_{\alpha \beta}|^2 \]

Transition amplitudes

Adiabatic transport: \[ |t_{\alpha \beta}|^2 = T_\alpha \delta_{\alpha \beta} \]

Transition probabilities

\[ G = \frac{2e^2}{h} \sum_{\alpha} T_\alpha \]

To get accurate quantization of conductance we need \( T \) to be very close to 1 for occupied modes and 0 for empty modes.
Edge states - 1D states in 2DEG

What happens near edges of 2DEG in magnetic field?

Skipping orbits, or edge states

Quantum mechanics: squeezing of wave function near edges leads to increase in the Landau level energy
Properties:

• Strictly one-dimensional

• All electrons near one edge move in the same direction while at the opposite edge move in the opposite direction

Current flow diagram of a Hall bar for two occupied Landau levels:

To be scattered back an electron has to traverse the whole sample, $T=1$.

Hall bar edge behaves as an ideal ballistic quantum wire!
Important:

If the edges are in an equilibrium, or there is no backscattering from contacts, then the potential difference between the edges is just the same as between the longitudinal contacts.

Then each channel contributes to the “Hall” current as an ideal ballistic quantum wire.

\[
R_{xx} = \frac{V_1 - V_2}{I_s} = \frac{V_3 - V_4}{I_s} = 0
\]
\[
R_{xy} = \frac{V_3 - V_1}{I_s} = \frac{V_4 - V_2}{I_s} = \frac{\hbar}{2je^2}
\]

This picture can be checked by inserting a gate causing backscattering in some channels.
The steps between Hall plateaus occur when electrons can percolate between the edges.

Simultaneously, transverse conductance becomes finite.

The steps are very accurate because backscattering is exponentially suppressed.
Structure of spinless edge states in the IQHE regime.

(a)-(c) One-electron picture of edge states. (a) Top view on the 2DEG plane near the edge. (b) Adiabatic bending of Landau levels along the increasing potential energy near the edge. (c) Electron density as a function of the distance to the boundary.

(d)-(t) Self-consistent electrostatic picture. (d) Top view of the 2DEG near the edge. (e) Bending of the electrostatic potential energy and the Landau levels. (f) Electron density as a function of distance to the middle of the depletion region.

Q: Why trajectories corresponding to different LLs do not mix?

A: Metallic screening

Chklovskii, Shklovskii, Glazman (1992)
A mechanically controlled break junction for observing conductance quantization in an Al QPC.

The elastic substrate is bent by pushing rod with a piezoelectric element.

The thin Al bridge, fabricated by electron beam lithography, can be broken and reconnected for many cycles.
Carbon nanotubes

Discovered by S. Iijima, 1991
Graphite sheet can be rolled in a tube, either single-wall, or multi-wall.

The graphene lattice and lattice vectors $\mathbf{a}_1$ and $\mathbf{a}_2$. A wrapping vector $n_1\mathbf{a}_1+n_2\mathbf{a}_2 = 4\mathbf{a}_1+2\mathbf{a}_2$ is shown. The shaded area of graphene will be rolled into a tube so that the wrapping vector encircles the waist of the NT. The chiral angle $\phi$ is measured between $\mathbf{a}_1$ and the wrapping vector.
The geometry of an unstrained NT is described by a wrapping vector. The wrapping vector encircles the waist of a NT so that the tip of the vector meets its own tail.

A NT with wrapping vector $5a_1+5a_2$, $\phi = 30^\circ$.

Chiral angle $\phi$ can vary between $0^\circ$ and $30^\circ$ (any wrapping vector outside this range can be mapped onto $0^\circ < \phi < 30^\circ$ by a symmetry transformation).
The most eye-catching features of these structures are their electronic, mechanical, optical and chemical characteristics, which open a way to future applications.

For commercial application, large quantities of purified nanotubes are needed.

**Electrical conductivity.** Depending on their chiral vector, carbon nanotubes with a small diameter are either semi-conducting or metallic.

The differences in conducting properties are caused by the molecular structure that results in a different band structure and thus a different band gap.

**Mechanical strength.** Carbon nanotubes have a very large Young modulus in their axial direction.

The nanotube as a whole is very flexible because of the great length. Therefore, these compounds are potentially suitable for applications in composite materials that need anisotropic properties.
Quantization around a graphene cylinder

In a cylinder such as a NT, the electron wave number perpendicular to the cylinder’s axial direction, $k_\perp$, is quantized.

For zero chiral angle

Electron states near $E_F$ are defined by the intersection of allowed $k$ with the dispersion cones at the $K$ points.
The quantized \( k_\perp \) are determined by the boundary condition

\[
\pi D k_\perp = 2\pi j
\]

where \( j \) is an integer and \( D \) is the NT diameter.

The exact alignment between allowed \( k \) values and the \( K \) points of graphene is critical in determining the electrical properties of a NT.

Consider NTs with wrapping indices of the form \((n_1, 0)\).

\[
k_\perp = \frac{2\pi j}{n_1}, \quad \mathbf{K} \rightarrow (k_\parallel, k_\perp) = \left(0, \frac{4\pi}{3}\right)
\]

When \( n_1 \) is a multiple of 3 \((n_1 = 3q \text{ where } q \text{ is an integer})\) there is an allowed \( k_\perp \) that coincides with \( \mathbf{K} \). Setting \( j = 2q \) one gets

\[
k_\perp = \frac{2\pi j}{n_1} = \frac{4\pi q}{3q} = \frac{4\pi}{3}.
\]
$n_1 = 3q+p$

(a) $p = 0$

(b) $p = 1$

(c) $p = -1$
The way in which nanotubes are formed is not exactly known. The growth mechanism is still a subject of controversy, and more than one mechanism might be operative during the formation of CNTs.

Potential applications of CNTs

• Hydrogen storage
• Composite materials
• Li intercalation (for Li-ion batteries)
• Electrochemical capacitors
• Molecular electronics
  - field emitting devices: flat panel displays, gas discharge tubes in telecom networks, electron guns for electron microscopes, AFM tips and microwave amplifiers;
  - transistors
  - nano-probes and sensors (AFM tips, etc)
Circuitry of ballistic wires and QPCs

Büttiker formula:

\[ I_p = \frac{2e}{\hbar} \sum_q (T_{qp}\mu_p - T_{pq}\mu_q) \]

Subscripts \( p \) and \( q \) label terminals

A very useful tool to study ballistic systems of non-interacting electrons
Series connection

Two QPCs in series is a non-Ohmic system: the series resistance is much smaller than the sum of the individual QPC resistances.

One needs quantum mechanics to understand such behavior - after the first QPC the electron beam gets collimated, and then it is prepared to pass through the second QPC without backscattering.
Parallel connection

If two QPCs are connected in parallel, special effect in magnetoresistance occur.

Shown are the so-called magnetic electron focusing.
Open questions

We avoided most interesting questions posed by electron-electron interaction in combination with quantum mechanical correlations.

In particular, we did not discuss:

• Fractional quantum Hall effect
• Kondo effect in QPC and quantum dots
• Luttinger liquid effects, important for 1D systems

These effects belong to cutting edge of modern nanoscience and nanotechnology.