COSMOS AND ITS FURNITURE

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Abstract. In this note I shall continue the study of the geometry of the moduli-space of pairs of points in 3 dimensions. I show that this space, $\underline{\tilde{H}}$, is the base space of a canonical family of associative k -algebras in dimension 4. The study of the corresponding family of derivations leads to a natural way of introducing an action of the gauge Lie algebras of the Standard Model, in $\underline{\tilde{H}}$. The results fit well with the set-up of the Standard Model, fusing our versions of General Relativity, Yang-Mills and Quantum Field Theory. It also furnishes a possible mathematical model for a Big Bang-scenario in cosmology. These subjects are all treated within the purely mathematical framework of [18].

Contents

Keywords: Representations, simple modules, phase spaces, Dirac derivations, time, quantum theory, time-space, stress-mass, heat-equation, Navier-Stokes equation, cosmology and the primordial fluid.

1. Introduction

Mindful of the well known quotation,

> Vos calculs sont corrects, mais votre physique est abominable: (Albert Einstein 1927 à George Lemaître),

I shall start by declaring my cards.

1.1. **Philosophy.** If we want to study a natural phenomenon, called **P**, we must in the present scientific situation, describe **P** in some mathematical terms, say as a mathematical object, X, depending upon some parameters, in such a way that the changing aspects of **P** would correspond to altered parameter-values for X. This object would be a *model for* **P** if, moreover, X with any choice of parameter-values, would correspond to some, possibly occurring, aspect of **P**.

Two mathematical objects X(1), and X(2), corresponding to the same aspect of **P**, would be called equivalent, and the set, $\mathcal{P}$, of equivalence classes of the objects **P**, would correspond to (possibly a quotient of) the *moduli space*, **M**, of the models, X, assumed to be algebraic schemes of some sort. The study of the natural phenomena **P**, and its changing aspects, would then be equivalent to the study of the *structure* of $\mathcal{P}$, and therefore to the study of the geometry of the moduli space **M**. In particular, the notion of *time* would, in agreement with Aristotle and St. Augustin, see, [1] and [16], correspond to some *metric* on this space.

It turns out that to obtain a complete theoretical framework for studying the phenomenon **P**, or the model X, together with its *dynamics*, we should introduce the notion of *dynamical structure*, defined for the moduli space, **M**, assumed to be a union of affine schemes. This is done via the construction of a universal non-commutative *Phase Space*-functor, $Ph(-) : Alg_k \to Alg_k$, where Alg_k is the category of k-algebras, k any field. It extends to the category of schemes, and its infinite iteration $Ph^\infty(-)$, is outfitted with a universal *Dirac derivation*, $\delta \in Der_k(Ph^\infty(-), Ph^\infty(-))$.

A dynamical structure defined on an associative k-algebra $A \in Alg_k$ is now a δ-stable ideal $\sigma \subset Ph^\infty(A)$, and its quotient $A(\sigma) := Ph^\infty(A)/(\sigma)$, with its induced Dirac derivation. The structure we are interested in is the *space* $\mathbf{U} := Ph^\infty(\mathbf{M})/\sigma$, corresponding to an open affine covering by algebras of the type, $A(\sigma)$, see [16], [33], see also [18].

But now we observe that there may be an action of a Lie algebra $\mathfrak{g}$, on **U**, such that the dynamics of $\mathcal{P}$, really corresponds to that of the quotient $\mathbf{U}/\mathfrak{g}_0$.

To any *open* subset O, of **U**, there would be associated a, not necessarily commutative, affine k-algebra, $A(\sigma) := \mathbf{O}_{\mathbf{U}}(O)$, with an action of the Lie algebra $\mathfrak{g}_0$, such that the non-commutative quotient $O/\mathfrak{g}_0$, represented by the system of simple representations of an algebra, $A(\sigma)(\mathfrak{g}_0)$, see (1.5), would contain all the available information about the structure of O. An element of $A(\sigma)$ would be called an *observable*, and wishing to measure the *value* of an observable, leads to the study of the eigenvectors, and their eigenvalues, of the $\mathfrak{g}_0$-invariant representations of this algebra, which as we shall see, is the same as the representations of $A(\sigma)(\mathfrak{g}_0)$.

With this philosophy in mind, and stimulated by the results in deformation theory, obtained in [10], and [11], we embarked, in a series of papers, see [12], [13], [14], on the study of moduli spaces of representations (modules) of associative algebras, in general, and on their quotients, modulo Lie algebra actions. Here is where *invariant theory* and *non-commutative algebraic geometry* enters the play. The Dirac derivation translates into a vector field on these moduli spaces, and give us the equations of motions that we need.

In [16], [33], see also [18], we introduced a *toy model*, used to illustrate the general theory, and to connect to present days physics. It was shown to generalize both general relativity and quantum field theory. In particular, the definition of time fits well with the notion of time in both quantum Yang-Mills theory and in General Relativity, where it made the space of velocities compact.

In this paper, this toy model has become the main figure, furnishing a (nice, but maybe not too realistic) mathematical model for a Big Bang scenario for the universe. The *Cosmos and its Furniture* of the title refer to this model, to its *geometry*, defined by a metric, it's dynamical structure defined by a Dirac derivation, and to its *material content*, called it's *furniture*.

The basic idea is that if one choose to take the Big Bang idea seriously, one would, probably, have to accept the presence of a *singularity* at the *start of the universe*. But then one might guess upon a mathematical model for this singularity, and use deformation theory, and the machinery described above to unravel the universe that we see. This is what we do, in chapter 4. We start with the basic singularity in dimension 3. It is composed of a single point, and a 3-dimensional tangent space, with affine algebra given by,

$$U = k[x_1, x_2, x_3]/(x_1, x_2, x_3)^2.$$

The base space of the miniversal deformation of this algebra, considered as an associative algebra, turns out to contain the above toy-model, and a lot of structure, with strange and maybe interesting interpretations in physics, in particular it seems that it contain a mathematically reasonable basis for the Standard Model.

The paper is organized as follows:

Section. 1. is an introduction to the general method proposed in [18]. There are three main ingredients, moduli theory, dynamical structures, and gauge theory. We include a short reminder of these ingredients, and of the most important technical tools handling them, such as the basic notion of the non-commutative *phase space functor* $Ph()$, including some purely mathematical consequences like, the co-simplicial structure of the infinitely iterated Ph^*, the resulting definition of a universal *Dirac* derivation, the corresponding relations to de Rham theory. We also recall the basics of non-commutative deformation theory for families of representations, its ring of *observables*, and its use in non-commutative algebraic geometry.

Section. 2. contains a reworking of the toy model, in particular including the notion of *furniture*, and the related Schrødinger equation, generalising the Heat equation and the Navier-Stokes.

Section. 3. contains a short introduction to an algebraic version of Entropy, and related to this, a computation of the moduli-space of finite dimensional representations of the infinite phase space of a polynomial algebra.

Section. 4. is then concerned with the deformation theory of the 4 -dimensional associative algebra U, geometrically a fat point in 3-space, and with the structure of its miniversal base space. The fact that this space contains the toy model, provides new and unsuspected structures for this model. The main result is the existence, produced by the deformation theory of associative algebras, of a canonical gauge Lie-algebra bundle, containing the gauge algebras of the Standard Model and a strange *super symmetry* relating the spaces of, our versions of, *bosonic* and *fermionic* fields.

Section. 5. contains worked out formulas that we need for the final chapters summing up:

Section. 6. sums up the Model. We propose a mathematical fusion of General Relativity, Quantum Field Theory and Yang-Mills, ending with a model of Elementary Particles, as furnished by the Toy Model..

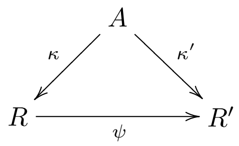
Section. 7. treats the notion of interaction and decay in the language of non-commutative deformation of families of modules. In particular we obtain a possible purely mathematical model of the Weak Interaction for quarks, with maybe interesting new . properties. Section. 8. contains some elementary examples, and some ideas.

Section. 9. takes a birds eye view of the present state of the Standard Model, compared with the Toy Model.

Finally we end with,

Section. 10. End Words.

1.2. **The Phase space functor Ph.** Given a k-algebra A, denote by $A/k-\underline{alg}$ the category where the objects are homomorphisms of k-algebras $\kappa : A \to R$, and the morphisms, $\psi : \kappa \to \kappa'$ are commutative diagrams,



and consider the functor

$$Der_k(A,-) : A/k-\underline{alg} \longrightarrow \underline{Sets}$$

defined by $Der_k(A,\kappa) := Der_k(A,R)$. It is representable by a k-algebra-morphism, $\iota : A \longrightarrow Ph(A)$ with a *universal family* given by a universal derivation $d : A \longrightarrow Ph(A)$. It is easy to construct Ph (A). In fact, let $\pi : F \to A$ be a surjective homomorphism of algebras, with $F = k < t_1, t_2, \ldots, t_r >$, freely generated by the t_i's, and put $I = ker\pi$. Let,

$$Ph(A) = k < t_1, t_2, \ldots, t_r, dt_1, dt_2, \ldots, dt_r > /(I, dI)$$

where dt_i is a formal variable, for i=1,...,r. Clearly there is a homomorphism, $i_0' : F \to Ph(A)$ and a derivation $d' : F \to Ph(A)$, defined by putting, $d'(t_i) = cl(dt_i)$, the equivalence class of dt_i. Since i_0' and d' both kill the ideal I, they define a homomorphism,

$$i_0 : A \to Ph(A)$$

and a derivation,

$$d : A \to Ph(A).$$

To see that i_0 and d have the universal property, let $\kappa : A \to R$ be an object of the category $A/k-\underline{alg}$. Any derivation $\xi : A \to R$ defines a derivation $\xi' : F \to R$, mapping t_i to $\xi'(t_i)$. Let $\rho_{\xi'} : k < t_1, t_2, \ldots, t_r, dt_1, dt_2, \ldots, dt_r > \to R$ be the homomorphism defined by

$$\rho_{\xi'}(t_i) = \kappa(\pi(t_i)), \quad \rho_{\xi'}(dt_i) = \xi(\pi(t_i))$$

$\rho_{\xi'}$ sends I and dI to zero, so defines a homomorphism $\rho_\xi : Ph(A) \to R$, such that the composition with $d : A \to Ph(A)$, is ξ. The unicity is a consequence of the fact that the images of i_0 and d generate $Ph(A)$ as k-algebra.

Clearly $Ph(-)$ is a covariant functor on $k - \underline{alg}$, and we have the identities,

$$d_* : Der_k(A, A) = Mor_A(Ph(A), A)$$

and

$$d^* : Der_k(A, Ph(A)) = End_A(Ph(A))$$

the last one associating d to the identity endomorphisme of $Ph(A)$. In particular we see that i_0 has a cosection, $\sigma_0 : Ph(A) \to A$, corresponding to the trivial (zero) derivation of A.

Let now V be a right A-module, with structure morphism

$$\rho(V) =: \rho : A \to End_k(V).$$

We obtain a universal derivation

$$u(V) =: u : A \longrightarrow Hom_k(V, V \otimes_A Ph(A))$$

defined by $u(a)(v) = v \otimes d(a)$. Let U and V be right A-modules, and consider the long exact sequences of Hochschild cohomology,

$$\begin{aligned} &0 \to Hom_A(U, V) \to Hom_k(U, V) \\ &\to^{\iota} Der_k(A, Hom_k(U, V)) \to^{\kappa} Ext^1_A(U, V) \to 0 \\ &0 \to Hom_A(V, V \otimes_A Ph(A)) \to Hom_k(V, V \otimes_A Ph(A)) \\ &\to^{\iota} Der_k(A, Hom_k(V, V \otimes_A Ph(A))) \to^{\kappa} Ext^1_A(V, V \otimes_A Ph(A)) \to 0 \end{aligned}$$

we obtain the non-commutative Kodaira-Spencer class

$$c(V) := \kappa(u(V)) \in Ext^1_A(V, V \otimes_A Ph(A))$$

inducing the Kodaira-Spencer morphism

$$g : \Theta_A := Der_k(A, A) \longrightarrow Ext^1_A(V, V)$$

via the identity d_*. If $c(V) = 0$, then the exact sequence above proves that there exist a $\nabla \in Hom_k(V, V \otimes_A Ph(A))$ such that $u = \iota(\nabla)$. This is just another way of proving that $c(V)$ is the obstruction for the existence of a connection

$$\nabla : Der_k(A, A) \longrightarrow Hom_k(V, V)$$

It is moreover well known (I think), that in the commutative case, the Kodaira-Spencer class gives rise to a Chern character by putting

$$ch^i(V) := 1/i! \; c^i(V) \in Ext^i_A(V, V \otimes_A Ph(A))$$

and that if $c(V) = 0$, the curvature $R(\nabla)$ of the connection ∇, induces a curvature class, in a generalized Lie-algebra cohomology

$$R_\nabla \in H^2(k, A; \Theta_A, End_A(V))$$

The same exact sequences furnish a proof for the following,

Lemma 1.1. *Let $\rho : A \to End_k(V)$, be an A-module, and let $\delta \in Der_k(A, End_k(V))$, map to 0 in $Ext^1_A(V, V)$, i.e. assume $\kappa(\delta) = 0$, then there exist an element, $Q_\delta \in End_k(V)$, such that for all $a \in A$,*

$$\delta(a) = [Q_\delta, \tilde{\rho}(a)].$$

If V is a simple A-module, $ad(Q_\delta)$ is unique.

Definition 1.2. *For any representation, $\rho : A \to End_k(V)$, put,*

$$\mathfrak{g}_V := \mathfrak{g}_\rho = \{\gamma \in Der_k(A)|\ g(\gamma) = \kappa(\delta\rho) = 0\},$$

and see that it is a Lie sub-algebra of $Der_k(A)$.

Obviously this means that there is always a connection,

$$\nabla : \mathfrak{g}_V \to End_k(V).$$

Any $Ph(A)$-module W, given by its structure map,

$$\rho(W)^1 =: \rho^1 : Ph(A) \longrightarrow End_k(W)$$

corresponds bijectively to an induced A-module structure $\rho : A \to End_k(W)$, together with a derivation $\delta_\rho \in Der_k(A, End_k(W))$, defining an element $[\delta_\rho] \in Ext^1_A(W, W)$. Fixing this last element we find that the set of $Ph(A)$-module structures on the A-module W is in one to one correspondence with $End_k(W)/End_A(W)$. Conversely, starting with an A-module V and an element $\delta \in Der_k(A, End_k(V))$, we obtain a $Ph(A)$-module V_δ. It is then easy to see that the kernel of the natural map

$$Ext^1_{Ph(A)}(V_\delta, V_\delta) \to Ext^1_A(V, V)$$

induced by the linear map

$$Der_k(Ph(A), End_k(V_\delta)) \to Der_k(A, End_k(V))$$

is the quotient

$$Der_A(Ph(A), End_k(V_\delta))/End_k(V)$$

and the image is a subspace $[\delta_\rho]^\perp \subseteq Ext^1_A(V, V))$, which is rather easy to compute, see examples below.

Example 1.3. *(i) Let $A = k[t]$, then obviously, $Ph(A) = k < t, dt >$ and d is given by $d(t) = dt$, such that for $f \in k[t]$, we find $d(f) = J_t(f)$ with the notations of* [14], *i.e. the non-commutative derivation of f with respect to t. One should also compare this with the non-commutative Taylor formula of loc.cit. If $V \simeq k^2$ is an A-module, defined by the matrix $X \in M_2(k)$, and $\delta \in Der_k(A, End_k(V))$, is defined in terms of the matrix $Y \in M_2(k)$, then the $Ph(A)$-module V_δ is the $k < t, dt >$-module defined by the action of the two matrices $X, Y \in M_2(k)$, and we find*

$$e^1_V := dim_k Ext^1_A(V, V) = dim_k End_A(V) = dim_k\{Z \in M_2(k)|\ [X, Z] = 0\}$$

$$e^1_{V_\delta} := dim_k Ext^1_{Ph(A)}(V_\delta, V_\delta) = 8 - 4 + dim\{Z \in M_2(k)|\ [X, Z] = [Y, Z] = 0\}$$

We have the following inequalities:

$$2 \le e^1_V \le 4 \le e^1_{V_\delta} \le 8$$

(ii) Let $A = k[t_1, t_2]$ then we find

$$Ph(A) = k < t_1, t_2, dt_1, dt_2 > /([t_1, t_2], [dt_1, t_2] + [t_1, dt_2]$$

In particular, we have a surjective homomorphism

$$Ph(A) \to k < t_1, t_2, dt_1, dt_2 > /([t_1, t_2], [dt_1, dt_2], [t_i, dt_i] - 1)$$

the right hand algebra being the Weyl algebra. This homomorphism exists in all dimensions. We also have a surjective homomorphism,

$$Ph(A) \to k[t_1, t_2, \xi_1, \xi_2]$$

i.e. onto the affine algebra of the classical phase-space.

Remark 1.4. *Since $Ext^1_A(V, V)$ is the tangent space of the miniversal deformation space of V as an A-module, we see that the non-commutative space $Ph(A)$ also parametrizes the set of generalized momenta, i.e. the set of pairs of an A- module V, and a tangent vector of the formal moduli of V, at that point.*

Ph(A) is relatively easy to compute. In particular, if $A = k[x_1, .., x_n]$ is the polynomial algebra, we have,

$$Ph(A) = k < x_1, .., x_n, dx_1, .., dx_n > /([x_i, x_j], [x_i, dx_j] + [dx_i, x_j]).$$

Notice that, any rank 1 representation of $Ph(A)$ is represented by a pair, (q,p), of a closed point, q of $Spec(k[\underline{x}])$, and a tangent, p at that point. We shall need the following formulas,

Theorem 1.5. *Given two such points, $(q_i, p_i), i = 1, 2$, we find,*

$$dim_k Ext^1_{Ph(A)}(k(q_1, p_1), k(q_2, p_2)) = 2n, for\ (q_1, p_1) = (q_2, p_2)$$
$$dim_k Ext^1_{Ph(A)}(k(q_1, p_1), k(q_2, p_2)) = n, for\ q_1 = q_2, , p_1 \neq p_2$$
$$dim_k Ext^1_{Ph(A)}(k(q_1, p_1), k(q_2, p_2)) = 1, for\ q_1 \neq q_2.$$

Moreover, there is a generator of,

$$Ext^1_{Ph(A)}(k(q_1, p_1), k(q_2, p_2)) = Der_k(Ph(A), Hom_k(k(q_1, p_1)), k(q_2, p_2)))/Triv,$$

uniquely characterized by the tangent line defined by the vector $\overline{q_1 q_2}$.

Proof. Assume for convenience that $n = 3$. Put $x_j(q_i, p_i) := q_{i,j},\ dx_j((q_i, p_i) := p_{i,j},\ \alpha_j = q_{1,j} - q_{2,j},\ \beta_j = p_{1,j} - p_{2,j}$.

See that for any element $\alpha \in Hom_k(k(q_1, p_1), k(q_2, p_2))$ we have,

$$x_j\alpha = q_{1,j}\alpha,\ \alpha x_j = q_{2,j}\alpha,\ dx_j\alpha = p_{1,j}\alpha,\ \alpha dx_j = p_{2,j}\alpha,$$

with the obvious identification. Any derivation

$$\delta \in Der_k(Ph(A), Hom_k(k(q_1, p_1), k(q_2, p_2)))$$

must satisfy the relations,

$$\delta([x_i, x_j]) = [\delta(x_i), x_j] + [x_i, \delta(x_j)] = 0$$
$$\delta([dx_i, x_j] + [x_i, dx_j]) = [\delta(dx_i), x_j] + [dx_i, \delta(x_j)] + [\delta(x_i), dx_j] + [x_i, \delta(dx_j)] = 0.$$

Using the above left-right action-rules, the result follows from the long exact sequence computing Ext^1_{PhA}. The two families of relations above give us two systems of linear equations.

The first, in the variables $\delta(x_1), \delta(x_2), \delta(x_3)$, with matrix,

$$\begin{pmatrix} -\alpha_2 & \alpha_1 & 0 \\ -\alpha_3 & 0 & \alpha_1 \\ 0 & -\alpha_3 & \alpha_2 \end{pmatrix}.$$

and the second, in the variables, $\delta(x_1), \delta(x_2), \delta(x_3), \delta(dx_1), \delta(dx_2), \delta(dx_3)$, with matrix,

$$\begin{pmatrix} -\beta_2 & \beta_1 & 0 & -\alpha_2 & \alpha_1 & 0 \\ -\beta_3 & 0 & \beta_1 & -\alpha_3 & 0 & \alpha_1 \\ 0 & -\beta_3 & \beta_2 & 0 & -\alpha_3 & \alpha_2 \end{pmatrix}$$

In particular we see that the *trivial* derivation given by,

$$\delta(x_i) = \alpha_i,\ \delta(dx_j) = \beta_j,$$

satisfies the relations, and the generator of $Ext^1_{PhA}(k(q_1, p_1), k(q_2, p_2))$ is represented by,

$$\delta(x_i) = 0,\ \delta(dx_j) = \alpha_j.$$

This is, in an obvious sense, the "tangent vector" $-\overline{q_1, q_2}$

It is easy to extend this result from dimension 3 to any dimension n. □

Remark 1.6. *Blowing Up and Desingularization*

The A-algebra $Ph(A)$ is graded by defining, for $a \in A$,

$$deg(a) = 0, deg(d(a)) = 1$$

By definition, any $Ph(A)$-representation V correspond to a representation of A together with a derivation of A into $End_k(V)$, which again induces a tangent direction in the moduli of representations of A at the point V. In complete generality we have a map,

$$\mathfrak{bu} : Simph_A(Ph(A)) \to Simp(A),$$

where $Simph_A(Ph(A))$ is the set of simple graded $Ph(A)$-modules and the mapping is onto the 0 Th. component.

The corresponding morphism in the commutative case is,

$$\mathfrak{bu} : Proj_A(Ph(A)) \to Spec(A),$$

which we shall call the General Blowing Up Map. By the universal property of $Ph(A)$ it is clear that the fibre of $\mathfrak{bu}$ at a (k-point) $x \in Spec(A)$ is $Proj(T(x)) \simeq \mathbf{P}^{n-1}$, where $T(x)$ is the tangent space of $Spec(A)$ at the point x, supposed to be of imbedding dimension n.

Therefore any vector field ξ, on $Spec(A)$, i.e. any derivation $\xi \in Der_k(A, A)$, defines a canonical section of $\mathfrak{bu}$,

$$\sigma(\xi) : D(\xi) \to Proj_A(Ph(A)),$$

defined in the open subscheme $D(\xi)$, where ξ is non-trivial. The blow-up of $Spec(A)$ defined by ξ is now the closure of the image of $\sigma(\xi)$,

$$Spec(A, \xi) = Spec(A(\xi)) \subset Proj_A(Ph(A)).$$

The relative notion, i.e. blowing up a subscheme in another scheme, is gotten by considering, for any morphism of algebras, $\pi : A \to B$, the blowing ups, as above, corresponding to the derivations, $\xi \in ker\{Der_k(A, A) \to Der_k(A, B)\}$.

Blowing up the origin in the affine n-space, would then correspond to the blowing up of $Spec(k[x_1, ..., x_n])$ defined by the derivation $\xi = \sum x_i \frac{\partial}{\partial x_i}$.

In general, if $A = k[x_1, ..., x_n]/(r_1, ..., r_s)$, consider the Jacobian matrix,

$$J = (\frac{\partial r_i}{\partial x_j})$$

and let J_α be a maximal sub-determinant. $J_\alpha \neq 0$ in an open subset $U := Spec(A) - Sing(A)$, supposed to be non-empty. Compute the solutions of the linear system of equations,

$$\sum_{j=1}^{n} \frac{\partial r_i}{\partial x_j} dx_j = 0, i = 1, ..., s.$$

We find solutions of the form,

$$dx_l = \sum_{i}^{d} c_i^l / J_\alpha dx_i, l = d + 1, ..., n$$

and the derivations of A of the form,

$$\xi_i := J_\alpha \frac{\partial}{\partial x_i} - \sum_{l=d+1}^{n} c_i^l \frac{\partial}{\partial x_l}$$

are all non-trivial in U. The corresponding blow-ups of A, looks like,

$$A(\xi_i) = Ph(A)/(J_\alpha dx_l - c_i^l dx_i, l = d + 1, ..., n, dx_j = 0, j \neq i)$$

This gives us a possible easier road to de-singularization since the Ph-operation is canonical and may be iterated.

1.3. **The iterated Phase Space functor, Ph^*, and the Dirac derivation.** The phase-space construction may, be iterated. Given the k-algebra A we may form the sequence, $\{Ph^n(A)\}_{0\leq n}$, defined inductively by

$$Ph^0(A) = A,\ Ph^1(A) = Ph(A), ..., Ph^{n+1}(A) := Ph(Ph^n(A)).$$

Let $i_0^n : Ph^n(A) \to Ph^{n+1}(A)$ be the canonical imbedding, and let $d_n : Ph^n(A) \to Ph^{n+1}(A)$ be the corresponding derivation. Since the composition of i_0^n and the derivation d_{n+1} is a derivation $Ph^n(A) \to Ph^{n+2}(A)$, corresponding to the homomorphism,

$$Ph^n(A) \to^{i_0^n} Ph^{n+1}(A) \to^{i_0^{n+1}} Ph^{n+2}(A)$$

there exist by universality a homomorphism $i_1^{n+1} : Ph^{n+1}(A) \to Ph^{n+2}(A)$, such that,

$$i_0^n \circ i_1^{n+1} = i_0^n \circ i_0^{n+1}$$

and such that,

$$d_n \circ i_1^{n+1} = i_0^n \circ d_{n+1}.$$

Notice that we here compose functions and functors from left to right. Clearly we may continue this process constructing new homomorphisms,

$$\{i_j^n : Ph^n(A) \to Ph^{n+1}(A)\}_{0\leq j\leq n},$$

such that,

$$i_p^n \circ i_0^{n+1} = i_0^n \circ i_{p+1}^{n+1}$$

with the property,

$$d_n \circ i_{j+1}^{n+1} = i_j^n \circ d_{n+1}.$$

We find, see [18], the following identities,

$$\begin{aligned} i_p^n i_q^{n+1} &= i_{q-1}^n i_p^{n+1},\ p < q \\ i_p^n i_p^{n+1} &= i_p^n i_{p+1}^{n+1} \\ i_p^n i_q^{n+1} &= i_q^n i_{p+1}^{n+1},\ q < p. \end{aligned}$$

To see this, compose with i_0^{n-1} and d_{n-1}, and use induction. Thus, the $Ph^*(A)$ is a semi-co-simplicial k-algebra with a co-section h_0, onto A. And it is easy to see that h_0 together with the corresponding co-sections $h_p : Ph^{p+1}(A) \to Ph^p(A)$, for $Ph^p(A)$ replacing A, form a trivializing homotopy for $Ph^*(A)$. Thus, we have,

$$H^n(Ph^*(A)) = 0, \forall n \geq 0,$$

i.e. Ph^{*+1} is a co-simplicial resolution of A. Therefore, for any object,

$$\kappa : A \to R \in A/k - \underline{alg}$$

the co-simplicial algebra above induces simplicial sets,

$$Mor_k(Ph^\star(A), R),\ Mor_A(Ph^\star(A), R),$$

and one should be interested in the homotopy. See also that this generalises to a canonical functor,

$$Spec : (k - alg^\Delta)^{op} \longrightarrow SPr(k)$$

where $(k - alg)^\Delta$ is the category of co-simplicial k-algebras, and $SPr(k)$ is the category of simplicial presheves on the category of k-schemes enriched by any Grothendieck topology. As usual, the imbedding of the category of k-algebras in the category of co-simplicial algebras is defined simply by giving any k-algebra a constant co-simplicial structure. The fact that $Ph^{\star}(A)$ is a resolution of A, is therefore simply saying that,

$$Spec(Ph^{\star}((A)) \rightarrow Spec(A),$$

is a week equivalence in $SPr(k)$.

This might be a starting point for a theory of homotopy for k-schemes. We may also consider, for any k-algebra R, the simplicial k-vectorspace,

$$Der_k(Ph^{\star}(A), R),$$

Consider this complex for $R = A$, i.e. $Mor_A(Ph^{\star}(A), A)$. Clearly,

$$Mor_A(Ph^{n+1}(A), A) = Der_k(Ph^n(A), A), n \geq 0,$$

and we have,

$$Mor_A(Ph^n(A), A) = \{\xi_0 \circ \xi_{i_1} \circ ... \circ \xi_{i_r} | 0 \leq i_l \leq i_{l+1} \leq n | \xi_0 = id_A, \xi_i \in Der_k(A), i \geq 1\}$$

Since Ph is a functor, and Ph^{*+1} is a co-simplicial resolution of A. we may apply this to any scheme X, given in terms of an affine covering $\mathbf{U}$, and obtain an algebraic homology (or cohomology), with converging spectral sequences,

$$E^1_{pq} = H_p({H_{\mathbf{U}}}^{-q}(Der_k(Ph^{\star}(A), A))), E_{q,p} = H^{-q}_{\mathbf{U}}(H_p(Der_k(Ph^{\star}(A), A)))$$

If we, in $Mor_A(Ph^n(A), A)$, identify $\xi \sim \alpha\xi, \alpha \in k^*$, we obtain a rational cohomology with converging spectral sequences,

$$E^{pq}_1 = H^p(H^q_{\mathbf{U}}(Mor_A(Ph^n(A), A), \mathbf{Q})), E^{q,p}_2 = H^q_{\mathbf{U}}(H^p(Mor_A(Ph^n(A), A), \mathbf{Q}))$$

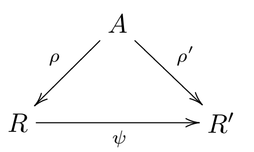
Remark 1.7. *The above suggests that we are closing in on Stacks and Motives. Any reasonable cohomology theory defined on the category of k-schemes, is now seen to be defined on the image category of Sec, so probably extendable to $SPr(k)$, therefore comes with a homotopy theory attached. Moreover, suppose we, instead of the example $R = A$ above, considered the category (actually an ordered set) of morphisms, $\mathfrak{a}(A) := \{A \rightarrow A/\mathfrak{p}_i\}$, for some family of irreducible bilateral ideals, corresponding in the commutative case to families of subschemes of $X = Spec(A)$. By the theorem (4.2.4) of* [10] *there is for any finite subcategory $\mathfrak{V} \subset \mathfrak{a}(A)$, a formal moduli $H(\mathfrak{V})$, for the deformation functor $Def(\mathfrak{V})$, of the category of morphisms $\mathfrak{V}$, with the algebra A trivially deformed, provided the corresponding cohomology groups of the deformation theory are countably generated.*

Moreover, we may globalise this to hold for any scheme, X, and in particular to any projective scheme over k, for which we know that the cohomology groups of the deformation theory will be of finite dimension, implying that the formal moduli $H(\mathfrak{V})$ will be finitely generated formal k-algebras. Formally the theory will be of the same nature for schemes as for algebras, and so to minimise the place and problems with hanging on to dull dual descriptions, we shall just describe the affine case. Of course in this case the formal moduli $H(\mathfrak{V})$ will not, in general, be finitely generated formal k-algebras, but this should not be of much trouble for most mathematicians.

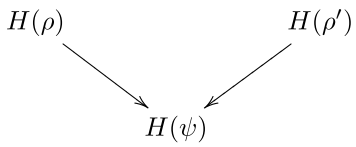
Obviously, if $\mathfrak{W} \subset \mathfrak{V}$ there is a natural morphism,

$$\pi(\mathfrak{V}, \mathfrak{W}) : H(\mathfrak{W}) \rightarrow H(\mathfrak{V}),$$

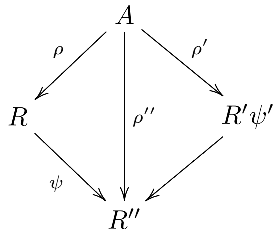
usually just called π, or name omitted. Given a commutative diagram,



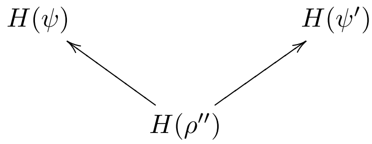
we have a diagram of canonical morphisms,



where we have put, $H(\psi) := H(\psi : \rho \to \rho')$. *And for any diagram like,*



we find a diagram of canonical morphisms,



with the same abbreviation as above.

A prime cycle in the motive of A should be any object $(\rho, H(\rho)), \rho \in Irr(A)$, *The set of which we shall call* $h(A)$. *A cycle should then be a linear combination over some abelian group of such prime cycles.*

We should like to define the intersection product of cycles, as a bilinear product of cycles. If ρ *and* ρ' *prime cycles, then we define the intersection as the sum,*

$$\rho \cup \rho' = \sum \alpha(\rho, \rho')\rho'', \ \alpha(\rho, \rho') := |H(\rho, \psi) \otimes_{H(\rho'')} H(\rho', \psi')|.$$

We would like to compare it to the Serre intersection formula,

$$\rho \cup \rho' = \sum_{0}^{\infty} (-1)^i Tor_i^A(R, R')$$

in the commutative case. But this demands a certain work which will be postponed.

Anyway, the notion of motive, over the rationals, should be given by the set of finite cycles,

$$M(A) = FinMap(h(A), \mathbf{Q})$$

divided out with some equivalence relation, which we shall come back to.

Consider now the co-simplicial algebra,

$$A \to^{i_0^0} Ph(A) \to^{i_p^1} Ph^2(A) \to^{i_p^2} Ph^3(A) \to^{i_p^3} ....$$

where, for each integer n, the symbol i_p^n, for $p = 0, 1, ..., n$ signify the family of A-morphisms between $Ph^n(A)$ and $Ph^{n+1}(A)$ defined above. The system of k-algebras and homomorphisms of k-algebras $\{Ph^n(A), i_j^n\}_{n, 0 \le j \le n}$ has an inductive (direct) limit,

$$Ph^\infty(A) = \varinjlim_{n \ge 0} \{Ph^n(A), i_j^n\}$$

together with homomorphisms

$$i_n : Ph^n(A) \longrightarrow Ph^\infty(A)$$

satisfying

$$i_j^n \circ i_{n+1} = i_n, \quad j = 0, 1, \ldots, n$$

Moreover, the family of derivations, $\{d_n\}_{0\leq n}$ define a unique derivation $\delta : Ph^\infty(A) \longrightarrow Ph^\infty(A)$, such that $i_n \circ \delta = d_n \circ i_{n+1}$. Put

$$Ph^{(n)}(A) := im\ i_n \subseteq Ph^\infty(A)$$

The k-algebra $Ph^\infty(A)$ has a descending filtration of two-sided ideals, $\{\mathbf{F}_n\}_{0\leq n}$ given inductively by

$$\mathbf{F}_1 = Ph^\infty(A) \cdot im(\delta) \cdot Ph^\infty(A)$$

and

$$\delta\mathbf{F}_n \subseteq \mathbf{F}_{n+1}, \quad \mathbf{F}_{n_1}\mathbf{F}_{n_2}\ldots\mathbf{F}_{n_r} \subseteq \mathbf{F}_n, \quad n_1 + \ldots + n_r = n$$

such that the derivation δ induces derivations $\delta_n : \mathbf{F}_n \longrightarrow \mathbf{F}_{n+1}$. Using the canonical homomorphism $i_n : Ph^n(A) \longrightarrow Ph^\infty(A)$ we pull the filtration $\{\mathbf{F}_p\}_{0\leq p}$ back to $Ph^n(A)$, obtaining a filtration of each $Ph^n(A)$ with,

$$\mathbf{F}_1^n = Ph^n(A) \cdot im(\delta) \cdot Ph^n(A)$$

and inductively,

$$\delta\mathbf{F}_p^n \subseteq \mathbf{F}_{p+1}^{n+1}, \quad \mathbf{F}_{p_1}^n\mathbf{F}_{p_2}^n\ldots\mathbf{F}_{p_r}^n \subseteq \mathbf{F}_p^n, \quad p_1 + \ldots + p_r = p.$$

Definition 1.8. *Let $\mathbf{D}(A) := \varprojlim_{n\geq 1} Ph^\infty(A)/\mathbf{F}_n$, the completion of $Ph^\infty(A)$ in the topology given by the filtration $\{\mathbf{F}_n\}_{0\leq n}$. The k-algebra $Ph^\infty(A)$ will be referred to as the k-algebra of higher differentials, and $\mathbf{D}(A)$ will be called the k-algebra of formalized higher differentials. Put*

$$\mathbf{D}_n := \mathbf{D}_n(A) := Ph^\infty(A)/\mathbf{F}_{n+1}$$

Clearly δ defines a derivation on $\mathbf{D}(A)$, and an isomorphism of k-algebras

$$\epsilon := exp(\delta) : \mathbf{D}(A) \to \mathbf{D}(A)$$

and in particular, an algebra homomorphism,

$$\tilde{\eta} := exp(\delta) : A \to \mathbf{D}(A)$$

inducing the algebra homomorphisms

$$\tilde{\eta}_n : A \to \mathbf{D}_n(A)$$

which, by killing, in the right hand algebra, the image of the maximal ideal, $\mathfrak{m}(\underline{t})$, of A corresponding to a point $\underline{t} \in Simp_1(A)$, induces a homomorphism of k-algebras

$$\tilde{\eta}_n(\underline{t}) : A \to \mathbf{D}_n(A)(\underline{t}) := \mathbf{D}_n/(\mathbf{D}_n\mathfrak{m}(\underline{t})\mathbf{D}_n)$$

and an injective homomorphism,

$$\tilde{\eta}(\underline{t}) : A \to \varprojlim_{n\geq 1} \mathbf{D}_n(A)(\underline{t})$$

see [15].

Remark 1.9. *Since $Ext_A^1(V,V)$ is the tangent space of the miniversal deformation space of V as an A-module, we see that the non-commutative space $Ph(A)$ also parametrizes the set of generalized momenta, i.e.the set of pairs of an A- module V, and a tangent vector of the formal moduli of V, at that point. Therefore the above implies that any representation, $\rho : Ph^\infty(A) \to End_k(V)$, corresponds to a family of $Ph^n(A)$-module-structures on V, for $n \geq 1$, i.e. to an A-module $V_0 := V$, an element $\xi_0 \in Ext_A^1(V,V)$, i.e. a tangent of the deformation functor of $V_0 := V$, as A-module, an element $\xi_1 \in Ext^1_{Ph(A)}(V,V)$, i.e. a tangent of the deformation functor of $V_1 := V$ as $Ph(A)$-module, an element $\xi_2 \in Ext^1_{Ph^2(A)}(V,V)$, i.e. a tangent of the deformation functor of $V_2 := V$ as $Ph^2(A)$-module, etc.*

All this is just $\rho_0 : A \to End_k(V)$, considered as an A-module, together with a sequence $\{\xi_n\}, 0 \leq n$, of a tangent, or a momentum, ξ_0, an acceleration vector, ξ_1, and any number of higher order momenta ξ_n. Thus, specifying a $Ph^\infty(A)$-representation V, implies specifying a formal curve through v_0, the base-point, of the miniversal deformation space of the A-module V. Formally, this curve is given by the composition of the homomorphism $\epsilon(\tau) := exp(\tau\delta)$ and ρ.

It is, however, impossible to prepare a physical situation such that a measurement, i.e. an object like ρ_0, is given by an infinite sequence $\{\xi_n\}$, of dynamical data. We shall have to be satisfied with a finite number of data, and normally with just the first one , i.e. the momentum ξ_0. This is the problem of Preparation and of the Time Evolution of a representation ρ, to be treated in the sequel.

1.4. **The generalized de Rham Complex.** Consider now the diagram,

$$\begin{array}{ccccccccc}
A & \xrightarrow{i_0^0} & Ph(A) & \xrightarrow{i_p^1} & Ph^2(A) & \xrightarrow{i_p^2} & Ph^3(A) & \xrightarrow{i_p^3} & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \mathfrak{m}_1^1 & \xrightarrow{i_p^1} & \mathfrak{m}_2^1 & \xrightarrow{i_p^2} & \mathfrak{m}_3^1 & \xrightarrow{i_p^3} &
\end{array}$$

where, for each integer n, the symbol i_p^n, for $p = 0, 1, ..., n$ signify the family of A-morphisms between $Ph^n(A)$ and $Ph^{n+1}(A)$ defined above, and where $\mathfrak{m}_n^1$ is the ideal of $Ph^n(A)$ generated by $im(d)$, which is the same as the ideal generated by the family, $\{i_p^{n-1}(i_p^{n-2}(...(i_p^1(d(A))...))\}$, for all possible p. And, inductively, let $\mathfrak{m}_n^m$ be the ideal generated by $\mathfrak{m}_n^1\mathfrak{m}_n^{m-1}$.

We find an extended diagram,

$$\begin{array}{ccccccccc}
A & \xrightarrow{i_0^0} & Ph(A) & \xrightarrow{i_p^1} & Ph^2(A) & \xrightarrow{i_p^2} & Ph^3(A) & \xrightarrow{i_p^3} & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
A & \xrightarrow{i_0^0} & A & \xrightarrow{i_p^1} & A & \xrightarrow{i_p^2} & A & \xrightarrow{i_p^3} & \cdots \\
 & \searrow^{d_0} & & \searrow^{d_1} & & \searrow^{d_2} & & \searrow^{d_3} & \\
 & & \mathfrak{m}_1^1/\mathfrak{m}_1^2 & \xrightarrow{i_p^1} & \mathfrak{m}_2^1/\mathfrak{m}_2^2 & \xrightarrow{i_p^2} & \mathfrak{m}_3^1/\mathfrak{m}_3^2 & \xrightarrow{i_p^3} & \cdots \\
 & & & \searrow^{d_1} & & \searrow^{d_2} & & \searrow^{d_3} & \\
 & & \mathfrak{m}_1^2/\mathfrak{m}_1^3 & \xrightarrow{i_p^1} & \mathfrak{m}_2^2/\mathfrak{m}_2^3 & \xrightarrow{i_p^2} & \mathfrak{m}_3^2/\mathfrak{m}_1^3 & \xrightarrow{i_p^3} & \cdots \\
 & & & \searrow^{d_1} & & \searrow^{d_2} & & \searrow^{d_3} & \\
 & & & & . & & . & & .
\end{array}$$

The diagonals are not necessarily complexes, but it suffices to kill d^2, to kill all d^n, $n \geq 2$, and for this it suffices to kill $d_1 d_0$, as one easily see, operating with the edge homomorphisms $i_p^n, n \geq 2$ on the elements, $d_1(d_0(a))$ for $a \in A$. Therefore we shall, in this general situation, make the following definition,

Definition 1.10. *The curvature $R(A)$ of the associative k algebra, A, is the k-linear map composition of d_0 and d_1,*

$$R(A) = d_0 d_1 : A \to \mathfrak{m}_2^2/\mathfrak{m}_2^3.$$

Now, kill the curvature $R(A)$, and all the terms under the first diagonal, beginning with $\mathfrak{m}_1^2/\mathfrak{m}_1^3$, together with all terms generated by the actions of the edge homomorphisms on these terms, and let, $\mathbf{\Omega}_n^m$ be the resulting quotient of $\mathfrak{m}_n^m/\mathfrak{m}_n^{m+1}$, for $n \geq 0$. Clearly, $\Omega_n^0 = A$ for all $n \geq 0$, and we have got a graded semi co-simplicial A-module, with a k-differential d, such that $d^2 = 0$, looking like,

$$\begin{array}{ccccccccc}
A & \xrightarrow{i_0^0} & Ph(A) & \xrightarrow{i_p^1} & Ph^2(A) & \xrightarrow{i_p^2} & Ph^3(A) & \xrightarrow{i_p^3} & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
A & \xrightarrow{i_0^0} & A & \xrightarrow{i_p^1} & A & \xrightarrow{i_p^2} & A & \xrightarrow{i_p^3} & \\
& \searrow^{d} & & \searrow^{d} & & \searrow^{d} & & \searrow^{d} & \\
& & \Omega_1^1 & \xrightarrow{i_p^1} & \Omega_2^1 & \xrightarrow{i_p^2} & \Omega_3^1 & \xrightarrow{i_p^3} & \\
& & & \searrow^{d} & & \searrow^{d} & & \searrow^{d} & \\
& & & & \Omega_2^2 & \xrightarrow{i_p^2} & \Omega_3^2 & \xrightarrow{i_p^3} &
\end{array}$$

It is a graded complex, in two ways. First as a complex induced from the semi-cosimplicial structure, with differential of bidegree (1,0), and second, as complex with differential d, of bidegree (1,1).

Lemma 1.11. *Suppose A is commutative, then there is a natural morphism of complexes of A-modules,*

$$\Omega_A^\star \subset \Omega_\star^\star,$$

with,

$$\Omega_A^n := \wedge^r \Omega_A \simeq \Omega_n^n.$$

Proof. Let, $a_i \in A, i = 1, ..., r$, and compute in $\Omega_\star^r$ the value of, $d^r(a_1 a_2 ... a_r)$. It is clear that this gives the formula,

$$\sum d_{i_1}(a_1) d_{i_2}(a_2) .. d_{i_r}(a_r) = 0,$$

the sum being over all permutation $(i_1, i_2, ..., i_r)$ of $(0, 1, ..., r-1)$. Here we consider A as a subalgebra of $Ph^n(A)$ via the unique compositions of the $i_0^s : Ph^s(A) \subset Ph^{s+1}(A)$. In particular, we have,

$$d_0(a_1) d_1(a_2) + d_1(a_1) d_0(a_2) = 0,$$

for all $a_1, a_2 \in A$. This relation and the relation $d_0(a_2) d_1(a_1) = d_1(a_1) d_0(a_2)$, which follows from commutativity, $d(a_2) a_1 = a_1 d(a_2)$, forcing the left and right A-action on Ω_A to be equal, immediately give us, $d_0(a_1) d_1(a_2) = -d_0(a_2) d_1(a_1)$.

Consider now the diagram,

$$\begin{array}{ccccccccc}
A & \xrightarrow{i_0^0} & Ph(A) & \xrightarrow{i_0^1} & Ph^2(A) & \xrightarrow{i_0^2} & Ph^3(A) & \xrightarrow{i_0^3} & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
A & \xrightarrow{i_0^0} & A \oplus \Omega_A^1 & \xrightarrow{i_0^1} & A \oplus \Omega_A^1 \oplus \Omega_A^2 & \xrightarrow{i_0^2} & A \oplus \Omega_A^1 \oplus \Omega_A^2 \oplus \Omega_A^3 & \xrightarrow{i_0^3} & \cdots,
\end{array}$$

where the bottom line is a sequence of Nagata-extentions of the k-algebra A, and the vertical homomorphisms correspond to the natural derivations among these, defined by the derivations of the de Rham complex, $\Omega^\star$, of A.

The universality of the two systems proves that there is a surjective map,

$$\alpha : \Omega_n^n \to \Omega_A^n := \wedge^n \Omega_A.$$

The map that sends the element $da_1 \wedge da_2 \wedge ... \wedge da_n \in \Omega_A^r$ to $d_0(a_1) d_1(a_2) .. d_{n-1}(a_r) \in \Omega_n^n$ is an inverse, proving that α is an isomorphism. □

It follows from this, that in the commutative case, for any scheme X considered as a covering of affine schemes in some sense, there are two spectral sequences converging to the same cohomology, first

$$E(1)^2_{p,q} = H^p(H^q_{dR}(X, \Omega^*_*))$$

then,

$$E(2)^2_{p,q} = H^q_{dR}(X, H^p(\Omega^*_*)).$$

Let now V be a right A-module, and assume $c(V) = 0$, such that there exist an element, $\nabla' \in Hom_k(V, V \otimes_A Ph(A))$ with $c = \iota(\nabla')$. This implies that for $a \in A$ and $v \in V$ we have $\nabla'(va) = \nabla'(v)a + v \otimes d_0(a)$. Composing ∇' with the projection, $o : Ph(A) \to A$, corresponding to the 0-derivation of A, we therefore obtain an A-linear homomorphism $P : V \to V$, *a potential.* Since $i^0_0 : A \to Ph(A)$ is a section of o, we find a k-linear map,

$$\nabla_0 := \nabla' - P : V \to V \otimes \mathfrak{m}^1_1$$

Using the property,

$$d_n \circ i^{n+1}_{j+1} = i^n_j \, d_{n+1},$$

we find well defined k-linear maps,

$$\nabla_1 : V \to V \otimes \Omega^1_2, \; \nabla_2 : V \to V \otimes \Omega^1_3, ..., \nabla_n : V \to V \otimes \Omega^1_{n+1} \; \forall n \geq 0,$$

given by ,

$$\nabla_{n+1} := \nabla_n \circ i^{n+1}_1, \; n \geq 0,$$

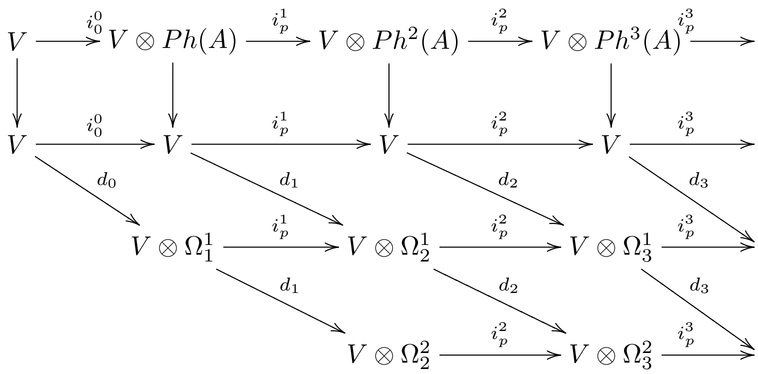
such that, for all, $v \in V, \omega \in \Omega^n_p$, the formula,

$$\nabla_n(v \otimes \omega) = \nabla_n(v)\omega + v \otimes d_n(\omega).$$

makes sense, and defines a sequence of *derivations*,

$$\nabla_n : V \otimes \Omega^p_n \to V \otimes \Omega^{p+1}_{n+1},$$

sometimes just denoted d_n, and called a *connection* ∇, on the A-module V. We obtain a situation just like above,



In general, there are no reasons for these derivations, $d_n := \nabla_n, \; n \geq 0$, to define complexes, and we shall make the following definition,

Definition 1.12. *The curvature $R(V, \nabla)$ of the connection ∇, defined on the right A-module V, is the k-linear map, composition of d_0 and d_1,*

$$R(V) = d_0 d_1 : V \to V \otimes \Omega^2_2.$$

The following Lemma is then easily proved,

Lemma 1.13. *Suppose A is commutative, and assume $c(V) = 0$. Let $\nabla : \Theta_A \to End_k(V)$ be the classical connection corresponding to ∇_0. Suppose moreover that the curvature R of ∇ is 0, then $R(V) = 0$, implying that $d^2 = 0$, and so the diagonals in the diagram above, are all complexes.*

Proof. We may put,

$$\nabla(v_i) = \sum_{j,k,} a_{i,j}^k v_j d_0(x_k)$$

and obtain,

$$\nabla_1(\nabla_0(v_i)) = \sum_{j,k,l} \frac{\partial a_{i,j}^k}{\partial x_l} v_j d_1(x_l) d_0(x_k) + \sum_{j,k,l,m} a_{i,j}^k a_{j,m}^l v_m d_1(x_l) d_0(x_k)$$

Now the classical curvature of ∇, may be defined as,

$$R_{k,l}^i = \sum_j \frac{\partial a_{i,j}^k}{\partial x_l} v_j + \sum_{j,m} a_{i,j}^k a_{j,m}^l v_m - \sum_j \frac{\partial a_{i,j}^l}{\partial x_k} v_j - \sum_{j,m} a_{i,j}^l a_{j,m}^k v_m,$$

so if, $R = 0$, and $d_1(x_l)d_0(x_k) = -d_1(x_k)d_0(x_l)$, we find that $\nabla_1(\nabla_0(v_i)) = 0$, from which it follows that $d^2 = 0$ □

1.5. **Non-commutative deformations of families of modules.** In [12], [13] and [14], we introduced non-commutative deformations of families of modules of non-commutative k-algebras, and the notion of *swarm* of right modules (or more generally of objects in a k-linear abelian category). Let $\underline{a}_r$ denote the category of r-pointed not necessarily commutative k-algebras R. The objects are the diagrams of k-algebras

$$k^r \xrightarrow{\iota} R \xrightarrow{\pi} k^r$$

such that the composition of ι and π is the identity. Any such *r-pointed k-algebra* R is isomorphic to a k-algebra of $r \times r$-matrices $(R_{i,j})$. The radical of R is the bilateral ideal $Rad(R) := ker\pi$, such that $R/Rad(R) \simeq k^r$. The dual k-vector space of $Rad(R)/Rad(R)^2$ is called the tangent space of R.

