Interaction of Lamb modes with two-level systems in amorphous nanoscopic membranes

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Using a generalized model of interaction between a two-level system (TLS) and an arbitrary deformation of the material, we calculate the interaction of Lamb modes with TLSs in amorphous nanoscopic membranes. We compare the mean free paths of the Lamb modes of different symmetries and calculate the heat conductivity \( \kappa \).

In the limit of an infinitely wide membrane, the heat conductivity is divergent. Nevertheless, the finite size of the membrane imposes a lower cutoff for the phonon frequencies, which leads to the temperature dependence \( \kappa \propto T(a + b \ln T) \). This temperature dependence is a hallmark of the TLS-limited heat conductance at low temperature.

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I. INTRODUCTION

The development of nanodetectors and the strict requirements on their performance triggered intense experimental and theoretical studies of their thermal properties. These detectors work usually in a temperature range around 1 K or below, and are supported by thin, insulating membranes. The thickness of such membranes is of the order of 100 nm, which, in the given temperature range, makes it comparable to the dominant thermal phonon wavelength. In problems where the phonon wavelength is comparable to or longer than some of the dimensions of the system in question, the three-dimensional phonon gas model cannot be applied anymore to calculate the system’s thermal properties. Instead, one has to use the phonon modes specific to the system. If, like in our case at subkelvin temperatures, the mean free path is much longer than the wavelength, these phonon modes are the eigenmodes of the elastic equation for the given geometry.

The membranes that support the detectors are made of amorphous, low stress silicon nitride (SiN\(_x\)) and their thermal properties have been measured in various geometries by different groups (see, for example, Refs. 1–6). Depending on the quality and the dimensions of the samples, and possibly the temperature range in which measurements were done, the heat flux along the membrane may be due to either diffusive\(^2\)–\(^4\) or radiative phonon transport.\(^3\),\(^5\) In the case of diffusive phonon transport, it was observed that the heat conductivity \( \kappa \) is roughly proportional to \( T^2 \).

For a better thermal insulation of the detector, the underlying membrane is sometimes cut. The result is a self-supporting structure with a wider area in the middle, connected to the bulk by long, narrow bridges, like in Fig. 1(b).\(^1\),\(^2\) The heat conductivity along such bridges has again a power law dependence on the temperature, \( \kappa \propto T^p \), where \( p \) takes values between 1.5 and 2.\(^1\),\(^2\) For the samples measured

![FIG. 1. (a) Full, dielectric membrane. (b) Cut membrane for better thermal insulation.](image-url)
An amorphous material contains dynamic defects which can be modeled by an ensemble of two-level systems (TLSs).\textsuperscript{10–12} A TLS can be understood as an atom or group of atoms which can tunnel between two close minima in the configuration space. Any deformation of the material disturbs the TLS, which can have a transition (an excitation or a deexcitation) to the other energy level. A passing phonon produces such a deformation and, therefore, may be scattered by the TLS. In bulk materials, the phonon modes are simple, transversally or longitudinally polarized plane waves, and the deformation field they produce can be described by only two parameters, the wave vector (or the wavelength) and the polarization. As a consequence, in the so-called standard tunneling model, the interaction Hamiltonian has a very simple structure.\textsuperscript{10–12} In a mesoscopic system, the deformation caused by the "displacement field" of a phonon mode is more complex and the TLS-phonon interaction Hamiltonian has to be modified accordingly. This was done in Ref. 13. Here, we use this more general Hamiltonian to calculate the interaction of the phonon modes of the membrane with the TLSs.

II. TWO-LEVEL SYSTEM--PHONON INTERACTION

A TLS is described by a Hamiltonian which has the form

$$H_{\text{TLS}} = \frac{\Delta}{2} \sigma_z - \frac{\Lambda}{2} \sigma_x$$  \hspace{1cm} (1)

when written in the basis formed by the ground states of the two potential wells between which the system tunnels.\textsuperscript{13} In Eq. (1), \( \Delta \) is the asymmetry of the potential, \( \Lambda \) is the tunnel splitting, and \( \sigma_x, \sigma_z \) are Pauli matrices. The Hamiltonian (1) can be diagonalized by an orthogonal transformation \( O \),

$$H'_{\text{TLS}} = O^T H_{\text{TLS}} O = \frac{\epsilon}{2} \sigma_z,$$  \hspace{1cm} (2)

and we obtain the excitation energy, \( \epsilon = \sqrt{\Delta^2 + \Lambda^2} \). Everywhere in this paper the superscript \( T \) denotes the transpose of a matrix.

The TLS parameters \( \Delta \) and \( \Lambda \) are not the same for all the TLSs in the material, but they are well modeled by the distribution \( P(\Delta, \Lambda) = P_0 / \Lambda \) in the unit volume of the material. We can rewrite the function \( P \) in terms of the more practical variables \( \epsilon \) and \( u = \Lambda / \epsilon \),

$$P(\epsilon, u) = \frac{P_0}{u \sqrt{1 - u^2}}.$$  \hspace{1cm} (3)

The TLSs that have an excitation energy comparable to \( k_B T \) are very efficient phonon scatterers.

The deformation due to the displacement field of a phonon is quantitatively described by the "strain field," which will be represented here by the six-component vector \( S \).\textsuperscript{8} If we denote the displacement field by \( u(r) \), then the strain is defined as the symmetric gradient of \( u(r) \), i.e., \( S^T = (\nabla S u)^T = (\partial_\mu u_x, \partial_\mu u_y, \partial_\mu u_z, \partial_\mu u_x, \partial_\mu u_y, \partial_\mu u_z) \). This deformation adds a time-dependent perturbation to the Hamiltonian (1), which we shall denote by \( H_1 \). The perturbation is assumed to be diagonal when written in the basis of the two ground states of the potential wells [like in Eq. (1)],\textsuperscript{10–12,14,15}

$$H_1 = \frac{\delta}{2} \sigma_z,$$  \hspace{1cm} (4)

and linear in the strain field at the location of the TLS,\textsuperscript{13,15}

$$\delta = 2 \gamma T \cdot [r] \cdot S.$$  \hspace{1cm} (5)

The other quantities in Eq. (5) are the coupling constant \( \gamma \), the six-component vector \( T \), which is defined by a generic orientation \( \hat{t} \) of the TLS as \( T^T = (t_x, t_y, t_z, 2 t_x t_y, 2 t_x t_z, 2 t_y t_z)^T \), and the \( 6 \times 6 \) matrix of the deformation potential parameters \( [r] \). For isotropic materials, the matrix \( [r] \) is

$$[r] = \begin{pmatrix} 1 & \xi & 0 & 0 & 0 \\ \xi & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$  \hspace{1cm} (6)

with the TLS potential parameters \( \xi \) and \( \zeta \) that satisfy the condition \( \zeta + 2 \xi = 1).\textsuperscript{13,14}

To calculate the scattering probabilities, we have to write \( H_1 \) in the second quantization. For this, we denote the TLS excited state by \( |\uparrow\rangle \) and its ground state by \( |\downarrow\rangle \), and we introduce the "creation" and "annihilation" operators \( a^\dagger \) and \( a \), respectively, so that \( a |\uparrow\rangle = |\downarrow\rangle \), \( a^\dagger |\downarrow\rangle = 0 \), and \( a |\downarrow\rangle = 0 \). The operators \( a \) and \( a^\dagger \) obey Fermi commutation relations, and in matrix form, we have \( \sigma_z = (2a^\dagger a - 1) \) and \( \sigma_x = (a^\dagger + a) \). The bosonic creation and annihilation operators for phonons will be denoted by \( b^\dagger \) and \( b \), respectively, where \( \mu \) stands, in general, for the quantum numbers of the phonon modes (see, for example, Refs. 13–16). Using these notations and applying the transformation \( O \) to the total Hamiltonian \( O^T (H_{\text{TLS}} + H_1) O = H'_{\text{TLS}} + H'_1 \), we obtain

$$H'_1 = \frac{\gamma A}{\epsilon} T^T \cdot [r] \cdot \sum_{\mu} [S_\mu b^\dagger_\mu + S^*_\mu b_\mu] (2a^\dagger a - 1)$$

$$- \frac{\gamma A}{\epsilon} T^T \cdot [r] \cdot \sum_{\mu} [S_\mu b^\dagger_\mu + S^*_\mu b_\mu] (a^\dagger + a).$$  \hspace{1cm} (7)

In the first order perturbation theory, the phonon absorption and emission rates are determined by the off-diagonal elements of \( H'_1 \), i.e., of the second row of Eq. (7). Higher order processes are not considered here.

Phonon modes in the membrane

The phonon modes of a freestanding, infinite membrane are divided into three groups, according to their symmetry properties. One group is formed of simple, transversally polarized modes, called the horizontal shear modes \( (h) \). The two other groups are the symmetric \( (s) \) and antisymmetric \( (a) \) Lamb modes. Together, these modes form a complete, orthonormal set of functions for the elastic displacement fields in the membrane, and their proper quantization has
been carried out in Ref. 16. In this paper, we shall use the results and notations from there, and we shall call the three different types of phonons (i.e., $h$, $s$, and $a$) polarizations.

We assume that the membrane of thickness $d$ is placed parallel to the $(xy)$ plane and its parallel surfaces cut the $z$ axis at $z = \pm d/2$. The phonons propagate in the $(xy)$ plane with the wave vector $k_i = k_i \hat{k}_i$, of real $k_i$. We use a hat to denote unit vectors.