For $r = 1$, there is an obvious inclusion of categories $\underline{l} \subseteq \underline{a}_1$, where $\underline{l}$, as usual, denotes the category of commutative local Artinian k-algebras with residue field k.

Fix a (not necessarily commutative) associative k-algebra A and consider a right A-module M. The ordinary deformation functor $Def_M : \underline{l} \to \underline{Sets}$ is then defined. Assuming $Ext_A^i(M, M)$ has finite k-dimension for $i = 1, 2$, it is well known, see [34], or [12], that Def_M has a pro-representing hull H, *the formal moduli of M*. Moreover, the tangent space of H is isomorphic to $Ext_A^1(M, M)$, and H can be computed in terms of $Ext_A^i(M, M)$, $i = 1, 2$ and their *matric* Massey products, see [12].

In the general case, consider a finite family $\mathbf{V} = \{V_i\}_{i=1}^r$ of right A-modules., and put $V := \oplus_{i=1}^r V_i$. Assume that $\dim_k Ext_A^1(V_i, V_j) < \infty$. Any such family of A-modules will be called a *swarm*. We shall define a deformation functor,

$$Def_{\mathbf{V}} : \underline{a}_r \to \underline{Sets},$$

generalising the functor Def_M above. Given an object $\pi : R = (R_{i,j}) \to k^r$ of $\underline{a}_r$, consider the k-vector space and left R-module $(R_{i,j} \otimes_k V_j)$. It is easy to see that

$$End_R((R_{i,j} \otimes_k V_j)) \simeq (R_{i,j} \otimes_k Hom_k(V_i, V_j))$$

Clearly π defines a k-linear and left R-linear map

$$\pi(R) : (R_{i,j} \otimes_k V_j) \to \oplus_{i=1}^r V_i$$

inducing a homomorphism of R-endomorphism rings,

$$\tilde{\pi}(R) : (R_{i,j} \otimes_k Hom_k(V_i, V_j)) \to \oplus_{i=1}^r End_k(V_i)$$

The right A-module structure on the V_i's is defined by a homomorphism of k-algebras,

$$\eta_0 : A \to \oplus_{i=1}^r End_k(V_i) \subset (Hom_k(V_i, V_j)) =: End_k(V)$$

Notice that this homomorphism also provides each $Hom_k(V_i, V_j)$ with an A- bi-module structure. Let $Def_{\mathbf{V}}(R) \in \underline{Sets}$ be the set of isoclasses of homomorphisms of k-algebras,

$$\eta' : A \to (R_{i,j} \otimes_k Hom_k(V_i, V_j))$$

such that, $\tilde{\pi}(R) \circ \eta' = \eta_0$, where the equivalence relation is defined by inner automorphisms in the R-algebra $(R_{i,j} \otimes_k Hom_k(V_i, V_j))$ inducing the identity on $\oplus_{i=1}^r End_k(V_i)$. One easily proves that $Def_{\mathbf{V}}$ has the same properties as the ordinary deformation functor and we may prove the following, see [12]:

Theorem 1.14. *The functor $Def_{\mathbf{V}}$ has a pro-representable hull, i.e. an object $H := H(\mathbf{V}))$ of the category of pro-objects $\hat{\underline{a}}_r$ of $\underline{a}_r$, together with a versal family*

$$\tilde{V} = (H_{i,j} \otimes V_j) \in \varprojlim_{n \geq 1} Def_{\mathbf{V}}(H/\mathfrak{m}^n)$$

where $\mathfrak{m} = Rad(H)$, such that the corresponding morphism of functors on $\underline{a}_r$

$$\kappa : Mor(H, -) \to Def_{\mathbf{V}}$$

defined for $\phi \in Mor(H, R)$ by $\kappa(\phi) = R \otimes_\phi \tilde{V}$, is smooth, and an isomorphism on the tangent level. H is uniquely determined by a set of matric Massey products defined on subspaces

$$D(n) \subseteq Ext^1(V_i, V_{j_1}) \otimes \cdots \otimes Ext^1(V_{j_{n-1}}, V_k)$$

with values in $Ext^2(V_i, V_k)$.

Moreover, the right action of A on $\tilde{V}$ defines a homomorphism of k-algebras,

$$\eta : A \longrightarrow O(\mathbf{V}) := End_H(\tilde{V}) = (H_{i,j} \otimes Hom_k(V_i, V_j))$$

The k-algebra $O(\mathbf{V})$, called the ring of observables of $\mathbf{V}$, acts on the family of A-modules $\{V_i\}_{i=1}^r$, extending the action of A.

If $dim_k V_i < \infty$, for all $i = 1, \ldots, r$, the operation of associating $(O(\mathbf{V}), \mathbf{V})$ to $(A, \mathbf{V})$ is a closure operation.

There is a crucial result, see [13], [14],

Theorem 1.15 (A Generalized Burnside Theorem)**.** *Let A be a finite dimensional k-algebra, k an algebraically closed field. Consider the family $\mathbf{V} = \{V_i\}_{i=1}^r$ of all simple A-modules, then*

$$\eta : A \longrightarrow O(\mathbf{V}) = (H_{i,j} \otimes Hom_k(V_i, V_j))$$

is an isomorphism. Moreover the k-algebras A and H are Morita-equivalent.

We also prove that there exists, in the non-commutative deformation theory, an obvious analogy to the notion of pro-representing (modular) substratum H_0 of the formal moduli H, see [10]. The tangent space of H_0 is determined by a family of subspaces

$$Ext_0^1(V_i, V_j) \subseteq Ext_A^1(V_i, V_j), \quad i \neq j$$

the elements of which should be called the almost split extensions (sequences) relative to the family $\mathbf{V}$, and by a subspace

$$T_0(\Delta) \subseteq \prod_i Ext_A^1(V_i, V_i)$$

which is the tangent space of the deformation functor of the full subcategory of the category of A-modules generated by the family $\mathbf{V} = \{V_i\}_{i=1}^r$, see [11]. If $\mathbf{V} = \{V_i\}_{i=1}^r$ is the set of all indecomposable's of some Artinian k-algebra A, we show that the above notion of *almost split sequence* coincides with that of Auslander, see [35].

Using this we consider, in [12] and [14], the general problem of classification of iterated extensions of a family of modules $\mathbf{V} = \{V_i, \}_{i=1}^r$, and the corresponding classification of filtered modules with graded components in the family $\mathbf{V}$, and extension type given by a directed representation graph Γ. This turns out to be the starting point for our treatment of *Interactions* in physics, and will be postponed, see section (7).

To any, not necessarily finite, swarm $\underline{c} \subset \underline{mod}(A)$ of right-A-modules, we have associated two associative k-algebras, see [13] and [14],

$$O(|\underline{c}|, \pi) = \varprojlim_{\mathbf{V} \subset |\underline{c}|} O(\mathbf{V})$$

and a sub-quotient $\mathbf{O}_\pi(\underline{c})$, together with natural k-algebra homomorphisms

$$\eta(|\underline{c}|) : A \longrightarrow O(|\underline{c}|, \pi)$$

and $\eta(\underline{c}) : A \longrightarrow \mathbf{O}_\pi(\underline{c})$ with the property that the A-module structure on $\underline{c}$ is extended to an $\mathbf{O}$-module structure in an optimal way. We then defined an *affine non-commutative scheme* of right A-modules to be a swarm $\underline{c}$ of right A-modules, such that $\eta(\underline{c})$ is an isomorphism. In particular we considered, for finitely generated k-algebras, the swarm $Simp^*_{<\infty}(A)$ consisting of the finite dimensional simple A-modules, and the *generic* point A, together with all morphisms between them. The fact that this is a swarm, i.e. that for all objects $V_i, V_j \in Simp_{<\infty}$ we have $dim_k Ext^1_A(V_i, V_j) < \infty$, is easily proved. We have in [14] proved the following result, (see (4.1), loc.cit. for the definition of the notion of *geometric* k-algebra, and compare with Lemma (2.5).)

Proposition 1.16. *Let A be a geometric k-algebra, then the natural homomorphism,*

$$\eta(Simp^*(A)) : A \longrightarrow \mathbf{O}_\pi(Simp^*_{<\infty}(A))$$

is an isomorphism, i.e. $Simp^*_{<\infty}(A)$ is a scheme for A.

In particular, $Simp^*_{<\infty}(k < x_1, x_2, \dots, x_d >)$, is a scheme for $k < x_1, x_2, \dots, x_d >$. To analyze the local structure of $Simp_n(A)$, we need the following, see [14],(3.23),

Lemma 1.17. *Let $\mathbf{V} = \{V_i\}_{i=1,..,r}$ be a finite subset of $Simp_{<\infty}(A)$, then the morphism of k-algebras,*

$$A \to O(\mathbf{V}) = (H_{i,j} \otimes_k Hom_k(V_i, V_j))$$

is topologically surjective.

Proof. Since the simple modules V_i $(i = 1, \dots, r)$ are distinct, there is an obvious surjection

$$\eta_0 : A \to \prod_{i=1,..,r} \mathrm{End_k(V_i)}$$

Put $\mathfrak{r} = ker\eta_0$, and consider for $m \geq 2$ the finite-dimensional k-algebra, $B := A/\mathfrak{r}^m$. Clearly $Simp(B) = \mathbf{V}$, so that by the generalized Burnside theorem, see [12], (2.6), we find,

$$B \simeq O^B(\mathbf{V}) := (H^B_{i,j} \otimes_k Hom_k(V_i, V_j))$$

Consider the commutative diagram

$$\begin{array}{ccccc} A & \longrightarrow & (H^A_{i,j} \otimes_k Hom_k(V_i, V_j)) =: O^A(\mathbf{V}) & & \\ \downarrow & & \downarrow & \searrow & \\ B & \longrightarrow & (H^B_{i,j} \otimes_k Hom_k(V_i, V_j)) & \xrightarrow{\alpha} & O^A(\mathbf{V})/\mathfrak{m}^m \end{array}$$

where all morphisms are natural. In particular α exists since $B = A/\mathfrak{r}^m$ maps into $O^A\mathbf{V}/rad^m$, and therefore induces the morphism α commuting with the rest of the morphisms. Consequently α has to be surjective, and we have proved the contention. $\square$

Example 1.18. *As an example of what may occur in rank infinity, we shall consider the invariant problem,* $\mathbf{A}^1_{\mathbf{C}}/\mathbf{C}^*$. *Here we are talking about the algebra* $A = \mathbf{C}[x](\mathbf{C}^*)$ *crossed product of* $\mathbf{C}[x]$ *with the group* $\mathbf{C}^*$. *If* $\lambda \in \mathbf{C}^*$, *the product in* A *is given by* $x \times \lambda = \lambda \times \lambda^{-1}x$. *There are two "points", i.e. orbits, modeled by the obvious origin* $V_0 := A \to \mathrm{End}_{\mathbf{C}}(\mathbf{C}(0))$, *and by* $V_1 := A \to End_{\mathbf{C}}(\mathbf{C}[x, x^{-1}])$. *We may also choose the two points* $V_0 := \mathbf{C}(0), V_1 := \mathbf{C}[x]$, *in line with the definitions of* [13]. *Obviously* $\mathbf{C}[x]$ *correspond to the closure of the orbit* $\mathbf{C}[x, x^{-1}]$. *This choice is the best if one want to make visible the adjacencies in the quotient, and we shall therefore treat both cases.*

We need to compute

$$Ext^p_A(V_i, V_j), \quad p = 1, 2, \quad i, j = 1, 2$$

Now,

$$Ext^1_A(V_i, V_j) = Der_{\mathbf{C}}(A, Hom_{\mathbf{C}}(V_i, V_j))/Triv, \quad i, j = 1, 2$$

and since x *acts as zero on* V_1, *and* $\mathbf{C}^*$ *acts as identity on* V_1 *and as homogenous multiplication on* V_0, *we find*

$$Der_k(A, Hom_k(V_0, V_0))/Triv = Der_k(A, Hom_k(V_0, V_0)) = Der_{\mathbf{C}}(A, \mathbf{C}(0))$$

Any $\delta \in Der_k(A, \mathbf{C}(0))$, *is determined by its values,* $\delta(x), \delta(\lambda) \in \mathbf{C}(0))|\lambda \in \mathbf{C}^*$. *Moreover since in* A *we have,* $(\lambda) \times (\lambda^{-1}x) = x \times (\lambda)$, *we find*

$$\delta(\lambda\mu) = \delta(\lambda) + \delta(\mu), \quad \delta((\lambda) \times (\lambda^{-1}x)) = \delta(x \times (\lambda))$$

The left hand side of the last equation is $\delta((\lambda^{-1}x)) = \lambda^{-1}\delta(x)$, *and the right hand side is* $\delta(x)$, *and since this must hold for all* $\lambda \in \mathbf{C}^*$, *we must have* $\delta(x) = 0$. *Moreover, since* $\delta(\lambda\mu) = \delta(\lambda) + \delta(\mu)$, *it is clear that continuity of* δ, *implies that* δ *must be equal to* $\alpha ln(|.|)$, *for some* $\alpha \in \mathbf{C}$. *(To simplify the writing, we shall put* $\log := ln(|.|)$.*)Therefore,*

$$Ext^1_A(V_0, V_0) = Der_k(A, Hom_{\mathbf{C}}(V_0, V_0)) = \mathbf{C}$$

The cup-product of this class, $\log \cup \log$, *sits in* $HH^2(A, \mathbf{C}(0)) = Ext^2_a(V_0, V_0)$, *and is given by the 2-cocycle*

$$(\lambda, \mu) \to \log(\lambda) \times \log(\mu)$$

This is seen to be a boundary, i.e. there exist a map $\psi : \mathbf{C}^* \to \mathbf{C}(0)$, *such that for all,* $\lambda, \mu \in \mathbf{C}^*$ *we have*

$$\log(\lambda) \times \log(\mu) = \psi(\lambda) - \psi(\lambda\mu) + \psi(\mu)$$

Just put $\psi_{1,1} := \psi_2 = -1/2 \log^2$. *Therefore the cup product is zero, and if we, in general, put*

$$\psi_n := \psi_{1,1,\ldots,1} = (-)^{n+1} 1/(n!) \log^n, \quad n \geq 1$$

where n is the number of 1*'s in the first index, then computing the Massey products of the element* $\log \in Ext^1_A(V_0, V_0)$, *we find the n.th Massey product*

$$[\log, \log, \ldots, \log] = \{(\lambda, \mu) \to \sum_{p=1,\ldots,n-1} \psi_p \psi_{n-p}$$

and this is easily seen to be the boundary of the 1-cochain

$$\psi_{n+1} = (-)^{n+2} 1/((n+1)!) \log^{n+1}$$

Therefore all Massey products are zero. Of course, we have not yet proved that they could be different from zero, i.e. we have not computed the obstruction-group $Ext^2_A(V_0,V_0)$ *and found it non-trivial! Now this is unnecessary.*

Now, assume first $V_0 = \mathbf{C}[x,x^{-1}]$, *then every*

$$\delta \in Ext^1_A(V_0,V_0) = Der_{\mathbf{C}}(A, Hom_{\mathbf{C}}(V_0,V_0))/Triv$$

is determined by the values of $\delta(x)$ *and* $\delta(\lambda),\ \lambda \in \mathbf{C}^*$. *Since* $Ext^1_{\mathbf{C}[x]}(V_0,V_1)=0$, *we may find a trivial derivation such that subtracting from* δ *we may assume* $\delta(x)=0$. *But then the formula*

$$\delta(x\times\lambda) = \delta(\lambda\times(\lambda^{-1}x))$$

implies

$$x\delta(\lambda) = \delta(\lambda)(\lambda^{-1}x)$$

from which it follows that

$$\delta(\lambda)(x^p) = (\lambda^{-1}x)^p\delta(\lambda)(1)$$

Now, since, $\lambda\mu = \mu\lambda$ *in* $\mathbf{C}^*$, *we find*

$$(\lambda^{-1}\mu x)^p\delta(\lambda)(1)(\mu x) = (\lambda\mu^{-1}x)^p\delta(\lambda)(1)(\lambda x)$$

which should hold for any pair of $\mu,\lambda \in \mathbf{C}^*$, *and any* p. *This obviously implies* $\delta = 0$.

This argument shows not only that

$$Ext^1_A(V_1,V_1) = Der_{\mathbf{C}}(A, Hom_{\mathbf{C}}(V_1,V_1))/Triv = 0$$

when $V_1 = \mathbf{C}[x,x^{-1}]$, *but also when* $V_1 = \mathbf{C}[x]$. *Finally we find that the formula above,*

$$x\delta(\lambda) = \delta(\lambda)(\lambda^{-1}x)$$

shows that for

$$\delta \in Ext^1_A(V_1,V_0) = Der_{\mathbf{C}}(A, Hom_{\mathbf{C}}(V_1,V_0))/Triv$$

we have $\delta(\lambda)(xx^p)=0$ *for all* p. *Therefore,*

$$Ext^1_A(V_1,V_0) = Der_{\mathbf{C}}(A, Hom_{\mathbf{C}}(V_1,V_0))/Triv = 0$$

when $V_1 = \mathbf{C}[x,x^{-1}]$. *However, when* $V_1 = \mathbf{C}[x]$, *we find that* δ *with,* $\delta(\lambda)(1)\neq 0$ *and with* $\delta(\lambda)(x^p)=0$ *for* $p\geq 1$, *survives. These will, as above give rise to a logarithm of the real part of* $\mathbf{C}^*$. *Therefore, in this case* $Ext^1_A(V_1,V_0)=\mathbf{C}$. *The miniversal families look like*

$$H = \begin{pmatrix} \mathbf{C}[[t]] & 0 \\ 0 & \mathbf{C} \end{pmatrix}$$

when $V_1 = \mathbf{C}[x,x^{-1}]$, *and like*

$$H = \begin{pmatrix} \mathbf{C}[[t]] & 0 \\ <\mathbf{C}> & \mathbf{C} \end{pmatrix}$$

when $V_1 = \mathbf{C}[x]$.

1.6. **Non-commutative Algebraic Geometry, and Moduli of Simple Modules.** The basic notions of affine non-commutative algebraic geometry related to a (not necessarily commutative) associative k-algebra, for k an arbitrary field, have been treated by many authors in several texts, see e.g.[25], [26], [12], [13], [14]. Given a finitely generated algebra A, we prove the existence of a non-commutative scheme-structure on the set of isomorphism classes of simple finite dimensional representations, i.e. right modules, $Simp_{<\infty}(A)$.

We show, in [14], that any *geometric k-algebra* A, see also [18], may be recovered from the (non-commutative) structure of $Simp_{<\infty}(A)$, and that there is an underlying quasi-affine (commutative) scheme-structure on each component $Simp_n(A) \subset Simp_{<\infty}(A)$, parametrizing the simple representations of dimension n. In fact, we have shown the following,

Theorem 1.19. *There is a commutative k-algebra $C(n)$ with an open subvariety $U(n) \subseteq Simp_1(C(n))$, an étale covering of $Simp_n(A)$, over which there exists a versal representation $\tilde{V} \simeq C(n) \otimes_k V$, a vector bundle of rank n defined on $Simp_1(C(n))$, and a versal family, i.e. a morphism of algebras,*

$$\tilde{\rho} : A \longrightarrow End_{C(n)}(\tilde{V}) \to End_{U(n)}(\tilde{V}),$$

inducing all isoclasses of simple n-dimensional A-modules.

$End_{C(n)}(\tilde{V})$ induces also a bundle, of *operators*, on the étale covering $U(n)$ of $Simp_n(\mathbf{A}(\sigma))$. Assume given a derivation, $\gamma \in Der_k(A)$. Pick any $v \in Simp_n(A)$ corresponding to the right A-module V, with structure homomorphism $\rho_v : A \to End_k(V)$, then γ composed with ρ_v, gives us an element,

$$\gamma_v \in Ext^1_A(V, V)).$$

Therefore, γ defines a unique one-dimensional distribution in $\Theta_{Simp_n(A)}$, which, once we have fixed a versal family, defines a vector field,

$$[\gamma] \in \Theta_{Simp_n(A)},$$

and, in good cases, a (rational) derivation,

$$[\gamma] \in Der_k(C(n)).$$

Notice also that we have the canonical isomorphism,

$$Der_k(A, A) \simeq Mor_A(Ph(A), A).$$

Therefore the derivation γ, and the A-module V, correspond to an $Ph(A)$-module V_γ.

1.7. **Dynamical Structures.** As we have seen, in the Remark (1.3), the dynamics of the space of representations of our algebra A, i.e. the dynamics of the space of measurements of the family of observables that A is assumed to represent, can be encoded in the category of representations of the k-algebra $Ph^\infty(A)$. We would therefore like to use the tools developed above, for the k-algebra $Ph^\infty(A)$.

However, $Ph^\infty(A)$ is rarely of finite type, and so the space of simple modules does not have a classical algebraic geometric structure. We shall therefore introduce the notion of *dynamical structure*, to reduce the problem to a situation we can handle. This is also what physicists do. They invoke a *parsimony principle*, or an *action principle*, originally proposed by Fermat, and later by Maupertuis, with the purpose of reducing the preparation needed, to be able to see ahead.

Definition 1.20. *A dynamical structure, σ, is a two-sided δ-stable ideal $(\sigma) \subset Ph^\infty(A)$, such that*

$$A(\sigma) := Ph^\infty(A)/(\sigma),$$

the corresponding, dynamical system, is of finite type. A dynamical structure, or system, is of order n if the canonical morphism,

$$\sigma : Ph^{(n-1)}(A) \to A(\sigma)$$

is surjective. If A is generated by the coordinate functions, $\{t_i\}_{i=1,2,...,d}$ *a dynamical system of order n may be defined by a force law, i.e. by a system of equations,*

$$\delta^n t_p = \Gamma^p(\underline{t}_i, \underline{d}t_j, \underline{d}^2 t_k, .., \underline{d}^{n-1} t_l), \ p = 1, 2, ..., d.$$

Put,

$$A(\sigma) := Ph^{\infty}(A)/(\delta^n t_p - \Gamma^p)$$

where $\sigma := (\delta^n t_p - \Gamma^p)$ *is the two-sided* δ*-ideal generated by the defining equations of* σ*. Obviously* δ *induces a derivation* $\delta_\sigma \in Der_k(A(\sigma), A(\sigma))$*, also called the Dirac derivation, and usually just denoted* δ*.*

Notice that if $\sigma_i, \ i = 1, 2$, are two different order n dynamical systems, then we may well have,

$$A(\sigma_1) \simeq A(\sigma_2) \simeq Ph^{(n-1)}(A)/(\sigma_*),$$

as k-algebras.

Assuming that the k-algebra A is finitely generated, and that the dynamical structure σ is such that also $A(\sigma)$ is finitely generated, we can now use the machinery of (1.3), with $\gamma = \delta$, the Dirac derivation. The following results, proved in [18], are important for the philosophy of this paper.

Theorem 1.21. *Formally, at any point* $v \in U(n) \subset Simp(C(n))$*, with local ring* $\hat{C}(n)_v$*, there is a derivation* $[\delta] \in Der_k(\hat{C}(n)_v)$*, and a Hamiltonian* $Q \in End_{\hat{C}(n)_v}(\tilde{V}_v)$*, such that, as operators on* $\tilde{V}_v$*, we have,*

$$\delta = [\delta] + [Q, -].$$

This means that for every $a \in A(\sigma)$*, considered as an element* $\tilde{\rho}(a) \in M_n(\hat{C}(n)_v)$*,* $\delta(a)$ *acts on* $\tilde{V}_v$ *as*

$$\tilde{\rho}(\delta(a)) = [\delta](\tilde{\rho}(a)) + [Q, \tilde{\rho}(a)].$$

In line with our general philosophy, we shall consider $[\delta]$ as measuring *time* in $Simp_n(\mathbf{A}(\sigma))$, respectively in $Spec(C(n))$. This is reasonable, since the last equation is equivalent to the following statement: The derivation δ induces an extension of $\tilde{V}_v$, as A-module, which is trivial. This is formally true for any derivation δ, by the definition of the versal family, i.e. the $\hat{C}(n)_v$-module $\hat{V}_v$. If $\hat{C}(n)_v$ had not been the versal base space, then we would have had to be careful, see the way time is defined in the configuration space of our Toy Model, in subsection 7.2.

Remark 1.22. *However, if we have given a representation,*

$$\rho : A \longrightarrow End_C(\tilde{V}),$$

and a derivation $\gamma \in Der_k(A)$ *such that for any* $a \in A$ *we have,* $\rho(\gamma(a)) = \nabla_\gamma(\rho(a)) + [Q_\gamma, \rho(a)]$*, where* $\nabla_\gamma \in Der_k(C)$ *and* $Q_\gamma \in End_C(\tilde{V})$*, this would mean that* γ *induces a k-derivation,* $\nabla_\gamma + ad([Q_\gamma)$*, of* $End_C(\tilde{V})$*, and a derivation, i.e. a connection,* $\nabla_\gamma + Q_\gamma$*, in* $\tilde{V}$*. This actually means that the corresponding first order changes of the A-module structure of* $\tilde{V}$*, can be considered, equivalently, as a Heisenberg-process, or as a Schrødinger-process.*

Notice also that $End_{C(n)}(\tilde{V}) \simeq M_n(C(n))$, and be prepared, in the sequel, to see this used without further warning. There are local (and even global) extensions of this result, where $[\delta]$ and Q may be assumed to be defined (rationally) on $C(n)$, see [18]. In this case, we may see that, provided the field k is (sufficiently) algebraically closed, any *quantum field*, $\psi \in End_{C(n)}(\tilde{V})$ can be expressed as a (finite) rational polynomial of generalized *creation* and *annihilation* operators, see (4.4), loc. cit. See also section (3), Entropy.

Assume for a while that $k = \mathbf{R}$, the real numbers, and that our constructions go through, as if k were algebraically closed. Let $v(\underline{\tau}_0) \in Simp_n(\mathbf{A}(\sigma))$ be an element, an *event*. Suppose there exist an integral curve $\mathbf{c}$ of $[\delta]$ through $v(\tau_0) \in Simp_1(C(n))$, ending at $v(\tau_1) \in Simp_1(C(n))$, given by the automorphisms $e(\tau) := exp(\tau[\delta])$, for $\tau \in [\tau_0, \tau_1] \subset \mathbf{R}$. The supremum of τ for which the corresponding point, $v(\tau)$, of $\mathbf{c}$ is in $Simp_n(\mathbf{A}(\sigma))$ should be called the *lifetime* of the particle. It is relatively easy to compute these lifetimes, and so to be able to talk about *decay*, when the fundamental vector field $[\delta]$ has been computed. In [18], we have also proposed a mathematically sound way of treating interaction, purely in terms of non-commutative deformation theory.

Let $\phi(\tau_0) \in \tilde{V}(v_0) \simeq V$ be a (classically considered) state of our *quantum system*, at the time τ_0, and consider the (uni)versal family,

$$\tilde{\rho} : \mathbf{A}(\sigma) \longrightarrow End_{C(n)}(\tilde{V})$$

restricted to $U(n) \subseteq Simp_1(C(n))$, the étale covering of $Simp_n(\mathbf{A}(\sigma))$. We shall consider $\mathbf{A}(\sigma)$ as our *ring of observables.* What happens to $\phi(\tau_0) \in V(0)$ when *time* passes from τ_0 to τ, along $\mathbf{c}$? This leads to a solution of the Schrödinger equation,

$$\frac{d\phi}{d\tau} = Q(\phi),$$

along $\mathbf{c}$, given by the next result, proving that ψ is completely determined, by the value of $\psi(\tau_0)$, for any $\tau_0 \in \mathbf{c}$. Here, we shall not go into the problem of *preparing* $\psi(\tau_0) \in V(\tau_0)$, i.e. of how to exactly determine *where we are*, at some chosen clock-time, τ, see [18].

Theorem 1.23. *The evolution operator $u(\tau_0, \tau_1)$ that changes the state $\psi(\tau_0) \in \tilde{V}(v_0)$ into the state $\phi(\tau_1) \in \tilde{V}(v_1)$, where $\tau_1 - \tau_0$ is the length of the integral curve $\mathbf{c}$ connecting the two points v_0 and v_1, i.e. the time passed, is given by,*

$$\phi(\tau_1) = u(\tau_0, \tau_1)(\phi(\tau_0)) = exp[\int_{\mathbf{c}} Q(\tau)d\tau]\ (\phi(\tau_0)),$$

where $exp \int_{\mathbf{c}}$ is the non-commutative version of the ordinary action integral, essentially defined by the equation,

$$exp[\int_{\mathbf{c}} Q(\tau)dt] = exp[\int_{\mathbf{c}_2} Q(\tau)d\tau] \circ exp[\int_{\mathbf{c}_1} Q(\tau)d\tau]$$

where $\mathbf{c}$ is $\mathbf{c}_1$ followed by $\mathbf{c}_2$.

In the situation of Theorem (1.23) we observe that, since $\delta = [\delta] + [Q, -]$. is a derivation defined in the algebra $End_{C(n)}(\tilde{V})$, the eigenvalues $\Lambda := \{\lambda\}$ of the eigenvectors a_λ of δ, will have a structure as an additive sub-monoide of the reals. Assume now that $[[\delta], Q] = 0$, (which will be the case for trivial metrics in (7.2)), and suppose that $[\delta](\psi) = \nu\psi$ and $[Q, \psi] = \epsilon\psi$, then,

$$exp(\delta)(\psi) = exp(ad(Q))exp([\delta])(\psi) = exp(ad(Q))(\psi(t + \nu)) = exp(\epsilon)exp(\nu)(\psi),$$

which means that if Λ has a generator h_0, then we have a Heisenberg relation,

$$\Delta E \times \Delta t \geq h := exp(h_0).$$

where $exp(\epsilon) = \Delta E,\ exp(\nu) = \Delta t$. Compare with [18], (4.4), where we consider the singular situation, corresponding to $\delta = ad(Q)$.

Remark 1.24. *Let A be any associative k-algebra, finitely generated by $\{t_i, i = 1, ..., d\}$, and let σ be a dynamical structure. Given any representation, $\rho : A(\sigma) \to End_k(V)$, we must have $g^{i,j} := \rho([dt_i, t_j]) = \rho([dt_j, t_i]) = g^{j,i}$, and moreover,*

$$\rho(\delta(g^{i,j})) = \rho([d^2(t_i), t_j] + [dt_i, dt_j]).$$

This looks, superficially, like a very generalized field equation, where $g^{i,j}$ is the inverse of a metric, $[d^2 t_i, t_j]$ is the action of a force, and $[dt_i, dt_j]$ is the curvature, of ρ. We shall return to this, but first we must introduced the third main ingredient in this story, the relations induced by "non-observable" infinitesimal automorphisms, the gauge groups.

1.8. **Gauge Groups and Invariant Theory.** We may use the above in an attempt to make precise the notion of *gauge group*, gauge fields, and gauge invariance, and thus to be able to understand why the physicists define their objects, the *fields* and *particles*, the way they do.

Suppose, in line with our philosophy, that we have uncovered the moduli space, $\mathbf{M}$, of the mathematical models X, of our phenomena $\mathbf{P}$, and that C is the *affine k-algebra* of (an affine open subset of) this space, assumed to contain all the parameters of our interest of the states of X.

Suppose, furthermore that we have identified a k-Lie algebra $\mathfrak{g}_0 \subset Der_k(C)$, of infinitesimal automorphisms, i.e. of derivations of C, a *global gauge groupe*, leaving invariant the physical properties of our phenomena $\mathbf{P}$. We would then be led to consider the *quotient space* $\mathbf{M}/\mathfrak{g}_0$, which in our non-commutative geometry, is equivalent to restricting our representations, $\rho : C \to End_k(V)$, to those representations V for which, $\mathfrak{g}_0 \subset \mathfrak{g}_V$, see (1.2), the Definition (1.2) and Lemma (1.1). This would then imply that the corresponding *Hamiltonians*, Q_γ define a $\mathfrak{g}_0$-connection on V,

$$Q : \mathfrak{g}_0 \longrightarrow End_k(V),$$

such that, for all $c \in C$, and for all $\gamma \in \mathfrak{g}_0$, $\rho(\gamma(c)) = [Q_\gamma, \rho(c)]$. This is usually written,

$$\rho(\gamma(c)) = [\gamma, \rho(c)].$$

The curvature,

$$R(\gamma_1, \gamma_2) := [Q_{\gamma_1}, Q_{\gamma_2}] - Q_{[\gamma_1,\gamma_2]} \in End_C(V),$$

corresponds to a *global force* acting on the representation ρ. These forces, *mediated* by the *gauge-particles*, $\lambda \in \mathfrak{g}_0$, will be the first to be studied in some details, see the next subsection. Put,

$$Rep(C, \mathfrak{g}_0) := \{\rho \in Rep(C) | \kappa(\gamma\rho) = 0, \forall \gamma \in \mathfrak{g}_0\} = \{\rho \in Rep(C) | \mathfrak{g}_0 \subset \mathfrak{g}_\rho\},$$

where $Rep(C)$ is the category of all representations of C, and notice that, in the commutative situation, if we consider the case where the gauge group $\mathfrak{g}_0 = Der_k(C)$ then $Rep(C, \mathfrak{g}_0)$ is the category of *C-Connections*, for which the space of isomorphism classes is discrete with respect to time. Notice that this is the situation in the classical quantum theory, where the *Hilbert Space* is always considered as the unique state space of interest.

Definition 1.25. *An object $V \in Rep(C, \mathfrak{g}_0)$ is called simple if there are no non-trivial sub-objects of V in $Rep(C, \mathfrak{g}_0)$. The generalized quotient $Simp(C)/\mathfrak{g}_0$, is by definition, the set, $Simp(C : \mathfrak{g}_0)$, of iso-classes of simple objects in $Rep(C, \mathfrak{g}_0)$.*

If the curvature also vanish, there is a canonical homomorphism,

$$\phi : U(\mathfrak{g}_0) \to End_k(V).$$

where $U(\mathfrak{g}_0)$ is the universal algebra of the Lie algebra $\mathfrak{g}_0$.

In the general case let,

$$C'(\mathfrak{g}_0) \subset End_k(C),$$

be the sub-algebra generated by C and $\mathfrak{g}_0$. Then we put, for all $c \in C, \gamma \in \mathfrak{g}_0$,

$$C(\mathfrak{g}_0) = C'(\mathfrak{g}_0)/(\gamma c - c\gamma - \gamma(c))$$

and we have an identification between the set of $\mathfrak{g}_0$-connections on V, and the set of k-algebra homomorphisms,

$$\rho_{\mathfrak{g}} : C(\mathfrak{g}_0) \to End_k(V),$$

since any such would respect the relation above, such that, for $c \in C, \gamma \in \mathfrak{g}_0$,

$$\rho_{\mathfrak{g}_0}(\gamma c) = \rho_{\mathfrak{g}_0}(\gamma)\rho_{\mathfrak{g}}(c) = \rho_{\mathfrak{g}_0}(c)\rho_{\mathfrak{g}_0}(\gamma) + \rho_{\mathfrak{g}_0}(\gamma(c)).$$

Therefore $Rep(C)/\mathfrak{g}_0 := Rep(C : \mathfrak{g}_0) \simeq Rep(C(\mathfrak{g}_0))$, and we note, for memory, the trivial,

Lemma 1.26. *In the above situation, we have the following isomorphisms,*

$$Rep(C)/\mathfrak{g}_0 := Rep(C : \mathfrak{g}_0) \simeq Rep(C(\mathfrak{g}_0))$$
$$Simp(C)/\mathfrak{g}_0 := Simp(C : \mathfrak{g}_0) \simeq Simp(C(\mathfrak{g}_0)).$$

Notice that the commutant of C in $C(\mathfrak{g}_0)$, of $\mathfrak{g}_0$ is the subring,

$$C^{\mathfrak{g}_0} := \{c \in C | \forall \gamma \in \mathfrak{g}_0, \gamma(c) = 0\} \subset C.$$

Notice also that the commutativisation, $C(\mathfrak{g}_0)^{com}$, of $C(\mathfrak{g}_0)$ is the quotient of $C(\mathfrak{g}_0)$ by an ideal containing $\{\gamma(c) | c \in C, \gamma \in \mathfrak{g}_0\}$. Therefore there is a natural map,

$$C^{\mathfrak{g}_0} \to C(\mathfrak{g}_0)^{com}.$$

However this map may not be injective, so we cannot, in general, identify the rank 1 points of $Simp(C, \mathfrak{g}_0)$, with $Simp_1(C^{\mathfrak{g}_0})$.

If C is assumed commutative, the classical invariant theory identifies the two schemes, $Spec(C)/\mathfrak{g}_0$ and $Spec(C^{\mathfrak{g}_0})$, which in the above light, is not entirely kosher. However, if $\mathfrak{a} \subset C$ is an ideal, stable under the action of $\mathfrak{g}_0$, then since any derivation γ of C acts on the multiplicative operators $a \in C$ as $\gamma(c) = \gamma c - c\gamma$, it is clear that the quotient $C/\mathfrak{a}$ is a $(C, \mathfrak{g}_0)$-representation. Moreover,

$$C/\mathfrak{a} \in Simp_1(C)/\mathfrak{g}_0$$

if and only if the subset $Simp_1(C/\mathfrak{a}) \subset Simp_1(C)$ is the closure of a *maximal integral subvariety* for $\mathfrak{g}_0$. The *space* of such integral subvarieties is what we, in [14], have termed the *non-commutative quotient*, $Spec(C)/\mathfrak{g}_0$.

Suppose now there is a C-Lie algebra $\mathfrak{g}_1$, acting C-linearly on those C-modules V, which we would consider of *physical* interest. $\mathfrak{g}_1$ should be called a *local gauge group*. One may then want to know wether the given action of $\mathfrak{g}_0$ moves $\mathfrak{g}_1$ in its formal moduli as C-Lie algebra. If so, the action of $\mathfrak{g}_1$ would not be invariant under the the gauge transformations induced by $\mathfrak{g}_0$, and we should not consider $(\rho, \mathfrak{g}_1)$ as physically kosher. If, on the other hand, the action of $\mathfrak{g}_0$ does not move $\mathfrak{g}_1$ in its formal moduli, it should follow that there is a relation between the $\mathfrak{g}_0$-action (i.e. the connection) on V, and the action of $\mathfrak{g}_1$. Now, in the case $\mathfrak{g}_0 \subset Der_k(C)$, it follows from the Kodaira-Spencer map,

$$ks : Der_k(C) \to A^1(C, \mathfrak{g}_1 : \mathfrak{g}_1),$$

see [3], Lemma(2.3), that we have the following result,

Lemma 1.27. *Assume $\mathfrak{g}_1$ as a C-module is such that $\mathfrak{g}_0 \subset \mathfrak{g}_{\mathfrak{g}_1}$, see Definition(1.2), let $c^k_{i,j} \in C$ be the structural constants of $\mathfrak{g}_1$ with respect to some C-basis $\{x_i\}$, and let $\delta : \mathfrak{F} \to \mathfrak{g}_1$ be a surjective morphism of a free C-Lie algebra $\mathfrak{F}$ onto $\mathfrak{g}_1$, mapping the generators $\mathfrak{x}_i$ of $\mathfrak{F}$ onto x_i. Let $\mathfrak{F}_{i,j} = [\mathfrak{x}_i, \mathfrak{x}_j] - \sum_k c^k_{i,j}\mathfrak{x}_k \in ker\delta$, and let $\gamma \in Der_k(C)$. Then, $ks(\gamma)$ is the element of $A^1(C, \mathfrak{g}_1 : \mathfrak{g}_1)$ determined by the element of $Hom_{\mathfrak{F}}(ker(\delta), \mathfrak{g}_1)$, given by the map,*

$$\mathfrak{F}_{i,j} \to -\sum_k \gamma(c^k_{i,j})x_k.$$

For $ks(\gamma)$ to be 0, there must exist a C- derivation $D_\gamma : \mathfrak{F} \to \mathfrak{g}_1$, (a potential), such that

$$D_\gamma(\mathfrak{F}_{i,j}) = D_\gamma([\mathfrak{x}_i, \mathfrak{x}_j] - \sum_k c^k_{i,j}\mathfrak{x}_k) = -\sum_k \gamma(c^k_{i,j})x_k$$

Let now $\nabla : \mathfrak{g}_0 \to Der_k(\mathfrak{g}_1)$ be a connection on the C-module $\mathfrak{g}_1$. Then $ks(\gamma) = 0$ iff there exist a $\mathfrak{g}_0$-Lie-connection of the form $\mathfrak{D} = \nabla - D$ on $\mathfrak{g}_1$, i.e. a k-linear map,

$$\mathfrak{D} : \mathfrak{g}_0 \to Der_k(\mathfrak{g}_1),$$

such that, for $\gamma \in \mathfrak{g}_0,\ c \in C,\ \kappa \in \mathfrak{g}_1$,

$$\mathfrak{D}_\gamma(c \cdot \kappa) = c\mathfrak{D}_\gamma(\kappa) + \gamma(c) \cdot \kappa.$$

If the curvature,

$$R(\gamma_1, \gamma_2) := [\mathfrak{D}\gamma_1, \mathfrak{D}\gamma_2] - \mathfrak{D}[\gamma_1, \gamma_2]$$

representing the 2. order action of $\mathfrak{g}_0$ on $\mathfrak{g}_1$, (the force, as a physicist might have said), vanish, the map,

$$\mathfrak{D} : \mathfrak{g}_0 \to Der_k(\mathfrak{g}_1),$$

would be a Lie-algebra morphism. And then this notion is related to the notion of a Lie-Cartan pair. In fact, the structure given by $\mathfrak{D}$ defines a Lie algebra structure on the sum

$$\mathfrak{g} := \mathfrak{g}_0 \oplus \mathfrak{g}_1,$$

with Lie-products of the sum defined as the product in each Lie algebra, and the cross-products defined for, $\gamma \in \mathfrak{g}_0, \xi \in \mathfrak{g}_1$, as,

$$[\gamma, \xi] = \mathfrak{D}_\gamma(\xi),$$

A structure like this, a Lie-Cartan pair, is now often called a Lie algebroid.

The situation above comes up when we have chosen a dynamical structure σ, with Dirac derivation δ. Assume that there exist, as above a global gauge group, $\mathfrak{g}_0 \subset Der_k(C)$, and suppose moreover that there is a C-Lie algebra $\mathfrak{g}_1 \subset Der_C(C(\sigma))$ that we would consider physically uninteresting, then we would be lead to consider the quotient of $C(\sigma)$ by both $\mathfrak{g}_0$ and $\mathfrak{g}_1$, i.e. we will for any representation V of $D(\sigma)/\mathfrak{g}$, $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, find a diagram,

$$\begin{array}{ccccc} \mathfrak{g}_0 & \xrightarrow{\mathfrak{D}} & Der_k(\mathfrak{g}_1) & \longleftarrow & \mathfrak{g}_1 \\ \downarrow{\scriptstyle \nabla_0} & & & & \downarrow{\scriptstyle \nabla_1} \\ End_k(V) & & \longleftarrow & & End_C(V) \end{array}$$

where $\mathfrak{D}, \nabla_0$, and ∇_1 are connections. If $\mathfrak{D}$ has vanishing curvature then there is a connection of the Lie algebroid $\mathfrak{g}$,

$$\nabla : \mathfrak{g} \to End_k(V).$$

The category of simple representations $\rho : C \to End_k(V)$ with this property vis a vis the Lie algebra $\mathfrak{g}$,

$$Simp(A)/\mathfrak{g}.$$

is, according to our philosophy, the object of study of mathematical physics.

Physicists have a way of classifying, or naming *states*, i.e. the elements of the representation vector space V, according to certain numbers associated to them, like spin, charge, hyperspin, etc. We find this in the situation above, as follows.

Consider the Cartan algebra $\mathfrak{h} \subset \mathfrak{g}_1$. It will operate on the above representation space V as diagonal matrices, and the eigenvectors may be labeled by the corresponding eigenvalues.

Notice also that if $V_i \in Rep(C, \mathfrak{g}), i = 1, 2$, then it follows from (1.26) that an extension of the $C(\mathfrak{g})$-module V_1 with V_2 will also sit in $Rep(C, \mathfrak{g})$.

Put, as above, $A := C(\sigma)$ where σ is a dynamical structure, and assume given an action of the Lie algebra $\mathfrak{g}_0$ on C. Notice that since $Ph()$ is a functor in the category of algebras and algebra morphisms, the action of $\mathfrak{g}_0$ extends to $Ph(C)$, but not necessarily to a dynamical system of the type $C(\sigma)$.

This will turn out to be important for our version of the Standard Model. There we will also meet the following extension of the situation above.

1.9. **The Generic Dynamical Structures associated to a Metric.** Now, let $k = \mathbf{R}$ be the real numbers, and consider a commutative polynomial k-algebra, $C = k[t_1, ..., t_n]$. Moreover, let

$$g = 1/2 \sum_{i,j=1,..,r} g_{i,j} dt_i dt_j \in Ph(C),$$

be a Riemannian metric. Recall the formula for the Levi-Civita connection,

$$\sum_l g_{l,k} \Gamma^l_{j,i} = 1/2 \left(\frac{\partial g_{k,i}}{\partial t_j} + \frac{\partial g_{j,k}}{\partial t_i} - \frac{\partial g_{i,j}}{\partial t_k}\right).$$

Since in $Ph^\infty(C)$, we have,

$$\delta(g) = \sum_{i,j,k=1,..,r} \frac{\partial g_{i,j}}{\partial t_k} dt_k dt_i dt_j + \sum_{i,j,=1,..,r} g_{i,j}(d^2 t_i dt_j + dt_i d^2 t_j),$$

we may plug in the formula,

$$\delta^2 t_l = -\Gamma^l := -\sum \Gamma^l_{i,j} dt_i dt_j.$$

on the right hand side, and see that we, in the commutative situation, i.e. for the dynamical structure $\sigma := \{[f, f'], f, f' \in Ph(C)\}$, have got a solution of the *Lagrange equation*,

$$\delta(g) = 0.$$

This solution has the form of a *force law*,

$$d^2 t_l = -\Gamma^l := -\sum \Gamma^l_{i,j} dt_i dt_j,$$

generating a dynamical structure $(\sigma) := (\sigma(g))$ of order 2. The dynamical system is the commutative algebra,

$$\mathbf{C}(\sigma) = k[\underline{t}, \underline{\xi}]$$

where ξ_j is the class of dt_j. The Dirac derivation now takes the form,

$$[\delta] = \sum_l \left(\xi_l \frac{\partial}{\partial t_l} - \Gamma^l \frac{\partial}{\partial \xi_l}\right),$$

coinciding with the fundamental vector field $[\delta]$ in $Simp_1(\mathbf{C}(\sigma)) = Spec(k[t_i, \xi_j])$.

The equation,

$$[\delta](g) = 0$$

imply that g is constant along the integral curves of $[\delta]$ in $Simp_1(Ph(C))$, and these integral curves projects into $Simp_1(C)$ to give the geodesics of the metric g, with equations,

$$\ddot{t}_l = -\sum_{i,j} \Gamma^l_{i,j} \dot{t}_i \dot{t}_j.$$

We may also look at this from another point of view. Suppose given any dynamical structure with Dirac derivation δ on $Ph(C)$. Consider $Simp_1(Ph(C))$. It is obviously represented by $C(1) := k[\underline{t}, \underline{\xi}]$, and the Dirac derivation induces a derivation $[\delta] \in Der_k(C(1))$, and the Hamiltonian must vanish. Therefore we have two options for *the same notion of time* in the picture, g and $[\delta]$. The last derivation must therefore be a *Killing vector field*, i.e. we must have a solution of Lagrange equation,

$$[\delta](g) = 0,$$

and we are left with the above solution for δ.