Along the $z$ direction, the phonon modes are stationary. As the $h$ modes are pure transversal waves, they have only one wave vector component along the $z$ direction, which we denote by $k_h$. The $s$ and $a$ waves are superpositions of transversal and longitudinal waves of wave vector components along the $z$ direction denoted by $k_s$ and $k_a$, respectively. Due to the boundary conditions, which demand that the membrane surfaces are stress-free, $k_s$ takes the discrete values $m \pi / d$, with $m$ taking all integer values between 0 and $\infty$, whereas $k_h$ and $k_a$ satisfy the more complicated relations

$$\frac{\tan(k_a d/2)}{\tan(k_h d/2)} = -\frac{4k_h k_a^2}{(k_h^2 - k_a^2)^2}$$

(8a)

for the symmetric modes and

$$\frac{\tan(k_a d/2)}{\tan(k_h d/2)} = -\frac{4k_h k_a^2}{(k_h^2 - k_a^2)^2}$$

(8b)

for the antisymmetric modes, Equations (8a) and (8b) plus Snell’s law, $\omega^2 = c_s^2 (k_s^2 + k_h^2) = c_t^2 (k_t^2 + k_h^2)$, enable us to write $k_i$, as a function of $k_i$, for each of the polarizations $s$ and $a$. (In Snell’s law, $c_s$ and $c_t$ are the respective transversal and longitudinal sound velocities for bulk media.) Like in the case of the $h$ modes, the dispersion relations for the symmetric and antisymmetric modes split into branches, i.e., Eqs. (8a) and (8b) and Snell’s law do not give only one function $k_i(k_i)$ for either set of modes, but produce an infinite, countable set of such functions. These functions will be called phonon branches and we shall number them with $m = 0, 1, \ldots$, as we did in Ref. 16, where branches of bigger $m$ lie above branches of smaller $m$ in the $(k_i, k_i)$ plane.

A simple way to express the quantum numbers of the phonon modes is to use Eqs. (8a) and (8b) and Snell’s law to write the functions $k_i(k_i)$ and $k_a(k_a)$. Then each branch of polarization $\sigma = s, a$ and branch number $m$ is going to be described by the continuous set of numbers $[k_i(k_i), k_a(k_a)]_{\sigma, m}$, with $k_i$ taking values from 0 to $\infty$. We, therefore, choose the set $\mu$ of quantum numbers that specify the phonon modes in Eq. (7) to be $\mu = (\sigma, m, k_i)$.

The functions $k_i(k_i)$ and $k_a(k_a)$ may take both real and imaginary values. To distinguish between these situations, we write the imaginary values of $k_i$ as $ik_i$ and the imaginary values of $k_a$ as $i k_a$. In these notations, $k_{i, s}$ and $k_{i, a}$ take always positive, real values.

In order to simplify the later discussion, we shall replace $k_i$ and $k_a$ with the complex quantities $\bar{k}_i = k_i + i k_i$ and $\bar{k}_a = k_a + i k_a$, respectively. Note, however, that $k_i$ and $k_a$ are never really complex, but they are either real or imaginary, as long as $k_i$ is real.

The displacement fields of the phonon modes are

$$u_h = N_h \cos[k_h(z - d/2)](\hat{x} \times \hat{z}) e^{i(k_h r - \omega t)},$$

(9a)

$$u_s = N_s \frac{i\bar{k}_s}{2} [2k_s^2 \cos(\bar{k}_s d/2) \cos(\bar{k}_s z) + [k_s^2 - k_h^2/2] \cos(k_h d/2)]$$

$$\times \cos(k_h z) \bar{k}_h - k_h 2k_s \bar{k}_s \cos(k_h d/2) \sin(k_h z) - [k_h^2 - k_s^2/2] \cos(\bar{k}_s d/2) \sin(\bar{k}_s z)] \hat{z} e^{i(k_h r - \omega t)},$$

(9b)

$$u_a = N_a [\bar{k}_a \sin(\bar{k}_a d/2) \sin(\bar{k}_a z) + [k_a^2 - k_h^2/2] \sin(k_h d/2)]$$

$$\times \sin(k_h z) \bar{k}_h + k_h 2k_s \sin(k_h d/2) \cos(k_h z) - [k_h^2 - k_s^2/2] \sin(\bar{k}_a d/2) \cos(\bar{k}_a z)] \hat{z} e^{i(k_h r - \omega t)}.$$

(9c)

As one can see, when $\bar{k}_i$ or $\bar{k}_a$ take imaginary values, the trigonometric functions in Eq. (9) will switch into hyperbolic functions. The normalization constants $N_h$, $N_s$, and $N_a$ are given by

$$N_{ih}^{-2} = \begin{cases} \sqrt{V}, & m = 0 \\ \sqrt{V/2}, & m > 0, \end{cases}$$

(10a)

$$N_{is}^{-2} = A \left\{ 4|\bar{k}_i|^2 |k_i|^2 \left[ \frac{\bar{k}_d}{2} \right]^2 \left[ \frac{(\bar{k}_i^2 + k_i^2)}{2k_i} \sinh(k_h d/2) - \frac{(\bar{k}_i^2 - k_i^2)}{2k_i} \sin(k_h d/2) \right] \right. $$

$$\times \left. \left[ \frac{(\bar{k}_i^2 + k_i^2)}{2k_i} \sinh(k_h d/2) + \frac{(\bar{k}_i^2 - k_i^2)}{2k_i} \sin(k_h d/2) \right] \right. $$

$$\times \left. \left[ \frac{(\bar{k}_i^2 + k_i^2)}{2k_i} \sinh(k_h d/2) - \frac{(\bar{k}_i^2 - k_i^2)}{2k_i} \sin(k_h d/2) \right] \right. $$

$$\times \left. \left[ \frac{(\bar{k}_i^2 + k_i^2)}{2k_i} \sinh(k_h d/2) + \frac{(\bar{k}_i^2 - k_i^2)}{2k_i} \sin(k_h d/2) \right] \right. $$

$$\times \left. \left[ \frac{(\bar{k}_i^2 + k_i^2)}{2k_i} \sinh(k_h d/2) - \frac{(\bar{k}_i^2 - k_i^2)}{2k_i} \sin(k_h d/2) \right] \right. $$

$$\times \left. \left[ \frac{(\bar{k}_i^2 + k_i^2)}{2k_i} \sinh(k_h d/2) + \frac{(\bar{k}_i^2 - k_i^2)}{2k_i} \sin(k_h d/2) \right] \right. $$

$$\times \left. \left[ \frac{(\bar{k}_i^2 + k_i^2)}{2k_i} \sinh(k_h d/2) - \frac{(\bar{k}_i^2 - k_i^2)}{2k_i} \sin(k_h d/2) \right] \right. $$

(10b)

$$N_{ia}^{-2} = A \left\{ 4|\bar{k}_i|^2 |k_i|^2 \left[ \frac{\bar{k}_d}{2} \right]^2 \left[ \frac{(\bar{k}_i^2 + k_i^2)}{2k_i} \sinh(k_h d/2) + \frac{(\bar{k}_i^2 - k_i^2)}{2k_i} \sin(k_h d/2) \right] \right. $$

$$\times \left. \left[ \frac{(\bar{k}_i^2 + k_i^2)}{2k_i} \sinh(k_h d/2) - \frac{(\bar{k}_i^2 - k_i^2)}{2k_i} \sin(k_h d/2) \right] \right. $$

$$\times \left. \left[ \frac{(\bar{k}_i^2 + k_i^2)}{2k_i} \sinh(k_h d/2) + \frac{(\bar{k}_i^2 - k_i^2)}{2k_i} \sin(k_h d/2) \right] \right. $$

$$\times \left. \left[ \frac{(\bar{k}_i^2 + k_i^2)}{2k_i} \sinh(k_h d/2) - \frac{(\bar{k}_i^2 - k_i^2)}{2k_i} \sin(k_h d/2) \right] \right. $$

$$\times \left. \left[ \frac{(\bar{k}_i^2 + k_i^2)}{2k_i} \sinh(k_h d/2) + \frac{(\bar{k}_i^2 - k_i^2)}{2k_i} \sin(k_h d/2) \right] \right. $$

(10c)
where $A$ is the area of the membrane and $V = dA$ is its volume. To obtain the expressions for $N_e$ and $N_0$ for the different combinations of real or imaginary $\tilde{k}_i$ and $\tilde{k}_0$, one has to take the limit to 0 of their redundant components in Eqs. \((10b)\) and \((10c)\). For example, if $\tilde{k}_i$ is real and $\tilde{k}_0$ is imaginary, we calculate the corresponding normalization factor by taking in Eq. \((10b)\) or \((10c)\) the limits $\kappa_i \to 0$ and $\kappa_0 \to 0$.\(^{16}\)

### III. Transition Rates

Now we have all the ingredients to calculate the TLS transition rates or phonon absorption and emission probabilities. We shall denote the phonon-TLS quantum states by $|n_{\mu}, \downarrow\rangle$ or $|n_{\mu}, \uparrow\rangle$, where we denoted the number of phonons on the mode $\mu$ by $n_{\mu}$. Using Eq. \((7)\), we write the emission amplitude of a phonon by a TLS as

$$\langle n_{\mu} | \hat{H}_t | n_{\mu} + 1, \downarrow \rangle = -\frac{\gamma \Lambda}{e} \sqrt{\frac{\hbar n_{\mu}}{2 \rho \omega_{\mu}}} M_{\mu}^{\ast}, \quad (11)$$

with $M_{\mu}$ is given by

$$M_{\mu}(t) = T^T \cdot [\tau] \cdot S_{\mu}.$$  

Explicitly, for the three phonon polarizations, we have

\[ M_{\mu}(t) = 2 \xi N_{\mu} (\cos(k d/2) \cos(k z) + \cos(k d/2) \cos(k z) + 8it_{\tilde{k}_0} \tilde{k}_i \tilde{k}_0 \sin(k d/2) \sin(k z) + 2t_{\tilde{k}_0} \tilde{k}_i \tilde{k}_0 \sin(k d/2) \sin(k z)), \quad (12c) \]