Since the metric is related to the gravitational force, the group of isometries, $O(g)$ of the metric g, i.e. the group of algebraic automorphisms of C leaving the metric g invariant would, in line with our philosophy, be an obvious real gauge group. We shall refer to its Lie algebra as,

$$\mathfrak{g}_0 := \mathfrak{o}(g)$$

Since $Ph(*)$ is a functor, $O(g)$ would also act on $Ph(C)$, and would induce an action of $\mathfrak{g}_0$ on $Ph(C)$, and so also on $Ph^\infty(C)$.

Example 1.28. *The Action of the Lie Algebra of Isometries*

Consider the metric, $g = 1/2 \sum_{i=1}^d g_{i,j} dt_i dt_j \in Ph(C)$ *and the corresponding Lie algebra of Killing vectors, i.e. the Lie algebra* $\mathfrak{o}(g)$ *of derivations,* $\gamma = \sum_i \gamma_i \frac{\partial}{\partial t_i} \in Der_k(C)$ *acting on* $Ph(C)$ *such that,* $\gamma(g) = 0$*, which is equivalent to,*

$$\sum_{i,j} \gamma(g_{i,j}) dt_i dt_j + \sum_{i,j,k} g_{i,j} \frac{\partial \gamma_i}{\partial t_k} dt_k dt_j + \sum_{i,j,k} g_{i,j} \frac{\partial \gamma_j}{\partial t_k} dt_i dt_k = 0.$$

implying, for all $i, j = 1, ..., d$*,*

$$\gamma(g_{i,j}) + \sum_k \frac{\partial \gamma_k}{\partial t_i} g_{k,j} + \sum_k \frac{\partial \gamma_k}{\partial t_j} g_{i,k} = 0.$$

There are two fundamental examples, the Euclidean and the Minkowski metrics. First, suppose all $g_{i,j}$ *are constants, and we are interested in the linear derivations. We obtain that the derivations are given in terms of matrices,* $(\gamma_{i,j}) := (\frac{\partial \gamma_j}{\partial t_i})$*, where* $\gamma_{i,j} g_{j,j} = -\gamma_{j,i} g_{i,i}$*. This gives in dimension 2, for the Euclidean resp. for the Minkowski metric,*

$$(\gamma_{i,j}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, (\gamma_{i,j}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The corresponding 1-dimensional Lie groups acting on C*, with coordinates* (t_1, t_2)*, are given by the exponential,*

$$O(g) = exp(\tau \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}) = \begin{pmatrix} cos(\tau) & sin(\tau) \\ -sin(\tau) & cos(\tau) \end{pmatrix},$$

respectively,

$$O(g) = exp(\tau \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}) = \begin{pmatrix} cosh(\tau) & sinh(\tau) \\ sinh(\tau) & cosh(\tau) \end{pmatrix}.$$

as we know.

As a contrast to the commutative case, (and the corresponding 1-dimensional representation), let us consider representations, ρ, for which $\mathfrak{g}_\rho = Der_k(C)$, i.e. representations that have no deformations, where the Dirac derivation $[\delta]$ vanishes, and the notion of time is taken care of by the Hamiltonian Q.

A non-degenerate metric, $g = 1/2\sum_{i=1}^d g_{i,j}dt_idt_j \in Ph(C)$ induces a duality, i.e. an isomorphism of C-modules

$$\Theta_C = Hom_C(\Omega_C, C) \simeq \Omega_C.$$

Consider the bilateral ideal (σ_g) of $Ph(C)$ generated by

$$(\sigma_g) = ([dt_i, t_j] - g^{i,j}),$$

and put,

$$C(\sigma_g) := Ph(C)/(\sigma_g).$$

Let, moreover,

$$T := -1/2(\sum_{k,l}\Gamma^k_{k,l}dt_l + \sum_{k,p,q} g^{k,q}\Gamma^p_{k,q}g_{p,l}dt_l),$$

and consider the inner derivation of $C(\sigma_g)$, defined by,

$$\delta := ad(g - T).$$

After a dull computation, we obtain, in $C(\sigma_g)$,

$$\delta(t_i) = dt_i,\ i = 1, ..., d.$$

Therefore, by universality, we have a well-defined dynamical structure (σ_g), with Dirac derivation, $\delta = ad(g - T)$. It is also easy to see that (σ_g) is invariant w.r.t. isometries, implying that $\mathfrak{o}(g) = \mathfrak{g}_0$ acts on $C(\sigma_g)$.

Any connection,

$$\nabla : \Theta_C \to End_k(E),$$

on a free C-module E, is given in terms of the operators,

$$\delta_{t_i} := \frac{\partial}{\partial t_i},\ \nabla_{\delta_{t_i}} = \delta_{t_i} + \nabla_i$$

where $\nabla_i \in End_C(E)$, is now a representation,

$$\rho_\nabla : C(\sigma_g) \to End_k(E),$$

defined by,

$$\rho_\nabla(t_i) = t_i,\ \rho_\nabla(dt_i) = \sum_{j=1}^d g^{i,j}\nabla\delta_j =: \nabla_{\xi_i}.$$

where we have put,

$$\xi_i := \delta^i = \sum_{j=1}^d g^{i,j}\delta_j.$$

Remark 1.29. *Notice that the representation, $\rho = \rho_\Theta$ of $C(\sigma_g)$, defined on Θ_C, by the Levi-Civita connection, has a Hamiltonian,*

$$Q := \rho(g - T) = 1/2\sum_{i,j} g^{ij}\nabla_{\delta_i}\nabla_{\delta_j},$$

i.e. the generalized Laplace-Beltrami operator, which is also invariant w.r.t. isometries, although the proof demands some algebra. This is our quantum version of the Einstein Field Equantion, g is the metric, Q is the quantum Mass-Energy-Stress Operator, and

$$T = T_ldt_l, T_l = -1/2(\sum_j(\Gamma^j_{j,l} + \bar{\Gamma}^j_{j,l}) = -1/2(trace\nabla_l + trace\bar{\nabla}_l),$$

see below for the computation, is our replacement of the Ric.

In this case, i.e. for the Levi-Civita connection, we shall, as above, denote by,

$$\rho_{\Theta}: C(\sigma_{g_{\Theta}}) \to End_k(\Theta_C),$$

the representation of $C(\sigma_{g_{\Theta}})$, and by,

$$D_{-}: \Theta_C \to End_k(\Theta_C),$$

the corresponding connection.

For the representation, ρ_{Θ_C}, and for an element (a state) $\xi \in \Theta_C$, we would, in line with classical Quantum Theory, assume the dynamics given by the Schrødinger equation,

$$\frac{d\xi}{d\tau} = Q(\xi),$$

where τ would be an ad hoc chosen time parameter. But, again, we have two, and only two options for the notion of time, namely ξ itself, or the metric g, measuring the time t, see the discussion above.

Since we have,

$$D_{\xi}(\xi) = \mu\frac{d\xi}{dt},$$

where D_{ξ} is the Levi-Civita connection applied to ξ, and $\mu = g(\xi,\xi)^{1/2}$, it seems reasonable to replace the classical Schrødinger equation, in our situation, by the following,

$$\frac{d\xi}{dt} = Q(\xi), \xi \in \Theta_C.$$

We shall return to this last equation, in the next section, where we shall refer to it as the *Furniture Equation.*

We find a general equation of motion for the representations of $C(\sigma_g)$,

Theorem 1.30 (The Generic Equation of Motion)**.** *Consider a $C(\sigma_g)$-representation, $\rho: C(\sigma_g) \to End_k(V)$, then the time development is, formally, given in terms of a parameter, τ, as*

$$\tilde{\rho}(\tau) = \rho(\epsilon(\tau)): C(\sigma_g) \to End_k(V),$$

where, $\epsilon(\tau)) = exp(\tau ad(g - T))$.

Now, to be able to handle this time development, we need to know formulas, in $C(\sigma_g)$, for $d^l t_i,\ l \geq 1,\ i = 1, ..., n$. To this end, put,

$$\bar{\Gamma}^i_{p,q} := \sum_{l,r} g^{r,i}\Gamma^l_{r,p}g_{l,q},\ \bar{\nabla}_l := (\bar{\Gamma}^j_{i,l}),\ T = \sum_l T_l dt_l$$

then,

$$T_l = -1/2(\sum_j(\Gamma^j_{j,l} + \bar{\Gamma}^j_{j,l}) = -1/2(trace\nabla_l + trace\bar{\nabla}_l)$$

$$\delta^2 t_i = [g - T, dt_i] = -1/2\sum_{p,q}(\bar{\Gamma}^i_{p,q} + \bar{\Gamma}^i_{q,p})dt_p dt_q$$

$$+ 1/2\sum_{p,q} g_{p,q}(R_{p,i}dt_q + dt_p R_{q,i}) + [dt_i, T],$$

where, as above, $R_{i,j} = [dt_i, dt_j]$. Put,

$$\Gamma^{j,i}_p = \sum_k g^{j,k}\Gamma^i_{k,p},$$

$$F_{i,j} := R_{i,j} - \sum_p(\Gamma^{j,i}_p - \Gamma^{i,j}_p)dt_p,$$

and recall (see [18], p. 82), that for any connection ρ, on a C-bundle, E, the corresponding representation of $C(\sigma_g)$ will map $F_{i,j}$ to the ordinary curvature of the connection In fact,

$$\rho(F_{i,j}) = [\rho(dt_i), \rho(dt_j)] - \sum_p (\Gamma_p^{j,i} - \Gamma_p^{i,j})\rho(dt_p) = [\nabla_{\xi_i}, \nabla_{\xi_j}] - \nabla_{[\xi_i,\xi_j]}.$$

Put, for short,

$$F(\xi_i, \xi_j) = F_{i,j} := \rho(F_{i,j}) \in End_C(E).$$

Computing, we find, see [18], for a proof,

Theorem 1.31 (The Generic Force Laws). *In $C(\sigma_g)$ we have the following force laws,*

$$\begin{aligned}
(1)\ d^2t_i =& -1/2\sum_{p,q}(\bar{\Gamma}^i_{p,q} + \bar{\Gamma}^i_{q,p})dt_p dt_q + 1/2\sum_{p,q} g_{p,q}(R_{p,i}dt_q + dt_p R_{q,i}) \\
&+ [dt_i, T] \\
(2)\ d^2t_i =& -\sum_{p,q}\Gamma^i_{p,q}dt_p dt_q - 1/2\sum_{p,q} g_{p,q}(F_{i,p}dt_q + dt_p F_{i,q}) \\
&+ 1/2\sum_{l,p,q} g_{p,q}[dt_p, (\Gamma_l^{i,q} - \Gamma_l^{q,i})]dt_l + [dt_i, T] \\
T_l =& -1/2(\sum_j(\Gamma^j_{j,l} + \bar{\Gamma}^j_{j,l}) = -1/2(trace\nabla_l + trace\bar{\nabla}_l).
\end{aligned}$$

Remark 1.32. *We shall consider the above formulas as general Force Laws, in $Ph(C)$, induced by the metric g. This means the following:*

First, assume given a representation,

$$\rho_0 : C \to End_k(V),$$

and pick any tangent vector (momentum) of the formal moduli of the A-module V, i.e. an extension of ρ_0,

$$\rho_1 : Ph(C) \to End_k(V),$$

then, if ρ_1 can be extended to a representation,

$$\rho_2 : Ph^2(C) \to End_k(V)$$

with, $\rho(d_2(d_1t_i)) = \rho_1(d^2t_i)$, given by the formula of the Force Law, this means that the force law has induced a second order momentum in the formal moduli space of the representation ρ_1, usually called $E \cdot \mathbf{a}(\rho_0)$, where E is the energy of the object in movement, and $\mathbf{a}$ is the acceleration, explaining the name Force Law.

We might also consider $(\mathfrak{c}_g)$, the δ- stable ideal generated by any one of these equation in $Ph^\infty(C)$. Since the force laws above holds in the dynamical system defined by (σ_g), we obviously have $(\mathfrak{c}_g) \subset (\sigma_g)$, and we might hope these new dynamical systems might lead to new Quantum Field Theories, as defined above, with equally new and interesting properties.

One immediate result is that the reduction of the force law (1), to the commutative case give us back the General Relativity, as we have seen above, since the tensors $\bar{\Gamma}^i_{p,q}$ are physically equivalent to $\Gamma^i_{p,q}$, so gives us the same results, i.e. the same geodesics, see (1.6). See [18], *and also (2.3) for an easy example to test this contention.*

For a connection ∇, on a free C-module E, the second Force Law above will now take the form, in $End_C(E)$,

$$\rho_E(d^2t_i) + \sum_{p,q} \Gamma^i_{p,q} \nabla_{\xi_p} \nabla_{\xi_q}$$
$$= 1/2 \sum_p F_{p,i} \nabla_{\delta_p} + 1/2 \sum_p \nabla_{\delta_p} F_{p,i} + 1/2 \sum_{l,q} \delta_q(\Gamma^{i,q}_l - \Gamma^{q,i}_l)\nabla_{\xi_l} + [\nabla_{\xi_i}, \rho_E(T)],$$

where we, as above, have put $\rho(dt_i) = \sum_j g^{i,j}\nabla_{\delta_j} =: \xi_i$..

Notice that considering the representation ρ_Θ, corresponding to the Levi-Civita connection, the above translate into,

$$\rho(dt_i) = [Q, t_i], \quad \rho(d^2t_i) = \sum_{j=1}^{d} [Q, \rho(dt_i)],$$

where Q is the Laplace-Beltrami operator.

Given any *observable* $f \in Ph(C)$, we would expect that the dynamics of the future *values* of f to be the spectrum of the operator,

$$f(\tau) := exp(\tau \cdot ad(Q))(f).$$

1.10. **The classical Gauge Invariance.** The space of representations, ρ of $C(\sigma)$ on a free (or projective) C-module V, is given as above, by

$$\rho_\phi(t_i) = t_i, \ \rho_\phi(dt_i) = \sum_{l=1}^{n} g^{il}\delta_l + \phi_i,$$

where $\phi_i \in End_C(V)$. The set of iso-classes is identified with the space of equivalence classes of the corresponding *potentials*, $\phi := (\phi_1, \phi_2, ..., \phi_n)$. It does not form an algebraic variety, but it has a nice structure.

The set of *potentials*, is a vector space, naturally isomorphic to,

$$\mathcal{P} := (End_C(V))^n.$$

The *tangent space* between any two representations, $\rho_l : C(\sigma) \to End_C(V_{\rho_l}), \ l = 1, 2$, represented by elements $\phi(l) \in \mathcal{P}, \ l = 1, 2$, may also be identified with a quotient of $\mathcal{P}$. In fact,

$$Ext^1_{C(\sigma)}(\rho_1, \rho_2) = Der_k(C(\sigma), End_k(V))/Triv.$$

Any derivation $\xi \in Der_k(C(\sigma), End_k(V))$, maps the relations of $C(\sigma)$ to zero, so we shall have,

$$[\xi(dt_i), t_j] + \rho_1(dt_i)\xi(t_j) - \xi(t_j)\rho_2(dt_i) = \xi(g^{i,j}).$$

Since V is a free C-module, such that $Ext^1_C(V, V) = 0$, there exists a linear map, $\Phi_0 \in End_k(V)$, such that $\xi(t_j) = t_j\Phi_0 - \Phi_0 t_j$, for all j. We may therefore, for a chosen ξ, assume all $\xi(t_i) = 0$, and it follows from the above equation, that the derivation ξ is determined by the family of elements, $\xi(dt_i) \in End_C(V)), i = 1, ..., n$.

We might see this as a reason behind the common physicists notation, simply denoting a field by $W^\mu_{a,b}$, assuming that one understands that the μ is a generic index for a basis of the tangent space.

In case $\rho_1 = \rho_2$, corresponding to $\phi \in \mathcal{P}ot$, we see that the trivial derivations, mapping t_i to 0, are exactly those given by the n-tuples,

$$((\sum_j^n g^{1,j}(\frac{\partial \Phi}{\partial t_j}) + [\phi_1, \Phi]), ..., (\sum_{j=1}^n g^{n,j}(\frac{\partial \Phi}{\partial t_j}) + [\phi_n, \Phi])),$$

for some $\Phi \in End_C(V)$. The expression,

$$\Phi(\phi) := (\xi_1(\Phi) + [\phi_1, \Phi], ..., \xi_n(\Phi) + [\phi_n, \Phi]),$$

therefore corresponds to an infinitesimal automorphism, Φ, of the space, $\mathcal{P}$, of representations of the algebra $C(\sigma_g)$. In fact, the Lie algebra $End_C(V)$, acts on a representation of $C(\sigma_g)$, on $\mathcal{P}$, in exactely the same way as the gauge groups acted in (1.6). The *physical* relevant space is therefore the quotient $\mathbf{P}$, of $\mathcal{P}$ with respect to this action.

As in the finite dimensional situation, the Dirac derivation induces a vector field,

$$[\delta] \in \Theta_{\mathbf{P}},$$

as long as we, by vector field, understand any map, which to an element ϕ in $\mathbf{P}$ associate an element in its tangent space, i.e. in $Ext^1_{C(\sigma)}(V_\rho, V_\rho)$. It must, however, vanish since the Dirac derivation $\delta = ad(g - T)$, necessarily must be mapped to a trivial derivation in $Der_k(C(\sigma), End_k(V))$, therefore to 0 in $Ext^1_{C(\sigma)}(V_\rho, V_\rho)$.

This may be interpreted as saying that time acts within each representation, $\rho : C(\sigma_g) \to End_k(V)$!

Remark 1.33. *The physicists usually write $\delta\phi := \Phi(\phi)$, not caring to mention Φ, taking for granted that $\delta\phi := \delta_\Phi(\phi)$ stands for an infinitesimal movement of ϕ in the direction of Φ, and call the transformation above, an infinitesimal gauge transformation. The literature on gauge theory, and its relation to non-commutativity of space, and to quantization of gravity, is huge. I think that the introduction of the non-commutative phase space, and in the metric case, the generic dynamical system,*

$$(\sigma_g) = ([dt_i, t_j] - g^{i,j}),$$

can, to some degree, elucidate the philosophy behind this effort. See e.g. the papers, [29]*, and* [22]*, where the authors initially introduce non-commutativity in the ring of observables generated by coordinates, $\hat{x}^\nu$, by imposing,*

$$[\hat{x}^\nu, \hat{x}^\mu] = \Theta^{\nu,\mu},$$

where $\Theta^{i,j}$ are constants.

The above treatment of the notion of gauge groups and gauge transformations may also explain why, in physics, one considers potentials as interaction carriers, thus as particles mediating force upon other particles. And maybe one can also see why the notion of Ghost particles of Faddeev and Popov, comes in. It seems to me that the introduction of ghost particles is linked to working with a particular section of the quotient map, $\mathcal{P} \to \mathbf{P}$.

The Dirac derivation, which is entirely dependent upon the notion of noncommutative phase space, is not (explisitely) found in present day physics. The parsimony principle is therefore introduced via the construction of a Lagrangian, and an Action Principle, i.e. a function of the (assumed physically significant) variables, the fields and their derivatives, defined in $\mathcal{P}$, assumed to to be invariant under the gauge transformations, so really defined in $\mathbf{P}$, and supposed to stay stable during time development, see [18]*. One choice in the context above, is,*

$$\mathbf{L}_{gf} = -1/4 F^{\mu\nu\alpha} F^{\alpha}_{\mu\nu}$$

1.11. **Yang-Mills Theory.** Given a metric, $g \in Ph(C)$, and assume there is a *gauge group*, i.e. a k- Lie algebra, $\mathfrak{g}_0$, operating on C, with extension to $C(\sigma)$, where σ is a dynamical structure. Normally σ would be σ_g, and $\mathfrak{g}_0 = \mathfrak{aut}(g)$, the Lie algebra of Killing vectors associated to the metric g.

According to our philosophy, we should consider (as moduli space, for our models), the invariant (or quotient) space,

$$Simp(C)/\mathfrak{g}_0 \simeq Simp(C(\mathfrak{g}_0)).$$

Fix a representation, $V \in Simp(C)/\mathfrak{g}_0$, given by the structure morphism, $\rho : C(\to End_k(V)$, then by definition, there is a $\mathfrak{g}_0$-connection,

$$\nabla := \nabla_V : \mathfrak{g}_0 \to End_k(V),$$

such that for all $c \in C$, and all $\xi \in \mathfrak{g}_0$, $\rho(\xi(c)) = [\nabla(\xi), \rho(c)]$, where $\nabla(\xi)$ correspond to the operator Q_ξ in Lemma (1.2).

Therefore any element $\xi \in \mathfrak{g}_0$ acts upon all representations of $C(\mathfrak{g}_0)$, and by the definition of gauge groups, this action should correspond to an equivalence relation.

In the general situation, when we also have a local gauge group, i.e. a C-Lie algebra $\mathfrak{g}_1$ insensitive to the action of $\mathfrak{g}_0$ see subsection (1.8), we also have a connection,

$$\mathfrak{D}_0 : \mathfrak{g}_0 \to Der_k(\mathfrak{g}_1),$$

which is a kind of a general *Coupling Morphism*, leading to a generalization of the Yang-Mills situation, where $\mathfrak{D}(\xi) = \nabla_\xi + [Q_\xi, -]$ with $Q_\xi \in \mathfrak{g}_1$. When the Lie algebra morphism, $ad : \mathfrak{g}_1 \to Der(\mathfrak{g}_1)$ is an isomorphism, this is automatic.

Recall that when the curvature of $\mathfrak{D}_0$ vanish, $\mathfrak{g} := \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a Lie-algebroid. Then if $\mathfrak{g}_1$ acts on V, the connection, $\mathfrak{D}_0$, extends to a connection,

$$\mathfrak{D} : \mathfrak{g} \to End_k(V)$$

When $\mathfrak{g}_V = Der_k(C)$ and the C-free C- Lie algebra $\mathfrak{g}_1$ acts on a C-free module, V, and is insensitive to $\mathfrak{g}_V$, there are connections,

$$\mathfrak{D}_0 : \Theta_C \to Der_k(\mathfrak{g}_1),\ \nabla : \Theta_C \to End_k(V)$$

leading to a family of representations,

$$\rho_W : C(\sigma_g) \to End_k(V),$$

parametrized by what physicists call *Gauge Fields*, the $W_i^l \in C$, in the following formula,

$$\rho_W(t_i) = t_i,\ \rho_W(dt_i) = \sum_{l=1}^{n} g^{il}\frac{\partial}{\partial t_l} + \sum_{l=1}^{r} W_i^l \gamma_l,$$

where $\{\gamma_i, i = 1, ..., r\}$ is a C-basis for $\mathfrak{g}_1$. Put, $W_i := \sum_{l=1}^r W_i^l \gamma_l$, and as above, $\xi_i = \sum_{l=1}^r g^{il}\dfrac{\partial}{\partial t_l}$, and denote by $c_{l,m}^p$ the structural constants of $\mathfrak{g}_1$, such that,

$$[\gamma_l, \gamma_m] = \sum_{p=1}^{r} c_{l,m}^p \gamma_p.$$

Now, consider the curvature of this representation,

$$\begin{aligned} F_{i,j} &= D_W(R_{i,j}) := D_W([dt_i, dt_j]) \\ &= [\xi_i, \xi_j] + \sum_{l=1}^{r}(\nabla_{\xi_i}(W_j^l\gamma_l) - \nabla_{\xi_j}(W_i^l\gamma_l) + \sum_{l,m,p=1}^{r} c_{l,m}^p W_i^l W_j^m \gamma_p. \end{aligned}$$

Notice that for the Euclidean metric, and for the case that the connection ∇ is trivial, one obtains, $F_{i,j} = \sum F_{i,j}^l \gamma_l$, with,

$$F_{i,j}^l := (\xi_i W_j^l - \xi_j W_i^l) + \sum_{p,m=1}^{r} c_{p,m}^l W_i^p W_j^m,$$

which is the classical expression for the curvature in this case. Due to the Coupling Morphism $\mathfrak{D}$, any gauge field W induces an action on all representation $\rho : C(\sigma) \to End_k(V)$, a *Gauge transformation*, which we shall return to later.

The Euler-Lagrange equations of the Lagrangian,

$$\mathbf{L}_{gf} = -1/4 F^{\mu\nu\alpha} F_{\mu\nu}^{\alpha},$$

mentioned above, then gives us the corresponding equation of motion,

$$\xi_\mu F^a_{\mu\nu} + c^c_{ab} W^{\mu b} F^c_{\mu\nu} = 0,$$

the *Yang-Mills equation*, corresponding to the vanishing of,

$$1/2 \sum_{j=1}^{n} [F_{i,j}, \xi_j + W_j].$$

With a *Source* added, it looks like,

$$\xi_\mu F^a_{\mu\nu} + c^c_{ab} W^{\mu b} F^c_{\mu\nu} = -J^a_\nu.$$

In the general metric case, with non-abelian gauge group, it is difficult to find gauge invariant Lagrangians of reasonable physical relevance, so we have to operate differently. Here is where the Generic Equation of Motion above, comes in and give us equations of motion, quite generally. Recall that the Force Law is given by,

$$d^2 t_i = -\sum_{p,q} \Gamma^i_{p,q} dt_p dt_q - 1/2 \sum_{p,q} g_{p,q}(F_{i,p} dt_q + dt_p F_{i,q})$$
$$+ 1/2 \sum_{l,p,q} g_{p,q}[dt_p, (\Gamma^{i,q}_l - \Gamma^{q,i}_l)] dt_l + [dt_i, T].$$

When the metric is trivial, or Minkowski, this reduces to,

$$d^2 t_i = -1/2 \sum_p g_{p,p}(F_{i,p} dt_p + dt_p F_{i,p}) = -\sum_p g_{p,p} F_{i,p} dt_p - 1/2 \sum_p g_{p,p}[dt_p, F_{i,p}].$$

Therefore,

$$\rho(d^2 t_i) = -\sum_{p,m} g_{p,p} F_{i,p} \rho(dt_p) - 1/2 \sum_{p,m} g_{p,p} \frac{\partial}{\partial t_p}(F^m_{i,p}) \gamma_m - 1/2 \sum_{p,l,m} g_{p,p} W^l_p F^m_{i,p} c^q_{l,m} \gamma_q,$$

where we, as usual, let the curvature conserve its name, $F_{i,j} := \rho(F_{i,j})$. The Yang-Mills equation above, is now seen to imply,

$$\rho(d^2 t_i) = -\sum_{p,m} g_{p,p} F_{i,p} \rho(dt_p).$$

This, however indicates that the tangent to the representation $\dot{\rho}$ given by,

$$\xi = (\xi_i) = (-1/2 \sum_{p,m} g_{p,p} \frac{\partial}{\partial t_p}(F^m_{i,p}) \gamma_m - 1/2 \sum_{p,l,m} g_{p,p} W^l_p F^m_{i,p} c^q_{l,m} \gamma_q),$$

is a classical 0-tangent, so the Yang-Mills equation just should tell us that there exist a potential $\Phi \in End_k(V)$, such that,

$$(\xi_i) = ((\sum_j^n g^{1,j} (\frac{\partial \Phi}{\partial t_j}) + [\phi_1, \Phi]), ..., (\sum_{j=1}^n g^{n,j} (\frac{\partial \Phi}{\partial t_j}) + [\phi_n, \Phi])).$$

Interpreting $\rho(d^2 t_i) = m\mathbf{a}_i$ and $\rho(dt_p) = mv_p$, we recover the classical equation of movement in an field. Compare with the Lorentz Force Law, for an electric field, see Example (6.4), and also [18], p. 115,

$$\mathbf{a}_i = \sum_{p=1}^n F_{i,p} v_p.$$

Remark 1.34. *As we shall see, in the next section, the gauge groups of the standard model, i.e. the Lie algebras,*

$$\mathfrak{g}' := \mathfrak{u}(1) \times \mathfrak{sl}(2) \subset \mathfrak{g}'_1 := \mathfrak{u}(1) \times \mathfrak{su}(2) \times \mathfrak{su}(3),$$

which is part of our toy model, see [18], *also pops up in our cosmological model,* $\underline{\tilde{H}}$, *see section 3, and now as a real local gauge group in the above sense. The elementary* *particles in that model should therefore, in line with the usage of present quantum theory, be the points of $Simp(Ph(\tilde{H})(\mathfrak{g}))$. In fact, we shall see that an impressive part of the structure of the Standard Model, is contained in this non-commutative quotient (or invariant) space.*

1.12. **Reuniting GR, Y-M and General Quantum Field Theory.** Let us have a look at the significance of the conclusion of the subsections (1.6), (1.7), (1.8), and the Remarks (1.5) and (1.19). We have, for every polynomial algebra C, outfitted with some metric g, proved that there exist a derivation,

$$\eta := ad(g - T) \in Der_k(Ph(C)),$$

such that it coincides with the canonical derivation $d : C \to C(\sigma_g)$, in the generic dynamical system (GDS). The corresponding force laws in $Ph(C)$, see the Theorem (1.31), generates equations of motions in General Relativity (GR), as well as in the generalized Yang-Mills (YM) theory introduced above. In fact, we find a very satisfying identity between the notions of *Time* in GDS, GR and in YM. The *Dirac derivation* $\delta = ad(g - T)$ in GDS, inducing $[\delta] = 0$ and a Hamiltonian equal to the Laplace-Beltrami operator, $Q = ad(g - T)$. Since T vanish in $Ph(C)^{com}$, the time in GR, reduces to the Dirac derivation,

$$[\delta] = \sum_l (\xi_l \frac{\partial}{\partial t_l} - \Gamma^l \frac{\partial}{\partial \xi_l}),$$

and a trivial Hamiltonian. The Schrødinger equation in GDS is given as,

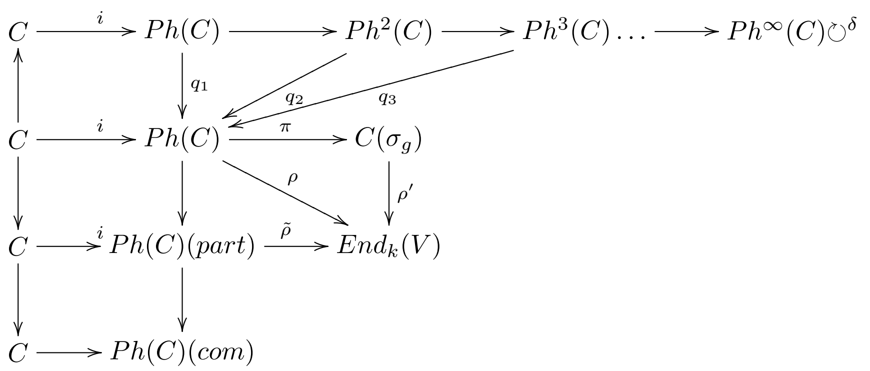
$$(Q - E)(\psi) = 0.$$

The corresponding notion in General Quantum Theory (GQT), including the Quantum Field Theory (QFT) and generalised Yang-Mills theory, turns out to be a composition of a Dirac derivation $[\delta]$ and a Hamiltonian $Q = ad(g - T)$.

To unify General Quantum Theory (GQT), including the Quantum Field Theory (QFT), and GR, we should just need to consider the dynamical system, $C(g)$, generated by, say, the Force Laws of type (1). Unluckily the structure of this system, in general, seems to be very complicated. It is, for example, not easy to decide whether or not $C(g)$ has finite dimensional representations at all.

However, we may study an obvious unification of GR, GDS, and GQT, where the algebra of observables is $Ph^\infty(C)$, and we shall show that we are able to classify, i.e. compute the moduli space of, the finite dimensional representations of $Ph^\infty(C)$, even though this space will turn out to be of infinite dimension. Thus we may hope to extend the method of subsection (1.3), and obtain a unified theory.There are, however, lots of problems involved in this scheme, one is the action of the gauge groups that turns up. Another is the philosophically difficult, but certainly reasonable, consequence of this restriction of the theory to just the finitely defined measurable entities: Our Space, and everything else modelled by the theory, would be discrete, simple objects would have pointlike structures.

We start by studying the structure of the derivation $ad(g - T)$ of $Ph(C)$, as deduced from the universal structure of the $Ph^\star(C)$. Consider the commutative diagram,



where $Ph(C)(part) := Ph(C/([dt_i, t_j]))$ and $Ph(C)(com) := Ph(C/([dt_i, t_j]), [dt_i, dt_j])$, uniquely defined by,

$$\begin{aligned} i \circ ad(g-T) &= d \circ q_1 : C \to Ph(C) \\ q_1 \circ ad(g-T) &= d \circ q_2 : Ph(C) \to Ph(C) \\ q_n \circ ad(g-T) &= d \circ q_{n+1} : Ph^n(C) \to Ph(C), n \geq 1 \end{aligned}$$

From the commutativity of the diagram, we observe that given a representation,

$$\rho : Ph(C) \to End_k(V),$$

its dynamical properties with respect to the derivation $\eta := ad(g-T)$, are equivalent to those induced by the infinite family of representations,

$$q_n \circ \rho : Ph^n(C) \to End_k(V),$$

with respect to the canonical Dirac derivation δ of $Ph^\infty(C)$. This follows from,

$$\eta^n \circ \rho = \delta^n \circ q_n \circ \rho.$$

The computation of the moduli space of finite dimensional representations of $Ph^\infty(C)$, mentioned above, and the further analysis of the resulting "Quantum Field Theory" will, for economical reasons, be postponed until subsections (3.3)and (6.2). However, this is the place for a remark, taking a new look at the framework of the theory covered up to now.

Remark 1.35. *In Ph(C) we have the relations,*

$$[dt_i, t_j] + [t_i, dt_j] = 0,$$

and the same relations will hold in any Dynamical System of the form,

$$C(\sigma) = Ph^\infty(C)/(\sigma)$$

see (1.4). In particular for those of order two we will have $\delta(t_i) = dt_i, d^2t_i := \delta^2(t_i) \in C(\sigma)$, *and we find the relations,*

$$[dt_i, t_j] = [dt_j, t_i] = g^{i,j}(\underline{t}) + \sum_{p\geq 1} h_p^{i,j}(\underline{t}, \underline{dt})$$

where the $h_p^{i,j}(\underline{t}, \underline{dt})$ *are (non-commutative) polynomials in the generators* t_i, dt_j *of degree p in the* dt_j*-s. And, of course, we may assume,*

$$g^{i,j} = g^{j,i}, h_p^{i,j} = h_p^{j,i}.$$

Above we have studied the situation where $g^{i,j}$ *is the inverse of a metric g, and* $h_p^{i,j} = 0$. *This led to force laws generalising Yang-Mills and General Relativity for representations* ρ'. *In (7.2) we shall see that a combination of this "global" set-up, with a localised "gauge version" based on the representations* $\tilde\rho$, *lead to a kind of*

Quantum Field Theory, containing generalisations of the classical Dirac equation. What happens if one tries out the possibility above with $h_p^{i,j} \neq 0$*, I do not know!*

2. Time-space and Space-times

2.1. **A Toy Model.** In the first paper on the problem of defining Time, see [14], we sketched a *toy model* in physics, where the space-time of classical physics became a section of a universal fibre space $\tilde{E}$, defined on the moduli space, $\mathbf{H} := Hilb^{(2)}(\mathbf{A}^3)$, of the physical systems we chose to consider, in this case the systems composed of an observer and an observed, both sitting in the affine real 3-space, $\mathbf{A}^3$. This moduli space, the Hilbert scheme of subschemes of length 2 in $\mathbf{A}^3(k)$, is easily computed, and has the form $\mathbf{H} = \underline{\tilde{H}}/Z_2$, where $H = k[t_1, ..., t_6], k = \mathbf{R}$ and $\underline{H} := Simp_1(H)$ is the space of all ordered pairs of points in $\mathbf{A}^3$, $\underline{\tilde{H}}$ is the blow-up of the diagonal, and Z_2 is the obvious group-action. Denote by $\gamma \in Z_2$ the generator, which we shall show will give us the parity operator, defining chirality in QT. The space $\mathbf{H}$, and by extension, $\underline{H}$ and $\underline{\tilde{H}}$, was called the *time-space* of the model.

Measurable time, in this mathematical model, turned out to be a metric g on the time-space, measuring all possible infinitesimal changes of *the state* of the objects in the family we are studying. This implies that the notion of relative velocity may be interpreted as an oriented line in the tangent space of a point of $\underline{\tilde{H}}$. Thus the space of velocities is compact.

This lead to a *physics* where there are no infinite velocities, and where the principle of relativity comes for free. The Abelian Lie-algebra of translations in $\mathbf{A}^3$ defines a 3-dimensional distribution, $\tilde{\Delta}$ in the tangent bundle of $\underline{\tilde{H}}$, corresponding to 0-velocities. Given a metric on $\underline{\tilde{H}}$, we defined the distribution $\tilde{c}$, corresponding to light-velocities, as the normal space of $\tilde{\Delta}$. We explained how the classical *space-time* can be thought of as a universal subspace, $\underline{\tilde{M}}(l)$, of $\underline{\tilde{H}}$, defined by a fixed line $l \subset \mathbf{A}^3$.

We also showed how the generator $\gamma \in Z_2$, above, is linked to the operators C, P, T in classical physics, such that $\gamma^2 = \gamma PT = id$. Moreover, we observed that the three fundamental gauge groups of current quantum theory $U(1)$, $SU(2)$ and $SU(3)$ are part of the structure of the fiber space,

$$\tilde{E} \longrightarrow \underline{\tilde{H}}.$$

In fact, introduce bases in affine 3-space, so that we may talk about the Euclidean 3-space $\mathbf{E}^3$. For any point $\underline{t} = (o, x)$ in $\underline{H}$, outside the diagonal $\underline{\Delta}$, we may consider the line l in $\mathbf{E}^3$ defined by the pair of points $(o, x) \in \mathbf{E}^3 \times \mathbf{E}^3$. We may also consider the action of $U(1)$ on the normal plane $B_o(l)$, of this line, oriented by the normal (o, x), and on the same plane $B_x(l)$, oriented by the normal (x, o). Using parallel transport in $\mathbf{E}^3$, we find an isomorphisms of bundles,

$$P_{o,x} : B_o \to B_x, P : B_o \oplus B_x \to B_o \oplus B_x,$$

the *partition isomorphism.* Using P we may write, (v, v) for $(v, P_{o,x}(v)) = P((v, 0))$. We have also seen, in loc.cit., that the line l defines a unique sub scheme $\underline{H}(l) \subset \underline{H}$. The corresponding tangent space at (o, x), is called $A_{(o,x)}$. Together this define a decomposition of the tangent space of $\underline{H}$,

$$T_{\underline{H}} = B_o \oplus B_x \oplus A_{(o,x)}.$$

If $\underline{t} = (o, o) \in \underline{\Delta}$, and if we consider a point $\underline{o}'$ in the exceptional fiber E_o of $\underline{\tilde{H}}$ we find that the tangent bundle decomposes into,

$$T_{\underline{\tilde{H}},o'} = C_{o'} \oplus A_{o'} \oplus \tilde{\Delta},$$

where $C_{o'}$ is the tangent space of E_o, $A_{o'}$ is the light velocity defining o' and $\tilde{\Delta}$ is the 0-velocities. Both B_o and B_x as well as the bundle $C_{(o,x)} := \{(\psi, -\psi) \in B_o \oplus B_x\}$, become complex line-bundles on $\underline{H}-\underline{\Delta}$. $C_{(o,x)}$ extends to all of $\underline{\tilde{H}}$, and its restriction to E_o coincides with the tangent bundle. Tensorising with $C_{(o,x)}$, we complexify all bundles. In particular we find complex 2-bundles $\mathbf{C}B_o$ and $\mathbf{C}B_x$, on $\underline{H}-\underline{\Delta}$, and we obtain a canonical decomposition of the complexified tangent bundle. Any real metric on $\underline{H}$ will decompose the tangent space into the light-velocities $\tilde{\mathbf{c}}$ and the 0-velocities, $\tilde{\Delta}$. This decomposition can, of course, also be extended to the complexified tangent bundle of $\underline{\tilde{H}}$,

$$T_{\underline{H}} = \tilde{\mathbf{c}} \oplus \tilde{\Delta}, \ \mathbf{C}T_{\underline{H}} = \mathbf{C}\tilde{\mathbf{c}} \oplus \mathbf{C}\tilde{\Delta}.$$

Clearly, $U(1)$ acts on $T_{\underline{H}}$, and $SU(2)$ and $SU(3)$ acts naturally on $\mathbf{C}B_o \oplus \mathbf{C}B_x$ and $\mathbf{C}\tilde{\Delta}$ respectively. Moreover $SU(2)$ acts also on $\mathbf{C}C_{o'}$, in such a way that the actions should be *physically* irrelevant. The groups, $U(1)$, $SU(2)$, $SU(3)$ are our elementary *gauge groups*, and we shall consider the corresponding Lie algebra,

$$\mathfrak{G} := \mathfrak{u}(1) \oplus \mathfrak{s}u(2) \oplus \mathfrak{s}u(3)$$

as the *gauge group*, in the sense of (1.5). We shall also see, in section 4, that the gauge group structure, introduced here actually follows from purely algebraic geometry, no need to introduce an Euclidean metric!

The reader is hereby warned, there will be no warnings when we suddenly go from a real situation, i.e.from $k = \mathbf{R}$ to a complexifyed situation, where $k = \mathbf{C}$. There are few problems involved and one should, unless mistakes, easily understand what is going on.

2.2. **Newton's and Kepler's Laws.** Let us study the geometry of $\mathbf{H}$. Recall that $\underline{\tilde{H}} \to \underline{H}$, is the (real) blow up of the diagonal $\underline{\Delta} \subset \underline{H}$, where $\underline{H}$ is the space of pairs of points in $\mathbf{E}^3$. Clearly any point $\underline{t} \in \underline{H}$ outside the diagonal, determines a vector $\xi(o,p)$ and an oriented line $l(o,p) \subset \mathbf{E}^3$, on which both the observer o and the observed p sits. This line also determines a subscheme $\underline{H}(l) \subset \underline{H}$, see above and [16], and in $\underline{H}(l)$ there is unique *light velocity curve* $\underline{l}(\underline{t})$, through $\underline{t}$, an integral curve of the distribution $\tilde{c}$, and this curve cuts the diagonal $\underline{\Delta}$ in a unique point $c(o,p)$, the *center of gravity of the observer and the observed*, and thus defines a unique point $\xi(\underline{t})$, of the blow-up of the diagonal, in the fiber of $\underline{\tilde{H}} \to \underline{H}$, above $c(o,x)$.

Recall that the subspace $\underline{\tilde{M}}(l) \subset \underline{\tilde{H}}$, corresponding to a line $l \subset \mathbf{A}^3$, referred to above, consists of all points $(o,p) \in \underline{\tilde{H}}$ for which $c(o,p) \in l$.

Any tangent $\eta := (\eta_1, \eta_2), \eta_2 = -\eta_1$, of $\underline{H}$ in $\tilde{c}$, at $\underline{t} = (o,x)$, normal to $\underline{l}(\underline{t})$, corresponds to a light velocity, to a *spin vector*, $\eta_1 \times \xi(o,p)$, in $\mathbf{E}^3$, with *spin axis*, the corresponding oriented line. The length of the spin vector is called the *spin momentum* of η.

There is a convenient parametrisation of $\underline{\tilde{H}}$. Consider, as above, for each $\underline{t} \in \underline{\tilde{H}}$ the length ρ, in $\mathbf{E}^3$, the Euclidean space, of the vector (o,p). Given a point $\underline{\lambda} \in \underline{\Delta}$, and a point $\xi \in E(\underline{\lambda}) = \pi^{-1}(\underline{\lambda})$, the fiber of,

$$\pi : \underline{\tilde{H}} \to \underline{H},$$

at the point $\underline{\lambda}$, for $o = p$. Since $E(\underline{\lambda})$ is isomorphic to S^2, parametrized by $\underline{\omega} = (\phi, \theta)$, any element of $\underline{\tilde{H}}$ is now uniquely determined in terms of the triple $\underline{t} = (\underline{\lambda}, \underline{\omega}, \rho)$, such that $c(\underline{t}) := c(o,p) = \underline{\lambda}$, and such that ξ is defined by the line $\overline{op}$, and the action of θ keeps ξ fixed. Here $\rho \geq 0$, see also the section *Cosmology, Cosmos and Cosmological Time.* Notice also that, at the exceptional fiber, i.e. for $\rho = 0$, the momentum corresponding to $d\rho$ is not defined.

Consider any metric on $\underline{\tilde{H}}$, of the form,

$$g = h_\rho(\underline{\lambda}, \underline{\phi}, \rho)d\rho^2 + h_{\underline{\phi}}(\underline{\lambda}, \underline{\phi}, \rho)d\underline{\phi}^2 + h_{\underline{\lambda}}(\underline{\lambda}, \underline{\phi}, \rho)d\underline{\lambda}^2,$$

where $d\underline{\phi}^2 = d\phi^2 + sin^2\phi d\theta^2$, is the natural metric in $S^2 = E(\underline{\lambda})$. Later, in section 6, we shall come back to the conditions that a metric should satisfy, given the fact that there are gauge groups acting. A first reasonable condition is that the component $h_{\underline{\phi}}(\underline{\lambda}, \underline{\phi}, \rho)$ should not depend on θ.

We shall also see that to be able to model black holes, and certainly the Big Bang event itself, we shall have to accept that the metric may degenerate in a certain subspace, from now on called the *Horizon*. We shall demand that everywhere else it is non-degenerate, and since the space $\underline{\tilde{H}}$ is smooth everywhere, locally isomorphic to an affine space, the spectrum of a polynomial algebra C, everything we have done in section (1) is valid outside the Horizon in this model.

It is reasonable to believe that the geometry of $(\underline{\tilde{H}}, g)$, might explain the notions like *energy*, mass, *charge*, etc. In fact, we tentatively propose that the source of mass and charge etc. is located in the *black holes* $E(\underline{\lambda})$. This would imply that mass, charge, etc. are properties of the 5-dimensional superstructure of our usual 3-dimensional Euclidean space, essentially given by a *density*, $h(\underline{\lambda}, \phi, \theta)$. This might bring to mind Kaluza-Klein-theory. However, it seems to me that there are important differences, making comparison very difficult.

Let us first treat the following simple case,

$$h_\rho = (\frac{\rho - h}{\rho})^2, \ h_\phi = (\rho - h)^2, \ h_\lambda = 1,$$

where h is a positive real number. This metric is everywhere defined in the subspace of $M(l)$, where we have reduced the spherical coordinates $\underline{\omega}$, to just ϕ. Notice that for $\rho = 0$, there are no tangent vectors in $d\rho$ direction.

It clearly reduces to the Euclidean metric far away from $\underline{\Delta}$, and it is singular on the *horizon* of the black hole, given by $\rho = h$, which in $\underline{H}$ is simply a sphere in the *light-space*, of radius h. Moreover it is clear that h is also the *radius of the exceptional fibre*, since the length of the circumference of $\rho = 0$, is $2\pi h$. Clearly, the exceptional fiber, the black hole itself, is not visible, and does not bound anything. However, the horizon bounds a piece of space. Moreover, if we reduce the horizon to a point in $\underline{H}$, then the circumference, or area of the exceptional fiber, as measured using the above metric, reduces to zero, and the metric becomes the usual Euclidean metric.

To understand the geometry of this space, including the notion of *gravity*, we shall reduce to a plane in the light directions, i.e. we shall just assume that $S^2 = E(\underline{\lambda})$, is reduced to a circle, with coordinate ϕ. This is actually an innocent restriction, as is easily seen.

The corresponding equations for the geodesics in $\underline{\tilde{H}}$ are, see [18],

$$\begin{aligned} \frac{d^2\rho}{dt^2} &= -(\frac{h}{\rho(\rho - h)})(\frac{d\rho}{dt})^2 + (\frac{\rho^2}{(\rho - h)})(\frac{d\phi}{dt})^2, \\ \frac{d^2\phi}{dt^2} &= -2/(\rho - h)\frac{d\rho}{dt}\frac{d\phi}{dt} \\ \frac{d^2\lambda}{dt^2} &= 0. \end{aligned}$$

where t is time. But time is, by definition, the distance function in $\underline{\tilde{H}}$, so we must have,

$$(\frac{\rho - h}{\rho})^2(\frac{d\rho}{dt})^2 + (\rho - h)^2(\frac{d\phi}{dt})^2 + (\frac{d\lambda}{dt})^2 = 1,$$

from which we find,

$$(\frac{d\rho}{dt})^2 = \rho^2(\rho - h)^{-2}(1 - (\frac{d\lambda}{dt})^2) - \rho^2(\frac{d\phi}{dt})^2.$$

From the third equation, we find that $\frac{d\lambda_j}{dt}$, $j = 1, 2, 3$, are constants, and $|\frac{d\lambda}{dt}|$ is the *rest-mass* of the system. Put $K^2 = (1 - |\frac{d\lambda}{dt}|^2)$, then K is the *kinetic energy of the system.* The definition of time therefore give us,

$$\rho^{-2}(\frac{d\rho}{dt})^2 = (\rho - h)^{-2}K^2 - (\frac{d\phi}{dt})^2.$$

Put this into the first equation above, and obtain,

$$\frac{d^2\rho}{dt^2} = -hK^2(\frac{\rho}{\rho - h})\frac{1}{(\rho - h)^2} + (\frac{\rho + h}{\rho - h})\rho(\frac{d\phi}{dt})^2.$$

Assume now $r := \rho - h \approx \rho$, we find,

$$\frac{d^2r}{dt^2} = -\frac{hK^2}{r^2} + r(\frac{d\phi}{dt})^2,$$

i.e. Keplers first law. The constant h, i.e. the radius of the exceptional fibre, is thus also related to mass. Recall that the Schwarzschild radius, the Einstein equivalent to h, is assumed to be,

$$r_s = 2GM/c^2,$$

where, G = Newton's gravitational constant, M = mass, c = speed of light, which here, of course, is put equal to 1. As we have hinted at above, this suggests that *mass*, is a property of the space $\underline{\tilde{H}}$. In this case it is a function of the surface of the exceptional fibre, i.e. the *black hole*, associated with the point $\underline{\lambda}$ in the ordinary 3-space $\underline{\Delta}$. In the same way, the second equation above gives us Keplers second law,

$$r(\frac{d^2\phi}{dt^2}) + 2(\frac{dr}{dt})(\frac{d\phi}{dt}) = 0.$$

Notice that with the chosen metric, time, in light velocity direction, is *standing still* on the *horizon* $\rho = h$, of the *black hole* at $\underline{\lambda} \in \underline{\Delta}$. Therefore no light can escape from the black hole. In fact, no geodesics can pass through $\rho = h$. Notice also that, for a photon with light velocity, we have $K = 1$, so we may measure h, by measuring the trajectories of photons in the neighborhood of the *black hole.* Finally, see that if the distance between the two interacting points is close to constant, i.e. if we have a circular movement, the left side of the time-equation becomes zero, and we therefore have the following equation,

$$(\rho - h)d\phi = Kdt,$$

which may be related to the perihelion precession, and also to the *Thomas Precession*, see [27] and [30].