where $k_i$ and $k_f$ are implicitly determined by the branch number $m$ and $k_0$. Using Fermi’s golden rule, we calculate the phonon absorption and emission rates $\Gamma_{ab}^{\mu}$ and $\Gamma_{em}^{\mu}$, respectively,

$$\Gamma_{ab}^{\mu} = \frac{\pi}{\rho \omega_{\mu}} \frac{\gamma^2 \Lambda^2}{e^2} |M_{\mu}|^2 n_{\mu} \delta(\hbar \omega_{\mu} - e), \quad (13a)$$

$$\Gamma_{em}^{\mu} = \frac{\pi}{\rho \omega_{\mu}} \frac{\gamma^2 \Lambda^2}{e^2} |M_{\mu}|^2 (n_{\mu} + 1) \delta(\hbar \omega_{\mu} - e), \quad (13b)$$

where $e$ is the energy of the TLS, as defined in Sec. II, and $\omega$ is the angular frequency of the phonon.

In an amorphous solid, the orientations of the TLSs are arbitrary, so the relevant quantities for our calculations are the averages of $|M_{\mu}|^2$ over the directions $\hat{t}$ of the TLSs. The only quantity that depends on $\hat{t}$ in Eqs. \((13)\) is $|M_{\mu}|^2$. Additionally, we assume that the distribution of TLSs in the membrane is uniform, which leaves again $|M_{\mu}|^2$, the only quantity dependent on $z$ in the expressions, for the absorption and emission rates. As we are interested in an average scattering probability rather than in a detailed description of where along the $z$ direction the scattering takes place, we also average $|M_{\mu}|^2$ along $z$. Denoting by $\langle \cdot \rangle$ the average over the TLS orientations and the $z$ variable, we obtain

$$\langle |M_{\mu}|^2 \rangle = \frac{C}{V} (k_i^2 + k_f^2), \quad (14a)$$

\[ \langle |M_{\mu}|^2 \rangle = \frac{N_0}{d} \left\{ 2C |\tilde{k}_i|^2 |\tilde{k}_0|^2 (\tilde{k}_i^2 + \tilde{k}_0^2)^2 \cos \left( \frac{k d}{2} \right) \right\} \left( \frac{\sinh(\kappa d)}{2\kappa_i} + \frac{\sin(\kappa d)}{2k_i} \right) + C^2 |\tilde{k}_i|^2 |\tilde{k}_0|^2 \left( \frac{\sinh(\kappa d)}{2\kappa_i} - \frac{\sin(\kappa d)}{2k_i} \right) - 2C \kappa_i (2\tilde{k}_i^6 - \tilde{k}_i^6) \left( \frac{\sinh(\kappa d)}{2\kappa_i} - \frac{\sin(\kappa d)}{2k_i} \right) - 2C \kappa_i (2\tilde{k}_i^6 - \tilde{k}_i^6) \left( \frac{\sinh(\kappa d)}{2\kappa_i} - \frac{\sin(\kappa d)}{2k_i} \right) - \tilde{k}_i^6 \left( \frac{\sinh(\kappa d)}{2\kappa_i} - \frac{\sin(\kappa d)}{2k_i} \right) \right\}, \quad (14b) \]
\[
\langle |M_\alpha|^2 \rangle = \frac{N_\alpha^2}{d} \left( 4C_i |k_\alpha|^2 |k_\alpha|^2 + |k_\alpha|^2 \right) \sin \left( \frac{k_\alpha d}{2} \right) \left[ \frac{\sinh(\kappa d)}{2 \kappa} - \frac{\sin(\kappa d)}{2k_i} \right] \sin \left( \frac{k_\alpha d}{2} \right) \right] \sin \left( \frac{k_\alpha d}{2} \right) \right]
\times \left( |k_\alpha|^2 + k_\alpha^2 \right) \sinh(\kappa d) \left[ \frac{\sinh(\kappa d)}{2 \kappa} - \frac{\sin(\kappa d)}{2k_i} \right] - 2C_i k_i (2k_i^6 + k_\alpha^4 |k_\alpha|^2 + |k_\alpha|^6) \sin \left( \frac{k_\alpha d}{2} \right) \right]
\times \sin(\kappa d) - 2C_i k_i (2k_i^6 + k_\alpha^4 |k_\alpha|^2 + |k_\alpha|^6) \left[ \frac{\sinh(\kappa d)}{2 \kappa} - \frac{\sin(\kappa d)}{2k_i} \right] \sin \left( \frac{k_\alpha d}{2} \right) \right].
\end{equation}

IV. LOW ENERGY EXPANSION: ASYMPTOTIC RESULTS

We can analytically calculate the scattering times or the thermal properties of the membrane only in the long-wavelength limit, i.e., for the branch \( m=0 \) for each of the three polarizations of the phonon modes and \( k_i \ll \frac{1}{d} \). The calculation of thermal properties in this limit corresponds to a temperature range in which \( k_B T \ll \hbar c_i / d \). First, we have to calculate the relaxation times for each polarization.
sions (10c) and (14c) are zero in the first and second leading orders. Therefore, we have to expand to the third leading order to get nonzero results. From Eq. (8b), we obtain

$$\omega_{a,0,k} = \frac{\hbar}{2m^*} \left( k_0^2 - d^2 \frac{27c_0^2}{90c_i^2} k_i^4 \right),$$

(24)

from which we finally get

$$\tau_{a,0,k}(\omega) = \frac{\hbar \rho c_i^2}{\pi^2 P_0 C_s (C_i c_i + C/2c_i^2)} \frac{1}{\coth(\beta \hbar \omega/2)} \frac{1}{\omega},$$

(25)

with $C_s = 4C_i$.

### D. Comparison of the scattering rates and mean free paths

The first thing to observe is that, although the dispersion relations for the $s$ and $a$ modes are different in the low $k_i$ limit, $\tau_{a,0,k}$ and $\tau_{a,0,k}$ are related by the simple equation $\tau_{a,0,k} = 4 \tau_{a,0,w}$ (we shall simply write $\tau_{a,0,w}$ for $\tau_{a,0,k}$). In other words, the scattering rate for the $s$ phonons is four times smaller than the scattering rate of $a$ phonons at the same $\omega$.

Let us now compare $\tau_{a,0,w}$ with $\tau_{a,0,w}$. For this, we calculate the ratio

$$\frac{\tau_{a,0,w}}{\tau_{a,0,w}} = \frac{C_i c_i^2 + C/2c_i^2}{C/4c_i^2} = 1 - (C_i/c_i)^2 \frac{1 - (C_i/C_i)(c_i/c_i)^2}{1 - (C_i/c_i)^2}. $$

(26)

In any normal material (i.e., with positive Poisson ratio), the ratio $c_i/c_i^2$ is restricted to $0 < c_i^2/c_i^2 \leq 1/2$, and for $C_i/C_i$, we have $15/4C_i = 15/4L/\xi + 8/3$ for $C_i/C_i = 3$. In Fig. 2, we plot the ratio $\tau_{a,0,w}/\tau_{a,0,w}$ as function of $c_i^2/c_i^2$ for different values of $C_i/C_i$.

We first remark that in the limit $c_i/c_i \rightarrow 0$, $\tau_{a,0,w}/\tau_{a,0,w} = 1$, independent of the value for $C_i/C_i$. Increasing $c_i/c_i$ will result in a decrease of $\tau_{a,0,w}/\tau_{a,0,w}$ until a minimum is reached at $c_i/c_i^2 = 1 - \sqrt{1 - (C_i/C_i)^2}$, which lies between 0 and 1/2. Afterwards, $\tau_{a,0,w}/\tau_{a,0,w}$ increases monotonically until it reaches the value $(C_i/C_i)/2$ for $(c_i/c_i)^2 = 1/2$. As $\tau_{a,0,w}/\tau_{a,0,w}$ increases monotonically with $C_i/C_i$, we conclude that $\tau_{a,0,w} < \tau_{a,0,w}$ for any $c_i/c_i$, as long as $C_i/C_i < 2$. For $C_i/C_i = 2$, $\tau_{a,0,w}$ can be either smaller or greater than $\tau_{a,0,w}$ depending on whether $(c_i/c_i)^2$ is smaller or greater than $C_i/C_i$, respectively. A typical value for $(c_i/c_i)^2$ is 0.36, which means that $\tau_{a,0,w} < \tau_{a,0,w}$ as long as $C_i/C_i$ is smaller than 2.78.

When comparing $\tau_{a,0,w}$ with $\tau_{a,0,w}$, we encounter a similar situation. As $\tau_{a,0,w} = 4 \tau_{a,0,w}$, the ratio $\tau_{a,0,w}/\tau_{a,0,w}$ has the same features as the ratio $\tau_{a,0,w}/\tau_{a,0,w}$ except for the fact that the critical value for $C_i/C_i$ is 8, i.e., $\tau_{a,0,w}$ is always smaller than $\tau_{a,0,w}$ if $C_i/C_i < 8$ and can be either smaller or greater than $\tau_{a,0,w}$ for $C_i/C_i = 8$. For the SiN, typical value, $(c_i/c_i)^2 = 0.36$, we have $\tau_{a,0,w} \approx \tau_{a,0,w}$ as long as $C_i/C_i \approx 17.6$.