Now suppose we use the quantum-theoretical general force law of our metric, reduced to the commutative case, then we obtain,

$$d^2t_i = -\sum_{p,q}\bar{\Gamma}^i_{p,q}dt_p dt_q.$$

We would have got the same Kepler's laws, except that the first law would have had the form,

$$\frac{d^2r}{dt^2} = -\frac{hK^2}{r^2} - r(\frac{d\phi}{dt})^2,$$

which is curious, since it corresponds to an (attractive) centripetal force.

Example 2.1. *Recall Example (1.28) and let us compute the Killing vectors, i.e. the global gauge group $\mathfrak{g}_0$, for this metric. With the above indexes, one obtains,*

$$\gamma = \gamma_\rho\frac{\partial}{\partial\rho} + \gamma_\phi\frac{\partial}{\partial\phi} + \gamma_\lambda\frac{\partial}{\partial\lambda} \in \mathfrak{g},$$

if and only if,

$$\gamma(g_{i,j}) + \frac{\partial \gamma_p}{\partial t_i} g_{p,j} + \frac{\partial \gamma_p}{\partial t_j} g_{i,p} = 0.$$

There is just one solution $\gamma_\rho = 0., \gamma_\phi = const$, $\gamma_\lambda = const$, so $\mathfrak{g}_0$ is just the rotations in ϕ and the translations in λ.

If we, in our metric, permit the radius of the black hole, h, to depend on λ, i.e. be given by a function of the form, $h(\lambda)$, so that the metric looks like,

$$(\frac{\rho - h(\lambda)}{\rho})^2 d\rho^2 + (\rho - h(\lambda))^2 d\phi^2 + \kappa(\lambda) d\lambda^2,$$

then the force law formulas above become more involved,

$$\frac{d^2\rho}{dt^2} = -(\frac{h(\lambda)}{\rho(\rho - h(\lambda))})(\frac{d\rho}{dt})^2 + (\frac{2}{(\rho - h(\lambda))})(\frac{dh}{d\lambda})(\frac{d\rho}{dt})(\frac{d\lambda}{dt}) + (\frac{\rho^2}{(\rho - h(\lambda))})(\frac{d\phi}{dt})^2,$$

$$\frac{d^2\phi}{dt^2} = -2/(\rho - h(\lambda))\frac{d\rho}{dt}\frac{d\phi}{dt} + 2/(\rho - h(\lambda))(\frac{dh}{d\lambda})(\frac{d\phi}{dt})(\frac{d\lambda}{dt})$$

$$\frac{d^2\lambda}{dt^2} = (\frac{(\rho - h(\lambda)}{\rho})(\frac{1}{\kappa(\lambda)})(\frac{dh}{d\lambda})(\frac{d\rho}{dt})^2 + (\rho - h(\lambda))(\frac{1}{\kappa(\lambda)})(\frac{dh}{d\lambda})(\frac{d\phi}{dt})^2 + 1/2(\frac{dln(\kappa)}{d\lambda})(\frac{d\lambda}{dt})^2$$

where t, as above, is time.

We find that, if $(\frac{dh}{d\lambda})$ is negative, then for $\rho \leq h(\lambda)$ the acceleration of ρ is positive, and unlimited for ρ close to $h(\lambda)$, and the acceleration of λ is negative. We shall come back to this in relation to the Big Bang and inflation.

Remark 2.2. *Recall the classical Schwarzschild metric,*

$$ds^2 = (1 - \frac{2GM}{rc^2})dt^2 - (1 - \frac{2GM}{rc^2})^{-1} dr^2 - r^2(d\phi^2 + sin^2(\phi)d\theta^2)$$

Promoting ds to be our $d\lambda$, and solving for dt, we obtain,

$$dt^2 = (\frac{r}{r-h})^2 dr^2 + r^2(\frac{r}{r-h})(d\phi^2 + sin^2(\phi)d\theta^2)$$

with $h = GMc^{-2}$. Consider the partition operator P, and put $r = h - \rho$, then,

$$(\frac{r}{r-h}) = (\frac{\rho - h}{\rho}), r^2 = (\rho - h)^2$$

and we are very close to our metric above, the difference maybe having to do with the definition of the material content of the Space. See next subsection.

2.3. **Thermodynamics, the Heat Equation and Navier-Stokes.** Let us now go back to Section (1.9), and consider the generic dynamical structure (σ_g), related to the metric on $\underline{\tilde{H}}$, of the form,

$$g = h_1(\underline{\lambda}, \underline{\phi}, \rho)d\rho^2 + h_2(\underline{\lambda}, \underline{\phi}, \rho)d\underline{\phi}^2 + h_3(\underline{\lambda}, \underline{\phi}, \rho)d\underline{\lambda}^2,$$

where $d\underline{\phi}^2$ is the natural metric in $S^2 = E(\underline{\lambda})$.

Recall for $C = H = k[t_1, ..., t_6]$, and a non-singular Riemannian metric $g = 1/2\sum_{i,j=1,..,r} g_{i,j} dt_i dt_j \in Ph(C)$, the notations,

$$\Gamma_p^{j,i} = \sum_k g^{j,k}\Gamma_{k,p}^i, \;\; R_{i,j} := [dt_i, dt_j], \;\; F_{i,j} := R_{i,j} - \sum_p (\Gamma_p^{j,i} - \Gamma_p^{i,j})dt_p,$$

and the general force Law in $Ph(C)$,

$$d^2 t_i = -\sum_{p,q} \Gamma_{p,q}^i dt_p dt_q - 1/2\sum_{p,q} g_{p,q}(F_{i,p}dt_q + dt_p F_{i,q})$$
$$+ 1/2\sum_{l,p,q} g_{p,q}[dt_p, (\Gamma_l^{i,q} - \Gamma_l^{q,i})]dt_l + [dt_i, T],$$

generating the dynamical structure $\mathfrak{c} := \mathfrak{c}(g)$.

Remark 2.3. *In principle, according to our philosophy, the natural common quantization of classical general relativity and Yang-Mills theory would be based on the dynamical properties of* $Simp_{\leq\infty}(\tilde{\underline{H}}(\mathfrak{c}))$*, with respect to the versal family,* $\tilde{\rho} : O_{\tilde{\underline{H}}(\mathfrak{c})} \to End_{\tilde{\underline{H}}(\mathfrak{c})}(\tilde{V})$*, where we have to consider* $O_{\tilde{\underline{H}}(\mathfrak{c})}$ *as a presheaf of associative* k*-algebras, defined in* $\tilde{\underline{H}}$*. As a first try, we shall concentrate on situations where the Dirac vectorfield* $[\delta]$ *vanish at a chosen representation, a situation that we have termed singular. In particular, we shall treat the structure of the Levi-Civita representation.*

It is reasonable to believe that the geometry of $(\tilde{\underline{H}}, g)$, might explain the notions like *energy*, mass, *charge*, etc. In fact, we tentatively propose that the source of mass and charge etc. *is located* in the *black holes* $E(\underline{\lambda})$. This would imply that mass, charge, etc. are properties of the 5-dimensional metric superstructure of our usual 3-dimensional Euclidean space, essentially given by a *density*, $h(\underline{\lambda}, \phi, \theta)$. This might bring to mind Kaluza-Klein-theory. However, it seems to me that there are important differences, making comparison very difficult.

Recall that at a point $\underline{t} = (o, x) \in \underline{H} - \underline{\Delta}$, the tangent space, $\Theta_{\tilde{\underline{H}}}(\underline{t})$, is represented by the space of all pairs of 3-vectors, $\xi(\underline{t}) = (\xi_o, \xi_x)$, ξ_o, fixed at o, and ξ_x, fixed at the point x in $\mathbf{E}^3$. Moreover, any such tangent vector may, depending only upon the choice of metric, be decomposed into the sum $\xi = \xi_1 + \xi_2$, with $\xi_1 \in \tilde{\Delta}$, and $\xi_2 \in \tilde{c}$.

Recall also that the *center of gravity of the observer and the observed*, $c(o, x) \in \underline{\Delta}$, defined in terms of a Euclidean structure on our 3-dimensional space, defines a unique point $\xi(\underline{t})$, of the blow-up of the diagonal, in the fiber of $\tilde{\underline{H}} \to \underline{H}$, above $c(o, x)$.

Now, consider the Levi-Civita connection,

$$\rho_\Theta : \tilde{H}(\sigma_g) \to End_R(\Theta_{\tilde{\underline{H}}}),$$

together with the Hamiltonian, i.e. the Laplace-Beltrami operator, $Q \in End_{\tilde{\underline{H}}}(\Theta_{\tilde{\underline{H}}})$.

Any *state* $\xi \in \Theta_{\tilde{\underline{H}}}$, may be interpreted as a (relative) momentum (ξ_o, ξ_x) of the pair of points $(o, x) \in \mathbf{E}^3$, defined for all pairs of points in the domain of definition for ξ. Write, as above,

$$\underline{\xi} = \underline{p} + \underline{m},\ \underline{p} = (p_1, p_2, p_3, 0, 0, 0) \in \tilde{c},\ \underline{m} = (0, 0, 0, m_1, m_2, m_3) \in \tilde{\Delta},$$

where we have introduced local coordinates, $(\lambda_1, \lambda_2, \lambda_3, x_1, x_2, x_3)$, such that

$$\tilde{\Delta} = (\delta_{\lambda_1}, \delta_{\lambda_2}, \delta_{\lambda_3}),\ \text{and}\ \tilde{c} = (\delta x_1, \delta x_2, \delta x_3)$$

The norms,

$$\mu := |\xi|,\ m := |\underline{m}|,\ \kappa := |\underline{p}|$$

defined by ξ, are the *energy-density*, the *density of mass*, and the *density of kinetic momentum*, respectively. We find that (p_1, p_2, p_3) is a classical relative momentum-vector, and, $v = (v_1, v_2, v_3) := \mu^{-1}(p_1, p_2, p_3)$, is a classical velocity vector.

Consider now the corresponding Schrødinger equation, our *Furniture equation*,

$$\frac{d}{dt}(\xi) = Q(\xi).$$

Time, here, is the notion used in quantum theory, and Q is the Laplace-Beltrami operator. We have, however, introduced another notion of time, the metric in general relativity theory, and they should be equal. Therefore we must have,

$$\frac{d}{dt} = \mu^{-1} D_\xi,$$

as operators on $\Theta_{\tilde{\underline{H}}}$.

Let us compute the left hand side of the Schrødinger equation. It is clear that,

$$\begin{aligned}\frac{d}{dt}(\xi) &= \frac{d}{dt}(\underline{m}) + \frac{d}{dt}(\underline{p})\\ &= \sum_1^3 \mu^{-1} m_j D_{\lambda_j}(\underline{m}) + \sum_1^3 \mu^{-1} m_j D_{\lambda_j}(\underline{p}) + \sum_1^3 v_j D_{x_j}(\underline{m}) + \sum_1^3 v_j D_{x_j}(\underline{p}).\end{aligned}$$

Reduce to the space-time like sub-scheme $\underline{\tilde{M}}(l) \subset \underline{\tilde{H}}$, corresponding to a chosen line $l \subset \mathbf{A}^3$, see (2.2), above, and consult [18]. Then the 3-dimensional vector $\underline{m} = \sum_{i=1}^3 m_i \frac{\partial}{\partial \lambda_i}$, reduces to, $m\frac{\partial}{\partial \lambda}$,, and the term, $\mu^{-1} m \frac{\partial}{\partial \lambda}$ may be compared to $\frac{\partial}{\partial \tau}$, where τ is the relativistic *proper time.*

The outcome of this is that, reduced to the subscheme $\underline{M}(l)$, the Schrødinger equation is, in a realistic classical Euclidean situation, where $D_{\lambda_j} = \frac{\partial}{\partial \lambda_j}$, the coupled equation, containing the general, relativistic Heat Equation (HE),

$$\frac{dm}{dt} = \frac{\partial m}{\partial \tau} + \sum_{j=1}^3 v_j \frac{\partial m}{\partial x_j} = \Delta(m)$$

and the "relativistic" Navier-Stokes Equation (NSE),

$$\frac{dp_i}{dt} = \frac{\partial p_i}{\partial \tau} + \sum_{j=1}^3 v_j \frac{\partial p_i}{\partial x_j} = \Delta(p_i),\ \ i = 1, 2, 3.$$

where $\Delta = Q = \frac{\partial^2}{\partial \lambda^2} + \sum_{j=1}^3 \frac{\partial^2}{\partial x_j^2}$.

Computing, we find for (NSE),

$$\frac{\partial p_i}{\partial \tau} = \mu \frac{\partial v_i}{\partial \tau} + \frac{\partial \mu}{\partial \tau} v_i,\ \ \sum_{j=1}^3 v_j \frac{\partial p_i}{\partial x_j} = \mu(v \nabla v_i) + (v \nabla \mu) v_i$$

where ∇ is the dimension 3 del-operator, with respect to the parameters $\underline{x} = (x_1, x_2, x_3)$. Let $\overline{\nabla}$ be the dimension 4 del-operator, with respect to the parameters (λ, x_1, x_2, x_3), put, $\overline{v} = (\nu/\mu, v_1, v_2, v_3)$, and compute,

$$\Delta(p_i) = \mu \Delta(v_i) + 2\overline{\nabla}\mu \overline{\nabla} v_i + \Delta(\mu) v_i,$$

from which we deduce an equation, close to the classical Navier-Stokes equation,

$$\mu(\frac{\partial v_i}{\partial \tau} + v \nabla v_i) = \mu \Delta(v_i) + 2\overline{\nabla}\mu \overline{\nabla} v_i + \Delta(\mu) v_i - (\overline{v}\overline{\nabla}\mu) v_i,\ \ i = 1, 2, 3.$$

Remark 2.4. *Any vector field $\xi \in \Theta_{\underline{\tilde{H}}}$ may be interpreted as a description of the relative state of the space, everywhere, a kind of mass-stress-tensor, describing the situation of all pairs of points in our 3-dimensional space, together with any corresponding pair of momenta. (The Furniture of our cosmos, referred to in the title of this paper, will be a state ξ of a much more complex representation, of the gauge groups, possibly an iterated extension of the simple "atoms" of $\Theta_{\underline{\tilde{H}}}$).*

In general, we might hope that knowing ξ, i.e. the 6 functions defined in $\underline{\tilde{H}}$, locally defining the vector ξ, the Schrødinger equation would determine the metric, g, i.e. the 6 functions $h_\rho, h_{\underline{\phi}}, h_{\underline{\lambda}}$. This would presumably lead to time-developments $\xi(T)$ and $g(T)$, determined by any given ground-state, $\xi_\star$, and clocked by some parameter T. This would again have as a consequence, that any cyclic behavior of the phenomenon modeled by $\xi(T)$, would lead to a gravitational wave defined by $g(T)$.

In particular, the collapsing of a star, or the Big Bang event, both usually modeled as a fluid depending on pressure, temperature, energy density, viscosity, entropy, would in the above scenario, define a generalised gravitational wave, $g(T)$.

We would therefore be tempted to consider the Schrødinger equation,

$$\frac{d}{dt}(\xi) = Q(\xi),$$

as our Field Equation, replacing the Einstein Field Equation. A solution would be a metric g determining the dynamics of the past and the future of our space. To make this reasonably understandable, we need a mathematical model of the beginning of it all, of the Big Bang. This is, however, the subject of the section 4.

As a first example, of the usefulness of the Schrødinger equation, in studying the geometry of our universe, consider the very special case of the metric of the last section, defined by,

$$h_\rho = (\frac{\rho - h}{\rho})^2, \; h_\phi = (\rho - h)^2, \; h_3 = 1,$$

where h is a positive real number.

Put $\rho = t_1, \phi = t_2, \lambda = t_3$, then we find,

$$\begin{aligned}
&\Gamma^1_{1,1} = h/\rho(\rho - h), \; \Gamma^1_{2,2} = -\rho^2/(\rho - h) \\
&\Gamma^2_{1,2} = 1/(\rho - h), \; \Gamma^2_{2,1} = 1/(\rho - h) \\
&\Gamma^3_{i,j} = 0
\end{aligned}$$

All other components vanish.

Since we now have the opportunity, let us just check that the contention of the Remark (1.14), i.e. in this case, the identities,

$$\overline{\Gamma}^k_{i,j} = \Gamma^k_{i,j},$$

proving that for this metric, the dynamical model of Yang Mills, i.e. for connections, and the corresponding for general relativity, is fused at the Ph^2-level.

Moreover we find the following formulas,

$$\begin{aligned}
D_\rho &:= \nabla_{\delta_1} = \frac{\partial}{\partial \rho} + \nabla_\rho, \\
D_\phi &:= \nabla_{\delta_2} = \frac{\partial}{\partial \phi} + \nabla_\phi, \\
D_\lambda &:= \nabla_{\delta_3} = \frac{\partial}{\partial \lambda} + \nabla_\lambda \\
Q &= \sum_{i=1}^{3} 1/h_i \nabla^2_{\delta_i} \\
\rho(\delta^2(t_i)) &= [Q, \rho(dt_i)] = 1/h_i [Q, \nabla_{\delta_i}].
\end{aligned}$$

Here the h_i is the function defined above, i.e. $g_{i,i}$ in our metric, and,

$$\nabla_\rho = \begin{pmatrix} h/\rho(\rho-h) & 0 & 0 \\ 0 & 1/(\rho-h) & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\nabla_\phi = \begin{pmatrix} 0 & -\rho^2/(\rho-h) & 0 \\ 1/(\rho-h) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\nabla_\lambda = 0$$

$$[\nabla_\rho, \nabla_\phi] = \begin{pmatrix} 0 & -1/\rho(\rho-h) & 0 \\ -\rho/(\rho-h) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\frac{\partial}{\partial\rho}\nabla_\rho = \rho^2(\rho-h)^{-2}\begin{pmatrix} -h\rho^{-2}(\rho-h)^{-1} - h\rho^{-1}(\rho-h)^{-2} & 0 & 0 \\ 0 & -(\rho-h)^{-2} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\frac{\partial}{\partial\rho}\nabla_\phi = (\rho-h)^{-2}\begin{pmatrix} 0 & 2h\rho & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The Schrødinger equation looks like,

$$D_\xi(\xi) = \mu\frac{\partial\xi}{\partial t} = \mu Q(\xi),$$

for a general vector field, $\xi = (f_1, f_2, f_3)$, we find,

$$\begin{aligned} D_\xi(\xi) =&(f_1\frac{\partial f_1}{\partial\rho}, f_1\frac{\partial f_2}{\partial\rho}, f_1\frac{\partial f_3}{\partial\rho}) + f_1(f_1h(\rho-h))^{-1}, f_2h(\rho-h)^{-1}, 0) \\ &+(f_2\frac{\partial f_1}{\partial\phi}, f_2(\frac{\partial f_2}{\partial\phi}, f_2\frac{\partial f_3}{\partial\phi}) + f_2(-f_2\rho^2(\rho-h)^{-1}, f_1h(\rho-h)^{-1}, 0) \\ &+(f_3\frac{\partial f_1}{\partial\lambda}, f_3\frac{\partial f_2}{\partial\lambda}, f_3\frac{\partial f_3}{\partial\lambda}), \end{aligned}$$

and, with the obvious simplified notations,

$$\begin{aligned} Q(\xi) =& -(\rho-h)^{-4}(\rho^4 f_{1:\rho\rho} + h^2 f_1 - \rho^2 f_1 + \rho^3 f_{2:\rho} + \rho^4 f_{1:\rho,\rho} + h^4 f_{1:\lambda,\lambda} - 2\rho^3 f_{2:\phi} \\ &+\rho^2 f_{1:\phi,\phi} + h^2 f1:\phi,\phi - 4h\rho^3 f_{1:\lambda,\lambda} + 6h^2\rho^2 f_{1:\lambda,\lambda} - 2h\rho f_1 - h^2\rho f_{1:\rho} \\ &-2h\rho^3 f_{1:\rho,\rho} + h^2\rho^2 f_{1:\rho,\rho} - 4h^3\rho f_{1:\lambda,\lambda} + 2h\rho^2 f_{2:\phi} - 2h\rho f_{1:\phi,\phi})D_\rho \\ &-(\rho-h)^{-4}(3\rho^3 f_{3:\rho} + 2\rho f_{1:\phi} - 2h f_{1:\phi} + \rho^2 f2:\phi,\phi + \rho^4 f_{2:\lambda,\lambda} + h^4 f_{2:\lambda,\lambda} \\ &+\rho^4 f_{2:\rho,\rho} h^2 f_{2:\phi,\phi} - 4h\rho^2 f_{2:\rho} - 2h\rho f_{2:\phi,\phi} + 6h^2\rho^2 f_{2:\lambda,\lambda} - 2h\rho f_2 + h^2\rho f_{2:\rho} \\ &+h^2\rho^2 f_{2:\rho,\rho} - 2h\rho^3 f_{2:\rho,\rho} - 4h^3\rho f_{2:\lambda,\lambda} - 4h\rho^3 f_{2:\lambda,\lambda})D_\phi \\ &+\rho_{-1}(\rho-h)^{-1}(\rho^2 f_{3:\rho} + \rho^3 f_{3:\rho,\rho} + \rho f_{3:\phi,\phi} + \rho^3 f_{3:\lambda,\lambda} \\ &-2h\rho^2 f_{3:\lambda,\lambda} + h^2\rho f_{3:\lambda,\lambda})D_\lambda \end{aligned}$$

Put,

$$\xi = (0, f(\rho), 0),$$

then $D_\xi(\xi) = (0,0,0)$, and the furniture equation reduces to the second order differential equation, $Q(\xi) = 0$, that simplifies to,

$$\rho^2(\rho-h)^2\frac{d^2f}{d\rho^2} + \rho(\rho-h)(3\rho-h)\frac{df}{d\rho} - 2h\rho f = 0,$$

with the easy solution,

$$f = (\rho-h)^{-2},$$

which means that the *fluid*, the content of the space, rotates about the Black Hole $\rho = 0$ with speed $(\rho - h)^{-1}$, so with infinite speed close to the horizon, almost standing still at great distances, and therefore with lots of *shear*. Notice that this fit well with Example (2.1), and notice also that,

$$\xi = (0, (\rho - h)^{-2}(\rho - h)^{-2}((1/2\rho - 2h)\rho + h^2 ln(\rho)), c),$$

with c a constant, is also a solution. Compare with the Remark (2.2).

3. Entropy

Consider an algebraic geometric object X, and let $aut(X)$ be the Lie algebra of automorphisms of X. The sub-Lie algebra $aut_0(X)$ that lifts to automorphisms of the formal moduli of X, is a Lie ideal. Put $\mathfrak{a}(X) := aut(X)/aut_0(X)$, then if $X(t)$ is a deformation of some X along a parameter t, we find $dim_k \mathfrak{a}(X(t)) \leq dim_k \mathfrak{a}(X)$. One may phrase this saying that an object X can never gain *information* when deformed. Moreover, deformation is, obviously, not a reversible process, so information can get lost. This measure of information losses, is related, as we shall see, to the notion of gain of entropy (en-ergy and tropos=transform) coined by Clausius (1865) and generalised by Boltzmann and Shannon.

3.1. **The classical commutative case.** In [21], studying moduli problems of singularities in (classical) algebraic geometry, we were led to consider the notion of *Modular Suite*. This is a canonical partition $\{\mathbf{M}_\alpha\}$, of the versal base space, $\mathbf{M}$, of the deformation functor of an algebraic object, X. The different *rooms*, $\mathbf{M}_\alpha$, correspond to the subsets of equivalence classes of deformations in $\mathbf{M}$, along which the Lie algebra $\mathfrak{a} := aut/aut_0$ deforms as Lie-algebras, and therefore conserves its dimension. Working with Thermodynamics, it occurred to me that the notion of entropy has an interesting parallel in deformation theory. In fact I have proposed the following,

Definition 3.1. *Fix an object X, and let $X(\underline{t})$ corresponds to the point $\underline{t} \in \mathbf{M}_\alpha$, then we shall term Entropy, of the state $\underline{t}$, the integer,*

$$S(\underline{t}) := dim_k(\mathbf{M}_\alpha)$$

In this classical situation, assuming that the field is algebraically closed, and that $\mathbf{M}$ is of finite Krull dimension, the modular suite $\{\mathbf{M}_\alpha\}$ is finite, with an inner room, the *modular substratum* and an *ambiant* (open) *maximal entropy* stratum. But the structure of the modular suite may be very complex, even for simple singularities X, see the example of the quasi homogenous plane curve singularity $x_1^5 + x_2^{11}$, in [21]. It is also clear that for any *algebraic dynamics* in $\mathbf{M}$, the entropy will always stay or grow, see again [21]. To be able to construct situations where the entropy is lowered, or the information goes up, we must leave classical algebraic geometry, and venture into non-commutative algebraic geometry. Here is where non-commutative deformation theory comes into play.

3.2. **The general case.** In the general situation, where our algebras of *observables* are associative but not necessarily commutative, the first interesting cases are deformations of associative algebras, A, or deformations of finite families of representations V_i of an associative algebra A. In [19] we touched upon the first case. Here we shall look at the second case, based upon the technique of the previous sections.

There we worked with a polynomial algebra, $A = k[t_1, ..., t_d]$, with a metric $g \in Ph(A)$, and the Levi-Civita connection, considered as a representation,

$$A := Ph(A)/(\sigma_g) \to End_k(\Theta_A).$$

The Dirac derivation, $[\delta]$, in this case, vanish and the corresponding Hamiltonian turned out to be the Laplace-Beltrami operator, $Q \in End_k(\Theta_A)$. We were then, in analogy with Quantum theory, led to consider, for every *state* $\xi \in \Theta_A$, the time development, or Schrødinger equation, and the corresponding *Furniture Equation*, see [19],

$$\frac{d\xi}{dt} = Q(\xi).$$

We have, in [19], shown that in the special case of our *toy model*, see [18] and [19], this equation amounts to a combined Heat and Navier-Stokes equations. The corresponding notion of entropy, in the above sense, might be defined by the modular suite of the versal base of the deformation functor of the corresponding representation,

$$\xi : Ph(A) \to A = End_A(A).$$

The formal moduli has a tangent space given by $Ext^1_{Ph(A)}(\xi,\xi)$, which is given by the classical long exact sequence,

$$\begin{aligned} 0 \to &Hom_{Ph(A)}(\xi,\xi) \to Hom_A(\xi,\xi) \\ &\xrightarrow{\iota} Der_A(Ph(A), Hom_A(\xi,\xi)) \xrightarrow{\kappa} Ext^1_{Ph(A)}(\xi,\xi) \to 0,. \end{aligned}$$

Here, $\iota = 0$, and $Hom_A(\xi,\xi) = A$, so that,

$$Ext^1_{Ph(A)}(\xi,\xi) = A^d \simeq \Theta_A,$$

as all relations $[dt_i, t_j] + [t_i, dt_j]$ in $Ph(A)$ are mapped to 0 by any A-derivation into A.

The miniversal base space of ξ, is therefore of infinite dimension, and since the Lie algebra of the automorphism group of A, is the Lie algebra $Der_k(A) = \Theta_A$, the Lie algebra of the automorphism group of the representation ξ, is,

$$aut(\xi) := \{\eta \in \Theta_A | [\eta, \xi] = 0\}.$$

We should now define the entropy of the state ξ, as

$$S(\xi) := dim\{\eta | aut(\xi) \vdash aut(\eta)\},$$

where $aut(\xi) \vdash aut(\eta)$ should mean that η is a deformation of ξ inducing a deformation of the Lie algebras of automorphisms. Obviously this is unrealistic, as most of the terms involved will be of infinite dimension.

We may try to overcome this difficulty by approximating the state ξ, as a representation, by a finite dimensional representation, defined for every point set object, $\mathcal{P} = \{P_p\}$, of $Simp_1(A)$ by,

$$\xi_{\mathcal{P}} : Ph(A) \to End_k(\mathcal{P}),$$

where,

$$\xi_{\mathcal{P}}(t_i) = \alpha_i^0(p),\ \xi_{\mathcal{P}}(dt_i) = \alpha_i^1(p)$$

and where we, in anticipation of the treatment via the technique of the next section, have put,

$$\alpha_i^0(p) = P_{p,i},\ \alpha_i^1(p) = \xi(P_p)_i.$$

We may look at the object $\xi_{\mathcal{P}}$ as a set of molecules in our observatory (or in the Universe), and ξ as the combined state of these, at the outset maybe considered independently. Extending the representation $\xi_{\mathcal{P}}$ to a representation,

$$\xi_{\mathcal{P}} : Ph^\infty(A) \to End_k(\mathcal{P}),$$

or cutting it down to a representation of some dynamical system,

$$\xi_{\mathcal{P},\sigma} : A(\sigma) := Ph^\infty(A)/(\sigma) \to End_k(\mathcal{P}),$$

for some reasonable dynamical structure (σ) of $Ph^{\infty}(A)$, we can now use part of the technique of the subsections (1.3) and (1.4), and look at the versal family $\mathbf{M}$ of the representation $\xi_{\mathcal{P},\sigma}$ for some fundamental *state* ξ, of the *Furniture* of the Universe. There will be a canonically defined *Moduli Suite*, $\{\mathbf{M}_\alpha\}$, and for any deformation η of $\xi_{\mathcal{P},\sigma}$, there will be one α such that,

$$\eta \in \mathbf{M}_\alpha$$

and the resulting definition of entropy, would be

$$S(\eta) = dim(\mathbf{M}_\alpha).$$

The goal is to show that for reasonable classical cases, this should come out close to the Boltzmann's definition,

$$S(\eta) := log\ Vol(M_\alpha), \eta \in M_\alpha$$

where M_α is the substratum of the corresponding *Coarse Graining* of the classical phase space of the situation, containing the *distribution* η, see again the very readable text of [24].

This approximation also makes it possible to return to the general theory of section (1), coupled with the section (4) of [18], and find a very natural way of introducing analogues of Fock space and the Fock representation, and so including what physicists call the *second quantisation*, in our picture. But first we have to take another look at the functor Ph^*.

3.3. **Representations of Ph^{∞}.** Now let $A = k[t_1, ..., t_d]$, and consider a representation of $Ph^{\infty}(A)$ as k-algebra,

$$\rho : Ph^{\infty}(A) \to End_k(V),$$

V a k-vector space. Put,

$$D_i^0 := \rho(t_i),\ D_i^p := \rho(d^p t_i), p \geq 1.$$

The composition of $exp(\tau\delta)$ and ρ is a homomorphisms of k-algebras,

$$Ph^{\infty}(A) \overset{\rho[\tau]}{\to} End_k(V) \otimes_k k[[\tau]],$$

for which we have,

$$X_i := \rho(\tau)(t_i) = \rho(exp(\tau\delta)) = \sum_{p\geq 0} \tau^p/p! D_i^p.$$

Since $[t_i, t_j] = 0$, we must have $[X_i, X_j] = 0$, and, since the relations in $Ph^{\infty}(A)$ are given by, $\sum_{p+q=n\geq 0} 1/p!q!\ [d^p t_i, d^q t_j] = 0$, this is the condition,

$$\sum_{p+q=n\geq 0} 1/p!q!\ [D_i^p, D_j^q] = 0,$$

for the family of matrices $\{D_i^p,\ p \geq 0,\ i = 1, ..., d\}$ to define a homomorphism, ρ, of k-algebras.

Clearly if $dimV = 1$ there are no conditions, and we may pick arbitrarily $D_i^p \in k$, and obtain formal power series,

$$X_i = \sum_n D_i^n/n! \tau^n,$$

which, when convergent, gives the dynamics of the point.

Example 3.2. *Assume $dim_k V = 2$, and put,*

$$\rho(t_i) = D_i^0 = \begin{pmatrix} x_i(1) & 0 \\ 0 & x_i(2) \end{pmatrix} =: \begin{pmatrix} \alpha_i^0(1) & 0 \\ 0 & \alpha_i^0(2) \end{pmatrix},$$

and, $\alpha_i^0(r,s) := x_i(r) - x_i(s)$, $r,s = 1,2$. Let, for $q \geq 0$,

$$D_i^q = \begin{pmatrix} \alpha_i^q(1) & r_i^q(1,2) \\ r_i^q(2,1) & \alpha_i^q(2) \end{pmatrix},$$

Put,

$$\alpha_i^l(r,s) : = \alpha_i^l(r) - \alpha_i^l(s), \ r,s = 1,2,$$

$$r_i^k(r,s) = \sum_{l=0}^{k} \binom{k}{l} \sigma_{k-l}(r,s)\alpha_i^l(r,s), \ r,s = 1,2.$$

where the sequence $\{\sigma_l(r,s)\}$, $l = 0,1,...$ is a sequence of arbitrary coupling constants, with $\sigma_0(r,s) = 0$. Then,

$$\rho(d^n t_i) := D_i^n,$$

defines a representation,

$$\rho : Ph^\infty(A) \to End_k(V),$$

if and only if,

$$\sum_{p+q=n\geq 0} 1/p!q! \ [D_i^p, D_j^q] = 0,$$

which is exactly when,

$$\sum_{p+q=n\geq 0} 1/p!q!(\alpha_i^p(r,s)r_j^q(s,r) - r_i^p(r,s)\alpha_j^q(s,r)) = 0.$$

Computing, we find the condition,

$$\sum_{p+q=n\geq 0} 1/p!q!\sigma_l(r,s)(\alpha_i^p(r,s)\alpha_j^{q-l}(s,r) - \alpha_i^{q-l}(r,s)\alpha_j^p(s,r)) = 0.$$

for $r,s = 1,2$, $l \geq 1$. The situation above arises when we consider two (different) points

$$P_1 = (\alpha_1^0(1),...,\alpha_d^0(1)), P_2 = (\alpha_1^0(2),...,\alpha_d^0(2)),$$

in space, with pre-described tangents, $\xi_1 = (\alpha_1^1(1),...,\alpha_d^1(1))$ and $\xi_2 = (\alpha_1^1(2),...,\alpha_d^1(2))$. For these two points, considered as dimension 1 representations of $Ph(A)$, we saw that there is a 1-dimensional space of tangents between the points, i.e. the $Ext^1_{Ph(A)}(k(P_1),k(P_2)) = k$. This leads to possibly non-zero elements $r_i^1(1,2)$, $r_i^1(2,1)$) in the matrix representation of the non-commutative deformation of the family $\{k(P_1),k(P_2)\}$ of $Ph(A)$-modules.

We now have a much more complete picture of the situation. The dynamics of the pair of points is described by the Dirac derivation. Assuming that for time $\tau = 0$, we know the position $\alpha^0(1),\alpha^0(2)$, and the momenta $\alpha^1(1),\alpha^1(2)$, of the two points, then the dynamics is described, in terms of the time, τ, by the matrices,

$$X_i = \rho(exp(\tau\delta))(t_i)$$

Putting

$$\alpha_i(r,s) = \sum_{n=0}^{\infty} \tau^n/n! \ \alpha_i^n(r,s), \ \sigma(r,s) = \sum_{n=0}^{\infty} \tau^n/n! \ \sigma_n(r,s),$$

we find the explicit formulas,

$$X_i = \begin{pmatrix} \alpha_i(1) & \sigma(1,2)\alpha_i(1,2) \\ \sigma(2,1)\alpha_i(2,1) & \alpha_i(2) \end{pmatrix}, \ i = 1,...,d.$$

The trace, and determinant are,

$$tr(X_i) = (\alpha_i(1) + \alpha_i(2))$$
$$det(X_i) = (\alpha_i(1)\alpha_i(2)) - \sigma(1,2))\alpha_i(1,2)\sigma(2,1))\alpha_i(2,1)$$

The spectrum of X_i, or the eigenvalues, are given as,

$$X_i(r) = 1/2(tr(X_i) \pm \sqrt{tr(X_i)^2 - 4det(X_i)}),\ r = 1,2.$$

From this we see that if all coupling constants vanish, i.e. if $\sigma(r,s) = 0, r,s = 1,2$, then we have undisturbed independent motions of the two points, $X_i(r) = \alpha_i(r),\ r = 1,2$. If the coupling constants are nonzero the representation becomes simple, with trivial automorphism group, and so, according to the definition, of maximal entropy.

Moreover, assume $\sigma_l(1,2) = \sigma_l(2,1) = 0$ for $l = 0,\ l \geq 2$, then the conditions above becomes, for $r,s = 1,2$, and for all $n \geq 0$,

$$\sigma_l(r,s) \cdot \sum_{p+q=n} 1/p!q!(\alpha_i^p(r,s)\alpha_j^{q-1}(s,r) - \alpha_i^{q-1}(r,s)\alpha_j^p(s,r)) = 0.$$

If $d = 3$ this simplifies to,

$$\sigma_1(1,2) \cdot \sum_{p+q=n} 1/p!q!(\alpha^p(1,2) \times \alpha^{q-1}(2,1)) = 0,$$

where, of course, $\alpha^m(r,s)) = -\alpha^m(s,r))$ for , $r,s = 1,2,\ m \geq 0$.

In general, considering an *Object*, $\mathcal{P}$ in space, consisting of r points $\{P_p\}_{p=1,...,r}$, we find that the dynamics is closely related to the interaction process, involving non-commutative deformation of families of representations, described in [18]. In fact, we easily obtain the following theorem,

Theorem 3.3. *Given an object, $\mathcal{P}$ consisting of r points $\{P_p\}_{p=1,...,r}$ in d-space, $Simp(A)$, where $A = k[t_1,...,t_d]$. With the notations above, in particular, $\alpha_i^n(p,q) = \alpha_i^n(p) - \alpha_i^n(q),\ i = 1,...,d$, consider the matrix,*

$$D_i^n := \begin{pmatrix} \alpha_i^n(1) & r_i^n(1,2) & ... & r_i^n(1,r) \\ r_i^n(2,1) & \alpha_i^n(2) & ... & r_i^n(2,r) \\ . & . & ... & . \\ r_i^n(r,1) & r_i^n(r,2) & ... & \alpha_i^n(r)) \end{pmatrix}$$

with,

$$r_i^0(p,q) = 0,\ r_i^n(p,q) = \sum_{l=0}^{n} \binom{n}{l} \alpha_i^l(p,q)\sigma_{n-l}(p,q),$$

where $\sigma_m(p,q) \in k$ are arbitrary coupling constants, with $\sigma_m = 0$, for $m \leq 0$. Then these operators define a representation,

$$\rho : Ph^\infty(A) \to M_r(k)$$

with,

$$\rho(d^n t_i) = D_i^n$$

if and only if, for all $n \geq 1$,

$$\sum_h \binom{n}{h} \sigma_{n-h}(p,q)(\alpha_i^h(p,q)\alpha_j^0(p,q) - \alpha_i^0(p,q)\alpha_j^h(p,q))$$
$$= \sum_{k,l,m,s} \frac{n!\sigma_{n-k-m}(p,s)\sigma_{k-l}(s,q)}{l!m!(k-l)!(n-k-m)!}(\alpha_j^m(p,s)\alpha_i^l(s,q) - \alpha_i^l(p,s)\alpha_j^m(s,q)),$$

Proof. Let us, as above, consider the matrix,

$$X_i = \rho(exp(\tau\delta))(t_i) = \sum_{n\geq 0} \tau^n/n! D_i^n$$

Putting

$$\alpha_i(r) = \sum_{n=0}^{\infty} \tau^n/n!\ \alpha_i^n(r),\ \alpha_i(r,s) = \sum_{n=0}^{\infty} \tau^n/n!\ \alpha_i^n(r,s),\ \sigma(r,s) = \sum_{n=0}^{\infty} \tau^n/n!\ \sigma_n(r,s),$$

we find the explicite formulas,,

$$X_i = \begin{pmatrix} \alpha_i(1) & \sigma(1,2)\alpha_i(1,2) & ... & \sigma(1,r)\alpha_i(1,r) \\ \sigma(2,1)\alpha_i(2,1) & \alpha_i(2) & ... & \sigma(2,r)\alpha_i(2,r) \\ ... & ... & ... & ... \\ \sigma(r,1)\alpha_i(r,1) & \sigma(r,2)\alpha_i(r,2) & ... & \alpha_i(r) \end{pmatrix},\ i = 1, ..., d.$$

Now, compute,

$$[X_i, X_j] = 0,$$

and see that the condition of the theorem emerges. □

Remark 3.4. *We may consider the space*

$$\mathbf{A}(r) = k[\alpha_i^n(p), \sigma_n(p,q)]/\mathfrak{a},$$

with coordinates $\{\alpha_i^n(p), \sigma_n(p,q), i = 1, ..., d,\ p, q = 1, ..., r,\ n \geq 0,\ \sigma_0(p,q) = 0\}$, *and where the ideal* $\mathfrak{a}$ *is generated by the equations above, as the versal base space for the versal family of the non-commutative deformation theory applied to the family of* $Ph^\infty(A)$ *modules defined by the object* $\mathcal{P}$, *and with Dirac derivation,* δ *acting as,* $[\delta](\alpha_i^n(p)) = \alpha_i^{n+1}(p)$.

Since $Ph^\infty(A)$ *is infinitely generated, there is, strictly speaking, no such thing, but we shall see that in special cases, we can overcome this difficulty by finding clever dynamical structures..*

In (3.2) we saw that to overcome another difficulty related to non-existing moduli spaces, we opted for approximating a state $\xi \in \Theta_A$, *by a finite dimensional representation, defined for every finite point set object,* $\mathcal{P} = \{P_p\}$, *of* $Simp_1(A)$ *by,*

$$\xi_{\mathcal{P}} : Ph(C) \to End_k(\mathcal{P}),$$

where,

$$\xi_{\mathcal{P}}(t_i) = \alpha_i^0(p),\ \xi_{\mathcal{P}}(dt_i) = \alpha_i^1(p).$$

We wanted to look at the object $\mathcal{P}$ *as the set of molecules in our observatory (or in the Universe), and at* $\xi_{\mathcal{P}}$ *as the combined state of these.*

Extending the representation $\xi_{\mathcal{P}}$ *to a representation,*

$$\xi_{\mathcal{P}} : Ph^\infty(A) \to End_k(\mathcal{P}),$$

or cutting it down to a representation of some dynamical system, $A(\sigma) := Ph^\infty(A)/(\sigma)$,

$$\xi_{\mathcal{P},\sigma} : A(\sigma) \to End_k(\mathcal{P}),$$

for some reasonable dynamical structure (σ) *we might be able to compute the versal family* $\mathbf{M}$ *of the representation* $\xi_{\mathcal{P},\sigma}$ *for some fundamental state* ξ, *of the Furniture of the Universe, and also the Moduli Suite,* $\{\mathbf{M}_\alpha\}$.

For any deformation η *of* $\xi_{\mathcal{P},\sigma}$, *represented by an equivalence class* $\tilde{\eta}$ *in the versal base space* $\mathbf{M}$ *there will be one* α *such that,*

$$\tilde{\eta} \subset \mathbf{M}_\alpha$$

and the entropy, of η, *would be*

$$S(\eta) = dim(\mathbf{M}_\alpha).$$

From the classification above, it follows that the minimal entropy would correspond to all $\sigma_l(p,q) = 0$*, i.e. to some dust-like furniture of the space. And the maximal entropy would correspond to all* $\sigma_l(p,q) \neq 0$*, i.e. to some black hole-like object,* $\mathcal{P}$*, containing a huge number of gravitational and/or other parameters.*

The goal is still to show that, for reasonable classical cases, this should come out close to the Boltzmann's definition,

$$S(\eta) := log\ Vol(M_\alpha), \eta \in M_\alpha$$

where M_α *is the substratum of the corresponding Coarse Graining of the classical phase space of the situation, containing the distribution* η*, see again* [24]*, chapter 2. In particular, consider Penrose's need (for) some clear-cut way of saying that "the gravitational degrees of freedom were not activated", and his need to identify the mathematical quantity that actually measures "gravitational degrees of freedom". I suggest that the above, coupled with my "Toy Model", see* [18]*, i.e. working on* $Hilb^{(2)}(\mathbf{A}^3)$ *instead of the trivial affine space* $\mathbf{A}^d = Spec(k[t_1, ..., t_d])$*, may be of interest for this quest.*

We have already looked at the case $r = 2$, and seen that the result makes physical sense. For $r = 3, n = 2$ we find,

$$\sigma_1(p,q)(\alpha_i^1(p,q)\alpha_j^0(p,q) - \alpha_i^0(p,q)\alpha_j^1(p,q)) =$$
$$\sigma_1(p,s)\sigma_1(s,q)(\alpha_j^0(p,s)\alpha_i^0(s,q) - \alpha_i^0(p,s)\alpha_j^0(s,q)).$$

In dimension $d = 3$ this has a particularly nice interpretation. Let $\alpha^0(i,j)$ be the vector starting at P_i and ending at P_j, and let ξ_i be a tangent vector at P_i for $i = 1, 2, 3$. Put $\alpha^1(i,j) = \xi_i - \xi_j$, then the condition above reads:

$$\sigma_1(p,q)(\alpha^1(p,q) \times \alpha^0(p,q)) = -\sigma_1(p,s)\sigma_1(s,q)(\alpha^0(p,s) \times \alpha^0(s,q)), \forall p,s,q = 1,2,3.$$

This says that for any two of the three points in space, the relative momentum must sit in the plane defined by the three points, the length being determined by the 3 coupling constants. Moreover, the sum of all three relative momenta must be 0. In fact, there are coefficients, $u, v \in k$ such that,

$$\alpha^1(p,q) = u\alpha^0(p,s) + v\alpha^0(s,q).$$

Put this into the left side of the formula above, and find,

$$\sigma_1(p,q)(v - u) = \sigma_1(p,s)\sigma_1(s,q)$$

assuming that the three points are not co-linear.

Assume now that the coupling constants are given as,

$$\sigma_1(p,q) := m(p,q)|\alpha^0(p,q)|^{-2},$$

then after some computation we find the following differential equations for the dynamics of the 3 points,

$$\alpha^1(p,q) = -m(q,p)|\alpha^0(p,q)|^{-1}\epsilon(p,q) - m(p,s)|\alpha^0(p,s)|^{-1}\epsilon(p,s) + \rho\alpha^0(p,q)$$
$$\alpha^1(q,s) = -m(s,q)|\alpha^0(s,q)|^{-1}\epsilon(q,s) - m(q,p)|\alpha^0(q,p)|^{-1}\epsilon(q,p) + \rho\alpha^0(q,s)$$
$$\alpha^1(s,p) = -m(p,s)|\alpha^0(p,s)|^{-1}\epsilon(s,p) - m(s,q)|\alpha^0(s,q)|^{-1}\epsilon(s,q) + \rho\alpha^0(s,p)$$

Here $\epsilon(i,j)$ is the unit vector from the point P_i to the point P_j.

Notice that there is a different set-up, related to Grothendieck's generalized differential algebra. For A commutative, consider an A-module E, and an extension of this representation to,

$$\rho : Ph^\infty(A) \to End_k(E).$$

We have seen that ρ must be given in terms of operators,

$$D_i^p := \rho(d^p t_i) \in End_k(E),$$

satisfying the conditions,

$$\forall n \geq 0, \sum_{p+q=n} 1/p!q![D_i^p D_j^q] = 0.$$

There is an obvious family of solutions of these equations, given by any differential operator,

$$Q \in Diff(E),$$

with, $D_i^0 = \rho(t_i) = t_i$, $D_i^p := ad(Q)^p(t_i) \in Diff(E)$, $p \geq 1$. Looking at the case of finite dimensional representations, treated above, one sees the difference between the two set-ups, and the much greater generality obtained by considering the representations of Ph^∞, the way we do.

4. Cosmology, Cosmos and Cosmological Time

4.1. **Background, and some Remarks on Philosophy of Science.**

> They say clearly that when the One had been constructed-whether *of planes or surface or seed or something they cannot express*-then immediately the nearest part of the Unlimited began "to be drawn and limited by the Limited"... giving it (the Unlimited) numerical structure: (Aristotle, on the creation of the world, as the Pythagoreans saw it, see [6], page 148.)

In the paper [16], we discussed the possibility of including a cosmological model in our toy-model of Time-Space, and thereby giving some sense to this age-old struggle to cope with the notion of creation. The 1-dimensional model we presented there was created by the deformations of the trivial singularity, $O := k[x]/(x)^2$. Using elementary deformation theory for algebras, we obtained amusing results, depending upon some rather bold mathematical interpretations of the, more or less accepted, cosmological vernacular. Here we shall go one step further on, and show that our toy-model, i.e. the moduli space, $\mathbf{H}$, of two points in the affine 3-space, or it's étale covering, $\underline{\tilde{H}}$, is *created* by the (non-commutative) deformations of the obvious singularity in 3-dimensions, $U := k < x_1, x_2, x_3 > /(x_1, x_2, x_3)^2$.