More interesting than the scattering rates is to compare the phonon mean free paths, since these can be directly measured experimentally. For this, let us first use the dispersion relations (20) and (23), and $\omega_{a,0,k} = \omega_{a,0,k}$ to write the expressions for the mean free paths:

$$l_{a,0,k}(\omega) = c_i \tau_{a,0,k} = \frac{\hbar \rho c_i^2}{\pi^2 P_0 C_i} \frac{1}{\beta \hbar \omega/2} \frac{1}{\omega},$$

(27a)

$$l_{a,0,k}(\omega) = \frac{\hbar \rho c_i^2}{\pi^2 P_0 C_i} \frac{1}{\beta \hbar c_i k_i/2} \frac{1}{\omega},$$

(27b)

$$l_{a,0,k}(\omega) = \frac{\hbar \rho c_i^2}{\pi^2 P_0 C_i} \frac{1}{\beta \hbar c_i k_i/2} \frac{1}{\omega},$$

(28a)

$$l_{a,0,k}(\omega) = \sqrt{\frac{2\hbar \omega}{m^* \tau_{a,0,k}}}$$

(28b)

$$l_{a,0,k}(\omega) = \frac{\hbar \rho c_i^2}{\pi^2 P_0 C_4} \frac{1}{\beta \hbar c_i k_i/2} \frac{1}{\omega},$$

(29a)

$$l_{a,0,k}(\omega) = \frac{\hbar \rho c_i^2}{\pi^2 P_0 C_i} \frac{1}{\beta \hbar c_i k_i/2} \frac{1}{\omega},$$

(29b)

A way to determine the mean free path of phonons is to measure the resonant attenuation of ultrasound propagating along the membrane. If this is experimentally impossible, another way to determine the material parameters is to make acoustic measurements on thicker and wider membranes. Note, however, that elastic waves attenuate not only because of resonant scattering of phonons [Eqs. (27)–(29)], but also due to energy relaxation. Nevertheless, since we are interested here in the thermal properties of the membranes, only the resonant scattering is important and we disregard the energy relaxation mechanism.
To analyze the results (27)–(29) in more detail, we expressed the mean free paths both in terms of the angular frequency and in terms of \( k_i \). If the elastic modes of different polarizations are produced with the same \( \omega \), then we should compare the mean free paths as given by the expressions (27a), (28a), and (29a), which we denote as \( l_{r,0,\omega} \). For example, \( l_{r,0,\omega}=\tau_{r,0,\omega}/(\tau_{r,0,\omega}+c_s/c_0) \), which is smaller than \( \tau_{r,0,\omega}/\tau_{r,0,\omega} \) in any material. The discussion we made above \( \tau_{r,0,\omega}/\tau_{r,0,\omega} \) applies here too.

The expressions (29) for \( l_{0,0,\omega} \) are very different from the ones for \( l_{h,0,\omega} \) and \( l_{r,0,\omega} \). Nevertheless, the expressions (27)–(29) are calculated for \( k_i \ll 1/d_s \) which means that \( \omega \ll (2\pi/k_i)\omega_c \), which implies \( \omega/\omega_c \ll 1 \). Taking this into account when we compare the expressions for \( l_{h,0,\omega} \), \( l_{r,0,\omega} \), and \( l_{0,0,\omega} \), at the same \( \omega \), we conclude that, as a function of frequency, for low enough frequencies the antisymmetric Lamb modes have the shortest mean free path. This is a consequence of the fact that the group velocity of the \( \alpha \) modes decreases to zero as \( k_i \) decreases.

If we compare \( l_{h,0,\omega} \), \( l_{s,0,\omega} \), and \( l_{r,0,\omega} \), we see that \( l_{h,0,\omega}/l_{s,0,\omega}=(\tau_{r,0,\omega}/\tau_{s,0,\omega})\coth(\beta hc_{\omega}k_i/2)/\coth(\beta hc_{\omega}k_i/2) \), which is bigger than \( \tau_{r,0,\omega}/\tau_{s,0,\omega} \) since \( c_i < c_s \). Comparing \( l_{s,0,\omega} \), \( l_{s,0,\omega} \), and \( l_{r,0,\omega} \), we observe, for example, that both \( l_{s,0,\omega}/l_{s,0,\omega} \) and \( l_{s,0,\omega}/l_{r,0,\omega} \) are proportional to \( \coth(\beta hc_{\omega}k_i/2)/\coth(\beta hc_{\omega}k_i/2) \). However, again, for long wavelengths, due to the quadratic dependence of \( \omega_{\alpha,0,\omega} \) on \( k_i \), we have \( \omega_{s,0,\omega} \ll \omega_{h,0,\omega} \). Moreover, \( \coth(x) \sim 1/x \) for \( x \to 0 \), so both \( l_{s,0,\omega} \), \( l_{s,0,\omega} \), and \( l_{r,0,\omega} \), \( l_{r,0,\omega} \), are proportional to \( 1/k_i \), and become very big in the limit of long wavelengths. In conclusion, as a function of \( k_i \) in the limit \( dk_i \ll 1 \), the antisymmetric modes have a much longer mean free path than the symmetric and horizontal shear modes with the same \( k_i \).

### E. Calculation of the heat conductivity

Now we calculate the heat conductivity in the limit of low temperature. In that limit, only the lowest branch of each polarization will be populated. Furthermore, in the low temperature range that we are interested in (typical temperatures are below 300 mK), the typical phonon frequency cutoff in amorphous materials can be neglected. In the \( k_i \) space, the density of states is \( \Lambda/(2\pi^2) \), and, after some changes of variables, we write Eq. (17) in the form

\[
\kappa = \frac{h^2}{16\pi k_i T^2} \sum \int_{\omega_{0}}^{\infty} d\omega \, \omega^2 \frac{\omega_{\alpha,0}(\omega)}{\sinh^2(\beta \hbar \omega/2)},
\]

where the lower limits \( \omega_{0,0} \) were introduced for the reasons that will become clear immediately. Using Eqs. (27)–(29) for the mean free paths, we express \( \kappa \) as a sum of three contributions:

\[
\kappa = \frac{k_B^2 \rho c_t^2}{16\pi^2 \hbar^2 T} \left( \frac{I(x_{0,0})}{C_t} + \frac{I(x_{0,0})}{C_s} + \frac{2I(x_{0,0})}{C_a} \right),
\]

where \( x_{\alpha,0} = \beta \hbar \omega_{\alpha,0} \) and by \( I(x) \) we denoted the integral

\[
I(x) = \int_{y}^{\infty} dy y^2 \coth(y/2) = \frac{4x^2 e^x}{(e^x - 1)^2} + \frac{8x}{e^x - 1} - 8 \ln(1 - e^{-x}).
\]

Note that, although the mean free paths for the \( h \) and \( s \) modes have different functional dependencies on \( \omega \) than the mean free path for the \( a \) modes, the integrand in Eq. (31) is the same for all three modes. The role of the lower cutoff in Eq. (30) becomes obvious when we look at Eq. (32): the integral \( I(x) \) has a logarithmic divergence in \( x=0 \).

If the cutoff is small enough, then we can approximate \( I(x) \) by

\[
I(x) \approx 12 - 8 \ln(x),
\]

and inserting this into Eq. (31), we obtain

\[
\kappa = \frac{k_B^2 \rho c_t^2}{16\pi^2 \hbar^2 T} \left[ \frac{3 - 2 \ln(\beta h \omega_{h,0})}{C_t} + \frac{3 - 2 \ln(\beta h \omega_{s,0})}{C_s} + \frac{2[3 - 2 \ln(\beta h \omega_{a,0})]}{C_a} \right],
\]

where the first, second, and third terms in the square brackets above give the contributions of the \( h \), \( s \), and \( a \) phonon modes to the heat conductivity. The above expression leads to the temperature dependence \( \kappa \propto \tau(a+b \ln T) \) and this dependence is a hallmark of the TLS-limited heat conductance at low temperature.

For a numerical estimate, let us use for the cutoff the finite size of the membrane, which limits the wave vectors to values of the order of \( 2\pi/\Lambda \). For the typical experimental parameters \( T=0.1 \) K, \( \Lambda=400 \) \( \mu \)m, and \( d=200 \) nm,12,7 we have \( \ln(x_{0,0,2\pi/\Lambda}) \approx -4.9 \), \( \ln(x_{0,0,2\pi/\Lambda}) \approx -4.4 \), and \( \ln(x_{0,0,2\pi/\Lambda}) \approx -11.4 \). However, since \( C_s=4C_t \) [see Eqs. (21) and (25)], the contributions of all the phonon polarizations to the heat conductivity are of the same order.

### V. CONCLUSIONS

We used the model introduced in Ref. 13 to calculate the scattering of the elastic modes in a thin, amorphous membrane. We modeled the scattering centers in the membrane by an ensemble of TLSs with the same properties and distribution over energy splitting and asymmetry as the TLSs in a bulk material. Whether this assumption is valid remains to be checked by experiment. We obtained the expressions for the TLS relaxation time [Eq. (15)], for the phonon scattering time [Eq. (16)], and for the heat conductivity \( \kappa \) [Eq. (17)].

For general temperatures, the heat conductivity and the scattering times have to be calculated numerically. We calculated analytical low temperature approximations and compared the mean free paths of different phonon polarizations. In this way, we observed that the contribution of the lowest branches of the phonon modes to the heat conductivity are logarithmically divergent at \( k_i \to 0 \). This could be a reason for which in some experiments a radiative heat transport is observed.5 Nevertheless, there is a natural lower cutoff of \( k_i \to 0 \) due to the finite size of the membrane. This cutoff renders \( \kappa \) finite, which, in the low temperature limit, behaves
like $\kappa \propto T(a + b \ln T)$. This behavior is a hallmark of the TLS-limited heat conductance at low temperature.

Due to the dispersion relations of the phonon modes, the TLS distribution in the low energy limit has a bigger impact on the heat conductivity in thin membranes than in bulk materials. If we, for instance, modify the distribution into $P_3 = P_0 (1 - u^2)$, with an extra energy dependence $\varepsilon^{-\alpha}$, we make the expression for $\kappa$ convergent even in the $\omega_{\tau,0} \rightarrow 0$ limit, which leads to a low temperature asymptotic dependence of $\kappa \propto T^{3+\alpha}$. However, whether this is the situation or not has to be decided experimentally.

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