The main axiom of the leading branches of cosmology, seems to be that the space-time of the existing universe can be described via a General Relativistic model, somehow given by Einstein's equation with respect to some mass-stress tensor, mass and energy being homogeneously and isotropically distributed in space. This leads to the assumption that the universe is a 4-dimensional space-time of a form commonly called a Friedman-Robertson-Walker model, or sometimes, the FLRW-model (where one includes Lemaître). In particular the space has an open ended time-coordinate, leaving out the Big Bang, but still assuming that the point-like Big Bang is in the closure of even the shortest complete history of the universe.

There are a lot of assumptions here. One is that the space-time is capable of containing something, and that these things can be described as independent upon the space, even though they curve space, and otherwise intervene in the dynamical process. For example, even in the very start of the universe, spin is assumed to be present. So, in mathematical terms, the space-time must be outfitted with a $su(2)$, or an $sl(2)$, action of the tangent structure, obviously determined by the Big Bang event. Moreover, since the space-time of the model does not contain the prime event, and the jump between that, supposedly singular, point-like event and the mathematically well defined space-time, is not part of the model, we do not have a model of the big bang itself, but rather of what may have happened in our usual space, a long time ago, with respect to a rather artificially chosen time parameter.

In this chapter I propose to show how the *time-space*, $\mathbf{H}$, can be thought of as an immediate product of a mathematical scenario incorporating a Big Bang event, making this event mathematically sound. Starting with the pure notion of 3-dimensionality, i.e. a k-scheme $\underline{U} = Spec(U)$ with only one point and a 3-dimensional tangent space, in algebraical terms, the singularity,

$$U := k[x_1, x_2, x_3]/(x_1, x_2, x_3)^2,$$

we shall see, in the next subsection, that we may construct, in a canonical way, a versal deformation base space, $\mathbf{M}$, and a corresponding versal family $\mathbf{U}_\star$, containing all isomorphism classes of deformations of U, as associative algebra.

The technique for this general deformation theory, can be found in [10], see also [21]. In the last book we introduced the notion of *Moduli-suite*. This corresponds here to a partition of the space $\mathbf{M}$, in a series of *rooms*, containing an inner room, composed of just one point $\star$, corresponding to the singularity we start with, U, and a very special component that turns out to be $\mathbf{H} := Hilb^2(\mathbf{A}^3)$, and where the family $\mathbf{U}$, the restriction of $\mathbf{U}_\star$ to $\mathbf{H}$ has, corresponding to a point $(o,p) \in \underline{H} - \underline{\Delta}$, the fibre,

$$U(o,p) := k < x_1, x_2, x_3 > /(x_i x_j - o_i x_j - p_j x_i + o_i p_j),$$

where we have used the coordinates x_1, x_2, x_3, to express the two points, o and p in 3-space $\mathbf{A}^3$, by coordinates, $\{o_i\}, \{p_j\}, i, j = 1, 2, 3$.

If $o = p$, $U(o,p)$ is isomorphic to U. But $\mathbf{U}$ has, never the less, a unique extension to all of $\underline{\tilde{H}}$, and the Z_2-action also extends.

There is also a special *room* in the moduli suite, defined by the tangent vector defined by the derivation that maps $x_i x_j$ to $\epsilon_{i,j,k}$. This implies that the Quatornians, $\mathbf{Q}$, is a deformation of U. In fact, we have,

$$\mathbf{Q} = k < x_1, x_2, x_3 > /(x_i x_j - \epsilon_{i,j,k} x_k + \delta_{i,j}),$$

where $\epsilon_{i,j,k}$ and $\delta_{i,j}$ are the usual notations for $0, 1$.

We have already in section (3) seen that the notion of moduli suite, in deformation theory, may serve as a model for both *Entropy* and *Information*, needed to connect to the present day Cosmology, see a readable text, in [24].

4.2. **Deformations of Associative Algebras.** The tangent space of the formal moduli of the singularity

$$U := k[x_1, x_2, x_3]/(x_1, x_2, x_3)^2,$$

as an associative k-algebra is, by deformation theory, see [10], and [21],

$$T_\star := A^1(k, U; U) = Hom_F(ker\rho, U)/Der,$$

where, $\rho : F \to U$ is any surjective homomorphism of a free k-algebra F, onto U, Hom_F means the F-bilinear maps, and Der denotes the subset of the restrictions to $I := ker\rho$ of the derivations from F to U. Choose, $F = k < x_1, x_2, x_3 >$, with the obvious surjection, making $ker\rho = (\underline{x})^2$, generated as F bimodule by the family $\{x_{i,j} := x_i x_j\}$.

Any F-bilinear morphism $\phi : (\underline{x})^2 \to U$, must be of the form,

$$\phi(x_{i,j}) = a^0_{i,j} + \sum_{l=1}^{3} a^l_{i,j} x_l$$

and the bilnearity is seen to imply that $a^0_{i,j} = 0$. Thus, the dimension of $Hom_F(I, U)$ is 27.

Any derivation $\delta \in Der$, must be given by,

$$\delta(x_i) = b^0_i + \sum_{l=1}^{3} b^l_i x_l$$

and the restriction of this map, to the generators of $I = (\underline{x})^2$, must have the form,

$$\delta(x_{i,j}) = b_j^0 x_i + b_i^0 x_j,$$

therefore determined by the $b_i^0 s$. It follows that the tangent space $T_\star$ is of dimension 27-3=24.

Let $o, p \in \mathbf{A}^3$, be two points, $o = (o_1, o_2, o_3), p = (p_1, p_2, p_3)$, with respect to the coordinate system, $\underline{x}$, and put,

$$\phi_{o,p}(x_{i,j}) = p_j x_i + o_i x_j,$$

then it is easy to see that the maps $\{\phi_{o,p}\}$ generate a 6-dimensional sub vector subspace T_0 of $T_\star$. Notice that, if $o = p$ then $\phi_{o,p}$, is a derivation, thus 0 in $T_\star$.

Now, the rather unexpected happens. We may integrate the tangent subspace T_0, and obtain a family of flat deformations of U. In fact, it is easy to see that,

$$U(o,p) := k < x_1, x_2, x_3 > /(x_i x_j - o_i x_j - p_j x_i + o_i p_j),$$

is an associative k-algebra of dimension 4, and a deformation of U, in a direction of $\underline{H}$. This defines a family of associative $H := k\,[o,p]$ algebras,

$$\mathbf{U} := H < x_1, x_2, x_3 > /(x_i x_j - o_i x_j - p_j x_i + o_i p_j).$$

where, normally $k = \mathbf{R}$, the real numbers. Let us put,

$$\mathbf{x}_{i,j} := (x_i - o_i)(x_j - p_j) = x_i x_j - o_i x_j - p_j x_i + o_i p_j, o := (o_1, o_2, o_3), p = (p_1, p_2, p_3) \in H^3,$$

Notice that if $o = p$ then $U(o,p)$ is isomorphic to U, as it should, and that, $U(o,p) \simeq U(-o,-p)$. Moreover, for any non-zero element $\kappa \in k$, and any 3-vector $c \in \mathbf{A}^3$, we have,

$$U(o,p) \simeq U(\kappa o, \kappa p),\ U(o,p) \simeq U(o-c, p-c).$$

Choosing $c = 1/2(p+o)$, we find $o' := o - c = -(p-c) =: -p'$, and it is easy to see that if $o' \neq 0$ the sub Lie algebra above generated by $\{x_1, x_2, x_3\}$ in $U(o', p')$, is isomorphic to the standard 3-dimensional Lie algebra with relations, $[y_1, y_2] = y_2, [y_1, y_3] = y_3, [y_2, y_3] = 0$. Moreover, choosing $c = (p+o)$, we find an isomorphism,

$$U(o,p) \simeq U(-p,-o) \simeq U(p,o),$$

which should be related to the action of Z_2 on $\underline{H}$, and thus, according to our philosophy, to the mathematical reason for the CPT-equivalence, see [18], (4.9).

Notice also that the algebra,

$$\mathbf{Q} := k < x_1, x_2, x_3 > /(x_i x_j - \epsilon_{i,j,k} x_k + \delta_{i,j}),$$

where $\epsilon_{i,j,k}$ and $\delta_{i,j}$ are the usual indices, the first one nonzero only for $\{i,j,k\} = \{1,2,3\}$, and the last one the usual delta function, is isomorphic to the quaternions, which therefore is another non-trivial deformation of U.

Consider now the restriction to the subscheme $\underline{H} - \underline{\Delta}$, of the family $\mathbf{U}$, denoted by,

$$\nu' : \mathbf{U}' \to \underline{H} - \underline{\Delta}.$$

Since for all non-zero $\kappa \in k$, we have $U(\lambda + \kappa u, -\kappa u + \lambda \simeq U(\kappa u, -\kappa u)) \simeq U(u, -u)$, this family extends uniquely to a family,

$$\nu : \tilde{\mathbf{U}} \to \underline{\tilde{H}}.$$

Let us compute the algebras $U(o,p)$, and their Lie algebras of derivations, $\mathfrak{g}(\underline{t}) := Der_k(U(\underline{t}))$.

First, the 4-dimensional k-algebra $U(o,p)$, with relation,

$$x_{i,j} = (x_i - o_i)(x_j - p_j),$$

obviously have only two simple representations, of dimension 1, call them k_o and k_p. By the generalised Burnside theorem, see [18], (3.2), (The O-construction), there is an isomorphisme,

$$\eta : U(o,p) \to \begin{pmatrix} H_{1,1} \otimes End(k_o) & H_{1,2} \otimes Hom_k(k_o,k_p) \\ H_{2,1} \otimes Hom_k(k_p,k_o) & H_{2,2} \otimes End_k(k_p) \end{pmatrix},$$

where, $H_{1,1}$ is a formal algebra with tangent space $Ext^1_{U(o,p)}(k_o,k_o)$, $H_{2,2}$ is a formal algebra with tangent space $Ext^1_{U(o,p)}(k_p,k_p)$, and $H_{1,2}$, respectively $H_{1,2}$, is a bi-module generated by $Ext^1_{U(o,p)}(k_o,k_p)^\star$, respectively by $Ext^1_{U(o,p)}(k_p,k_o)^\star$. There are no problems computing the Ext-groups. Recall that

$$Ext^1_{U(o,p)}(V_1,V_2) = Der_k(U(o,p), Hom_k(V_1,V_2))/Triv,$$

and that $u \in U(o,p)$ operates on $\phi \in Hom_k(V_1,V_2)$, as,

$$(u\phi)(v_1) = u\phi(v_1), (\phi u)(v_1) = \phi(uv_1).$$

In the general case (one may test it in the interesting case, $o = (1,0,0), p = (0,0,0)$ above), we obtain,

$$Ext^1_{U(o,p)}(k_o,k_o) = Ext^1_{U(o,p)}(k_p,k_p) = Ext^1_{U(o,p)}(k_o,k_p) = 0, Ext^1_{U(o,p)}(k_p,k_o) = k^2.$$

Therefore,

$$\eta : U(o,p) \to \begin{pmatrix} k & 0 \\ < \xi_1, \xi_2 > & k \end{pmatrix}$$

is an isomorphism. Here $\xi_i \cdot 1 = \xi_i$. We may pick generators of this algebra,

$$x_1 := \begin{pmatrix} 0 & 0 \\ \xi_1 & 0 \end{pmatrix}, x_2 := \begin{pmatrix} 0 & 0 \\ \xi_2 & 0 \end{pmatrix}, x_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and obtain the relations corresponding to the choice of $o = (0,0,-1), p = (0,0,1)$. We have therefore obtained an algebraic subspace $\underline{\tilde{H}}$, of the miniversal base space $\mathbf{M}$ of the algebra U, corresponding to the algebras $U(o,p)$ that are all isomorphic. This subspace is therefore a *trivial section* of this miniversal base space.

Using the same technique as above, computing the deformations of one of these isomorphic algebras, we may show that the tangent space of $\mathbf{M}$ at such a point, has a 10 dimensional subspace not contained in the tangent space of $\underline{\tilde{H}}$. The significance of this is not obvious! If one wants to dream, one could consider this as a thickening of our universe, i.e. as a local extra structure. Moreover a dull calculation shows that at least 4 of these dimensions are obstructed at second order, providing $\underline{\tilde{H}}$ with a *hairy* infinitesimal structure!

4.3. **Spin, Isospin, and SUSY.** Consider the Lie algebra,

$$\mathfrak{g} := Der_H(\mathbf{U})$$

as a principal Lie algebra bundle on the space, $\underline{\tilde{H}}$. Any element $\delta \in Der_H(\mathbf{U})$ must be given by its values on the coordinates,

$$\delta(x_i) = \delta_i^0 + \delta_i^1 x_1 + \delta_i^2 x_2 + \delta_i^3 x_3,\ \delta_i^j \in H.$$

Now, let us define,

$$\tilde{\Theta}_{\tilde{H}} := \{\kappa \in End_{\tilde{H}}(\mathbf{U}), \delta(\mathbf{1}) = 0\}.$$

Oviously,

$$\mathfrak{g} \subset \tilde{\Theta}.$$

Any $\kappa \in \tilde{\Theta}_{\tilde{H}}$ will correspond to $\kappa_i := \kappa(x_i) \in U(o,p), i = 1,2,3$, i.e. to a matrix of the type,

$$M := \begin{pmatrix} 0 & 0 & 0 & 0 \\ \kappa_1^0 & \kappa_1^1 & \kappa_1^2 & \kappa_1^3 \\ \kappa_2^0 & \kappa_2^1 & \kappa_2^2 & \kappa_2^3 \\ \kappa_3^0 & \kappa_3^1 & \kappa_3^2 & \kappa_3^3 \end{pmatrix},$$

where, $\kappa_i := (\kappa_i^0, \kappa_i^1, \kappa_i^2, \kappa_i^3)$. Moreover, it is clear that $\tilde{\Theta}$ is a Lie algebra, and that $\mathfrak{g}$ is a natural sub-Lie algebra, of this matrix algebra.

Put,

$$\bar{o} = (1, o_1, o_2, o_3), \bar{p} = (1, p_1, p_2, p_3),$$

and consider now the 4-vectors,

$$\delta_i = (\delta_i^0, \delta_i^1, \delta_i^2, \delta_i^3), i = 1,2,3.$$

Suppose $\delta \in \mathfrak{g}$, then we find the formula, in $\mathbf{U}$,

$$\delta(x_i x_j - o_i x_j - p_j x_i + o_i p_j) = (\delta_i \cdot \bar{o})x_j - (\delta_i \cdot \bar{o})p_j + x_i(\delta_j \cdot \bar{p}) - o_i(\delta_j \cdot \bar{p})$$

which leads to,

$$\delta \in Der_H(\mathbf{U})$$

if and only if,

$$\delta_i \cdot \bar{o} = \delta_i \cdot \bar{p} = 0, i = 1,2,3.$$

Consider, from now on,

$$\mathbf{x}_{i,j} = x_i x_j - o_i x_j - p_j x_i + o_i p_j \in H < x_1, x_2, x_3 >=: \mathbf{F}$$

We find that if $\delta \in Der_H(\mathbf{U})$ then, in the free algebra $\mathbf{F}$, we have,

$$\delta(\mathbf{x}_{i,j}) = \sum_{p=1}^{3} \delta_i^p(\mathbf{x}_{p,j}) + \sum_{p=1}^{3} \delta_j^p(\mathbf{x}_{i,p}).$$

Let $\xi \in A^1(H, \mathbf{U}; \mathbf{U})$ be represented by $\xi(\mathbf{x}_{i,j}) = \xi_{i,j}^0 + \sum_{p=1}^3 \xi_{i,j}^p x_p$. Recall that ξ is zero if it is of the form,

$$\xi(\mathbf{x}_{i,j}) = -(\eta_i.\bar{o})p_j - o_i(\eta_j.\bar{p}) + (\eta_i.\bar{o})x_j + (\eta_j.\bar{p})x_i,$$

for some derivation $\eta \in Der_H(\tilde{F}, \mathbf{U})$.

Choose $\underline{t} = (o,p) \in \underline{H} - \underline{\Delta}$, and consider the relation of $U(\underline{t})$,

$$x_{i,j} := x_i x_j - o_i x_j - p_j x_i + o_i p_j,$$

Let us compute the Lie algebra $\mathfrak{g}(\underline{t}) := Der_k(U(\underline{t}))$. Any element $\delta \in Der_k(U(\underline{t}))$ must have the form,

$$\delta(x_i) = \delta_i^0 + \delta_i^1 x_1 + \delta_i^2 x_2 + \delta_i^3 x_3, \ \delta_i^p \in k.$$

Put, as above, $\bar{o} = (1, o_1, o_2, o_3), \bar{p} = (1, p_1, p_2, p_3)$, and consider the 4-vectors $\delta_i = (\delta_i^0, \delta_i^1, \delta_i^2, \delta_i^3), i = 1,2,3$.

As above, we find that,

$$\delta \in Der_k(U(\underline{t}))$$

if and only if,

$$\delta_i \cdot \bar{o} = \delta_i \cdot \bar{p} = 0, i = 1,2,3.$$

Moreover we find that if $\xi \in A^1(k, U(\underline{t}), U(\underline{t}))$ is represented by $\xi_{i,j} := \xi(x_{i,j}) = \xi_{i,j}^0 + \sum_{p=1}^3 \xi_{i,j}^p x_p$, it is easy to see that the action of any $\delta \in Der_k(U(\underline{t}))$ on the tangent space, $A^1(k, U(\underline{t}), U(\underline{t}))$, of the versal deformation space, of $U(\underline{t})$, is given as follows:

$$\delta(\xi)(x_{i,j}) := \xi(\delta(x_{i,j})) - \delta(\xi(x_{i,j})) = \sum_{p=1}^{3}\delta_i^p(\xi_{p,j}) + \sum_{p=1}^{3}\delta_j^p(\xi_{i,p}) - \delta(\xi_{i,j})$$
$$= \sum_{p=1}^{3}\delta_i^p\xi_{p,j}^0 + \sum_{p=1}^{3}\delta_j^p\xi_{i,p}^0 - \sum_{p=1}^{3}\delta_p^0\xi_{i,j}^p$$
$$+ \sum_{r=1}^{3}(\sum_{p=1}^{3}\delta_i^p\xi_{p,j}^q + \sum_{p=1}^{3}\delta_j^p\xi_{i,p}^q - \sum_{p=1}^{3}\delta_p^q\xi_{i,j}^p)x_q.$$

If we consider the particular interesting part of the tangent space, given by the $\xi \in A^1(k, U(\underline{t}), U(\underline{t}))$, of $U(\underline{t})$ represented by,

$$\xi(x_{i,j}) = \xi_{i,j}^0 + \xi_i x_j + \nu_j x_i,$$

then we find that the action above simplifies to,

$$\delta(\xi)(x_{i,j}) = \sum_{p=1}^{3}\delta_i^p\xi_{p,j}^0 + \sum_{p=1}^{3}\delta_j^p\xi_{i,p}^0 - \delta_i^0\xi_j - \delta_j^0\nu_i + \sum_{p=1}^{3}\delta_i^p\xi_p x_j + \sum_{p=1}^{3}\delta_j^p\nu_p x_i.$$

This translates into the following. Given a point $\underline{t} \in \underline{H}$, the tangent space is represented by the space of all pairs of 3-vectors, (ξ, ν) and the action of $\delta \in \mathfrak{g}(\underline{t})$ is $\delta(\xi, \nu) = (\delta(\xi), \delta(\nu))$, where in each coordinate, the action is,

$$\delta(\mu) = (\sum_{i=1,2,3}\delta_1^i\mu_i, \sum_{i=1,2,3}\delta_2^i\mu_i, \sum_{i=1,2,3}\delta_3^i\mu_i).$$

Even though, as we have seen, all tangents (ξ, ν) to $\underline{\tilde{H}}$ at a point $\underline{t} = (o, p)$ will be trivial in the tangent space of the formal moduli of $U(o, p)$, since the family of algebras $\mathbf{U}$ is constant, the above conclusion is correct. In fact, consider such a tangent, and the corresponding $k[\epsilon]$-algebra $U(o + \xi\epsilon, p + \nu\epsilon)$. There is necessarily an isomorphism,

$$\eta : U(o, p) \otimes k[\epsilon] \to U(o + \xi\epsilon, p + \nu\epsilon),$$

commuting with the projection onto $U(o, p)$. It must be given by formulas,

$$\eta(x_i) = x_i + \kappa(x_i)\epsilon,\ \kappa(x_i) = \kappa_i^0 + \kappa_i^1 x_1 + \kappa_i^2 x_2 + \kappa_i^3 x_3 \in U(o, p), i = 1, 2, 3.$$

Put $\kappa_i := (\kappa_i^0, \kappa_i^1, \kappa_i^2, \kappa_i^3)$, then,

$$\kappa \in \tilde{\Theta}_{\tilde{H}}.$$

A little computation now shows that we must have,

$$\xi_i = \kappa_i \cdot \overline{o},\ \nu_i = \kappa_i \cdot \overline{p}, i = 1, 2, 3.$$

therefore, Given a point $\underline{t} = (o, p)$, and the corresponding generators $\{x_i, i = 1, 2, 3\}$ of $\mathbf{U}$, any $\kappa \in \tilde{\Theta}_{\tilde{H}}$ will correspond to $\kappa_i := \kappa(x_i) \in U(o, p), i = 1, 2, 3$, and therefore to a tangent of $\underline{H}$ at the point $\underline{t} = (o, p)$,

$$(\xi = \overline{\kappa} \cdot \overline{o},\ \nu = \overline{\kappa} \cdot \overline{p}) \in T_{\underline{H}, \underline{t}}.$$

We therefore find an exact sequence of bundles on $\underline{\tilde{H}}$,

$$0 \to \mathfrak{g} \to \tilde{\Theta}_{\tilde{H}} \to \Theta_{\tilde{H}} \to 0.$$

The Lie algebra $\mathfrak{g}$, is now seen to operate naturally on $\tilde{\Theta}_{\tilde{H}}$, corresponding to exactly the operation above, drawn from the deformation theory of algebras. Any $\delta \in \mathfrak{g}$, operates on $\kappa \in \tilde{\Theta}_{\tilde{H}}$ as $\delta(\kappa) = \delta \cdot \kappa - \kappa \cdot \delta$. Since $\delta \cdot \overline{o} = \delta \cdot \overline{p} = 0$, we find

$$\delta(\xi, \nu) = (\delta(\xi), \delta(\nu)) := (\delta(\kappa)\overline{o}, \delta(\kappa)\overline{p}).$$

Observe also that, since $(\bar{o}-\bar{p}) = (o-p)$, the Lie algebra representation of $\mathfrak{g}$ on the tangent space $\Theta_{\tilde{H}}$, at the point $\underline{t} = (o,p)$, kills the subspace generated by the vectors $(\xi = \alpha(o-p), \nu = \beta(o-p))$.

If $o \neq p$, it follows that $\bar{o}$ and $\bar{p}$, are linearly independent, in a 4-dimensional vector space, therefore each vector $\delta_i, i = 1,2,3$ is confined to a 2-dimensional vector space. Consequently, $\mathfrak{g}(\underline{t}) := Der_k(U(\underline{t}))$ is of dimension 6. Using the isomorphism, $U(o,p) \simeq U(o-c, p-c)$, mentioned above, we may choose coordinates such that $o = (0,0,0), p = (1,0,0)$. In fact, we may first put $c = o$, and reduce to the situation where $o = 0$, and p is a non zero 3-vector. Any $\delta \in Der_k(U(o,p))$ will then be represented by a matrix of the form,

$$M := \begin{pmatrix} \delta_1^1 & \delta_1^2 & \delta_1^3 \\ \delta_2^1 & \delta_2^2 & \delta_2^3 \\ \delta_3^1 & \delta_3^2 & \delta_3^3 \end{pmatrix},$$

where $M(p) = 0$, and we know that the Lie structure is the ordinary matrix Lie-products. Now, clearly we may find a nonsingular matrix N such that $N(p) = (1,0,0)$, and the Lie algebra of matrices M, will be isomorphic to the Lie-algebra of the matrices, NMN^{-1}, which are those corresponding to $p = e_1 := (1,0,0)$, and we are working with $U(0, e_1)$. Notice that in this picture, the fundamental vector $\overline{op} = (1,0,0)$. With this it is easy to see that, $\delta \in \mathfrak{g}(\underline{t})$ imply,

$$\delta_i^0 = \delta_i^1 = 0, i = 1,2,3.$$

The following result is now easily seen.

Theorem 4.1. *The Lie algebra $\mathfrak{g}(\underline{t})$ is isomorphic to the Lie algebra of matrices of the form,*

$$\begin{pmatrix} 0 & \delta_1^2 & \delta_1^3 \\ 0 & \delta_2^2 & \delta_2^3 \\ 0 & \delta_3^2 & \delta_3^3 \end{pmatrix}$$

The radical $\mathfrak{r}$, is generated by 3 elements, $\{u, r_1, r_2\}$, with,

$$u = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, r_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, r_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

where $u \notin [\mathfrak{g}, \mathfrak{g}]$, $[u, r_i] = -r_i, [r_1, r_2] = 0$, and the quotient,

$$\mathfrak{g}(\underline{t})/\mathfrak{r} = \mathfrak{sl}(2).$$

with the usual generators u_0, u_1, u_2,

$$u_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, u_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, u_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}.$$

In particular, we find that $\mathfrak{sl}(2) \subset \mathfrak{g}(\underline{t})$.

Remark 4.2. *At this point, it may be useful to refer to the examples and the explicit calculations,coming up in subsections (5.3) and (5.4). Given a point $\underline{t} \in \underline{H}$, the tangent space at this point is, of course, nicely represented by the space of all pairs of 3-vectors, (ξ, μ) and, as we have seen, it is easy to compute the action of $\mathfrak{g}(\underline{t})$ on this 6-dimensional vector space. Just as in the case of the action of $Der_k(U)$ on $\underline{H}$, any $\delta \in \mathfrak{g}(\underline{t})$ with $\delta_i^0 = 0$, $i = 1,2,3$, acts as $\delta(\xi, \mu) = (\delta(\xi), \delta(\mu))$, and in each coordinate, the action is that of the matrix algebra above.*

The Lie algebra $\mathfrak{sl}_2(\underline{t})$ acts as follows. There are natural 3-dimensional subundles Θ_o, Θ_p of the tangent bundle of $\underline{H}' := \underline{H} - \underline{\Delta}$, such that $\Theta_{\underline{H}'} = \Theta_o \oplus \Theta_p$. We may find a natural basis for both components, for Θ_o as well as for Θ_p, $\{\mathfrak{l}, \nu_1, \nu_2\}$, where $\mathfrak{l}$ is the special tangent vector given by $\overline{po}$, i.e. the tangent direction in our Euclidean

3-space, in which we are looking. It is obvious from the above matrix-bases of $\mathfrak{g}(\underline{t})$, *that* $\mathfrak{g}(\underline{t})$ *kills* $\mathfrak{l}$. *Therefore, assuming some metric given, in this basis,* $\mathfrak{sl}(2)$ *acts on the planes normal to* $\mathfrak{l} = \overline{po}$. *As a consequence, if we pick any line* $\mathbf{l} \subset \mathbf{A}^3$, *then the tangent space of* $\underline{H}(\mathbf{l})$ *is killed by* $\mathfrak{g}(\underline{t})$, *for all* $\underline{t} \in \underline{H}(\mathbf{l})$. *We have therefore seen that for any point* $\underline{t} \in \underline{\tilde{H}}$ *the* $\mathfrak{sl}(2)$ *component of the Lie algebra of infinitesimal automorphisms of the universal algebra* $U(\underline{t})$, *act on* $\Theta_{\underline{\tilde{H}}}$ *in a particular nice way. The generators* u_0, u_1, u_2, *(the generators of* $\mathfrak{sl}(2)$, *normally denoted* h, e, f*), acting on sections of the sub bundle* $B_o \oplus B_p$ *of the tangent bundle* $\Theta_{\underline{\tilde{H}}}$, *just as we described, geometrically, in the Introduction and in* [16] *and* [18]. *This* $\mathfrak{sl}(2)$*-action takes, as we shall see later on, care of the Isospin and the Electromagnetic Forces, and the action of* $\mathfrak{r}$ *corresponds to the Electroweak sector, the generators* $\{u, r_1, r_2\}$ *corresponding to the Bosons* $\{Z, W_+, W_-\}$, *responsible for the Electroweak Forces, acting on the quarks, see section 8.*

Now, as in [16], consider the Kodaira-Spencer map of the family,

$$\nu : \mathbf{U} \longrightarrow \underline{\tilde{H}},$$

It is the linear map,

$$\eta : \Theta_H = Der_k(\tilde{H}, \tilde{H}) \to A^1(H, \mathbf{U}, \mathbf{U}),$$

defined by,

$$\eta(\frac{\partial}{\partial o_i}) = \{\mathbf{x}_{i,j} = (x_i x_j - o_i x_j - p_j x_i + o_i p_j) \mapsto \frac{\partial}{\partial o_i}(r_{i,j}) = -x_j + p_j\}$$
$$\eta(\frac{\partial}{\partial p_j}) = \{\mathbf{x}_{i,j} = (x_i x_j - o_i x_j - p_j x_i + o_i p_j) \mapsto \frac{\partial}{\partial p_i}(r_{i,j}) = -x_i + o_i\}.$$

Recall that,

$$A^1(H, \mathbf{U}, \mathbf{U}) = Hom_{H,H}(\tilde{I}, \mathbf{U})/Der$$

where we have picked a surjection ρ of a free H-algebra $\tilde{F}$ onto $\mathbf{U}$ and $\tilde{I} = ker(\rho)$. Der is then the sub module of $Hom_{H,H}(\tilde{I}, \mathbf{U})$ generated by the elements of $Der_H(\tilde{F}, \mathbf{U})$, restricted to $\tilde{I}$. In our case $\tilde{I}$ is generated by $(\mathbf{x}_{i,j} := x_i x_j - o_i x_j - p_j x_i + o_i p_j), i, j = 1, 2, 3$. Therefore, since any derivation $\delta \in Der_H(\tilde{F}, \mathbf{U})$ has the form,

$$\eta(x_i) = \eta_i^0 + \eta_i^1 x_1 + \eta_i^2 x_2 + \eta_i^3 x_3,\ \eta_i^l \in H, i = 1, 2, 3, l = 0, 1, 2, 3,$$

we find that,

$$\eta(r_{i,j}) = \eta(x_i)x_j + x_i\eta(x_j) - o_i\eta(x_j) - \eta(x_i)p_j = (\eta_j \cdot \bar{o})(x_i - o_i) + (\eta_i \cdot \bar{o})(x_j - p_j).$$

Since we have seen that all $\mathfrak{g}(\underline{t})$, $\underline{t} \in \underline{\tilde{H}}$ are isomorphic, it is reasonable that the Kodaira-Spencer map η vanishes on $\underline{\tilde{H}}$, implying,

Lemma 4.3. *The kernel of the Kodaira-Spencer map, also called the Gauss-Manin-kernel of the family* ν, *is,*

$$\tilde{\mathbf{G}} := ker(\eta) = \{\xi = \sum_{i=1}^{3}(\eta_i \cdot \bar{o})\frac{\partial}{\partial o_i} + \sum_{j=1}^{3}(\eta_i \cdot \bar{p})\frac{\partial}{\partial p_j} \mid \eta \in Der_H(\tilde{F}, \mathbf{U})\} \simeq \Theta_{\underline{\tilde{H}}}.$$

This is equivalent to $\mathfrak{g}_{\mathfrak{g}} = \Theta_H$.

As we shall see later, we would like to have an isomorphism,

$$\Theta_{\underline{\tilde{H}}} \to Der_{\tilde{H}}(\mathfrak{g}) \simeq \mathfrak{g}$$

Notice that, so far, we have not been forced to choose an origin in our affine space $\mathbf{A}^3$. All results regarding the vectors $o, p, \bar{o}, \bar{p}$, have been independent upon this choice. However, for the next Theorem we shall have to fix an origin. This may seem strange, but as it will be explained in the next section, this is a natural consequence of the introduction of a Big Bang event, the singularity U, and its versal base space. Notice, that this choice of origin in the affine space $\mathbf{A}^3$, has a consequence for our definition of the cosmological 0-velocity sub-bundle, $\tilde{\Delta}$, that we shall have to come back to. Compare also the next result with the discussion of Bosonic and Fermionic representations, in section (6).

Theorem 4.4. *There is a well defined $\tilde{H}$-linear map,*

$$\kappa : \tilde{\mathfrak{g}} \to \tilde{\Delta} \subset \Theta_{\underline{\tilde{H}}},$$

defined by,

$$\kappa(\delta) = \sum_{i=1}^{3}((\delta_i \cdot o)\frac{\partial}{\partial o_i} + (\delta_i \cdot p)\frac{\partial}{\partial p_i}) = -\sum_{i=1}^{3}\delta_i^0(\frac{\partial}{\partial o_i} + \frac{\partial}{\partial p_i}).$$

where, $\delta_i \cdot o = -\delta_i^0 = \delta_i \cdot p$.

Assume the vectors $o := (o_1, o_2, o_3), p := (p_1, p_2, p_3)$ are linearly independent, and let $\alpha := (\alpha_1, \alpha_2, \alpha_3) = o \times p$. Then,

$$\tilde{\mathfrak{k}} := ker(\kappa).$$

is a rank 3 $\tilde{H}$- sub Lie algebra of $\tilde{\mathfrak{g}}$ generated by elements $\{u, v, w\}$, with Lie structure,

$$[u, v] = -\alpha_2 u + \alpha_1 v,\ \ [u, w] = -\alpha_3 u + \alpha_1 w,\ \ [v, w] = -\alpha_3 v + \alpha_2 w,$$

which, at each point $\underline{t} \in \underline{\tilde{H}}$, is isomorphic to $rad(\mathfrak{g}(\underline{t})) \subset \mathfrak{g}(\underline{t})$.

If we for any $\delta \in \tilde{\mathfrak{g}}$, define,

$$\tilde{\delta} := \delta + \kappa(\delta),$$

then, applying the adjoint (regular) representation of $\mathfrak{g}$, we have the following,

$$\begin{aligned} &\tilde{\delta} \in End_k(\tilde{\mathfrak{g}}) \\ &\kappa(\tilde{\delta}) = \kappa(\delta) \\ &\kappa([\tilde{\delta}, \tilde{\eta}] - \widetilde{[\delta, \eta]}) = [\kappa(\delta), \kappa(\eta)]. \end{aligned}$$

Finally κ defines an isomorphism of $\tilde{H}$-modules,

$$\tilde{\kappa} : \tilde{\mathfrak{g}}/\tilde{\mathfrak{k}} \to \tilde{\Delta}.$$

where $\mathfrak{g}/\mathfrak{k} \simeq \mathfrak{sl}(2)$, and we may identify the base-elements,

$$\tilde{\kappa}(h) = d_3,\ \ \tilde{\kappa}(e) = d_1,\ \ \tilde{\kappa}(f) = d_2.$$

Proof. Put,

$$\begin{aligned} \bar{\delta}_i &: = (\delta_i^0, \delta_i^1, \delta_i^2, \delta_i^3) \\ \bar{o} &: = (1, o_1, o_2, o_3) \\ \bar{p} &: = (1, p_1, p_2, p_3). \end{aligned}$$

Let $\delta \in Der$, and compute, as above,

$$\delta(r_{i,j}) = \delta(x_i)x_j + x_i\delta(x_j) - o_i\delta(x_j) - \delta(x_i)p_j = (\bar{o}\cdot\bar{\delta}_i)x_j + (\bar{p}\cdot\bar{\delta}_j)x_i - (\overline{o}\cdot\bar{\delta}_i)p_j - (\bar{p}\cdot\bar{\delta}_j)o_i.$$

From this we learn two things. First we see that $\delta \in Der_H(\mathbf{U}, \mathbf{U})$, if and only if, $\bar{o} \cdot \bar{\delta}_i = 0,\ \bar{p} \cdot \bar{\delta}_i = 0$, for all $i = 1, 2, 3$, which we already knew, second, we see that the definition of κ is well defined. The computation of $[\tilde{\delta}, \tilde{\eta}]$, and its image by κ, is left as an exercise.

But then we see that $\tilde{\mathfrak{k}} := ker(\kappa) = \{\delta \in Der_H(\mathbf{U}, \mathbf{U}) | \delta_i^0 = 0,\ i = 1, 2, 3\}$, is a sub Lie algebra, though not necessarily a Lie ideal. By definition of $\tilde{\mathfrak{k}}$, if $\delta \in \tilde{\mathfrak{k}}$, we must have $\delta_i^0 = 0$, for all $i = 1, 2, 3$, and so also,

$$o \cdot \delta_i = 0,\ \ p \cdot \delta_i = 0, \forall i = 1, 2, 3.$$

It follows that,

$$(\delta_i^1, \delta_i^2, \delta_i^3) = c_i(\alpha_1, \alpha_2, \alpha_3),\ \ c_i \in H.$$

If δ, δ' corresponds to the two vectors $c := (c_1, c_2, c_3)$ and $c' := (c'_1, c'_2, c'_3)$ then $[\delta, \delta']$ is seen to correspond to the vector, $c \times c' \times \alpha$, from where the structural constants may be read off. Moreover, since the determinant of the corresponding matrix,

$$\begin{pmatrix} -\alpha_2 & \alpha_1 & 0 \\ -\alpha_3 & 0 & \alpha_1 \\ 0 & -\alpha_3 & \alpha_2 \end{pmatrix}$$

is zero, it is clear that $[\mathfrak{k}, \mathfrak{k}]$, is of dimension 2, and abelian. Since $\mathfrak{k}$ is solvable, and obviously maximal, the contention follows.

Notice that the map κ, obviously is not a Lie algebra morphism. Never the less the kernel is an $\tilde{H}$-sub–Lie algebra, identified, at every point $\underline{t} \in \underline{\tilde{H}}$, with the radical of $\mathfrak{g}(\underline{t})$. Therefore we may, for every $\delta \in \mathfrak{g}$ talk about its class, modulo $\mathfrak{k}$, i.e. in $\mathfrak{sl}(2)$. The last contention therefore follows from the surjectivity of κ. □

Now, recall that $\tilde{\Delta} \subset \Theta_{\tilde{H}}$, is the subbundle defined, at the point $\underline{t}$, as the set of tangents of the form (ξ, ξ). Given a metric on $\underline{\tilde{H}}$, then $\mathfrak{so}(3)$ acts on $\tilde{\Delta}$, as a gauge group. We may also look at the action of $\mathfrak{su}(3)$ on the complexified tangent fields, $\tilde{\Delta} \otimes \mathcal{C}$. It acts in the obvious way on the lower right corner of $\Theta = \tilde{c} \oplus \tilde{\Delta}$, like

$$\gamma = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & * & * & * \end{pmatrix}.$$

Since the 0-velocity direction defined at (o, p), is $(o - p, o - p)$, which here is d_3, we may in an essential unique way decompose the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{su}(3)$, into the Cartan subalgebra $\mathfrak{h}_1$, for the $\mathfrak{su}(2)$- component leaving δ_3 invariant, and the part $\mathfrak{h}_2 \subset \mathfrak{h}$ perpendicular, in the Killing metric, to $\mathfrak{h}_1$. They act as,

$$\mathfrak{h}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},\ \mathfrak{h}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2/3 \end{pmatrix},$$

in the basis $\{c_1, c_2, c_3, d_1, d_2, d_3\}$. Recall also that $\mathfrak{so}(3) \simeq \mathfrak{su}(2)$

See [18]. It is clear that this, together with the formulas above give good reasons to believe that there is a relation to the Standard Model. Moreover, here all ingredients are universally given by the information contained in the singularity U, the Big Bang, in my tapping. The choice of metric, i.e. time and so gravitation, will have to be made on the basis of the nature of what I have called the Furniture of the model, see above, at the end of Subsection (2.3), where the simple metric proposed for the purpose of comparing our gravitation laws with the theory of Kepler and Newton, seems to fit with the present theory of black holes.

This result is going to have several important consequence, in particular for the derivation of a general analogy of the Dirac equation, and for the construction, in our case, of a kind of supersymmetry. But first, let us consider the *singular subscheme* of the morphism κ above, i.e. the subspace $\underline{M}(B)$ of $\underline{\tilde{H}}$, where o and p are not linearly independent. It turns out to be a 4-dimensional subspace of $\underline{\tilde{H}}$, fibered over the exceptional fiber $E(\underline{0})$, associated to the Big Bang, $\underline{0} \in \underline{\Delta}$. We have the following diagram,

$$\begin{array}{ccccccc} \underline{M}(B) & \xrightarrow{\mu} & E(\underline{0}) \hookrightarrow & & \underline{\tilde{\Delta}} \hookrightarrow & & \underline{\tilde{H}} \\ & & \downarrow & & \downarrow & & \downarrow \\ & & \{\underline{0}\} \hookrightarrow & & \underline{\Delta} \hookrightarrow & & \underline{H} \end{array}$$

Where μ is a map that for all directed lines $l \subset \mathbf{A}^3$, through the Big Bang, i. e. such that, $\underline{0} \in l$, maps the subspace $\underline{\tilde{H}}(l) \subset \underline{\tilde{H}}$ into the point of $E(\underline{0})$ corresponding to l.

Remark 4.5. *$\underline{M}(B)$ looks like the product, $\underline{c}(\underline{0}) \times \mathbf{R}$, where $\underline{c}(\underline{0})$ is the light space of the Big Bang, and where the $\mathbf{R}$-coordinate is the unique zero velocity coordinate of $\underline{\tilde{H}}(l)$, which we have called λ_0. If we make a picture of this, with λ pointing upwards, we have, as we shall see better at the end of the next section, the usual cosmological 4-dimensional space time picture, with the not so unimportant difference that the usual time coordinate, pointing upwards, is replaced by proper time.*

Recall now, for later use, that if we start with $U = U(0,0)$, the Big Bang, we observe that the Dirac derivation, $\delta = id$ acts on the subspace, T_0 of its tangent space, $T_* = A^1(k, U, U)$, corresponding to the creation of $\underline{\tilde{H}}$, like the vector field,

$$[\delta] := \sum_{i=1,2,3} o_i \frac{\partial}{\partial o_i} + \sum_{j=1,2,3} p_j \frac{\partial}{\partial p_j}.$$

Evaluated at a point $\underline{t} = (o, p)$, we see that,

$$\delta(o,p) = (-1/2(\overline{op}), 1/2(\overline{op})) + (1/2(o+p), 1/2(o+p)).$$

The last tangent is in $\tilde{\Delta}$, i.e. it is a 0-velocity, and the only one contained in $M(B)$. This statement is, however, not entirely kosher, since the notion of $\tilde{\Delta}$ in this case where we have fixed a point in $\underline{\Delta}$, is not well defined. One might say that standing still in $M(B)$ means to be carried along by the cosmic stream defined by the Dirac derivation δ of U. The first part of δ is a classical light-direction, and represents the extension of the visible Universe. Moreover, any point $(o,p) \in M(B)$, considered as an observer o observing an observed p, does so in the direction of the Big Bang event.

Having shown that the gauge Lie algebra of the Standard Model (SM), operates naturally on the non-commutative space $Ph(\mathbf{H})$, and that there is defined on $\underline{\tilde{H}}$, a canonical principal bundle,

$$\mathfrak{g} := Der_H(\mathbf{U}),$$

we have now a good measure of the ingredients of the SM. In fact, we see that the choice of a metric g, defines a complex structure on $\Theta_{\mathbf{H}}$. Moreover as we have seen that $su(2)$, and also complexified $sl(2)$, acts naturally on complexified $B_o \oplus B_p$, and that $su(3)$ acts on complexified $\tilde{\Delta}$. Going back to (1.8), we see now that we have available most of the ingredients of a *canonical Yang-Mills Theory*, defined on $\mathbf{H}$.

It is therefore tempting to propose that the SM, itself, is concerned with the geometry of the non-commutative quotient scheme,

$$\tilde{H}(\sigma_g)/\mathfrak{G}, with\ \mathfrak{G} := \mathfrak{g}_0 \oplus \mathfrak{g}_1,\ \mathfrak{g}_1 =\subset \mathfrak{g} \oplus \mathfrak{su}(3).$$

Put,

$$\mathbf{GQR} := Simp(\tilde{H}(\sigma_g), \mathfrak{G}).$$

Taking this as a model for a combined GR and SM, let us make sure that the main ingredients of SM are available here.

First of all, we have the necessary *Gauge Fields*, since we have the principal bundles,

(1) $$\mathfrak{u}(1) \simeq B_o \simeq B_p$$

(2) $$\mathfrak{g} \simeq Der_{\tilde{H}}(\mathbf{U})$$

(3) $$\mathfrak{s}u(3) := \mathfrak{su}_{\tilde{H}}(\tilde{\Delta}_{\mathbf{C}}),$$

where $\tilde{\Delta}_{\mathbf{C}} := \tilde{\Delta} \otimes \mathbf{C}$. Since we have shown that we may operate with any field k, and certainly go from the reals $\mathbf{R}$ to the complexes $\mathbf{C}$, without any mathematical problems, we shall normally omit the $\mathbf{C}$ in $\tilde{\Delta}_{\mathbf{C}}$.

The canonical action of these principal bundles on the complexified tangent bundle of $\mathbf{H}$, give us a lot of possible *force and matter fields.* Moreover, we have the $\tilde{H}$-linear isomorphism,

(4) $$\kappa : \tilde{\mathfrak{s}l}(2) \simeq \mathfrak{g}/\mathfrak{k} \simeq \tilde{\Delta},$$

which defines the two obvious $\tilde{H}$-endomorphisms,

(5) $$Q_i \in End_{\tilde{H}}(\tilde{\mathfrak{s}l}(2) \oplus \tilde{\Delta}),\ i = 1, 2,$$

corresponding to $(\kappa, 0)$, respectively $(0, \kappa^{-1})$, such that,

(6) $$\{Q_1, Q_2\} = id,\ [Q_i, P_\mu] = 0,$$

where P_μ, is *the infinitesimal translation operator*, in the physicists notation.

We have got a graded Lie algebra,

$$\mathfrak{G}* \subset End_{\tilde{H}}(\tilde{\mathfrak{s}l}(2) \oplus \tilde{\Delta})$$

generated by the complexified adjoint operations of $\mathfrak{sl}(2)$ and $\mathfrak{s}u(3)$, together with $Q_i,\ i = 1, 2$. An element is *even*, or *odd*, according to whether it contains an even or odd number of factors of the type Q_i. Even operators, takes "Bosonic states", $\mathfrak{s}l(2)$, into bosonic states, and also "Fermionic states", $\tilde{\Delta}$, into fermionic states. And, obviously, odd operators takes bosons into fermions, and vice versa. With this interpretation, our model has aquired an $N = 1$, SUSY-like structure, defined in $\underline{\tilde{H}}$ outside of $\underline{M}(B)$. On this singular subset, the symmetry is "broken".

But, of course, in quantum field theory, Bosons and Fermions are observables of type a^+ or a, having the correct "statistics", i.e. being eigen-operators of $ad(Q)$, where Q is our Hamiltonian, the Laplace-Beltrami operator on $\Theta_{\underline{\tilde{H}}}$, the least eigenvalue of which is the Planck's constant. See the section (6.3), and also [18], (4.6), p.70.

From this point of view, Bosons are observables corresponding to even elements in $\mathfrak{G}*$, and Fermions are observables corresponding to odd elements.

We have in the [18] discussed the notions of Chirality, the PST invariance, stemming from the Hilbert scheme structure of $\mathbf{H} = \underline{\tilde{H}}/Z_2$, and Spinors, with the action on the tangent bundle, of two copies of $\mathfrak{s}l(2)$, together with $\mathfrak{s}u(3)$. We saw how the charges of the up and down quarks where defined by the split form of the Cartan sub-algebras $\mathfrak{h}_1 \times \mathfrak{h}_2$ of the the Lie algerbras $\mathfrak{s}u(2) \subset \mathfrak{s}u(3)$, respectively, canonically defined at any point in $\underline{\tilde{H}}$, see (6.4), and further remarks in the End Words.

Here we have made all this a unique consequence of the Big Bang event, mathematically played by the versal family of associative noncommutative 4-dimensional k-algebras, deforming the algebra $U := k[x_1, x_2, x_3]/(x_1, x_2, x_3)^2$. As we have seen, U (and thus also $\mathbf{M}$) contains a lot of information, in its 9-dimensional Lie algebra of derivations, although it is just modelling a single point, together with a 3-dimensional tangent space.

We are tempted to express the content of this subsection, by saying that a substantiel part of SM, including its spin structure and a canonical SUSY structure, turns out to be an immediate consequence of a Big Bang scenario.

Nothing less.

4.4. **The Universe as a Versal Base Space.** So, where was the Big Bang, in relation to our Time-Space, and what on earth is the meaning of the terms; cosmological time, expansion of the universe, red-shift? How can one fill into this geometric picture the more down to earth notions like; matter, stress, pressure, charge, and forces, like; gravitation, electromagnetism, weak and strong forces, acting on; elementary particles, quarks and their multiple combinations?

We should not have to goose-feed the Big Bang-created geometric picture, with this additional structure. It should all be part of the Creation! Otherwise it must be difficult to believe in the existence of this prime event.

Going back to the constructed family, the universal family of the Hilbert scheme of sub-schemes of length 2 in $\mathbf{A}^3$,

$$\pi : \mathbf{E} \longrightarrow \mathbf{H},$$

we have just proved that this family may be complemented with another family, no longer a universal one, but just part of a versal family,

$$\nu : \mathbf{U} \longrightarrow \mathbf{H},$$

of 4-dimensional associative algebras. The 3-dimensional space $\underline{\Delta}$ is not a subspace of $\mathbf{H}$, in fact, any point of this ghost space correspond to the same 4-dimensional algebra, namely to U, the Big Bang (BB) itself. A metric defined on $\underline{\Delta}$ therefore measures time at BB, before the creation of the universe, when "God did nothing", see St. Augustin [1]!

So let us fix a point $* \in \underline{\Delta}$, the origin of the coordinate system (x_1, x_2, x_3), used to define U, thereby fixing the whereabouts of BB, clearly outside of our universe, even though time is already there, as the metric in $\underline{\Delta}$, measuring 0-velocities of U.

Observe that the component $\tilde{\Delta} \subset \Theta_{\tilde{H}}$, of the canonical 0-velocity momenta, is no longer uniquely defined, since the action of the additive group k^3 on $\underline{\tilde{H}}$ creating $\tilde{\Delta}$, do not keep the $* \in \underline{\Delta}$ fixed.

However, as we have seen, $\mathfrak{g}(*) = Der_k(U) = \mathfrak{g}l_3(k)$ acts on the tangent space of the versal base space, T_*, and in fact on the sub space T_0 identified with $\underline{H}$, creating a very special derivation, again the *Dirac Derivation, (maybe a model for Black Energy) in this situation,*

$$\delta \in \mathfrak{g}(*), \delta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

the unit element. We have seen above, that δ acts on $\underline{H}$ as,

$$\delta := \sum_{i=1,2,3} o_i \frac{\partial}{\partial o_i} + \sum_{j=1,2,3} p_j \frac{\partial}{\partial p_j},$$

so this operation will stretch all vectors in $\underline{\Delta}$ proportional to their length (in the Euclidean metric). The value of δ at a point (o,p) is the tangent,

$$\overline{(o,p)} = (1/2(\overline{op}), -1/2(\overline{op})) + (1/2((o+p), 1/2((o+p)).$$

The first summand,

$$(1/2(\overline{op}), -1/2(\overline{op})) = \rho \frac{\partial}{\partial \rho}$$

sits in $\tilde{c}$ and the last,

$$(1/2((o+p),1/2((o+p)) = \lambda\frac{\partial}{\partial\lambda}$$

is in the subspace we have called $\tilde{\Delta}$, i.e. it is a relative "0-velocity", and

$$\delta = \rho\frac{\partial}{\partial\rho} + \lambda\frac{\partial}{\partial\lambda}$$

in $M(B)$. IIntroduce polar coordinates in $\underline{\Delta}$, with $\lambda = |\underline{\lambda}|, \alpha_1, \alpha_2$, and introduce ,

$$\tilde{\Delta}_u = \{\delta, \frac{\partial}{\partial\alpha_1}, \frac{\partial}{\partial\alpha_2}\}.$$

It is going to be our general 0-velocity tangent space, in cosmology, replacing $\tilde{\Delta}$.

Now, as a start, assume we concentrate on the first question of this sub-section, and assume the metric is the trivial one, so that mass, stress, and charge etc. can be neglected.

Then we find that the velocity associated to the direction of the tangent vector $(\rho o, \rho p)$ at $\underline{t}$ is given as $v = sin(\theta)$, where,

$$tg(\theta) = |1/2(\rho\cdot\overline{op}), -1/2(\rho\cdot\overline{op}))|/|1/2(\rho\cdot(o+p), 1/2(\rho\cdot(o+p))|.$$

From this we deduce two versions of the Hubble formula,

$$v = |\overline{op}|/\sqrt{|\overline{op}|^2 + |(o+p)|^2} = r/t,$$

and,

$$v/\sqrt{1-v^2} = |\overline{op}|/|(o+p)| = r/T,$$

where T is cosmological time, and t is "real time" since the BB. In fact, $r = \rho = 1/2|\overline{op}|/$, $T = \lambda = 1/2|(o+p)|$, and $t = 1/2\sqrt{|\overline{op}|^2 + |(o+p)|^2}$ is the distance in $\underline{H}$ from $*$ to (o,p) in the Euclidean metric. Of course, this is purely formal, since t is the distance covered by a point in our space left alone, carried away by the intrinsic expansion of space. The term r/T in the last formula is, in an obvious sense, the speed of the expansion of the Universe, with respect to cosmological time. It is seen to approach infinity when the real speed of the expansion v comes close to maximum, 1. This lead us to think about the inflation-scenario, more or less accepted in cosmology, and we shall have reasons to return to the problem, shortly.

Now we need to introduce a metric that would fit with the one we introduced in section (2), inducing a Dirac operator $[\delta]$, defining the gravitation of Cosmos. To be able to say something non-nonsensical about the choice of this metric, one should have to guess about a content of the Universe, about our furniture, call it ξ, and deduce the metric from the equation,

$$\frac{d\xi}{dt} = Q(\xi),$$

see (2.3). This seems to be what cosmologists are trying out, and I shall, hopefully, be able to return to the question at a later time. There are several nontrivial problems to ponder. The light-component of the furniture propagates, of course, in time but also in cosmological time. It looks like the propagation of a light wave, ψ, which, due to the structure imposed on our "toy model", by the choice of a Big Bang event, now is characterised by propagating perpendicular to, $\tilde{\Delta}_u$, and so perpendicular to δ.

This will have an important consequence on the metric, and the history of our "Present Universe".

At this point we shall, as an example, try out the metric,

$$(\frac{\rho - h(\lambda)}{\rho})^2 d\rho^2 + (\rho - h(\lambda))^2 d\phi^2 + \kappa(\lambda)d\lambda^2,$$

introduced in section 2. for the simplified space, in which $\underline{\omega}$ is reduced to the angle ϕ, and the coordinates $\underline{\lambda}$ reduced to one parameter $\lambda = |\underline{\lambda}|$. This corresponds to reducing our universe to the sub-universe of $M(B)$, parametrized by (λ, ϕ, ρ). And the decomposition $\Theta = \tilde{c} \oplus \Delta$, is now replaced by, $\Theta = \tilde{c}_u \oplus \tilde{\delta}$, where $\tilde{\delta} =< \delta > \subset \tilde{\Delta}_u$, and where, the summands should be orthogonal in the above metric.

Now, choose $h(\lambda) = h/\lambda$, and $\kappa(\lambda) = 1$. An easy calculation shows that, $\xi = \frac{\partial}{\partial \rho} - (\rho/\lambda)(1-(h/\rho\lambda))^2 \frac{\partial}{\partial \lambda}$ is orthogonal to δ, and we may choose,

$$\tilde{c}_u =< \frac{\partial}{\partial \rho} - (\rho/\lambda)(1-(h/\rho\lambda))^2 \frac{\partial}{\partial \lambda}, \frac{\partial}{\partial \phi} > .$$

At a point of the horizon we observe that the light directions are given by,

$$\tilde{c}_u =< \frac{\partial}{\partial \rho}, \frac{\partial}{\partial \phi} > .$$

just as in the Kepler-Newton case. Moreover an integral curve of ξ will never intersect the horizon for $\lambda > 0$. As a consequence we might say that an observer observing an observed not too far away, would do it in the same way as within the Kepler-Newton space. However, he/she will receive light signals from all the way back to the infinite horizon of the Big Bang.

We have already computed the Force Laws, and they look like,

$$\frac{d^2\rho}{dt^2} = -(\frac{h(\lambda)}{\rho(\rho-h(\lambda))})(\frac{d\rho}{dt})^2 + (\frac{2}{(\rho-h(\lambda))})(\frac{dh}{d\lambda})(\frac{d\rho}{dt})(\frac{d\lambda}{dt}) + (\frac{\rho^2}{(\rho-h(\lambda))})(\frac{d\phi}{dt})^2,$$

$$\frac{d^2\phi}{dt^2} = -2/(\rho-h(\lambda))\frac{d\rho}{dt}\frac{d\phi}{dt} + 2/(\rho-h(\lambda))(\frac{dh}{d\lambda})(\frac{d\phi}{dt})(\frac{d\lambda}{dt}),$$

$$\frac{d^2\lambda}{dt^2} = -(\frac{(\rho-h(\lambda)}{\rho})(\frac{1}{\kappa(\lambda)})(\frac{dh}{d\lambda})(\frac{d\rho}{dt})^2 - (\rho-h(\lambda))(\frac{1}{\kappa(\lambda)})(\frac{dh}{d\lambda})(\frac{d\phi}{dt})^2$$
$$+1/2(\frac{dln(\kappa)}{d\lambda})(\frac{d\lambda}{dt})^2,$$

where t, as above, is time.

In the general $\underline{\tilde{H}}$-case, we put $h(\underline{\lambda}) = h_0/\lambda$, with h_0 positive, then the area of the exceptional fibre, $E(\underline{\lambda})$, is $4\pi h(\lambda)^2$. The integral of this area over any sphere $S(\lambda) \subset \underline{\Delta}$ with center at BB, comes out as $16\pi^2 h_0^2$. So considering λ as the cosmological time, and the area of the Black hole $E(\underline{\lambda})$, as the (gravitational) mass-density, this corresponds to the conservation of (gravitational) mass in our universe, with respect to cosmological time. Of course, looking back in cosmological time, the gravitational mass-density h_0/λ increases, so it may seem like the universe contains more mass than we can, optically, account for.

In the simplified "universe" above, the corresponding constant mass is, of course, $2\pi h_0$. We observe that $(\frac{dh}{d\lambda})$ is always negative. This means that for $\rho \leq h(\lambda)$ the acceleration of ρ is positive, and unlimited close to the Horizon, i.e. for ρ close to $h(\lambda)$. In the same region, assuming that $\kappa(\lambda)$ is constant, the acceleration of λ is negative, vanishing on the Horizon. In particular we see that gravitation is an expanding force inside the Horizon, and a contracting one outside, giving ideas about inflation in cosmology. Moreover $h(0)$ is infinite, and we have found a startling analogy to the present day assumption of Inflation. Time is 0 on the Horizon, so creation of photons at the Horizon of the BB, the assumed origin of the Cosmic Microwave Radiation (CMR) (that reaches us from "everywhere", which in $M(B)$ means from BB, or really from $E(*)$), "happens at the same time". This might be the reason why CMR seems to be in coherent phase, an argument for Inflation brought forward by, among others, Dodelson (arXiv.sep 03).

In the general case, by definition, $\tilde{c}_u$ *and* $\tilde{\Delta}_u$ *are orthogonal. We would , however, have liked to find coordinates* t_i *such that* $\tilde{c} =< \frac{\partial}{\partial t_1}, \frac{\partial}{\partial t_2}, \frac{\partial}{\partial t_3} >$*, and* $\tilde{\Delta} =< \frac{\partial}{\partial t_4}, \frac{\partial}{\partial t_5}, \frac{\partial}{\partial t_6} >$*, such that, for the corresponding metric,* $g = \sum g_{p,q} dt_p dt_q$*, we have,*

$$\frac{\partial}{\partial t_i}(g^{j,q}) = \frac{\partial}{\partial t_j}(g^{i,p}) = 0, \ \forall p, q, 1 \le i \le 3, 4 \le j \le 6.$$

Then we would have called $\tilde{c}$ *and* $\tilde{\Delta}$*, normal, or normally orthogonal, a notion we shall come back to in section (6), in relation to the general Dirac equation.*

We shall leave the situation here. The rest of the story depends on the usefulness of the Furniture Equation, to which we shall return.

Remark 4.6. *We observe, from the results of section 2. and 3. and the above, that gravitation in this model, is a force related to pairs of point-like objects in 3-space. There is no problem with modelling a situation with an arbitrary number of objects, but the calculations are not very easy, so we shall postpone the treatment of this "many-body problem" until we have a better understanding of the situation studied in (3.3).*

5. Worked out Formulas

5.1. **Some examples.** *As we have seen,* $Ext^1_A(V, V)$ *is the tangent space of the miniversal deformation space of* V *as an* A*-module, so that the non-commutative space* $Ph(A)$ *also parametrizes the set of generalised momenta, i.e. the set of pairs of an A- module* V*, and a tangent vector of the formal moduli of V, at that "point". In particular, any rank 1 representation of* $Ph(A)$ *is represented by a pair, (q,p), of a closed point, q of Spec(A), and a tangent, p at that point. For* $A = k[x_1, ..., x_d]$*, and two such points,* $(q_i, p_i), i = 1, 2$*, we proved in (1.3),*

$$dim_k Ext^1_{Ph(A)}(k(q_1,p_1), k(q_2,p_2)) = 2n, for\ (q_1,p_1) = (q_2,p_2)$$
$$dim_k Ext^1_{Ph(A)}(k(q_1,p_1), k(q_2,p_2)) = n, for\ q_1 = q_2, , p_1 \neq p_2$$
$$dim_k Ext^1_{Ph(A)}(k(q_1,p_1), k(q_2,p_2)) = 1, for\ q_1 \neq q_2.$$

Moreover, there is a generator of,

$$Ext^1_{Ph(A)}(k(q_1,p_1), k(q_2,p_2)) = Der_k(Ph(A), Hom_k(k(q_1,p_1)), k(q_2,p_2)))/Triv,$$

uniquely characterized by the tangent line defined by the vector $\overline{q_1 q_2}$*.*

Consider the following example.

Example 5.1. *Let* $A = k[x_1, x_2, x_3]$*, and consider now the space of 2-dimensional representation of* $Ph(A)$*. It is an easy computation that any such is given by the actions,*

$$x_1 = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}, \ x_2 = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}, \ x_3 = \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix},$$

and,

$$dx_1 = \begin{pmatrix} \alpha_1 & \sigma(a_1 - a_2) \\ \sigma(a_2 - a_1) & \alpha_2 \end{pmatrix},$$
$$dx_2 = \begin{pmatrix} \beta_1 & \sigma(b_1 - b_2) \\ \sigma(b_2 - b_1) & \beta_2 \end{pmatrix},$$
$$dx_3 = \begin{pmatrix} \gamma_1 & \sigma(c_1 - c_2) \\ \sigma(c_2 - c_1) & \gamma_2 \end{pmatrix}$$

The angular momentum is now given by,

$$L_{1,2} := x_1 dx_2 - x_2 dx_1 = \begin{pmatrix} (a_1\beta_1 - b_1\alpha_1) & \sigma(a_2b_1 - a_1b_2) \\ \sigma(a_1b_2 - a_2b_1) & (a_2\beta_2 - b_2\alpha_2) \end{pmatrix}.$$

And the isospin, has the form,

$$I_1 := [x_1, dx_1] = \begin{pmatrix} 0 & \sigma(a_1 - a_2)^2 \\ \sigma(a_2 - a_1)^2 & 0 \end{pmatrix},$$

.

Use the result, (1.6), and obtain for the k-algebra, $Ph(A)$, the existence of a k-algebra $C(2)$, an open subscheme $\underline{U}(2) \subset Spec(C(2))$, an étale morphism,

$$\pi : \underline{U}(2) \to Simp_2(Ph(A))$$

and a versal family,

$$\rho : Ph(A) \to M_2(C(2)).$$

Any section ψ, of the $\underline{C}(2)$-bundle $M_2(C(2))$ defined in $\underline{U}(2)$, determines a vector field $\xi \in \Theta_{\underline{\tilde{H}}}$. In fact , ψ defines for every point in $\underline{C}(2)$, the points,

$$p_i := (a_i, b_i, c_i) \in \mathbf{A}^3,\ i = 1, 2.$$

together with the vectors,

$$\xi_i := (\alpha_i, \beta_i, \gamma_i),\ i = 1, 2,$$

which together defines a vector field in $\underline{H}$. The extension to $\underline{\tilde{H}}$, is left as an exercise!

The appearance of the of the coupling constant σ in the formulas for ρ, which is just dependent upon the pair of points, p_1, p_2, shows that a quantum field theory, in the language of the Introduction, defined by $Simp_2(Ph(A))$, and a Dirac derivation defined in $Ph(A)$, tell us much more about the dynamics of the vector fields, than the covariant derivations of a connection. The next results, below, will show that this model contains an infinitely more complex extension, which someone should take seriously.

Example 5.2. *Let $A = M_2(k)$, and let us compute $Ph(A)$. Clearly the existence of the canonical homomorphism, $i : M_2(k) \to Ph(M_2(k))$ shows that $Ph(M_2(k))$ must be a matrix ring, generated, as an algebra, over $M_2(k)$ by $d\epsilon_{i,j},\ i, j = 1, 2$, where $\epsilon_{i,j}$ is the elementary matrix. A little computation shows that we have the following relations,*

$$d\epsilon_{1,1} = \begin{pmatrix} 0 & (d\epsilon_{1,1})_{1,2} = -(d\epsilon_{2,2})_{1,2} \\ (d\epsilon_{1,1})_{2,1} = -(d\epsilon_{2,2})_{2,1} & 0 \end{pmatrix}$$
$$d\epsilon_{2,2} = \begin{pmatrix} 0 & (d\epsilon_{2,2})_{1,2} = -(d\epsilon_{1,1})_{1,2} \\ (d\epsilon_{2,2})_{2,1} = -(d\epsilon_{1,1})_{2,1} & 0 \end{pmatrix}$$
$$d\epsilon_{1,2} = \begin{pmatrix} \epsilon_{1,2}(d\epsilon_{2,2})_{2,1} & (d\epsilon_{1,2})_{1,2} = -(d\epsilon_{2,1})_{2,1} \\ 0 & -(d\epsilon_{2,2})_{2,1}\epsilon_{1,2} \end{pmatrix}$$
$$d\epsilon_{2,1} = \begin{pmatrix} (d\epsilon_{2,2})_{1,2}\epsilon_{2,1} & 0 \\ (d\epsilon_{2,1})_{2,1} = -(d\epsilon_{1,2})_{1,2} & \epsilon_{2,1}(d\epsilon_{1,1})_{1,2} \end{pmatrix}$$

From this follows that any cosection, $\rho : Ph(M_2(k)) \to M_2(k)$, of $i : M_2(k) \to Ph(M_2(k))$, is given in terms of an element $\phi \in M_2(k)$, such that $\rho(da) = [\phi, a]$.

5.2. **Action of the gauge group $\mathfrak{g} \oplus \mathfrak{su}(2)$ on the tangent space.** *Let us go back to the subsection (4.3), the theorem (4.1), and the remark (4.2).*

If $o = (0,0,0), p = (1,0,0)$ have seen that the Lie algebra $\mathfrak{g}(\underline{t})$ comes out isomorphic to the Lie algebra of matrices of the form,

$$\begin{pmatrix} 0 & \delta_1^2 & \delta_1^3 \\ 0 & \delta_2^2 & \delta_2^3 \\ 0 & \delta_3^2 & \delta_3^3 \end{pmatrix}$$

The radical $\mathfrak{r}$, is generated by 3 elements, $\{u, r_1, r_2\}$, with,

$$u = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, r_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, r_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

where $u \notin [\mathfrak{g}, \mathfrak{g}]$, $[u, r_i] = -r_i, [r_1, r_2] = 0$, and the quotient,

$$\mathfrak{g}(\underline{t})/\mathfrak{r} = \mathfrak{sl}(2).$$

with the usual generators h, e, f,

$$h = u_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, e = u_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, f = u_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

In particular, we find that $\mathfrak{sl}(2) \subset \mathfrak{g}(\underline{t})$.

Notice also that, in this case, the unique 0-tangent line at the point $\underline{t}_0 = (o, p)$, $o = (0,0,0), p = (1,0,0)$, killed by $\mathfrak{g}$, is represented by the pair $d_3 := ((1,0,0),(1,0,0))$, and the unique light-velocity line is represented by $c_3 := ((1,0,0),(-1,0,0))$. Let, $d_1 := ((0,1,0),(0,1,0))$, $d_2 := ((0,0,1),(0,0,1))$ and let, $c_1 := ((0,1,0),(0,-1,0))$, $c_2 := ((0,0,1),(0,0,-1))$. Then $\{c_1, c_2, c_3, d_1, d_2, d_3\}$ is a basis for the tangent space $\Theta_{\underline{t}_0}$, and $\{d_1, d_2, d_3\}$ is a basis for $\tilde{\Delta}_{\underline{t}_0}$.

We observe that the generator h of the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ acts in this basis as,

$$h = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

which makes the choice of basis above canonical, i.e. determines $\{c_1, c_2, d_1, d_2\}$ as (± 1) eigenvectors of h, in $\tilde{c}$, resp. in $\tilde{\Delta}$. The actions of the gauge fields $\tilde{\mathfrak{g}}$ can then be given canonically: The generators, $h, e, f \in \mathfrak{sl}(2) \subset \mathfrak{g}$ act, in the above basis, like,

$$h = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$e = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$f = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The generators, $u, r_1, r_2 \in rad(\mathfrak{g})$ act, in the above basis, like,

$$u = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

$$r_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

$$r_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Now, go back to Lemma (4.3), recall that $\tilde{\Delta} \subset \Theta_{\tilde{H}}$, is the sub-bundle defined, at the point $\underline{t}$ as the space of tangents of the form (ξ, ξ). Given a metric on $\underline{\tilde{H}}$, we may look at the action of $\mathfrak{su}(3)$ on $\tilde{\Delta} \otimes \mathbf{C}$. Knowing that $\Theta = \tilde{c} \oplus \tilde{\Delta}$, it acts in the obvious way on the lower right corner, like

$$\gamma = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & * & * & * \end{pmatrix}.$$

Since the 0-velocity direction defined at (o, p), by the affine line $\overline{op}$ which here is $d_3 = (o - p, o - p)$, is unique, see (6.1), we have seen in section (4) that we may, in an essential unique way, decompose the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{su}(3)$, into the Cartan subalgebra $< \mathfrak{h}_1 >$, for the sub-Lie algebra $\mathfrak{su}(2) \subset \mathfrak{su}(3)$, leaving δ_3 invariant, and the part $< \mathfrak{h}_2 > \subset \mathfrak{h}$ perpendicular, in the Killing metric, to $\mathfrak{h}_1$. Obviously, we have,

$$\mathfrak{h}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathfrak{h}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2/3 \end{pmatrix},$$

Classically one denotes the other 6 base elements of $\mathfrak{su}(3) \otimes \mathbf{C}$, *as,* $\mathfrak{e}^i_{\pm}, i = 1, 2, 3$. *The restriction to* $\tilde{\Delta}$ *of these operators, in the basis* $\{d_1, d_2, d_3\}$ *are given by,*

$$\mathfrak{e}^1_+ = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \mathfrak{e}^2_+ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \mathfrak{e}^3_+ = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

and their duals,

$$\mathfrak{e}^1_- = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \mathfrak{e}^2_- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \mathfrak{e}^3_- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

We observe that the action of $\mathfrak{e}^1_+$ *and* $\mathfrak{e}^1_-$ *on* $\tilde{\Delta}$ *coincide with that of* e *and* f *of* $\mathfrak{g}$.
Let us note the commutators,

$$[\mathfrak{h}_1, \mathfrak{h}_2] = 0, [\mathfrak{h}_1, \mathfrak{e}^1_{\pm}] = \pm\mathfrak{e}^1_{\pm}, [\mathfrak{h}_1, \mathfrak{e}^2_{\pm}] = \mp 1/2\mathfrak{e}^2_{\pm}, [\mathfrak{h}_1, \mathfrak{e}^3_{\pm}] = \pm 1/2\mathfrak{e}^3_{\pm}$$

together with the following ones,

$$[\mathfrak{h}_2, \mathfrak{e}^1_{\pm}] = 0, [\mathfrak{h}_2, \mathfrak{e}^2_{\pm}] = \pm\mathfrak{e}^2_{\pm}, [\mathfrak{h}_2, \mathfrak{e}^3_{\pm}] = \pm\mathfrak{e}^3_{\pm}$$

Notice, for later use, the Casimir element,

$$C = \mathfrak{e}^1_+ \cdot \mathfrak{e}^1_- + \mathfrak{e}^2_+, \mathfrak{e}^2_- + \mathfrak{e}^3_- \mathfrak{e}^3_+ = 1.$$

Notice also that, since the direction of light at the point $\underline{t}$ *is given by* c_3, *we find that the action of* $\phi \in u(1)$ *is,* $exp(\phi \cdot (e - f))$, *on the photon, i.e. on the* $\mathfrak{g}$*-representation on* $< c_1, c_2 >$, *is the (transverse light-wave) given by,*

$$A(\phi) = \begin{pmatrix} cos\phi & sin\phi & 0 & 0 & 0 & 0 \\ sin\phi & cos\phi & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

usually just expressed as a state function, $\psi(\tau) = \exp(\imath \cdot \nu\tau)$, *by plugging in* $\phi =: \nu\tau$.
This completes the computation of the action of $\mathfrak{G}$ *on* $\Theta_{\tilde{H}}$. *Since the tangent space of* $\underline{H}(\overline{op})$, *generated by* c_3, d_3, *is uniquely determined, say by (6.1), the effective gauge group of our system turns out to be the real ,*

$$\mathfrak{g}* := \mathfrak{g} \oplus < \mathfrak{h}_2 > .$$

We shall be interested in the algebra, $End_{\tilde{H}}(\Theta_{\tilde{H}})$, *with the adjoint action of* $\mathfrak{g}$. *For* $\lambda \in \mathfrak{g}$, *and for* $\psi \in End_{\tilde{H}}(\Theta_{\tilde{H}})$, *put,*

$$\lambda(\psi) := [\lambda, \psi].$$

Together these formulas show that the quotients of of the $\mathfrak{g}*$*-representation* $\Theta_{\tilde{H}}$ *are the following,*

- $\tilde{c}$ *and therefore also the photon* $\{c_1, c_2\}$ *and a singleton,* $\{c_3\}$, *both simple.*
- $\tilde{\Delta}$ *and therefore the "electron"* $\{d_1, d_2\}$ *and a singleton,* $\{d_3\}$, *both simple.*
- *Weyl spinors,* B_o, B_p, *and Dirac spinors,* $B_o \oplus B_p$

The non-trivial simple quotients of of the $\mathfrak{su}(3)$*-representation* $\Theta_{\tilde{H}}$ *are reduced to,*

- *The quarks,* $\tilde{\Delta}$

It is now easy to see that the the Pauli matrices are found as follows,

$$\sigma^1 = e + f = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\sigma^2 = i(e + f) = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix},$$

$$\sigma^3 = h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Moreover, the parity operator P, the generator γ of the symmetry group $\mathbf{Z}_2$, operating on $\underline{\tilde{H}}$, acts on $\tilde{c}$, as multiplication by (-1), see [18]. *Therefore, it maps the basis $\{(c_1 + d_1), (c_2 + d_2)\}$ into $\{(-c_1 + d_1), (-c_2 + d_2)\}$. Consequently, we have an isomorphism,*

$$P : B_o \to B_p,$$

Chirality, in the physicists language, is explained as follows. The morphism P, extended to $B_o \oplus B_p$, in the basis chosen above, is given by the matrix,

$$\gamma^5 = \begin{pmatrix} 0 & id \\ id & 0 \end{pmatrix},$$

which turns left-handedness to right-handedness, with respect to the direction (o, p), resp. (p, o).

This shows that it is meaningful to consider the representations given by the Dirac matrices,

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ \gamma^k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix}, \ k = 1, 2, 3.$$

as well as the new operators,

$$\gamma^{k+3} = \begin{pmatrix} \sigma^k & 0 \\ 0 & -\sigma^k \end{pmatrix}, \ k = 1, 2, 3,$$

acting on, $B_o \oplus B_p$, such that,

$$\forall p \neq q, \ \gamma^p \gamma^q = -\gamma^q \gamma^p, \gamma^p \gamma^p = 1, \ p, q = 1, 2, 3, 4, 5, 6.$$

There are faithful representations of $\mathfrak{g}$ in the bundle $B_o \oplus B_p$, and $\mathfrak{g}$ kills $A_{o,p}$. The representations, B_o and B_p, the 2-component Weyl-Spinors of the physicists., and the space $B_o \oplus B_p$ the Dirac-Spinors are points of the non-commutative quotient of the moduli space $\tilde{H}$, by the gauge group $\mathfrak{g}$. It is clear that this, together with the formulas above give good reasons to believe that there is a relation between this model, and the Standard Model (and so also to the 8-fold way of Gell-Mann). Moreover, here all ingredients are universally given by the information contained in the singularity U, the Big Bang, in my tapping. The choice of metric, i.e. time and so gravitation, will have to be made on the basis of the nature of what I have called the Furniture of the model..

5.3. **Adjoint actions of $\mathfrak{g}$.** *Let's have a look at the Boson Fields, and let us start with the adjoint action of the family $\mathfrak{g}$. Consider the basis, used above, h, e, f, u, r_1, r_2, given by the basis for the $\mathfrak{sl}(2)$,*

$$h = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, e = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, f = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

and the basis for the radical, $\mathfrak{r}$,

$$u = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, r_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, r_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

where $u \notin [\mathfrak{g}, \mathfrak{g}]$, $[u, r_i] = -r_i, [r_1, r_2] = 0$, *and the quotient,*

$$\mathfrak{g}(\underline{t})/\mathfrak{r} = \mathfrak{sl}(2).$$

The adjoint action of $\mathfrak{g}$, *in the above basis, is given as,*

$$ad(h) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$ad(e) = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$

$$ad(f) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$ad(u) \; = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

$$ad(r_1) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$ad(r_2) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

From this we deduce the Killing form of $\mathfrak{g}$, *given, in the above basis, as,*

$$< h, h >= 10 < h, e >= 0, \; < h, f >= -0 \; < h, u >= 0, \; < h, r_1 >= 0, \; < h, r_2 >= 0$$
$$< e, e >= 0 < e, f >= -1, \; < e, u >= 0, \; < e, r_1 >= 0, \; < e, r_2 >= 1$$
$$< f, f >= 0 < f, u >= 0, \; < f, r_1 >= 0, \; < f, r_2 >= 0$$
$$< u, u >= 2 < u, r_1 >= 0, \; < u, r_2 >= 0$$
$$< r_i, r_i >= 0 < r_i, r_j >= 0,$$

showing that, $\mathfrak{g}^{\perp} = \{r_1, r_2\} \subset \mathfrak{g}$, *and that* $rad(\mathfrak{g})^{\perp} \simeq \mathfrak{sl}_2$

Now, use (4.2), and see that we can, outside the subscheme $\underline{M}(B)$, *identify* $\mathfrak{sl}(2)$ *with* $\tilde{\Delta}$, *such that* h, e, f *corresponds to* d_3, d_1, d_2, *respectively.*

6. Summing up the Model

Recall the philosophy of this model-building. We are assumed to have a moduli space, $\mathbf{M}$*, the points of which represent the different aspects of a physical phenomenon that we want to study.* $\mathbf{M}$ *should be outfitted with a metric* g*, the Time of our model. Its dynamical extension,* $Ph^\infty(\mathbf{M})$ *with the canonical Dirac derivation* δ*, and together with its moduli space of representations, is the non-commutative algebraic geometric space, representing all possible measurable changes in the moduli space* $\mathbf{M}$*.*

Let us focus on an affine covering of $\mathbf{M}$*, the general object of which is the* k*-algebra* C*. To be able to work with this model, we need a choice of a dynamical structure, i.e. a* δ*-stable ideal* (σ) *of* $Ph^\infty(C)$*, such that the dynamical system,* $A := C(\sigma) := Ph^\infty(C)/(\sigma)$*, becomes finitely generated.*

We may have global and local gauge groups, $\mathfrak{g}_0$ *and* $\mathfrak{g}_1$*, denote the direct sum of these,* $\bar{\mathfrak{g}}$*. The global one, being the isometry Lie algebra* $\mathfrak{g}_0$ *of the metric* g*, and the local one, the Lie algebra* $\mathfrak{g}_1$*, of inessential automorphisms of the relevant representations of this dynamical system. The goal is to classify the representations of* $C(\sigma)/\bar{\mathfrak{g}}$*, i.e. compute the non-commutative algebraic quotient space, the set of representations insensitive to the action of* $\bar{\mathfrak{g}}$*, thereby classifying the possible outcomes of measurements of the observables, i.e. of the elements of* $C(\sigma)$*, prepared at time* τ_0 *as time, clocked by the Dirac derivation* δ*, or some* $[\delta]$*, goes by.*

Recall also the definition, and the significance, of the notion of quotient in non-commutative algebraic geometry, see subsection (1.5), in particular Lemma (1.11). Given a representation $\rho : C(\sigma) \to End_k(V)$*, there is the exact sequence,*

$$0 \to End_{C(\sigma)}(V) \to End_k(V) \to Der_k(C(\sigma), End_k(V)) \xrightarrow{\kappa} Ext^1_{C(\sigma)}(V) \to 0$$

Any derivation, $\gamma \in Der_k(C(\sigma))$ *induces a derivation*

$$\rho(\gamma) := \gamma\rho \in Der_k(C(\sigma), End_k(V)).$$

If $\kappa(\rho(\gamma)) \in Ext^1_{C(\sigma)}(V)$ *is 0, then this means that the infinitesimal change of* ρ *produced by the action of* γ*, is trivial.*

Therefore, if the combined global and local gauge group, the Lie-algebra of inessential derivations $\bar{\mathfrak{g}} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$*, is given, with its Lie-Cartan structure, i.e. with the existence of a flat connection,*

$$\mathfrak{D} : \mathfrak{g}_0 \to Der_k(\mathfrak{g}_1),$$

the non-commutative quotient "space",

$$C(\sigma)/\bar{\mathfrak{g}}$$

is the category of representations,

$$C(\sigma)/\bar{\mathfrak{g}} := \{(\rho, \mu) : \rho : C(\sigma) \to End_k(V),\ \mu : \bar{\mathfrak{g}} \to End_C(V)\}$$

where, $\forall \gamma \in \mathfrak{g}_0, \kappa(\rho(\gamma)) = 0$*, and* μ *is a* ρ*-connection.*

We shall use the notation,

$$\rho \in Simp(C(\sigma))/\bar{\mathfrak{g}}$$

to mean that the representation ρ *is a simple object, in this category.*

In our Toy Model situation, $\mathbf{M} = \underline{\tilde{H}}$*, and we have introduced a metric* g *in* $\underline{\tilde{H}}$*, taking care of Time, via the dynamical structure,* $\sigma_g =< [dt_i, t_j] - g^{i,j} >$*, the Dirac derivation* $\delta := ad(g - T)$ *of* $\tilde{(}\sigma_g)$*, the Dirac derivation* $[\delta]$ *of General Relativity (GR), and so also gravitation. We have computed the global gauge group, the isometry Lie algebra* $\mathfrak{g}_0$*. The local gauge group* $\mathfrak{g}_1$ *is split into two parts, the* $\underline{\tilde{H}}$*-bundle of Lie algebras* $\mathfrak{g}$*, acting on* $\Theta_{\tilde{H}}$*, and the Lie algebra bundle* $\mathfrak{su}(3)$ *acting on* $\tilde{\Delta}_{\mathbf{C}}$*.*

Recall also the abbreviations, $\tilde{H}(com) := Ph(\tilde{H})/\sigma_0$ *where* $\sigma_0 = ([dt_i, t_j], [dt_i, dt_j])$, *and* $H(part) := Ph(\tilde{H})/\sigma_1$ *where* $\sigma_1 = ([dt_i, t_j])$.

We do not have a unique dynamical system, furnishing a unique model for all forces and fields. But we have a very special vector space of states, $\Theta_{\tilde{H}}$, *containing the states of our elementary particles. Moreover, we propose that the notion of particle, as we hinted at in subsection (1.6), is linked to those representations of* $Ph(\tilde{H})$ *induced by representations of* $\tilde{H}(part)/\bar{\mathfrak{g}}$, *on iterated extensions,* $\mathfrak{V}$ *of sub-extensions of* $\Theta_{\tilde{H}} = \tilde{c} \oplus \tilde{\Delta}$.

This situation leads to 3 linked models, which we shall now describe in detail. We have already introduced 2 of them.

The general theory, of subsections (1.5-1.8) has furnished,

- *a generalised Y-M model, where the global gauge group,* $\mathfrak{g}_0$, *is the Lie algebra of infinitesimal isometries, and the local gauge group,* $\mathfrak{g}_1$ *has a flat* $\mathfrak{g}_0$-*connection,*
$$\mathfrak{D} : \mathfrak{g}_0 \to Der_k(\mathfrak{g}_1).$$
We obtain the representations,
$$\rho : Ph(\tilde{H})/(\sigma_g) =: \tilde{H}(\sigma_g) \to End_k(\mathfrak{V}), \rho_0 := \rho/\tilde{H},$$
where there is a ρ_0-*connection,*
$$\mu : \bar{\mathfrak{g}} \to End_k(\mathfrak{V}).$$
Here, $[\delta] = 0$, *and the Hamiltonian is equal to the Dirac derivation,* $Q = \rho(g - T)$.
- *a GR model, for the scheme* $Simp_1(\tilde{H}(com))$, *i.e. for representations,*
$$\rho : \tilde{H}(com) \to End_{\tilde{H}(com)}(\tilde{H}(com)) \simeq \tilde{H}(com)$$
with Dirac derivation,
$$[\delta] = \sum(-\Gamma^i)\frac{\partial}{\partial \xi_i} + \xi_i \frac{\partial}{\partial t_i} \in Der_k(\tilde{H}(com)),\ \xi_i := dt_i.$$
The global gauge group $\mathfrak{g}_0$, *is the Lie-algebra of infinitesimal isometries, the local gauge group* $\mathfrak{g}_1$ *is trivial, and the Hamiltonian* $Q = 0$.

The link between these two is given by the generalised Einstein Field Equations, see Remark (1.29), and refer to the problem of defining the content = our furniture of the Universe. This must, if we shall be true to our philosophy, be a universal, bundle of $\mathfrak{g}_1$-*representations, and a result of the Big Bang event! (Recall that this is a paper about mathematical models).*

The 3. model, which also will serve as a link between the two above, and introduce a kind of a Quantum Field Theory (QFT), linking global force fields ρ, *i.e. connections, and Gauge Fields,* ρ_1 *will turn out to have the form,*

- *a QFT model for gauge fields, i.e. for those* $\tilde{H}$-*representations that we have considered, particles,*
$$\rho_1 : Ph(\tilde{H}) \to End_{\tilde{H}}(\mathfrak{V}),\ \rho_1(dt_q) = \sum_p W_p,\ W_p \in \mathfrak{g}_1,$$
with the global gauge groupe $\mathfrak{g}_0$, *and local gauge group,* $\mathfrak{g}_1$ *as before, and a flat* ρ_0-*connection,* $\bar{\mathfrak{g}} \to End_k(\mathfrak{V})$.
Here we have a generalised Dirac derivation,
$$[\delta] = \sum_{p,q} g^{p,q}\gamma_p(\frac{\partial}{\partial t_q} + A_q) \in End_k(\mathfrak{V}),\ A_q \in End_{\tilde{H}}(\mathfrak{V}).$$
where $\{\gamma_p\}$ *is an* $\tilde{H}$-*basis for* $\mathfrak{g}$. *The Hamiltonian is as before,* $Q := \rho(g-T)$,

Notice that the local gauge group, $\mathfrak{g}_1$ operates on $\tilde{H}(part) := Ph(\tilde{H})/([t_i, dt_j])$, as derivations, killing $\tilde{H}$, so $\mathfrak{g}_1 \subset Der_{\tilde{H}}(\tilde{H}(part))$. Moreover, since the structural constants of $\mathfrak{g}_1$ are just that, i.e. constants, elements of k, there exists, a flat. connection $\mathfrak{D}$, in the diagram,

$$\begin{array}{ccccc} \mathfrak{g}_0 & \xrightarrow{\mathfrak{D}} & Der_k(\mathfrak{g}_1) & \longleftarrow & \mathfrak{g}_1 \\ \downarrow{\scriptstyle \nabla_0} & & & & \downarrow{\scriptstyle \nabla_1} \\ End_k(\mathfrak{V}) & & \longleftarrow & & End_{\tilde{H}}(\mathfrak{V}) \end{array}$$

Therefore the non-commutative quotient, $\tilde{H}(part)/\bar{\mathfrak{g}}$, is the category of representations,

$$\tilde{H}(part) := Ph(\tilde{H})/([t_i, dt_j]) \to End_{\tilde{H}}(\mathfrak{V}),$$

with a $\mathfrak{g}_0$-connection ∇_0, as well as a $\mathfrak{g}_1$-connection, ∇_1 (recall that $\mathfrak{g}_1$ kills $\tilde{H}$) and a generalised Dirac derivation,

$$[\delta] = \sum_p \rho_1(dt_p)\rho(dt_p).$$

This operator in $\mathfrak{V}$ is a generalized momentum-operator, taking care of the two types of momenta, operating on $\mathfrak{V}$, stemming from the actions of the global fields and the local fields, respectively.

The operator $[\delta]$ will be important for defining time-derivatives of representations. This is a problem since neither the Force Laws (1) nor (2), define dynamical systems with Dirac derivations in $Ph(\tilde{H})$, see Remark(1.4) and [18], *page 83. A way out here, is the next point on the agenda.*

6.1. **Time in Quantum Field Theory.** *Now, let us recall from (1.12), that general relativity is, as we have shown above, related to our model via the canonical diagram,*

$$\begin{array}{ccccc} \tilde{H} & \xrightarrow{i} & Ph(\tilde{H}) & \xrightarrow{\pi} & \tilde{H}(\sigma_g) \\ \downarrow & & \downarrow & & \\ \tilde{H} & \xrightarrow{i} & Ph(\tilde{H})_{com} & & \end{array}$$

defined by a choice of a metric $g = 1/2\sum_{i,j} g_{i,j} \in Ph(\tilde{H})$.

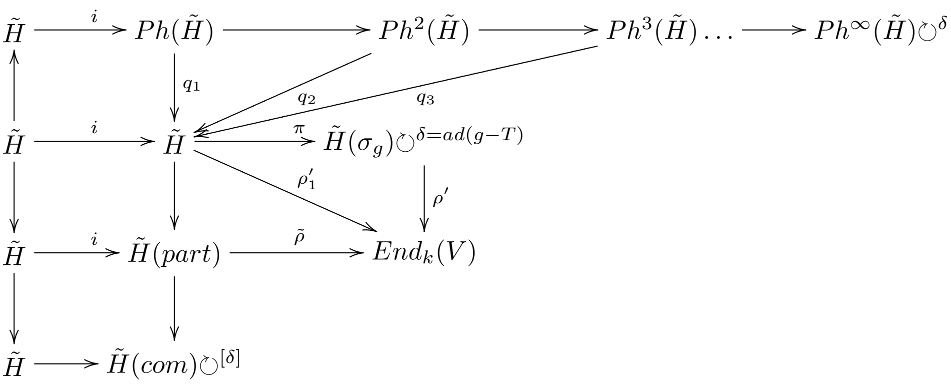
According to Theorem (1.13) there are in $Ph(\tilde{H})$ force laws, defining dynamical structures, generated by,

$$\begin{aligned} (1)\ d^2t_i =& -1/2\sum_{p,q}(\bar{\Gamma}^i_{p,q} + \bar{\Gamma}^i_{q,p})dt_pdt_q + 1/2\sum_{p,q} g_{p,q}(R_{p,i}dt_q + dt_pR_{q,i}) \\ &+ [dt_i, T] \\ (2)\ d^2t_i =& -\sum_{p,q}\Gamma^i_{p,q}dt_pdt_q - 1/2\sum_{p,q} g_{p,q}(F_{i,p}dt_q + dt_pF_{i,q}) \\ &+ 1/2\sum_{l,p,q} g_{p,q}[dt_p, (\Gamma^{i,q}_l - \Gamma^{q,i}_l)]dt_l + [dt_i, T] \\ T_l =& -1/2(\sum_j(\Gamma^j_{j,l} + \bar{\Gamma}^j_{j,l}) = -1/2(trace\nabla_l + trace\bar{\nabla}_l) \end{aligned}$$

and consistent with the Dirac derivation, $ad(g - T)$ in $\tilde{H}(\sigma_g)$, as well as with the Dirac derivation $[\delta] = \sum(-\Gamma^i)\frac{\partial}{\partial \xi_i} + \xi_i\frac{\partial}{\partial t_i}$ in $\tilde{H}(com)$.

Notice in relation with the diagram above, that even though in $Ph(\tilde{H})$ *we have the relation* $[dt_i, t_j]+[t_i, dt_j] = 0$*, we cannot conclude that* $[d^2t_i, t_j]+[dt_i, dt_j]+[dt_i, dt_j]+[t_i, d^2t_j] = 0$*, since in* $Ph(\tilde{H})$*, the derivation* $ad(g-T)$*, is not the Dirac derivation* δ*, and* $ad(g-T)(t_i) \neq dt_i$*. However,* $[\nu^2 t_i, t_j] + [\nu t_i, \nu t_j] + [\nu t_i, \nu t_j] + [t_i, \nu^2 t_j] = 0$*, for* $\nu := ad(g-T)$*.*

Let us now go back to subsection (1.12) and the diagram,



uniquely defined by,

$$\begin{aligned} i \circ ad(g-T) &= d \circ q_1 : \tilde{H} \to Ph(\tilde{H}) \\ q_1 \circ ad(g-T) &= d \circ q_2 : Ph(\tilde{H}) \to Ph(\tilde{H}) \\ q_n \circ ad(g-T) &= d \circ q_{n+1} : Ph^n(\tilde{H}) \to Ph(\tilde{H}), n \geq 1 \end{aligned}$$

outside the horizon, i.e. outside of the subscheme where the metric g *is degenerate. Recall that,* $\tilde{H}(part) := Ph(\tilde{H}/([dt_i, t_j]))$ *and* $\tilde{H}(com) := Ph(\tilde{H}/([dt_i, t_j]), [dt_i, dt_j])$*. In* $\tilde{H}(\sigma_g)$ *we have,* $dt_i := \delta(t_i) = \nu(t_i)$*, so we may write,*

$$\nu^n(t_i) = d^n t_i = \delta^n \circ q_n$$

We observe that given a representation,

$$\rho_1' : Ph(\tilde{H}) \to End_k(V),$$

the above implies that the dynamical properties with respect to the derivation $\nu = ad(g-T)$*, in* $Ph(\tilde{H})$*, are expressed as the dynamical properties of the infinite family of representations,*

$$\rho_n := q_n \circ \rho_1' : Ph^n(\tilde{H}) \to End_k(V),$$

with respect to the canonical Dirac derivation δ *of* $Ph^\infty(\tilde{H})$*. There may even be a limit representation,*

$$\rho^\infty : Ph^\infty(\tilde{H}) \to End_k(V),$$

where the dynamical properties of ρ_1' *and* $\nu = ad(g-T)$ *are taken care of by* ρ^∞ *and the Dirac derivation* $\delta : Ph^\infty(\tilde{H}) \to Ph^\infty(\tilde{H})$*.*

This is the situation we would have liked to have, to be able to use the general theory of Quantum Field Theory, of section (1.4), Theorems (1.12) and (1.13). However, as we have seen, the algebra $Ph^\infty(\tilde{H})$ *is far from finitely generated, and the representations we want to look at, are far from finite dimensional.*

It seems that we have only two obvious options. Either to reduce our space to a discrete set, and work with finite dimensional representations of $Ph^\infty(\tilde{H})$*, like in Subsection (3.3) or, given interesting Force Laws, try to make do with a reasonable approximation of this ideal situation. Let us first look at this last one.*

According to Remark (1.19), given a representation,

$$\rho_0 : \tilde{H} \to End_k(V),$$

considered as a point v, in its moduli space (there is always a formal moduli space $\mathbf{H}_V$), we may choose a tangent (or momentum) of this moduli space, at the point v, defined by an extension of ρ_0,

$$\rho_1 : Ph(\tilde{H}) \to End_k(V),$$

and if now the Force Law we choose gives us an extension of ρ_1,

$$\rho_2 : Ph^2(\tilde{H}) \to End_k(V),$$

such that $\delta^2\rho_2 = d^2$ we have at hand a second order momentum of ρ_0, a real classical force law. Now, let us see how to use this in our special situation.

Pick a representation, $\mathfrak{V}$, of the local gauge group, $\mathfrak{g}_1$ (say a sub-quotient of $\otimes^n\Theta$). $\mathfrak{V}$ is now the state space for a particle,

$$\tilde{\rho}_1 : Ph(\tilde{H}) \to \tilde{H}(part) \to End_{\tilde{H}}(\mathfrak{B}),$$

defined by,

$$\tilde{\rho}_1(t_i) = t_i, \tilde{\rho}_1(dt_i) = W_i,$$

where $W_i \in U(\mathfrak{g})$, $i = 1, ..., 6$, is what the physicists call a gauge field. This is obviously well defined, since $\tilde{\rho}([dt_i, t_j]) = 0$, for all i, j.

The representation $\tilde{\rho}_1$ extends (essentially uniquely) to,

$$\tilde{\rho}_2 : Ph^2(\tilde{H}) \to End_{\tilde{H}}(\mathfrak{B})$$

defined by,

$$\begin{aligned} \tilde{\rho}_2(t_i) &= t_i, \\ \tilde{\rho}_2(d_0t_i) &= W_i, \tilde{\rho}_2(d_1t_i) = \xi_{0i} \in End_{\tilde{H}}(\mathfrak{B})^{\mathfrak{g}}, \\ \tilde{\rho}_2(d_1(d_0t_i)) &= \tilde{\rho}_2(d^2t_i). \end{aligned}$$

since these equations will, of course, depending upon which Force Law we choose, respect the relations of $Ph^2(\tilde{H})$,

$$\begin{aligned} 0 &= [d_0t_i, t_j] + [t_i, d_0t_j] \\ 0 &= [d_1d_0t_i, t_j] + [d_0t_i, d_1t_j] + [d_1t_i, d_0t_j] + [t_i, d_1d_0t_j]. \end{aligned}$$

Recall from (1.7),

$$\begin{aligned} \bar{\Gamma}^i_{p,q} &= \sum_{l,r} g^{r,i}\Gamma^l_{r,p}g_{l,q}, \\ \nabla_l &= (\Gamma^j_{i,l}),\ \bar{\nabla}_l := (\bar{\Gamma}^j_{i,l}),\ T = \sum_l T_l dt_l \\ T_l &= -1/2(\sum_j(\Gamma^j_{j,l} + \bar{\Gamma}^j_{j,l}) = -1/2(trace\nabla_l + trace\bar{\nabla}_l) \\ \Gamma^{j,i}_p &= \sum_k g^{j,k}\Gamma^i_{k,p}, \\ F_{i,j} &= R_{i,j} - \sum_p(\Gamma^{j,i}_p - \Gamma^{i,j}_p)dt_p, \end{aligned}$$

where, $R_{i,j} = [dt_i, dt_j]$, and the Force Law (1),

$$\begin{aligned} d^2t_i = &- 1/2\sum_{p,q}(\bar{\Gamma}^i_{p,q} + \bar{\Gamma}^i_{q,p})dt_pdt_q + 1/2\sum_{p,q} g_{p,q}(R_{p,i}dt_q + dt_pR_{q,i}) \\ &+ [dt_i, T] \end{aligned}$$

The above fit into the following, creating a time-operator in $End_k(\mathfrak{V})$,

Theorem 6.1. *Let*

$$\tilde{\rho}_1 : Ph(\tilde{H}) \to End_{\tilde{H}}(\mathfrak{B})$$

be a particle given as above, and put $\tilde{\rho}_0 = i \circ \rho_1$. Assume that all W_p are constants, and suppose moreover, $\bar{\Gamma} = \Gamma$, (see (2.4) for an example). Then the force law (1) above, induces a representation,

$$\tilde{\rho}_2 : Ph^2(\tilde{H}) \to End_{\tilde{H}}(\mathfrak{B})$$

with Dirac derivation, and Hamiltonian given by,

$$[\delta] = \sum_{j,k} g^{j,k} W_j \frac{\partial}{\partial t_k}, \text{ respectively, } Q = \tilde{\rho}(g - T),$$

such that,

$$\tilde{\rho}_0(dt_i) = [\delta](\tilde{\rho}_1(t_i)) + [Q\tilde{\rho}_0(t_i)]$$
$$\tilde{\rho}_2(d^2 t_i) = [\delta](\tilde{\rho}_1(dt_i)) + [Q, \tilde{\rho}_1(dt_i)].$$

Thus, $[\delta] + ad(Q)$ is the Time Operator, up to order 2, in $End_k(\mathfrak{V})$.

Proof. The particle induces a field,

$$\tilde{\rho}_2(d_1(d_0 t_i)) = -1/2 \sum_{p,q} (\bar{\Gamma}^{p,q,i} + \bar{\Gamma}^{p,i,q}) W_p W_q - 1/2 \sum_p (R_{i,p} W_p + W_p R_{i,p}) + \tilde{\rho}([dt_i, T])$$
$$= -1/2 \sum_{p,q} (\bar{\Gamma}^{p,q,i} + \bar{\Gamma}^{p,i,q}) W_p W_q + \tilde{\rho}([g - T, t_i])$$

Let us now compute, $[\delta](\tilde{\rho}(dt_i))$. There is a well known formula,

$$\frac{\partial g_{i,k}}{\partial t_l} = 1/2 \sum_p (g_{p,k} \Gamma^p_{i,l} + g_{i,p} \Gamma^p_{k,l}),$$

from which we obtain the equally well known,

$$\frac{\partial g^{r,m}}{\partial t_q} = -1/2 \sum_k (g^{r,k} \Gamma^m_{k,q} + g^{k,m} \Gamma^r_{k,q}).$$

Plug it into the formula for $[\delta](\tilde{\rho}(dt_i))$, and calculate,

$$[\delta](\tilde{\rho}(dt_m)) = \sum_{p,q,r} g^{p,q} W_p \frac{\partial}{\partial t_q} (g^{r,m} W_r)$$
$$= -1/2 \sum_{p,q,r} g^{p,q} (g^{r,k} \Gamma^m_{k,q} + g^{k,m} \Gamma^r_{k,q}) W_p W_r$$
$$= -1/2 \sum_{p,r} (\Gamma^{r,m,p} + \Gamma^{m,r,p}) W_p W_r.$$

Now, compute,

$$\tilde{\rho}(d^2 t_i)) = -1/2 \sum_{p,q} (\bar{\Gamma}^i_{p,q} + \bar{\Gamma}^i_{q,p}) \tilde{\rho}(dt_p) \tilde{\rho}(dt_q) + \tilde{\rho}(1/2 \sum_{p,q} g_{p,q} (R_{p,i} dt_q + dt_p R_{q,i}) + [dt_i, T])$$
$$= -1/2 \sum_{p,q} (\bar{\Gamma}^i_{p,q} + \bar{\Gamma}^i_{q,p}) \tilde{\rho}(dt_p) \tilde{\rho}(dt_q) + [\tilde{\rho}(g - T), \tilde{\rho}(dt_i]$$
$$= -1/2 \sum_{p,q} (\Gamma^{p,i,r} + \Gamma^{p,r,i}) W_p W_r + [\tilde{\rho}(g - T), \tilde{\rho}(dt_i]$$
$$= [\delta](\tilde{\rho}(dt_i)) + [Q, \tilde{\rho}(dt_i)],$$

□

Remark 6.2. *Looking back to our model for quantum fields, (1.4) - (1.11), the form of the Dirac derivation* $[\delta]$ *may seem strange. In fact, it is not a Dirac derivative of the kind we have seen before. There is no reasons for that, since we have no Dirac derivation defined on any one of the rings of observables,* $Ph^n(\tilde{H})$*. It is, however, more reasonable to consider* $[\delta]+ad(Q)$ *as the time operator in* $End_{\tilde{H}}(\mathfrak{B})$*, at least up to order 2.*

Another way of understanding the above result, is the following. Start with a point $\underline{t}\in\underline{\tilde{H}}$*, and the corresponding finite dimensional representation,*

$$\rho : Ph(\tilde{H}) \to End_k(\mathfrak{B}_{\underline{t}})$$

The gauge group $\mathfrak{g}$ *acts on* $\mathfrak{B}_{\underline{t}}$*, and we shall consider the deformations of this representation conserving this action (see section 7. for an extension of this, and the related introduction of the Weak Force). The tangent space of the formal moduli of such objects is given as*

$$Ext^1_{Ph(\tilde{H})}(\mathfrak{B}_{\underline{t}},\mathfrak{B}_{\underline{t}})^{\mathfrak{g}} = (Der_k(Ph(\tilde{H}), End_k(\mathfrak{B}_{\underline{t}})/Triv)^{\mathfrak{g}}.$$

Any derivation, $\xi\in Der_k(Ph(\tilde{H}), End_k(\mathfrak{B}_{\underline{t}}))$ *is determined by the values,*

$$\xi(t_i) =: \xi_0^i \in End_k(\mathfrak{B}_{\underline{t}}), \xi(dt_i) =: \xi_1^i \in End_k(\mathfrak{B}_{\underline{t}})$$

with relations,

$$0 = \xi([t_i, dt_j] + [dt_i, t_j]) = [\xi_0^i, W_j] + [t_i, \xi_1^j] + [\xi_1^i, t_j] + [W_i, \xi_0^i]$$

Invariance under the action of $\mathfrak{g}$ *implies that for all* $\gamma\in\mathfrak{g}$ *there is an* $\phi_\gamma\in End_k(\mathfrak{B}_{\underline{t}})$*, such that, for all* $i=1,...,6$*,*

$$[\gamma, \xi_0^i] = 0,\ [\gamma, \xi_1^i] = [\phi_\gamma, W_i].$$

If $\mathfrak{B}_{\underline{t}}$ *is a simple* $\mathfrak{g}$ *module, then we have solutions,*

$$\xi_0^i = \alpha_i \cdot id,\ \xi_1^i = W_i,\ \alpha_i \in k$$

proving that the completion of the local ring $\tilde{H}_{\underline{t}}$ *is a quotient of the formal moduli of* ρ *as a* $\mathfrak{g}$*-representation. Then the form of* $[\delta]=\sum_{j,k} g^{j,k} W_j \dfrac{\partial}{\partial t_k} = \sum_j W_j\xi_j$*, becomes more reasonnable, and more in line with physicists language, "a gauge field associates a group element to each path in space-time" (Wikipedia: magnetic Monopoles).*

We have now got an example resembling what we have termed a Quantum Field Theory, where the objects are the quantum fields, $\psi\in End_{\tilde{H}}(\mathfrak{B})$*, and the equation of motion, and energy, should be given by,*

$$([\delta] + Q(\phi) = E\phi, \phi \in \mathfrak{V}.$$

However, since the operator $([\delta]+adQ)$*, is not a derivation of* $End_k(\mathfrak{V})$*, the eigenfunctions do not multiply, and we may not define Planck's constants, and play with the corresponding creation and annihilation operators, as we did above, see also* [18]*, p. 60.*

The gravitational force is given in terms of g*, the Time, but the other forces present themselves as fields,* ψ*, which added to* $Q=\tilde{\rho}_1(g-T)$*, give us the classical Hamiltonians. The gauge fields,* $\gamma\in\tilde{\mathfrak{g}*}$ *act on all the other fields* $\psi\in\Theta$ *as* $\gamma(\psi)=[\gamma,\psi]$*. In this sense the gauge fields are mediating forces, acting locally.*

The energy-mass-equation looks very much like an equations of Dirac type, and in fact, assuming the metric trivial, and having picked constant gauge fields $\{\gamma_i\}$

properly, we find that $Q = \sum_i \gamma_i^2$ *is the identity, and then Theorem (1.9) actually produces the Weyl equation, for Weyl-spinors,*

$$[\delta](\xi) := \sum_i \gamma_i \frac{\partial}{\partial t_i}(\xi) = E\xi,\ \xi \in B_o,$$

and the Dirac equation, for Dirac-spinors,

$$[\delta](\xi) = E\xi,\ \xi \in B_o, \oplus B_p,$$

but beware, the parity operator P *does not preserve the Weyl spinors.*

In complete generality, we have also a way of distinguishing type of energy. Clearly the derivation, $[\delta]$ *may be cut up into two parts,*

$$[\delta] := \sum_{q=1,2,3} g^{p,q}\gamma_p \frac{\partial}{\partial t_q} + \sum_{q=4,5,6} g^{p,q}\gamma_p \frac{\partial}{\partial t_q},$$

where the first term is related to the kinetic energy, and the second to the mass. This follows from the definition, of $\tilde{c}$ *and* $\tilde{\Delta}$*, being orthogonal. Now, consider, as above (5.2), the Pauli matrices,*

$$\sigma^1 = e + f = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\sigma^2 = -ie + if = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix},$$

$$\sigma^3 = h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and see the discussion in the next section. The parity operator, P*, induces the Dirac matrices,*

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ \gamma^k = \begin{pmatrix} 0 & \sigma^k \\ \sigma^k & 0 \end{pmatrix},\ k = 1,2,3.$$

as well as the new operators,

$$\gamma^{k+3} = \begin{pmatrix} \sigma^k & 0 \\ 0 & -\sigma^k \end{pmatrix},\ k = 1,2,3,$$

acting on, $B_o \oplus B_p$*, such that,*

$$\forall p \neq q,\ \gamma^p\gamma^q = -\gamma^q\gamma^p, \gamma^p\gamma^p = 1,\ p,q = 1,2,3,4,5,6.$$

Put,

$$\kappa := \sum_{p,(q=1,2,3)} g^{p,q}\gamma^p \frac{\partial}{\partial t_q},\ \mu := \sum_{p,(q=4,5,6)} g^{p,q}\gamma^p \frac{\partial}{\partial t_q},$$

and notice that since, by definition $\tilde{c}$ *and* $\tilde{\Delta}$ *are normal, we have* $\sum_p g^{p,q} g^{p,q'} = 0$*, for* $\frac{\partial}{\partial t_q} \in \tilde{c}, \frac{\partial}{\partial t'_q} \in \tilde{\Delta}$*, moreover,* $\kappa(\tilde{\Delta}) = \mu(\tilde{c}) = 0$*, see subsection (4.4).*

Notice that normality of $\tilde{c}$ *and* $\tilde{\Delta}$ *is implied by the condition that the geometry of space* $\tilde{c}$ *stays constant with respect to* $\underline{\lambda}$*, and clocks* $\tilde{\Delta}$ *are not affected by movement in space with light velocity!*

Then we find,

$$\{\kappa, \mu\} = 0,\ [\delta]^2 = (\kappa + \mu)^2 = \kappa^2 + \mu^2$$

proving that, $[\kappa^2, \mu^2] = 0$*. So if* ω *is a nonsingular eigenvector for either* κ *or* μ*, then it must be an eigenvector for both* κ^2 *and* μ^2*, and we find,*

$$\kappa(\omega) = \sum_{p,(q=1,2,3)} g^{p,q}\gamma^p \frac{\partial}{\partial t_q}(\omega) = K\omega,\ \mu(\omega) = \sum_{p,(q=4,5,6)} g^{p,q}\gamma^p \frac{\partial}{\partial t_q}(\omega) = m\omega,$$

and consequently,,

$$[\delta]^2(\omega) = E^2\omega = (K^2 + m^2)\omega.$$

This is, in our model, the combined Einstein, Klein-Gordon and Dirac equation. Notice that, for the trivial metric, i.e. for constant $g^{i,j}$,

$$[\delta]^2 = \sum_p \xi_p^2,$$

is the Laplace operator, see (1.29).

Summarising, we observe that our Toy Model has furnished, 3 nicely interrelated models;

- *a Field Theory (including a Y-M model) for connections on* $\tilde{H}$*-bundles* $\mathfrak{B}$*, i.e. for representations,*

$$\rho : Ph(\tilde{H})/(\sigma_g) =: \tilde{H}(\sigma_g) \to End_k(\mathfrak{B})$$

with Dirac derivation for representations, $[\delta] = 0$ *and Hamiltonian* $Q = \rho(g - T)$.

- *a Quantum Field Theory for gauge fields, i.e. for representations,*

$$\rho : Ph(\tilde{H})/([dt_i, t_j]) =: Ph(\tilde{H})_{part} \to End_{\tilde{H}}(\mathfrak{B})$$

with Dirac derivation $[\delta] = \sum_{p,q} g^{p,q} W_p \dfrac{\partial}{\partial t_q}$ *and Hamiltonian* $Q := \tilde{\rho}(g-T)$.

- *and a General Relativistic model, for the scheme* $Spec(Ph(\tilde{H})_{com})$*, i.e. for representations,*

$$\rho : \tilde{H}(\sigma_0) := Ph(\tilde{H})_{com} \to End_{\tilde{H}(\sigma_0)}(\tilde{H}(\sigma_0)) \simeq \tilde{H}(\sigma_0)$$

with Dirac derivation, $[\delta] = \sum(-\Gamma^i)\dfrac{\partial}{\partial \xi_i} + \xi_i \dfrac{\partial}{\partial t_i}$*, and Hamiltonian* $Q = 0$,

The exterior force fields in physics are in our Field Theory Model induced by representations like $\rho : \tilde{H}(\sigma_g) \to End_k(\mathfrak{B})$*. But now, we see that any ordinary particle will also induce a field. In fact the dynamics of* ρ *is given by the time derivative* $\dot{\rho}$ *of* ρ*, being defined by,*

$$\begin{aligned}
\dot{\rho}(dt_i) = \rho(d^2 t_i) &= \rho(-1/2\sum_{p,q}(\bar{\Gamma}^i_{p,q} + \bar{\Gamma}^i_{q,p})dt_p dt_q \\
&\quad + 1/2\sum_{p,q} g_{p,q}(R_{p,i}dt_q + dt_p R_{q,i}) + [dt_i, T]) \\
&= -\sum_{p,q}\Gamma^i_{p,q}\nabla_p\nabla_q - 1/2\sum_{p,q} g_{p,q}(F_{i,p}\nabla_q + \nabla_p F_{i,q}) \\
&\quad + 1/2\sum_{l,p,q} g_{p,q}[\nabla_p, (\Gamma^{i,q}_l - \Gamma^{q,i}_l)]\nabla_l + [\nabla_i, T],
\end{aligned}$$

where $\nabla_p := \rho(dt_p)$.

In the classical case, $\nabla_p = \xi_p = \sum_s g^{p,s}\dfrac{\partial}{\partial t_s}$ *is the momentum, and for a trivial metric (or for the Minkowski metric), the equation reduces to,*

$$\dot{\rho}(dt_i) = \rho(d^2 t_i) = -\sum_p (F_{i,p}\nabla_p + 1/2\frac{\partial}{\partial t_p}(F_{i,p})),$$

and the Hamiltonian $Q = \rho(g-T) = \Delta$*, reduces to the Laplace differential operator.*

Example 6.3. *It is not difficult to see that this last equation, restricted to the subspace $M(B)$ contains Maxwells equation. Interpreting,*

$$\mathfrak{B} = \theta_{M(B)},\ v_p = \dot{t}_p := \nabla_p = (\frac{\partial}{\partial t_p} + A_p)(\underline{t}),\ A_p \in End_{M(B)}(\Theta),\ \dot{v}_p := E \cdot \rho(d^2 t_i)(\underline{t})$$

we find that the observed electromagnetic fact, i.e. the Lorentz force law,

$$\dot{v}_i = E \cdot \sum_p F_{i,p} v_p$$

implies that the tangent to $\dot{\rho}$, given by the potential $\{(1/2 \sum_p \frac{\partial}{\partial t_p}(F_{i,p})), i = 1, ..., 4\}$, must be zero, (or as physicists would say, gauge). This means, see Subsections (1.8)-(1.11), that there exist $\Phi \in End_{M(B)}(\Theta_{M(B)})$, such that,

$$\{1/2 \sum_p \frac{\partial}{\partial t_p}(F_{i,p}), i = 1, ..., 4\} = \{\frac{\partial}{\partial t_i}(\Phi) + [1/2 \sum_p \frac{\partial}{\partial t_p}(F_{i,p})), \Phi], i = 1, ..., 4\}$$

Assuming that the electromagnetic field potentials, A_i are scalars, the curvatures, $F_{i,p} = \frac{\partial A_i}{\partial t_p} - \frac{\partial A_p}{\partial t_i}$ are also scalars, so all commutators $[1/2 \sum_p \frac{\partial}{\partial t_p}(F_{i,p})), \Phi]$ vanish, implying,

$$1/2 \sum_p \frac{\partial F_{i,p}}{\partial t_p} = \frac{\partial \Phi}{\partial t_i}, i = 1, ..4.$$

Now, let us choose coordinates such that $t_4 =: t_0 = \lambda$ (the proper time for physicists) is the only 0-velocity coordinate of $M(B)$, and use the classical notations,

$$E_i = F_{i,0} = \frac{\partial A_i}{\partial t_0} - \frac{\partial A_0}{\partial t_i},\ B_{p \times q} = F_{i,p} = \frac{\partial A_p}{\partial t_q} - \frac{\partial A_q}{\partial t_p}.$$

Simple computation then shows,

$$1/2 \sum_p \frac{\partial F_{0,p}}{\partial t_p} = -1/2 \nabla_s E$$

$$-1/2 \sum_p \frac{\partial F_{1,p}}{\partial t_p} = -1/2(\frac{\partial F_{1,0}}{\partial t_0} + \frac{\partial F_{1,2}}{\partial t_2} + \frac{\partial F_{1,3}}{\partial t_3}) = -1/2 \frac{\partial E_1}{\partial t_0} - 1/2(\nabla_s \times B)_1$$

$$-1/2 \sum_p \frac{\partial F_{2,p}}{\partial t_p} = -1/2(\frac{\partial F_{2,0}}{\partial t_0} + \frac{\partial F_{2,1}}{\partial t_2} + \frac{\partial F_{2,3}}{\partial t_3}) = -1/2 \frac{\partial E_2}{\partial t_0} - 1/2(\nabla_s \times B)_2$$

$$-1/2 \sum_p \frac{\partial F_{3,p}}{\partial t_p} = -1/2(\frac{\partial F_{3,0}}{\partial t_0} + \frac{\partial F_{3,1}}{\partial t_1} + \frac{\partial F_{3,2}}{\partial t_2}) = -1/2 \frac{\partial E_3}{\partial t_0} - 1/2(\nabla_s \times B)_3$$

from which the Maxwell equations follows, with $\Phi = \nabla \cdot A$, electric current $\mathfrak{J} = \nabla_s(\Phi)$, and electric charge $\rho = \frac{\partial \Phi}{\partial t_0}$. Together, $(\rho, \mathfrak{J})$ is a 0-tangent to the representation ρ, the first component in $\tilde{c}$, i.e. in space (or light) direction, and the second in $\tilde{\Delta}$, or in 0-velocity direction. Note that if we use the classical Minkowski, relativistic, metric in the same space we find exactly the same result, although the time derivative, $\dot{\rho}$, now must be thought of as the derivative with respect to proper time.

Notice also that we might have looked at the perfectly symmetric situation, where we concentrate on the subspace of $\underline{\tilde{H}}$ parametrized by $\{c_3, d_1, d_2, d_3\}$. We would then have got a massy force with properties analogue to the electromagnetic force, and with equations just like Maxwell's.

However, neither the above example, nor the dual case just mentioned have philosophically sense. The only time-space that we have at hand is the Toy Model, and *this is the quotient space $\tilde{H}/(Z_2)$ where the generator of Z_2 is the parity operator P, operating on $B_o \oplus B_p$, given by the matrix,*

$$\gamma^5 = \begin{pmatrix} 0 & id \\ id & 0 \end{pmatrix},$$

in the base $\{e_1 = (c_1 + d_1), e_2 = (c_2 + d_2), e_1* = (-c_1 + d_1), e_2* = (-c_2 + d_2)\}$, which turns left-handedness to right-handedness, with respect to the direction (o,p), resp. (p,o).

Now, in this general situation, let coordinates t_i be chosen such that,

$$dt_0, dt_1, dt_2, dt_3, dt_4, dt_\infty$$

are dual to,

$$c_3, e_1, e_2, e_1*, e_2*, d_3$$

and pick potentials A_i symmetric with respect to P. Then, just as above, put,

$$E_i = F_{0,i}, H_i = F_{\infty,i}, B_1 = F_{3,2}, B_2 = F_{4,1}, B_3 = F_{1,2}, B_4 = F2,3$$

and assume a reasonable Lorentz Force Law holds, i.e.

$$\dot{v}_i = E \cdot \sum_{j=0}^{\infty} F_{i,j} v_j$$

and that the tangent to this vanish, like above. Then there exists an element $\Phi \in End_{\tilde{H}}(\Theta)$, such that, for $i = 0, 1, ..., 4, \infty$,

$$1/2 \sum_p \frac{\partial F_{i,p}}{\partial t_p} = \frac{\partial \Phi}{\partial t_i},$$

and we find,

$$\frac{\partial F_{3,2}}{\partial t_2} = \frac{\partial F_{3,4}}{\partial t_4}$$
$$\frac{\partial F_{4,1}}{\partial t_1} = \frac{\partial F_{4,3}}{\partial t_3}$$
$$\frac{\partial F_{1,2}}{\partial t_2} = \frac{\partial F_{1,4}}{\partial t_4}$$
$$\frac{\partial F_{2,1}}{\partial t_1} = \frac{\partial F_{2,3}}{\partial t_3}$$

and,

$$\frac{\partial \Phi}{\partial t_0} = 1/2 \sum \frac{\partial E_p}{\partial t_p} + \frac{\partial H_0}{\partial t_\infty}$$
$$\frac{\partial \Phi}{\partial t_\infty} = 1/2 \sum \frac{\partial H_p}{\partial t_p} + \frac{\partial E_\infty}{\partial t_0}$$
$$\frac{\partial \Phi}{\partial t_1} = \frac{\partial B_3}{\partial t_2} - 1/2(\frac{\partial E_1}{\partial t_0} + \frac{\partial H_1}{\partial t_\infty})$$
$$\frac{\partial \Phi}{\partial t_2} = \frac{\partial B_4}{\partial t_3} - 1/2(\frac{\partial E_2}{\partial t_0} + \frac{\partial H_2}{\partial t_\infty})$$
$$\frac{\partial \Phi}{\partial t_3} = \frac{\partial B_1}{\partial t_4} - 1/2(\frac{\partial E_3}{\partial t_0} + \frac{\partial H_3}{\partial t_\infty})$$
$$\frac{\partial \Phi}{\partial t_4} = \frac{\partial B_2}{\partial t_1} - 1/2(\frac{\partial E_4}{\partial t_0} + \frac{\partial H_4}{\partial t_\infty})$$

In our Quantum Field Theory, we find that any particle, given by a representation,

$$\rho : \tilde{H}(part) \to End_k(\mathfrak{B})$$

also induces a field. In fact, the dynamics of ρ *is given by the time derivative* $\dot{\rho}$ *of* ρ*, being defined just like above,*

$$\begin{aligned}
\dot{\rho}(dt_i) = \rho(d^2t_i) &= \rho(-1/2\sum_{p,q}(\bar{\Gamma}^i_{p,q} + \bar{\Gamma}^i_{q,p})dt_p dt_q \\
&+ 1/2\sum_{p,q} g_{p,q}(R_{p,i}dt_q + dt_p R_{q,i}) + [dt_i, T]) \\
&= -\sum_{p,q}\Gamma^i_{p,q}\nabla_p\nabla_q - 1/2\sum_{p,q} g_{p,q}(F_{i,p}\nabla_q + \nabla_p F_{i,q}) \\
&+ 1/2\sum_{l,p,q} g_{p,q}[\nabla_p, (\Gamma^{i,q}_l - \Gamma^{q,i}_l)]\nabla_l + [\nabla_i, T],
\end{aligned}$$

where now $\nabla_p := \rho(dt_p) = \sum_s g^{p,s}W_s$*, should be considered as elements in* $\mathfrak{g}*$*, therefore derivations, so forces, acting upon the sections of* $\mathfrak{B}$*.*

6.2. **Elementary Particles, Bosons and Fermions.** *The elementary particles of the Standard Model, are, in our QFT-model identified with the gauge representations of the form,*

$$\rho : \tilde{H}(part) := Ph(\tilde{H})/([dt_i, t_j]) \to End_{\tilde{H}}(\mathfrak{B}),$$

where $\mathfrak{B}$ *is a sub-quotient of the* $\mathfrak{g}_1$*-representation* $\Theta^{\otimes n}$*, the n-th tensor product bundle, and the corresponding representation,* $\rho : \tilde{H}(part) \to End_k(\Theta^{\otimes n})$*, is defined by* $\rho(dt_i)(\omega_1 \otimes \omega_2 \otimes ... \otimes \omega_n) = \sum_j \omega_1 \otimes ... \otimes \rho(dt_i)(\omega_j) \otimes ... \otimes \omega_n$*. It is clear that if the element* $\gamma \in \mathfrak{g}_1$ *has eigenstates* $\omega_j \in \Theta_{\tilde{H}}$ *with eigenvalues* κ_j*, then the product state* $(\omega_1 \otimes \omega_2 \otimes ... \otimes \omega_n)$ *will be an eigenvector for the action of* γ *on the tensor product, with eigenvalue* $\sum_{j=1}^n \kappa_j$*.*

The states in $\Theta_{\tilde{H}}$ *are classified according to being particular eigenvectors, for the different markers, the elements of the Cartan sub-algebra* $\mathfrak{h}$ *of the local gauge group,* $\mathfrak{g}_1 := \mathfrak{g} \oplus su(3)$*. Clearly, if the particle* $\rho : \tilde{H}(part) \to End_{\tilde{h}}(\mathfrak{V})$ *is defined in terms of* $\rho(dt_p) = W_p \in \mathfrak{g}_1$ *then the sub-*$\tilde{H}$*-module of* $\mathfrak{V}$ *given by any specification of the eigenvalues of the induced representation of* $\mathfrak{h}$ *will be a sub-representation if and only if,* $[W_p, \mathfrak{h}] = 0$*.*

Consider now the list of such markers used by physicists, the ordinary intrinsic spin h or (as we shall see), $1/2h$*, the electric charge* $= \mathfrak{h}_2$*, the weak isospin* $I_3 = 3/4\mathfrak{h}_2 \pm 1/2\mathfrak{h}_1$*, and the weak hypercharge,* $Y_W = 2(\mathfrak{h}_2 - I_3) = 1/2\mathfrak{h}_2 \pm \mathfrak{h}_1$*. Since the direction* d_3 *is universally determined, the gauge group* $\mathfrak{g}_1$ *is basically reduced to* $\mathfrak{g}$*. Moreover, the change of sign in the last formula is, as we shall see later, related to what the physicists are calling Chirality, choosing a Left Hand or a Right Hand orientation, of the coordinate system for* $\tilde{\Delta}$*. In my choice above, the coordinate system* $\{d_1, d_2, d_3\}$ *is a right handed one, and exchanging* d_1 *and* d_2*, make the system left-handed. It turns out that the concept of left-right in physics is the opposite, as the following display will show.*

$Markers:$	$1/2\cdot h$	u	$\mathfrak{h}_1^{\pm}$	$\mathfrak{h}_2$	$I_3^{\pm}$	$Y_W^{\pm}$
c_1	$1/2$	1	0	0	0	0
c_2	$-1/2$	1	0	0	0	0
c_3	0	0	0	0	0	0
d_1	$1/2$	1	$\pm 1/2$	$-1/3$	$0(-1/2)$	$-2/3(1/3)$
d_2	$-1/2$	1	$\mp 1/2$	$-1/3$	$-1/2(0)$	$1/3(-2/3)$
d_3	0	0	0	$2/3$	$1/2$	$1/3$

We have the following (intrinsic) spin operator,

$$e-f=\begin{pmatrix}0&1&0\\-1&0&0\\0&0&0\end{pmatrix}$$

which is the rotation in $\tilde{c}$ about the axe c_3. It is also defining a rotation, or spin, in $\tilde{\Delta}$ about the axe d_3. Moreover, we have two folds, the first,

$$f-r_2=\begin{pmatrix}0&0&0\\1&0&0\\0&-1&0\end{pmatrix}$$

in $\tilde{c}$, (as well as in $\tilde{\Delta}$) mapping c_1 to c_2 and c_2 to c_3, killing c_3, and finally,

$$e-r_1=\begin{pmatrix}0&1&0\\0&0&0\\-1&0&0\end{pmatrix},$$

in $\tilde{c}$, (as well as in $\tilde{\Delta}$) mapping c_2 to c_1 and c_1 to c_3 and killing c_3.

The corresponding Elementary Bosons are given by the following list,

	Intrinsic spin	U
Markers	$1/2 \cdot h$	u
h	0	0
e	1	0
f	-1	0
u	0	0
r_1	$-1/2$	-1
r_2	$1/2$	-1

Here there is also a spin structure. The action of,

$$ad(e-f) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix},$$

induces a spin structure in $rad(\mathfrak{g})$ *about the axe* u.

Given these tables, we should explain why it seems that the physicists use the marker $1/2 \cdot h$ *for the (classical) angular momentum J, and why J and the isospin measured by* $\mathfrak{h}_1$ *coincide for* d_1, d_2. *The ordinary spin J, induced by action of the global gauge group* $\mathfrak{g}_0$, *see Example (2.1), for the metrics of interest here, i.e. our metric g of the Toy Model (as well as the Euclidean, the Minkowski, or the Schwarzschild metrics), was seen to be given as a rotation in* ϕ. *Now, if we consider a vector field* ψ *of* $\mathbf{H} := Hilb^2(\mathbf{A}^3)$, *it is clear that, pulled up to a vector field* ψ' *of* $\underline{\tilde{H}}$, *a rotation of* ψ' *by* $\phi = 2\pi$, *corresponds to a double rotation of* ψ, *since the parity operator, the generator of* Z_2, *is the rotation by* $\phi = \pi$.

This seems to be the reason why physicists introduce the notion of fermions, for globally defined states of $\Theta_{\mathbf{H}}$. *These are, after all, the only physically well defined states of our Toy Model. In this sense, all representations of the ring of observables* $Ph^\infty(\tilde{H})$, *invariant under* $Z_2 =< P >$ *and vectors, are Fermions, and thus of Spin 1/2. As in* [18], *(4.6), we show that the parity operator, P, cuts up the space* $\Theta_{\underline{\tilde{H}}}$ *into the sum of two pieces,* $\Theta^L_{\underline{\tilde{H}}} := im(1/2(1-P)) \simeq \tilde{c}$ *and* $\Theta^R_{\underline{\tilde{H}}} := im(1/2(1+P)) = \tilde{\Delta}$, *in physics referred to as the left hand states, respectively the right hand states. This is the basis for the notion of chirality, to which we shall return. But, clearly, there is a natural isomorphism,*

$$\Theta^R_{\underline{\tilde{H}}} := im(1/2(1+P)) \to \tilde{\Delta} \subset \Theta_{\underline{\tilde{\mathbf{H}}}},$$

identifying $\Theta^R_{\underline{\tilde{H}}} := im(1/2(1+P))$ *with the zero-velocity states. Since, evidently the metric must be invariant under the Parity operator, we find that the Levi-Civita representation,*

$$\rho : \tilde{H}(\sigma_g) \to End_k(\Theta_{\tilde{H}})$$

has a Hamiltonian $Q = \rho(g - T) \in End_k(\Theta_{\tilde{H}})$ *that respects the decomposition*

$$\Theta_{\underline{\tilde{H}}} = \Theta^R_{\underline{\tilde{H}}} \oplus \Theta^L_{\underline{\tilde{H}}}$$

The eigenvectors of $ad(Q)$ *in* $End_k(\Theta_{\tilde{H}})$ *are therefore divided into two groups,*

$$Bosons \subset End_k(\tilde{\Delta}) \oplus End_k(\tilde{c})$$

and,

$$Fermions \subset Hom_k(\tilde{\Delta}, \tilde{c}) \oplus Hom_k(\tilde{c}, \tilde{\Delta}),$$

or, considering the obvious graded structure on $\Theta_{\underline{\tilde{H}}}$, *Bosons being the degree 0 endomorphisms, and Fermions the degree* ± 1 *endomorphisms.*

This takes care of a Super Symmetry, and makes it possible to introduce the usual structures related to Planck's constant, vacuum state, creation and annihilation operators. Thus we find what we have termed a Quantum Field Theory, see [18], *section (4.6).*

We should, at this point try to model the most obvious particles we meet in physics. But first, what is a particle? Is it point-like, or is it a bumpy field defined in our space? The literature concerned with this question is formidable, and the points of views are quite different. Let us have a look at what comes out of the philosophy above.

In all 3 models, we have a vector space of states, $\Theta_{\tilde{H}}$ *producing the elements of the representation spaces* $\mathfrak{V}$, *for all representations,* $\rho : Ph(\tilde{H}) \to End_k(\mathfrak{V})$ *that we would consider as (gauge) particles. Moreover we have a Hamiltonian* $Q = \rho(g - T)$ *acting on this space, and therefore acting on the ring of fields,* $End_{\tilde{H}}(\Theta_{\tilde{H}})$ *by the adjoint action. Assume for a moment that this* $ad(Q)$ *acts naturally on* $End_k(\mathfrak{V})$. *Since* $ad(Q)$ *is a derivation in* $End_k(\mathfrak{V})$, *the spectrum must be an additive monoid* $\Lambda \subset k^+$. *Suppose further that there exist a (positive) generator* $\hbar$ *of* Λ. *In* [18], *(4.6) this was called our Planck's constant , and we let* $f_\hbar$ *be the corresponding eigenvector, or maybe eigenfield, of* $ad(Q)$. *The corresponding spectrum of* Q *in* $\mathfrak{V}$ *would be contained in the set of differences of* Λ, *see* [18], *(4.6).*

Now, for the Bosonic representations, we then find that a simple representation is in fact composed of one field $\psi_n = f^n_\hbar \in End_k(\mathfrak{V})$ *in each particular energy-state* $n\hbar$ *and that* $\mathfrak{V}$ *is composed of one state* ϕ_n *in each energy-state* $n\hbar$, *if there is a vacuum state* ϕ_0 *with*

$$Q(\phi_0) = 0.$$

So a Bosonic particle, obviously corresponding to a point in the moduli space of representations, turns out to be related with just one object, as a field, and as a state, but both carrying a bag of discrete energies $E^2_n = n\hbar$. *If* $f^2_\hbar = 0$ *we have a Fermionic representation, and no states are occupied by more than one energy quantum (at a time!), i.e. the Pauli principle, see again* [18], *(4.6).*

Moreover, as we have seen in Subsection (6.2) above, under the condition that $\tilde{c}$ *and* $\tilde{\Delta}$ *are normally orthogonal (which seems not to be true, cosmologically), this implies that there are eigenstates* ψ_n *for* $\kappa + \mu$, *such that*

$$\mu(\psi_n) = \mu_n\psi_n, \kappa(\psi_n) = \kappa_n\psi, \ \mu(\phi_n) = \mu_n\phi_n, \kappa(\phi_n) = \kappa_n\phi$$

with,

$$\mu^2_n + \kappa^2_n = E^2_n.$$

Then each one eigenstate for Q, also carries a sequence of mass and kinetic energy, given by its decomposition as sum of the ψ_n's, related to the MNS or PMNS matrices.

So which are the most elementary particles we could meet in physics? They should be simple representations of of the algebra of observables, so points in the noncommutative algebraic geometry $Simp(Ph(\tilde{H}))/\mathfrak{g}$. Let us look at the possible doublets, i.e. representations of $\mathfrak{g}$ that are locally simple representations of dimension 2.

The obvious first candidate is light, which we have to model as the obvious representation on the subspace of $\Theta_{\tilde{H}}$ generated by $< c_1, c_2 >$. The solutions of the Dirac equation above give us a transversal wave ψ, considered as a state of a photon, in the representation treated in Subsection (6.2), with an energy spectrum determined by the the Laplace-Beltrami operator, see loc.cit.

$$Q(\psi) = \Delta(\psi) = E^2 \cdot \psi.$$

Perfectly dual to light is the representation of $\mathfrak{g}$ on the subspace generated by $< d_1, d_2 >$ which should be considered as a transversal wave, a matter-state of an electron, in zero velocity direction. Notice also that the singletons $< d_3 >$, the up-quark, and $< c_3 >$ would generate the state-space of a particle, maybe linked to the recent rumours about observed X and Y particles by CDF at Fermilab. It would then have been natural to promote $< d_3 >$ to the role of proton, and $B_o \oplus B_p$ to the role of electron, and $< d_1, d_2, d_3 >$ to the role of neutron, with charge respectively $+2/3$, $-2/3$, and 0, but then I am afraid, the physicists would scream. However, it seems to me that both the Zitterbewegung as well as the Weak Force, would then be reasonably explained.

Finally, considering the Levi-Civita connection, the 3 covariant derivations, see (1.9),

$$D_{d_i} \in End_k(\Theta_{\tilde{H}}), i = 1, 2, 3,$$

seems to give us different mass operators, that could possibly explain the 3 (generations) of elementary particles. The classical relativity theory has only one mass-energy operator available, namely derivation with respect to time, and only two quarks, the up and down quarks.

Given that we have found the elementary particles, how do we construct the more complicated atoms and molecules that seem to be the constituents of our world, the atoms and molecules, in short the "Furniture of our Universe"? And what are the forces acting upon these elementary particles, performing the changes we see?

The iterated extensions of a family of representations, see (1.16) may produce the new representations from the primeval ones (thus new particles, of all complexities, from our elementary ones).

We have also seen, in section (1.4) that extensions of the representations (corresponding to elementary particles), induce momenta. In (1.9), higher order momenta are considered local forces. Decay of a simple representation has been treated in (1.23), and in (6.4), and since we already know that the Lorenz Force Law, the Dirac equation and the equation for geodesics in general relativity, are related, we should suspect that there exist a natural mathematical framework "explaining" this unity. The construction of such framework, is the purpose of the next section. And we shall have to start with a general theory of Interaction for Particles in a general Quantum Field situation, based on the notion of non-commutative deformations of families of modules, referred to in (1.16) and (1.17).

7. Interactions

Now go back to the Introduction, sections (1.7) and (1.8). Given a dynamical system $\mathbf{A}(\sigma)$*, we have seen that any particle,* $v \in Simp_n(\mathbf{A}(\sigma))$ *corresponding to a representation,*

$$\rho : \mathbf{A}(\sigma) \to End_k(V)$$

that we know "occurred" at some point $\underline{t} \in U(n) \subset Spec(C(n))$*, see Theorem (1.19) and Theorem (1.23), will after some time* τ *have developed into the particle sitting at a point on the integral curve* $\mathbf{c}$ *through* $\underline{t}$ *defined by the vector field* $[\delta]$ *of* $U(n)$*, at a time "distance"* τ*. Recall that the point* $\underline{t}$ *corresponds to an A-representation* ρ_0*, together with a formal curve of representations through* ρ_0*, We are, of course, assuming that the field k is contained in the real numbers. Now, this may well be a point on the border of* $Simp_n(\mathbf{A}(\sigma))$*, i.e. in* $\mathbf{\Gamma}_n = Simp(C(n)) - U(n)$*, where it decays into a decomposable representation, i.e. into an iterated extension of two or more new particles* $\{V_i \in Simp_{n_i}(\mathbf{A}(\sigma), n = \sum n_i\}$*. See* [18]*, subsection (3.3).*

What happens next is dependent on the dynamics of these iterated extensions. And, according to our philosophy, this is a again dependent on the geometry of the moduli space of all such objects. Assume that the world is populated by a maybe infinite family of elementary particles $\mathbf{V} = \{V_i\}_{i=1}^{\infty}$*, such that the Furniture of Cosmos is composed of the finitely iterated extensions of these elementary particles, forming atoms, molecules, and the myriads of different stuff we observe around us, then the changing world would be described as a dynamical structure defined on the moduli space of finitely iterated extensions of the particles in* $\mathbf{V}$*.*

7.1. **Interaction and Non-commutative Deformations.** *The purpose of this section is to sketch a mathematical model for the above scenario. Recall from the Introduction, and see* [18]*, (3.3), that a family of* $\mathbf{A}(\sigma)$*-representations,* $\mathbf{V} = \{V_i\}_{i=1}^{r}$ *is called a swarm, if*

$$\dim_k Ext^1_{\mathbf{A}(\sigma)}(V_i, V_j) < \infty \quad \text{for all } i, j = 1, \dots, r.$$

Consider now, for any swarm, $\mathbf{V} := \{V_i\}, i = 1, ..., r,$ *of* $\mathbf{A}(\sigma)$*-representations, the deformation functor,*

$$Def_{\mathbf{V}} : \underline{a}_r \longrightarrow \underline{Sets},$$

and its formal moduli,

$$H(\mathbf{V}) := \begin{pmatrix} H_{1,1} & ... & H_{1,r} \\ & ... & . \\ H_{r,1} & ... & H_{r,r} \end{pmatrix},$$

together with the versal family, i.e. the essentially unique homomorphism of k-algebras,

$$\tilde{\rho} : \mathbf{A}(\sigma) \longrightarrow O(\mathbf{V}) := \begin{pmatrix} H_{1,1} \otimes End_k(V_1) & ... & H_{1,r} \otimes Hom_k(V_1, V_r) \\ . & ... & . \\ H_{r,1} \otimes Hom_k(V_r, V_1) & ... & H_{r,r} \otimes End_k(V_r) \end{pmatrix}.$$

Recall the form of this homomorphism. Let $\Psi_{i,j}$ *be the k-dual of* $rad(H)_{i,j}$*, and let* $\{\psi^s_{i,j}\}$ *and* $\{h^s_{i,j}\}$ *be dual k-bases of* $\Psi_{i,j}$ *and* $rad(H)_{i,j}$*, then for* $v_i \in V_i$

$$\tilde{\rho}(a)(v_i) = v_i a + \sum_{j,s} h^s_{i,j} \otimes \psi^s_{i,j}(a)(v_i), \psi^s_{i,j}(a) \in Hom_k(V_i, V_j),$$

This is, in an obvious sense, the universal interaction. The elements $\hat{h}_{i,j}$ *of the k-dual,* $rad(H)^*$*, of* $rad(H)$*, correspond to linear maps* $\mathbf{A}(\sigma) \to Hom_k(V_i, V_j)$*, which we should consider forces (mediated by elements of* $\mathbf{A}(\sigma)$*), acting by mapping the states of* V_i *into states of* V_j*.*

However, we need a way of specifying which interactions we want to consider. This is the purpose of the following tentative definition,

Definition 7.1. *An interaction mode for the swarm* $\mathbf{V} = \{V_i\}, i = 1, ..., r,$ *is a finite dimensional right* $H(\mathbf{V})$*-module* M*.*

An interaction mode is a kind of preparation, see the Introduction. It consists of a rule, telling us which interactions we expect to occur, for the given swarm, and therefore how to, ideally, prepare their interactions for experiments. The structure morphism $\phi : H(\mathbf{V}) \to End_k(M)$*, fixes all relevant higher order momenta, i.e.* ϕ *evaluates all the tangents between these modules, and creates a new* $A(\sigma)$*-module, see (1.15) and (1.16).*

In fact, since $M \simeq \oplus_{i=1}^{r} M\epsilon_i$*, where* $\epsilon_i \in H_{i,i}$ *is a unit, an interaction mode induces a homomorphism,*

$$\kappa(M) : A(\sigma) \longrightarrow End_k(V),$$

where $V := \oplus_{i=1,...,r} M_i \otimes V_i$*.*

Thus, we have, given the swarm, $\mathbf{V}$ *and the interaction mode* M*, constructed a new* $A(\sigma)$*-representation (a 1-member swarm of modules),* V*.*

Suppose the interaction mode is reduced to a finite dimensional $H(\mathbf{V})$ *module* M*, killed by* $rad(H)^2$*, then for each tuple* (i, j) *consider the linear map,*

$$\psi_{i,j} : A(\sigma) \to Hom_k(M_i \otimes V_i, M_j \otimes V_j.).$$

Since the vector space $Hom_k(M_j, M_i)$ *is canonically isomorphic to the dual of* $Hom_k(M_i, M_j)$*, any element* $\hat{\psi}_{i,j} \in Hom_k(M_j, M_i)$ *induces a derivation, therefore an element in* $Ext^1_{A(\sigma)}(V_i, V_j)$*, so a force,*

$$\psi_{i,j} : A(\sigma) \to Hom_k(V_i, V_j).$$

Thus we have got a family of interaction forces, defined for each (i, j)*, and mediated by the elements* $a \in A(\sigma)$*,*

$$\psi_{i,j}(a) : V_i \to V_j.$$

mapping states of V_i *to states of* V_j*.*

Moreover, we have now returned to the starting point of this paper, i.e. to subsection (1.6). We could expect that the Dirac derivation of $A(\sigma)$*, together with the preparation of the dynamical evolution of the swarm* $\mathbf{V} = \mathbf{V} = \{V_i\}, i = 1, ..., r,$ *give us the time-development of the representation* V*, at least in the case* V_i *are finite dimensional.*

Obviously V *is an iterated extension and, in general, we have to work with deformations of such, see next subsection. But first, let us look at the following interesting and simple example.*

Example 7.2. *The first example we may think of is the case of one representation, i.e. for r=1. This is the case of the self-interactions we have treated above. Assuming given a dynamical structure and a Dirac derivation* δ*, we have associated to any representation* $\rho : A(\sigma) \to Hom_k(V)$ *its unique extension, or force acting on* V *via,*

$$\psi(a) : V \to V, \psi = \delta \circ \rho.$$

For gauge fields, given as above by a representation of $\tilde{H}$ *of the form* $\mathfrak{B}$*, where* $\mathfrak{B}$ *is a representation of the principal Lie-bundle* $\mathfrak{g} \oplus \mathfrak{su}(3)$ *defined on* $\underline{\tilde{H}}$*, we have considered self-interactions of the form,*

$$\rho : Ph(\tilde{H})/[dt_i, t_j] \to End_{\tilde{H}}(\mathfrak{B}),$$

and the force fields resulting. However, there are, in this case, other forces to be considered. The $\tilde{H}$ *module* $\mathfrak{B}$ *may be a free, and* $\mathfrak{sl}(2)$ *and* $\mathfrak{su}(3)$ *being semi-simple,* *do not have nontrivial extensions, but the* $rad(\mathfrak{g})$ *part of* $\mathfrak{g} = \mathfrak{sl}(2) \oplus rad(\mathfrak{g})$*, has non-trivial extension modules, and as we shall see, this give us a model for the Weak Interaction.*

Lemma 7.3. *Consider the action of* $\mathfrak{g} = \mathfrak{sl}(2) \oplus rad(\mathfrak{g})$*, on* $\tilde{c}$ *and* $\tilde{\Delta}$ *then there is an isomorphisme,*

$$Ext^1_{\tilde{H}(\mathfrak{g})}(\Theta,\Theta) \simeq Ext^1_{\mathfrak{g}}(\Theta,\Theta) \simeq Ext^1_{rad(\mathfrak{g})}(\Theta,\Theta)$$

and,

$$Ext^1_{rad(\mathfrak{g})}(\Theta,\Theta) = Ext^1_{rad(\mathfrak{g})}(\tilde{c},\tilde{c}) \oplus Ext^1_{rad(\mathfrak{g})}(\tilde{\Delta},\tilde{\Delta}) \oplus Ext^1_{rad(\mathfrak{g})}(\tilde{c},\tilde{\Delta}) \oplus Ext^1_{rad(\mathfrak{g})}(\tilde{\Delta},\tilde{c})$$

Moreover,

$$Ext^1_{rad(\mathfrak{g})}(\tilde{c},\tilde{c}) \simeq Ext^1_{rad(\mathfrak{g})}(\tilde{\Delta},\tilde{\Delta})$$

and, any $\psi \in Ext^1_{rad(\mathfrak{g})}(\tilde{\Delta},\tilde{\Delta})$ *is given by the derivation,*

$$\psi : rad(\mathfrak{g}) \to End_{\tilde{H}}(\tilde{\Delta},\tilde{\Delta})$$

in the bases, $\{d_1,d_2,d_3\}$ *of* $\tilde{\Delta}$*, the quarks, and* $\{u,r_1,r_2\}$ *of* $rad(\mathfrak{g})$*, modulo the trivial derivations, given by an element* $a := (a_{i,j}) \in End_{\tilde{H}}(\tilde{\Delta},\tilde{\Delta})$*. We find,*

$$\psi(u) = \begin{pmatrix} \alpha_{1,1} & 0 & \alpha_{1,3} \\ 0 & \alpha_{2,2} & \alpha_{2,3} \\ \alpha_{3,1} & \alpha_{3,2} & \alpha_{3,3} \end{pmatrix} mod \begin{pmatrix} 0 & 0 & -a_{1,3} \\ 0 & 0 & -a_{2,3} \\ a_{3,1} & a_{3,2} & 0 \end{pmatrix}$$

where, $\alpha_{1,1} = \alpha_{2,2} = \alpha_{3,3}$*, and*

$$\psi(r_1) = \begin{pmatrix} -\alpha_{1,3} & 0 & 0 \\ -\alpha_{2,3} & 0 & 0 \\ \alpha^1_{3,1} & \alpha^1_{3,2} & \alpha_{1,3} \end{pmatrix} mod \begin{pmatrix} a_{1,3} & 0 & 0 \\ a_{2,3} & 0 & 0 \\ a_{3,3}-a_{1,1} & -a_{1,2} & -a_{1,3} \end{pmatrix}$$

$$\psi(r_2) = \begin{pmatrix} 0 & -\alpha_{1,3} & 0 \\ 0 & -\alpha_{2,3} & 0 \\ \alpha^2_{3,1} & \alpha^2_{3,2} & \alpha_{2,3} \end{pmatrix} mod \begin{pmatrix} 0 & a_{1,3} & 0 \\ 0 & a_{2,3} & 0 \\ -a_{2,1} & a_{3,3}-a_{2,2} & -a_{2,3} \end{pmatrix}$$

for some matrix $(a_{i,j})$*.*

Proof. Recall the structure of $rad(\mathfrak{g})$. We have $[u,r_i] = \pm r_i, [r_1,r_2] = 0$, and the actions on Θ, in the basis $\{c_1,c_2,c_3,d_1,d_2,d_3\}$, given by,

$$u = \begin{pmatrix} 1&0&0&0&0&0 \\ 0&1&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&1&0&0 \\ 0&0&0&0&1&0 \\ 0&0&0&0&0&0 \end{pmatrix}.$$

$$r_1 = \begin{pmatrix} 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 1&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&1&0&0 \end{pmatrix}.$$

$$r_2 = \begin{pmatrix} 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&1&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&1&0 \end{pmatrix}.$$

Now, $Ext^1_{rad(\mathfrak{g})}(V_i,V_j) = Der(rad(\mathfrak{g}), End_k(V_i,V_j))/Triv$. The rest is just computation. $\square$

In case (1), choose $a=\alpha$. Then we find,

$$\psi(u) = \begin{pmatrix} \alpha_{1,1} & 0 & 0 \\ 0 & \alpha_{2,2} & 0 \\ 2\alpha_{3,1} & 2\alpha_{3,2} & \alpha_{3,3} \end{pmatrix}$$

where, $\alpha_{1,1}=\alpha_{2,2}=\alpha_{3,3}$, and

$$\psi(r_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \alpha^1_{3,1} & \alpha^1_{3,2} & 0 \end{pmatrix}$$

$$\psi(r_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \alpha^2_{3,1} & \alpha^2_{3,2} & 0 \end{pmatrix}$$

since $\alpha_{1,2}=\alpha_{2,1}=0$.

If, in case (2), we choose $a=-\alpha$, we find,

$$\psi(u) = \begin{pmatrix} \alpha_{1,1} & 0 & 2\alpha_{1,3} \\ 0 & \alpha_{2,2} & 2\alpha_{2,3} \\ 0 & 0 & \alpha_{3,3} \end{pmatrix}$$

where, $\alpha_{1,1}=\alpha_{2,2}=\alpha_{3,3}$, and

$$\psi(r_1) = \begin{pmatrix} -2\alpha_{1,3} & 0 & 0 \\ -2\alpha_{3,2} & 0 & 0 \\ \alpha^1_{3,1} & \alpha^1_{3,2} & 2\alpha_{1,3} \end{pmatrix}$$

$$\psi(r_2) = \begin{pmatrix} 0 & -2\alpha_{1,3} & 0 \\ 0 & -2\alpha_{2,3} & 0 \\ \alpha^2_{3,1} & \alpha^2_{3,2} & 2\alpha_{2,3} \end{pmatrix}$$

since $\alpha_{1,2}=\alpha_{2,1}=0$.

Case (1) can be interpreted as follows: $Z_0 := \psi(u)$ mediates a force, mapping $\{d_1,d_2\}$ to d_3, $W_- := \psi(r_1)$ and $W_+ := \psi(r_2)$ do the same, but do not permute (d_1,d_2). In case (2), Z_0 maps d_3 to $\{d_1,d_2\}$, and W_-, W_+ sends $\{d_1,d_2\}$ to d_3 and permute (d_1,d_2), i.e. they change the orientation, see (6.3), and the relationship to left-and right-handedness, that is, to chirality.

This seems to be very close to the weak interaction on quarks, mediated by the weak interaction bosons, Z, W_+, W_-, identified above with the Bosons $u, r_1, r_2 \in \mathfrak{g}$, respectively.

We have, above, omitted the full story, the action of Z, W_+, W_- in $Hom_{\tilde{H}(\mathfrak{g})}(\tilde{c},\tilde{\Delta})$ and in $Hom_{\tilde{H}(\mathfrak{g})}(\tilde{\Delta},\tilde{c})$. This may be interesting, so we add the corresponding matrices for Z_0, and r_1, without further comments.

$$\psi(u) = \begin{pmatrix} \alpha_{1,1} & 0 & \alpha_{1,3} & \alpha_{1,4} & 0 & \alpha_{1,6} \\ 0 & \alpha_{2,2} & \alpha_{2,3} & 0 & \alpha_{1,4} & \alpha_{2,6} \\ \alpha_{3,1} & \alpha_{3,2} & \alpha_{3,3} & \alpha_{3,4} & \alpha_{3,5} & \alpha_{1,4} \\ \alpha_{4,1} & 0 & \alpha_{4,3} & \alpha_{4,4} & 0 & \alpha_{4,6} \\ 0 & \alpha_{4,1} & \alpha_{5,2} & 0 & \alpha_{5,5} & \alpha_{5,6} \\ \alpha_{6,1} & \alpha_{6,2} & \alpha_{4,1} & \alpha_{6,4} & \alpha_{6,5} & \alpha_{6,6} \end{pmatrix}$$

where,$\alpha_{1,1}=\alpha_{2,2}=\alpha_{3,3}=\alpha_{4,4}=\alpha_{5,5}=\alpha_{6,6}$, and modulo matrices of the form,

$$\begin{pmatrix} 0 & 0 & a_{1,3} & 0 & 0 & a_{1,6} \\ 0 & 0 & a_{2,3} & 0 & 0 & a_{2,6} \\ -a_{3,1} & -a_{3,2} & 0 & -a_{3,4} & -a_{3,5} & 0 \\ 0 & 0 & a_{4,3} & 0 & 0 & a_{4,6} \\ 0 & 0 & a_{5,2} & 0 & 0 & a_{5,6} \\ -\alpha_{6,1} & -\alpha_{6,2} & 0 & -a_{6,4} & -a_{6,5} & 0 \end{pmatrix},$$

and,

$$\psi(r_1) = \begin{pmatrix} -\alpha_{1,3} & 0 & 0 & -\alpha_{1,6} & 0 & 0 \\ -\alpha_{2,3} & 0 & 0 & -\alpha_{2,6} & 0 & 0 \\ \alpha^1_{3,1} & \alpha^1_{3,2} & \alpha_{1,3} & \alpha^1_{3,4} & \alpha^1_{3,5} & \alpha_{1,6} \\ -\alpha_{4,3} & 0 & 0 & -\alpha^1_{4,6} & 0 & 0 \\ -\alpha_{5,3} & 0 & 0 & 0-\alpha_{5,6} & 0 & 0 \\ \alpha^1_{6,1} & \alpha^1_{6,2} & \alpha_{4,3} & \alpha^1_{6,4} & \alpha^1_{6,5} & \alpha_{4,6} \end{pmatrix}$$

modulo matrices of the form,

$$\begin{pmatrix} a_{1,3} & 0 & 0 & a_{1,6} & 0 & 0 \\ a_{2,3} & 0 & 0 & a_{2,6} & 0 & 0 \\ b_{3,1} & -a_{1,2} & -a_{1,3} & b_{3,4} & -a_{1,5} & -a_{1,6} \\ a_{4,3} & 0 & 0 & a_{4,6} & 0 & 0 \\ a_{5,3} & 0 & 0 & a_{5,6} & 0 & 0 \\ b_{6,1} & -a_{4,2} & -a_{4,3} & b_{6,4} & -a_{4,5} & -a_{4,6} \end{pmatrix},$$

where, $b_{3,1} = a_{3,3} - a_{1,1}, b_{3,4} = a_{3,6} - a_{1,4}, b_{6,1} = a_{6,3} - a_{4,1}, b_{6,4} = a_{6,6} - a_{4,4}$*, for some matrix* $a = (a_{i,j})$.

7.2. **Graphs and Subcategories Generated by a Family of Modules.** *Let* A *be any associative* k*-algebra,, and assume given a swarm,* $\mathbf{V} = \{V_i\}$ *of* A *representations. Let* Γ *be an ordered graph with set of nodes* $|\Gamma| = \{V_i\}$*. Starting with a first node* V_{i_1}*,of* Γ*, we can construct, in many ways, an extension of the module* V_{i_1} *with the module* V_{i_2}*, corresponding to the end point of the first arrow of* Γ*, then continue, choosing an extension of the result with the module corresponding to the endpoint of the second arrow of* Γ*, etc. until we have reached the endpoint of the last arrow. Any finite length module can be made in this way, for some* Γ *corresponding to a decomposition of the module into simple constituencies, by peeling off one simple sub-module at a time, i.e. by picking one simple sub-module and forming the quotient, picking a second simple sub-module of the quotient and taking the quotient, and repeating the procedure until it stops.*

The ordered k*-algebra* $k[\Gamma]$ *of the ordered graph* Γ *is the quotient algebra of the usual algebra of the graph* Γ *by the ideal generated by all admissible words which are not "intervals" of the ordered graph. Say* $...\gamma_{i,j}(n-1)\gamma_{j,j}(n)\gamma_{j,k}(n+1)...$ *is an interval of the ordered graph, then* $\gamma_{i,j}(n-1).\gamma_{j,k}(n+1) = 0$ *in* $k[\Gamma]$.

The main result in this context, is now the following result, see [14],

Proposition 7.4. *Let* A *be any associative* k*-algebra,* $\mathbf{V} = \{V_i\}_{i=1}^r$ *any swarm of right* A*-modules. Consider an iterated extension* E *of* $\mathbf{V}$*, with extension graph* Γ*. Then there exists a morphism of* k*-algebras*

$$\phi : H(\mathbf{V}) \to k[\Gamma]$$

such that, as right A*-modules,*

$$E \simeq k[\Gamma] \otimes_\phi \tilde{V}$$

Here $\tilde{V}$ *is the versal deformation, of the family* $\mathbf{V}$*, as left* $H(\mathbf{V})$*-, and right,* A*-module. Moreover the set of equivalence classes of iterated extensions of* $\mathbf{V}$ *with* *representation graph* Γ*, is a quotient of the set of closed points of the affine algebraic variety*

$$\underline{A}[\Gamma] = Mor(H(\mathbf{V}), k[\Gamma])$$

There is a versal family $\tilde{V}[\Gamma]$ *of* A*-modules defined on* $\underline{A}[\Gamma]$*, containing as fibres all the isomorphism classes of iterated extensions of* $\mathbf{V}$ *with extension graph* Γ*.*

Let $Mod_A^{\mathbf{V}}$ *denote the full abelian subcategory of* Mod_A *generated by the iterated extensions of the objects in* $\mathbf{V}$*, and let* $Mod_{H(\mathbf{V})}$ *be the category of finite dimensional* H*-modules. Then we have the following structure theorem, generalising a result, of Beilinson, see* [2].

Theorem 7.5. *Let* A *be any* k*-algebra, and fix a swarm,* $\mathbf{V} = \{V_i\}_{i=1}^r$*, of* A*-modules, then there exists a functor,*

$$\iota(\mathbf{V}) : Mod_{H(\mathbf{V})} \to Mod_A^{\mathbf{V}}$$

which is an isomorphism on equivalence-classes of objects, and injective on morphisms. If $\mathbf{V}$ *consists of simple modules, then* κ *is an equivalence.*

Proof. Any right $H(\mathbf{V})$-module M, is a k^r-module, so it can be decomposed as, $M = \oplus M_i$, where $M_i := Me_i$. The structure map is therefore given as,

$$\rho_0 : H(\mathbf{V}) \to End_k(M) = (Hom_k(M_i, M_j)).$$

Here, ρ_0 maps $H_{i,j}$ into $Hom_k(M_i, M_j)$, and therefore the formal family may be decomposed to give us the following k-algebra homomorphisms,

$$\begin{aligned}\rho : A &\xrightarrow{\tilde{\rho}} (H_{i,j} \otimes Hom_k(V_i, V_j)) \xrightarrow{\rho_0} (Hom_k(M_i, M_j) \otimes_k Hom_k(V_i, V_j)) \\ &= (Hom_k(M_i \otimes_k V_i, M_j \otimes V_j)) = End_k(W).\end{aligned}$$

Here $W = \oplus_{i=1}^r (M_i \otimes V_i)$, and by definition,

$$\iota(\mathbf{V})(M) := W$$

Since M by definition of $Mod_{H(\mathbf{V})}$ is of finite dimension as k-vector space, and therefore is an iterated extension of the simple modules k_i, it is clear that W is an iterated extension of the modules V_i in the family $\mathbf{V}$. Moreover we have seen that for any object being an iterated extension, along a graph Γ, of the modules in $\mathbf{V}$, we can find, a morphism

$$\phi : H(\mathbf{V}) \to k[\Gamma]$$

which obviously defines $k[\Gamma]$ as a finite dimensional $H(\mathbf{V})$-module, such that,

$$E \simeq k[\Gamma] \otimes_\phi \tilde{V} = \iota(\mathbf{V})(k[\Gamma])$$

Thus, the first part of the theorem follows from Proposition (4.1). The rest is more or less evident. □

Example 7.6 (The case of $Coh(\mathbf{P}^1)$)**.** *Consider the category of coherent* $O_{\mathbf{P}^1}$*-modules. Pick the finite swarm given by* $\mathbf{V} = \{V_1 := \omega := O_{\mathbf{P}^1}(-2), V_2 := O_{\mathbf{P}^1}(1)\}$*. Recall that* $Ext^1_{O_{\mathbf{P}^1}}(O_{\mathbf{P}^1}(m), O_{\mathbf{P}^1}(n)) = H^1(\mathbf{P}^1, O_{\mathbf{P}^1}(n-m))$*. From this, and the trivial* $Ext^2_{O_{\mathbf{P}^1}}(-,-) = 0$*, we compute the formal moduli,*

$$H(\mathbf{V}) = \begin{pmatrix} k & 0 \\ k^2 & k \end{pmatrix}.$$

According to the theorem above, we have an injection of categories,

$$Mod_{H(\mathbf{V})} \to Coh(\mathbf{P}^1)^{\mathbf{V}}$$

This should be compared to the classical Beilinson result, stating that,

$$RHom(F, -) : D^b(Coh(\mathbf{P}^1)) \to D^b(Mod_{H(\mathbf{V})}),$$

where $F = O_{\mathbf{P}^1} \oplus O_{\mathbf{P}^1}(1)$*, is an equivalence.*

Now, for every finite swarm $\mathbf{V}$*, we obtain a quiver,* $Q := Q(\mathbf{V})$*, by associating to the set* $|\mathbf{V}| = \{V_i\}$ *a set of nodes* $\{v_i \in Q\}$*, and for any two such nodes, let arrows* $\nu_{i,j} : v_i \to v_j \in Q$ *correspond to a dual base of* $Ext^1_A(V_i, V_j)$*. The free quiver-algebra* $k[Q]$ *is then defined in the obvious way, and by construction there is a surjective algebra homomorphism,* $\kappa : k[Q(\mathbf{V})] \to H(\mathbf{V})$*, and, of course, if there are no obstructions, for example if all* $Ext^2_A(V_i, V_j) = 0$*, then* κ *is an isomorphism. If the quiver* $Q = Q_1 \sqcup Q_2$*, i.e. if there are no arrows from* Q_i *to* Q_j *for* $i \neq j$*, then, obviously,* $k[Q] \simeq k[Q_1] \times k[Q_2]$*, and, if* $Q_i = Q(\mathbf{V}_i)$*, and* $\mathbf{V} = \mathbf{V}_1 \cup \mathbf{V}_2$*, then,*

$$H(\mathbf{V}) = H(\mathbf{V}_1) \times H(\mathbf{V}_2).$$

Notice that, in this case, any right-module, M*, of* $H(\mathbf{V})$ *is a sum of a* $H(\mathbf{V}_1)$*-module* M_1 *and a* $H(\mathbf{V}_2)$*-module* M_2*.*

There is in the literature a notion of mutation of a quiver Q*, without cycles, which in this general context, correspond to the situation in which* $\mathbf{S}$ *is the star (source) of a node in the quiver* Q*, considered as the swarm of all simple modules* $\mathbf{V}$ *of* $k[Q]$*. Replacing* Q *by* $(Q - \mathbf{S}) \cup \{V\}$*, where* V *is the* $k[Q]$*-module corresponding to the projective system with value* k *at any node of* $\mathbf{S}$ *and with arrows between them being identities, should now be called a mutation.*

Mutations are, in the above sense, a discrete process. In the general case we need a real dynamical theory for the time evolution of iterated extensions of our swarm $\mathbf{V}$ *of elementary particles. This is the purpose of the next subsection.*

7.3. **Interactions and Dynamics.** *Go back to the start-scenario of this section, and to our assumption that all the stuff in our world is represented by iterated extensions of the modules representing the elementary particles,* $\mathbf{V} = \{V_i\}$*. Assume that* $\mathbf{V}$ *is a swarm of* $\mathbf{A}(\sigma)$*-modules. Then* $H(\mathbf{V}) = (H_{i,j})$ *is finitely generated, but maybe not algebraic. To simplify the situation a little, we shall therefore assume there is an algebraisation,* $\mathbf{H}(\mathbf{V}) = (\mathbf{H}_{i,j})$*, for which* $H(\mathbf{V}) = (H_{i,j})$ *is the formalisation at the canonical module, given by the structure morphism* $\pi : \mathbf{H}(\mathbf{V}) \to W_0 := k^r$*. Moreover we assume there exist an extension of the versal family,*

$$\tilde{\rho} : A \to O(\mathbf{V}) = (\mathbf{H}_{i,j} \otimes_k Hom_k(V_i, V_j))$$

Now, any elementary particle, represented by the simple representation,

$$\rho : \mathbf{A}(\sigma) \to End_k(V)$$

evolves, as time goes by, maybe ending up by decaying into a composite particle, represented by the module V_∞ *which by our assumption must be an iterated extension of the simple modules in* $\mathbf{V}$*. Use the Beilinson-Theorem, above, and identify* V_∞ *with a* $\mathbf{H}(\mathbf{V}) = (\mathbf{H}_{i,j})$*-module* M_0*. There may be many extension graphs corresponding to this module. Pick one* Γ*, and consider the moduli space,* $\underline{A}[\Gamma]$ *of such iterated modules. The dynamics of* $\underline{A}[\Gamma]$*, is now taken care of by the same method that we applied at the outset of this study. We must look for a dynamical structure for* $\mathbf{H}(\mathbf{V})$*, i.e. look at* δ*-stable ideals,*

$$(\sigma) \subset Ph^\infty(\mathbf{H}(\mathbf{V}))$$

and we must be prepared to study the set of morphisms,

$$\mathbf{H}(\mathbf{V})(\sigma) := Ph^\infty(\mathbf{H}(\mathbf{V}))/(\sigma) \to k[\Gamma].$$

Any such will, as we know, give us a formal curve of iterated modules with extension graph Γ*. Then we may try to copy the use of Theorem (1.21) and Theorem (1.23). In the general situation this does not seem so easy, but in our Toy Model case, it should be possible, as we have had a glimpse of in the Example (7.2), where something looking very much like a mathematical equivalent to the Weak Force, popped up.*

One question that comes up in relation to a more or less serious interpretation of our mathematical model is, can we have an interaction of several particles the outcome of which is an elementary particle? It is reasonable to conjecture the following result.

Theorem 7.7. *Suppose, in the situation above, that there is a deformation, W, of W_0, as $\mathbf{H}(\mathbf{V})$-module, such that,*

$$\pi' : \mathbf{H}(\mathbf{V}) \to End_k(W) \simeq M_r(k),$$

deforming π, is surjective. Then $\iota(\mathbf{V})(W)$ is a simple A module.

Notice that if k is algebraically closed, any simple deformation of W_0 as $\mathbf{H}(\mathbf{V})$-module must necessarily be given by a surjective structure map,

$$\rho_W : \mathbf{H}(\mathbf{V}) \to End_k(W) = M_r(k).$$

The existence of a surjective homomorphism of k-algebras, ρ_W, is now a problem we have to address. Consider for this purpose, the following definition, see [11], *and* [4],

Definition 7.8. *Let Q be the quiver associated to the family $\mathbf{V}$ of A-modules. A subgraph, Γ of Q is called Massey-trivial if all Massey products containing non-trivial subgraphs of Γ vanish.*

We may then prove the following result,

Theorem 7.9. *Suppose there is a Massey-trivial connected complete cycle, Γ, of Q. Then there is a simple deformation, W, of W_0, as $\mathbf{H}(\mathbf{V})$-module.*

Obviously, this result make the interaction scenario sketched above quite involved, even though we may concentrate on a finite number of particles in the theory we have chosen,

Example 7.10 (Spontaneous Interaction and Evolution)**.** *Suppose given a sub-family of finite-dimensional simple modules, $\mathbf{S} \subset \mathbf{V}$, and suppose there is a surjective homomorphism of k-algebras,*

$$\mu : \mathbf{H}(\mathbf{S}) \to M_s(k),$$

where s is the number of elements in $\mathbf{S}$. Consider now the composition,

$$\tilde{\mu} : A \xrightarrow{\tilde{\rho}} O(\mathbf{S}) \overset{\mu\otimes id}{\to} M_n(k) \otimes End_k(\bigoplus_{V\in\mathbf{S}} V)) = End_k(\bigoplus_{V\in\mathbf{S}} V) = End_k(U),$$

where, $U = \iota(\mathbf{S})(k^s)$. According to the results above, we see that this composition defines a new simple A-module. The process of replacing $\mathbf{V}$ by $(\mathbf{V} - \mathbf{S}) \cup \{U\}$, is now a mutation.

7.4. **Bialgebras and Quantum Groups in our Models.** *In physics, interactions are often represented by tensor products of the representations involved. For this to fit into the philosophy we have followed here, we must give reasons for why these tensor products pop up, seen from our moduli point of view. It seems to me that the most natural point of view might be the following: Suppose A is the moduli algebra parametrizing some objects $\{X\}$, and B is the moduli algebra for some other objects $\{Y\}$, then considering the product, or rather, the pair, (X, Y), one would like to find the moduli space of these pairs. A good guess would be $A \otimes_k B$ since it algebraically defines the product of the two moduli spaces. However, this is, as we know, too simplistic. There are no reasons why the pair of two objects, should deform independently, unless we assume that they do not fit into any ambiant space, i.e. unless the two objects are considered to sit in totally separate universes, and then we have done nothing, but doubling our model in a trivial way. In fact, we* *should, for the purpose of explaining the role of the product, assume that our entire universe is parametrized by the moduli algebra A, and accepting, for two objects X and Y in this universe, that the superposition, or the pair, (X, Y), correspond to a collection of, maybe new, objects parametrized by A. This is basically what we do, when we consider two particles, modelled by the representations, V and W, of some moduli k-algebra A, and then consider the tensor product of the representations, say $V \otimes_k W$, as a new representation, modelling a collection of new particles. As above we observe that the obvious moduli algebra of tensor-products of representations, $V \otimes W$, is $A \otimes_k A$. But since these representations should be of the same nature as any representation of A, this would, by universality, lead to a homomorphism of moduli algebras,*

$$\Delta : A \to A \otimes_k A,$$

i.e. to a bialgebra structure on the moduli algebra A. This is just one of the reasons why mathematical physicists are interested in the vast theory of Tannaka categories and quantum groups. See now that, if A_0 is commutative, and if we put $A = Ph(A_0)$, then there exist a canonical homomorphism,

$$\Delta : A \to A \otimes_{A_0} A.$$

In fact, the canonical homomorphism $i : A_0 \to Ph(A_0)$ identifies A_0 with the a sub-algebra of $A \otimes_{A_0} A$. Moreover, $d \otimes 1 + 1 \otimes d$ is a natural derivation, $A_0 \to A \otimes_{A_0} A$., so by universality, Δ is defined. Thus, for representations of $A := Ph(A_0)$ there is a natural tensor product, $- \otimes_A -$. See also [18], *Example (4.14), for an effort to use this in explaining some interactions in physics.*

8. More Examples and Some Ideas

Example 8.1. *Let us consider the notion of interaction between two particles, $V_i := k(v_i) \in Simp_1(k[x_1, ..., x_r])$, $i = 1, 2$, in the above sense. Look at the $A_0 := k[x_1, ..., x_r]$-module $V := V_1 \oplus V_2$, i.e. the homomorphism of k-algebras, $\rho_0 : A_0 \to End_k(V)$, and let us try to extend this module-structure to a representation,*

$$\rho : Ph^\infty A_0 \to End_k(V).$$

We have the following relations in $Ph^\infty A_0$:

$$[x_i, x_j] = 0$$
$$[dx_i, x_j] + [x_i, dx_j] = 0$$
$$....$$
$$\sum_{l=0}^{p} \binom{p}{l} [d^l t_i, d^{p-l} t_j] = 0.$$

Put,

$$\rho_0(x_i) = \rho_0(d^0 x_i) = \begin{pmatrix} x_i(1) & 0 \\ 0 & x_i(2) \end{pmatrix} =: \begin{pmatrix} \alpha_i^0(1) & 0 \\ 0 & \alpha_i^0(2) \end{pmatrix},$$

and, $\alpha_i^0(r, s) := x_i(r) - x_i(s)$, $r, s = 1, 2$. Let, for $q \geq 0$,

$$\rho(d^q x_i) = \begin{pmatrix} \alpha_i^q(1) & r_i^q(1,2) \\ r_i^q(2,1) & \alpha_i^q(2) \end{pmatrix},$$

Now, compute, for any $p \geq k$,

$$[\rho(d^k x_i), \rho(d^{p-k} x_j)] =$$
$$\begin{pmatrix} r_i^k(1,2) r_j^{p-k}(2,1) - r_j^{p-k}(1,2) r_i^k(2,1) & r_j^{p-k}(1,2)\alpha_i^k(1,2) + r_i^k(1,2)\alpha_j^{p-k}(2,1) \\ r_i^k(2,1)\alpha_j^{p-k}(1,2) + r_j^{p-k}(2,1)\alpha_i^k(2,1) & r_i^k(2,1) r_j^{p-k}(1,2) - r_j^{p-k}(2,1) r_i^k(1,2) \end{pmatrix}$$

and observe that,

$$[\rho(d^p x_i),\rho(x_j)]+[\rho(x_i),\rho(d^p x_j)]=$$
$$\begin{pmatrix} 0 & r_i^p(1,2)\alpha_j^0(2,1)+r_j^p(1,2)\alpha_i^0(1,2)\\ r_i^p(2,1)\alpha_j^0(1,2)+r_j^p(2,1)\alpha_i^0(2,1) & 0\end{pmatrix}$$

After some computation we find the following condition for these matrices to define a homomorphism ρ, independent of the choice of diagonal forms,

$$r_i^k(r,s)=\sum_{l=0}^{k}\binom{k}{l}\sigma_{k-l}\alpha_i^l(r,s),\ \ r,s=1,2,$$

where the sequence $\{\sigma_l\},\ l=0,1,...$ is an arbitrary sequence of coupling constants, with $\sigma_0=0$ and σ_l of order l. By recursion, we prove that this is true, for $k\le p-1$, therefore $r_i^k(1,2)=-r_i^k(2,1)$, and so the diagonal elements above vanish, i.e.

$$r_i^k(1,2)r_j^{p-k}(2,1)-r_j^{p-k}(1,2)r_i^k(2,1)=0.$$

The general relation is therefore proved if we can show that with the above choice of $r_i^k(r,s)$ we obtain, for every $p\ge 0$,

$$\sum_{k=0}^{p}\binom{p}{k}(r_j^{p-k}(1,2)\alpha_i^k(1,2)+r_i^k(1,2)\alpha_j^{p-k}(2,1))=0,$$

and this is the formula,

$$\sum_{k=0}^{p}\binom{p}{k}\sum_{l=0}^{p-k}\binom{p-k}{k}\sigma_{p-k-l}\alpha_j^l(1,2)\alpha_i^k(1,2)+$$
$$\sum_{k=0}^{p}\binom{p}{k}\sum_{l=0}^{k}\binom{k}{l}\sigma_{k-l}\alpha_i^l(1,2)\alpha_j^{p-k}(2,1)=$$
$$\sum_{k=0}^{p}\binom{p}{k}\sum_{l=0}^{k}\binom{k}{l}\sigma_{k-l}(\alpha_j^l(1,2)\alpha_i^{p-k}(2,1)-\alpha_i^l(1,2)\alpha_j^{p-k}(2,1))=0,$$

Notice that the relations above are of the same form for any commutative coefficient ring C, i.e. they will define a homomorphism,

$$\rho: Ph^\infty A_0\to M_2(C),$$

for any commutative k-algebra C. Now, consider the Dirac time development $D(\tau)=exp(\tau\delta)$ in $\mathbf{D}$, the completion of $Ph^\infty A_0$. Composing with the morphism ρ defined above, we find a homomorphism,

$$\rho(\tau): Ph^\infty(A_0)\to M_2(k[[\tau]]),$$

where,

$$X_i:=\rho(\tau)(t_i)=\begin{pmatrix}\Phi_i(1) & \Phi_i(1,2)\\ \Phi_i(2,1) & \Phi_i(2),\end{pmatrix}$$

and

$$\Phi_i(r)=\sum_{n=0}^{\infty}1/(n!)\tau^n\cdot\alpha_i^n(r),\ \ r=1,2,$$
$$\Phi_i^0(r,s)=\sum_{n=0}^{\infty}1/(n!)\tau^n\cdot\alpha_i^n(r,s),\ \ r,s=1,2,$$
$$\sigma=\sum_{n=0}^{\infty}1/(n!)\tau^n\cdot\sigma_n,$$
$$\Phi_i(r,s)=\sigma\cdot\Phi_i^0(r,s),\ \ r,s=1,2.$$

This must be the most general, Heisenberg model, of motion of our two particles, clocked by τ. Observe that the interaction acceleration $\Phi_i(r,s)$ is pointed from r to s, just like in physics! The formula above, is now seen to be a consequence of the obvious equality of the two products of the formal power series, $\sigma\cdot(\Phi_i^0(1,2)\Phi_j^0(1,2))$ and $\sigma\cdot(\Phi_j^0(1,2)\Phi_i^0(1,2))$, just compare the coefficients of the resulting power series. What we have got is nothing but a formula for commuting matrices $\{X_i\}_{i=1}^d$ in $M_2(k[[\tau]])$, since for such matrices we must have,

$$\frac{d^n}{d\tau^n}[X_i,X_j]=0,\ n\geq 1.$$

The eigenvalues $\lambda_i(1,\tau)$, and $\lambda_i(2,\tau))$ of X_i describes points in the space that should be considered the trajectories of the two points under interaction. This is OK, at least as long as we are able to label them by 1 and 2 in a continuous way with respect to the clock time τ. If all coupling constants, $\sigma_n,\ n\geq 0$, vanish, then the system is simply given by the two curves $\Phi(r):=(\Phi_1(r),\Phi_2(r),...,\Phi_d(r)),\ r=1,2$, where d is the dimension of A_0. In general, the eigenvalues of X_i are given by,

$$\begin{aligned}\lambda_i(r)=&\,1/2\cdot(\Phi_i(1)+\Phi_i(2))\\ &(-1)^r 1/2\sqrt{(\Phi_i(1)+\Phi_2(2))^2-4(\Phi_i(1)\cdot\Phi_i(2)+\sigma^2(\Phi_i(1)-\Phi_i(2))^2)},\end{aligned}$$

for $r=1,2$. Clearly,

$$\begin{aligned}1/2(\lambda_i(1)+\lambda_i(2))&=1/2(\Phi_i(1)+\Phi_i(2))\\ (\lambda_i(1)-\lambda_i(2))^2&=(\Phi_i(1)+\Phi_2(2))^2-4(\Phi_i(1)\cdot\Phi_i(2)+\sigma^2(\Phi_i(1)-\Phi_i(2))^2)\\ &=(1-4\sigma^2)(\Phi_i(1)-\Phi_i(2))^2.\end{aligned}$$

Denote by, $\lambda(r)=(\lambda_1(r),\lambda_2(r),...,\lambda_d(r)),\ r=1,2$, the vectors corresponding to the eigenvalues, and by,

$$o:=1/2(\lambda(1)+\lambda(2))=1/2(\Phi(1)+\Phi(2))$$

the common median, and put,

$$\begin{aligned}R_0:&=|(\Phi(1)-\Phi(2)|\\ R:&=|\lambda(1)-\lambda(2)|,\end{aligned}$$

then

$$R=\sqrt{(1-4\sigma^2)}R_0.$$

Choose coupling constants such that,

$$\frac{d^2}{d\tau^2}R=rR^{-2},$$

where r is a constant we should expect Newton like interaction. If $\sigma\geq 1/2$, the point particles are confounded, the eigenvalues of X_i become imaginary, and the result is no longer obvious. If we pick $\sigma=\sqrt{1-r^2R_0^{-2}}$, the relative motion will be circular, with constant radius r about o.

Example 8.2. *Let B be the free k-algebra on two non-commuting symbols, $B=k<x_1,x_2>$, and see (7.2). Let P_1 and P_2 be two different points in the (x_1,x_2)-plane, and let the corresponding simple B-modules be V_1,V_2. Then, $Ext^1_B(V_1,V_2)=k$. Let Γ be the quiver,*

$$V_1\longleftrightarrow V_2,$$

then an interaction mode is given by the following elements: First the formal moduli of $\{V_1,V_2\}$,

$$H:=\begin{pmatrix}k<u_1,u_2> & <t_{1,2>}\\ <t_{2,1>} & k<v_1,v_2>\end{pmatrix},$$

then the k-algebra,

$$k\Gamma := \begin{pmatrix} k & k \\ 0 & k \end{pmatrix},$$

and finally a homomorphism,

$$\phi : H \longrightarrow k\Gamma$$

specifying the value of $\phi(t_{1,2}) \in Ext^1_B(V_1, V_2)$. Since $Hom_k(V_i, V_j) = k$, we obtain $V = k^2$, and we may choose a representation of $\phi(t_{1,2})$ as a derivation, $\psi_{1,2} \in Der_k(B, Hom_k(V_1, V_2))$, such that the B-module $V = k^2$ is defined by the actions of x_1, x_2, given by,

$$X_1 := \begin{pmatrix} \alpha_1 & 1 \\ 0 & \alpha_2 \end{pmatrix},\ X_2 := \begin{pmatrix} \beta_1 & 0 \\ 0 & \beta_2 \end{pmatrix},$$

where $P_1 = (\alpha_1, \beta_1)$ and $P_2 = (\alpha_2, \beta_2)$. V is therefore an indecomposable B-module, but not simple. If we had chosen the following quiver,

$$V_1 \longleftrightarrow^{\epsilon_{1,2}}_{\epsilon_{2,1}} V_2,$$

where $\epsilon_{i,j}\epsilon_{j,i} = 0,\ i, j = 1, 2$, then the resulting B-module $V = k^2$ would have been simple, represented by,

$$X_1 := \begin{pmatrix} \alpha_1 & 1 \\ 0 & \alpha_2 \end{pmatrix},\ X_2 := \begin{pmatrix} \beta_1 & 0 \\ 1 & \beta_2 \end{pmatrix}.$$

In general, if $B = \mathbf{A}(\sigma)$, where (σ) is a dynamical system with Dirac derivation δ, any interaction mode producing a simple module V, thus a point $v \in Simp(\mathbf{A}(\sigma))$, represents a creation of a new particle from the information contained in the interacting constituencies. Moreover, any family of state-vectors $\psi_i \in V_i$, produces a corresponding state-vector $\psi := \sum_{i=1,\ldots} \psi_i \in V$, and Theorem (3.3) then tells us how the evolution operator acts on this new state-vector. If the created new particle V is not simple, the Dirac derivation $\delta \in Der_k(\mathbf{A}(\sigma))$, will induce a tangent vector $[\delta](V) \in Ext^1_{PhA}(V, V)$ which may or may not be modular, or pro-representable, see section (3), which means that the particles V_i, when integrated in this direction, may or may not continue to exist as distinct particles, with a non-trivial endomorphism ring, or, with a Lie algebra of automorphisms, equal to k^2. If they do, this situation is analogous to the case which in physics is referred to as the super-selection rule. Or, if $[\delta] \in Ext^1_{\mathbf{A}(\sigma)}(V, V)$ does not sit (or stay) in the modular stratum, the particle V looses automorphismes, and may become indecomposible, or simple, instantaneously. We may thus create new particles, and we have in Example (4.7) discussed the notion of lifetime for a given particle. In particular we found that the harmonic oscillator had ever-lasting particles of k-rank 2. If, however, we forget about the dynamical system, and adopt the more physical point of view, picking a Lagrangian, and its corresponding action, we may easily produce particles of finite lifetime.

Example 8.3. *Let, as in (4.6) $A := PhA_0 = k < x, dx >$, with $A_0 = k[x]$ and put $x =: x_1,\ dx =: x_2$. Consider the curve of two-dimensional simple A-modules,*

$$X_1 = \begin{pmatrix} 0 & 1+t \\ 0 & t \end{pmatrix}$$
$$X_2 = \begin{pmatrix} t & 0 \\ 1+t & 0 \end{pmatrix},$$

either as a free particle, with Lagrangian $1/2dx^2$, or as a harmonic oscillator with Lagrangian $1/2dx^2 + 1/2x^2$. The action is, in the first case, $S = 1/2TrX_2^2 = t^2$, and in the second case, $S = 1/2Tr(X_2^2 + 1/2X_1^2) = 2t^2$. Thus the Dirac-derivation *becomes* $\nabla S = t\frac{\partial}{\partial t}$*, or* $\nabla S = 2t\frac{\partial}{\partial t}$*. Computing the Formanek center* f*, see (3.6), we find,*

$$f(t) = t^2(1+t)^2 - (1+t)^4.$$

The corresponding particle, born at $t < 0$*, decays at* $t = -1/2$*, and thus has a finite lifetime. Of course, the parameter* t *in this example, is not our time, and the curve it traces is not an integral curve of the dynamic system of the harmonic oscillator, see (3.7). This shows that one has to be careful about mixing the notions of dynamic system, and the dynamics stemming from a Lagrangian-, or from a related action-principle.*

Example 8.4. *Suppose we are given an element* $v \in Simp(A(\sigma))$*, and consider the monodromy homomorphism,*

$$\mu(v) : \pi_1(v; Simp_n(A(\sigma))) \to Gl_n(k).$$

In some physics literature v *is called Fermionic, or Anyonic, if there exist a loop in* $Simp_n(A(\sigma))$ *for which the monodromy is non-trivial. Assume the tangent of this loop at* v *is given by* $\xi \in Ext^1_{A(\sigma)}(V,V)$*. Since this tangent has no obstructions it is reasonable to assume that there is a quotient,*

$$H(\{V,V\}) \to \begin{pmatrix} k & f^- \\ f^+ & k \end{pmatrix},$$

with $f^-f^+ = f^+f^- = 1$*. This would give us an* $A(\sigma)$*-module, with structure map,*

$$A(\sigma) \to \begin{pmatrix} End_k(V) & f^- \otimes_k End_k(V) \\ f^+ \otimes_k End_k(V) & End_k(V) \end{pmatrix},$$

i.e. a simple $A(\sigma)$*-module of Fermionic type, see* [18]*, Chapter 4, Grand Picture etc. Notice that, for a given connection on the vector-bundle* $\tilde{V}$*, the correct monodromy group to consider for the sake of defining Bosons and Fermions, etc., should probably be the infinitesimal monodromy group generated by the derivations of the curvature tensor,* R*.*

9. The present state of the Standard Model

Let us end with a bird's-eye view of the current state of the Standard Model, and compare it with our simple toy model (and not with the general cosmological model considered in Subsection (4.4). The ingredients are quite different. First, the SM is build upon the 4-dimensional Minkowski space, where proper time must be compared to our λ_3*, and our two other null-directions are missing. Then, our gauge groups, which turns out to contain the gauge group of the SM, are no longer only related to the metric, transporting the symmetry to the physicists Holy Grail, the Hilbert space, where the different states pop up. Our gauge groups are the Lie algebras responsible for the equivalence relation in our non-commutative algebraic space of representations of the affine algebra* $Ph(\tilde{H})$*, basically on* $\Theta_{\tilde{H}}$*. Our basic elementary particles comes out naturally; the weak bosons as the generators of* $\mathfrak{g}$*, and the gluons as the generators of* $\mathfrak{su}(3)$*, the massless states are* $\{c_1, c_2, c_3\}$*, and the massive and charged ones are* $\{d_1, d_2, d_3\}$*. The corresponding simple modules generate all particles, by tensor products, and by iterated extensions. The chirality, massiveness, charge, etc. all comes out canonically, via reasonably canonical choices of markers.*

There are also many similarities. Both the Weyl- and the Dirac-Spinors have the same dimension and the same symmetries in SM as in our model.

We have the possibility to define 3 different mass terms, namely, via,

$$\rho : \tilde{H}(\sigma_g) \to End_k(\Theta_{\tilde{H}})$$

the operators,

$$\rho(d\lambda_i) \in End_k(\Theta_{\tilde{H}})$$

We already know that there are, at any point $(o,p) \in \underline{\tilde{H}}$*, three tangent planes, called* $B_o, B_p, A_{o,p}$*, the two first "orthogonal " to the* (op)*-direction, and the third containing the canonical light and 0-velocity directions at that point. There are natural bases for these planes,* $\{(c_1+d_1),(c_2+d_2)\}$*,* $\{(-c_1+d_1),(-c_2+d_2)\}$*, and* $\{c_3, d_3\}$*, respectively, and the action of* $\mathfrak{g}$ *on these planes, can be read from the the table in Subsection (6.2). We have also seen that the the Pauli matrices turns up as,*

$$\sigma^1 = e+f = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
$$\sigma^2 = -ie+if = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix},$$
$$\sigma^3 = h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and that the parity operator, P *induces the Dirac matrices,*

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ \gamma^k = \begin{pmatrix} 0 & \sigma^k \\ \sigma^k & 0 \end{pmatrix},\ k = 1,2,3.$$

as well as the new operators,

$$\gamma^{k+3} = \begin{pmatrix} \sigma^k & 0 \\ 0 & -\sigma^k \end{pmatrix},\ k = 1,2,3,$$

acting on, $B_o \oplus B_p$*, such that,*

$$\forall p \neq q,\ \gamma^p\gamma^q = -\gamma^q\gamma^p, \gamma^p\gamma^p = 1,\ p,q = 1,2,3,4,5,6.$$

Chirality, in the physicists language, was explained as follows. The morphism P*, extended to* $B_o \oplus B_p$*, is in the basis chosen above, given by the matrix,*

$$\gamma^5 = \begin{pmatrix} 0 & id \\ id & 0 \end{pmatrix},$$

which turns left-handedness to right-handedness, with respect to the direction (o,p)*, respectively* (p,o)*. Recall also that there are faithful representations of* $\mathfrak{g}$ *in the bundle* $B_o \oplus B_p$*, and that* $\mathfrak{g}$ *kills* $A_{o,p}$*. We see that the representations,* B_o *and* B_p*, the points in the non-commutative quotient of the moduli space* $\tilde{H}(\sigma_g)$*, by the gauge group* $\mathfrak{g}$*, are the 2-component Weyl-Spinors of the physicists., and the space* $B_o \oplus B_p$ *is the space of Dirac-Spinors, and the Hamiltonian, the operator representing time and energy* Q*, looks the same in both models.*

We may now cook up a list of particles vaguely resembling the ingredients of the Standard Model from our arsenal of representations of the local gauge group, $\mathfrak{g}* = \mathfrak{g} \oplus < h_2 > \subset \mathfrak{g} \oplus \mathfrak{su}(3)$*. Let us start with the simple sub-representations of* $\Theta_{\tilde{H}}$*. They coincide with the elementary simple representations of* $\mathfrak{sl}(2)$*,*

$$L_c(1) =< c_1, c_2 >, L_d(1) =< d_1, d_2 >$$

The tensor products of these will produce lots of simple representations, writing d_id_j *for* $1/2(d_i \otimes d_j + d_j \otimes d_i)$*, and in general,* $d_id_jd_k$ *for the symmetric tensor product, the first tensor (or symmetric) products of* $L_d(1)$*, will give us the simple representations,*

$$L_d(2) =< d_1d_1, d_1d_2, d_2d_2 >, L_d(3) =< d_1d_1d_1, d_1d_1d_2, d_1d_2d_2, d_2d_2d_2, >$$

We find, of course, the same results for the tensor products of $L_c(1)$*, and using elementary representation theory of the Lie algebras* $\mathfrak{g}_0, \mathfrak{g}, \mathfrak{su}(3)$*, or* $\mathfrak{so}(3)$*, we may,* *in principle classify all the possible representations that would qualify for our Furniture of the Universe, as defined in the previous sections. Since the following is meant to show the relations between the Toy Model and the Standard Model, we shall concentrate our attention to the build up of the so called elementary particles, which are defined in terms of quarks. The Bosons are already treated.*

Put, Isospin=$\mathfrak{h}_1$*, Charge=*$\mathfrak{h}_2$*, Weak Isospin=* $I_3 = 3/4\mathfrak{h}_2 + 1/2\mathfrak{h}_1$*, Weak Hypercharge =*$Y_W = 1/2\mathfrak{h}_2 - \mathfrak{h}_1$*, and use these numbers as markers.*

Then consider the diagram:

$Markers:$	$1/2 \cdot h$	u	$\mathfrak{h}_1$	$\mathfrak{h}_2$	I_3	Y_W
$L_d(0):$						
$uq = d_3$	0	0	0	$2/3$	$1/2$	$1/3$
$L_d(1):$						
$dq_1 = d_1$	$1/2$	1	$1/2$	$-1/3$	0	$-2/3$
$dq_2 = d_2$	$-1/2$	1	$-1/2$	$-1/3$	$-1/2$	$1/3$
$L_d(1):$						
$dq_2^- = d_1d_3$	$1/2$	1	$1/2$	$1/3$	$1/2$	$-1/3$
$dq_1^- = d_2d_3$	$-1/2$	1	$-1/2$	$1/3$	0	$2/3$
$L_d(1):$						
$p_1 = d_1d_3d_3$	$1/2$	1	$1/2$	1	1	0
$p_2 = d_2d_3d_3$	$-1/2$	1	$-1/2$	1	$1/2$	1

$Markers:$	$1/2 \cdot h$	u	$\mathfrak{h}_1$	$\mathfrak{h}_2$	I_3	Y_W
$L_d(2):$						
d_1d_1	1	2	1	$-2/3$	0	$-4/3$
$uq^- = d_1d_2$	0	2	0	$-2/3$	$-1/2$	$-1/3$
d_2d_2	-1	2	-1	$-2/3$	-1	$2/3$
$L_d(2):$						
$\nu_e = d_1d_1d_3$	1	2	1	0	$1/2$	-1
$n = d_1d_2d_3$	0	2	0	0	0	0
$\nu_e^- = d_2d_2d_3$	-1	2	-1	0	$-1/2$	1
$L_d(3):$						
$d_1d_1d_1$	$3/2$	3	$3/2$	-1	0	-2
$e_L = d_1d_1d_2$	$1/2$	3	$1/2$	-1	$-1/2$	-1
$e_R = d_1d_2d_2$	$-1/2$	3	$-1/2$	-1	-1	0
$d_2d_2d_2$	$-3/2$	3	$-3/2$	-1	$-3/2$	1

To go from left to right handedness, in the physics language, comes out by just exchanging d_1 and d_2. Here one may see that, $e_L + \nu_L \rightarrow e_R$, and one easily find reasons for the decay,

$$n \rightarrow p + e + \nu_e.$$

Notice also that we may, in an obvious way, identify ν_e with the element f and $e + f$ with interchanging d_1, d_2, and notice also that particles and antiparticles ,which usually are denoted by an "overline" and related to the reversing of time, in our model add up to n, the "neutron" in the model.

The part of the SM that the physicists call; The Electroweak Sector, is concerned with representations of $\tilde{H}(\sigma_g)/\mathfrak{g}$, on the complexified bundles, $B_o^c := \mathbf{C} \otimes_{\mathbf{R}} B_o$, $B_p^c := \mathbf{C} \otimes_{\mathbf{R}} B_p$, and the sum, $B_o^c \oplus B_p^c$, i.e. it is concerned with representations on

Weyl Spinors or Dirac Spinors. The possible representations on spinors, i.e. on B_o^c or B_p^c, are then usually written,

$$D_p = \xi_p + \nabla_p, \nabla_p \in U(\mathbf{C} \otimes \mathfrak{sl}(2)).$$

The typical example is, for trivial metric g,

$$D_\mu = \imath\delta_\mu - g'1/2Y_W B_\mu - g1/2\overline{\tau}_L \overline{W}_\mu.$$

Here, B_μ is strange, and it seems that,

$$Y_W = 2(\mathfrak{h}_2 - I_3) = (1/2\mathfrak{h}_2 \pm \mathfrak{h}_1),\ \overline{\tau}_L \overline{W}_\mu = \sum_{k=1}^{3} W_\mu^k \sigma^k,$$

where $\pm$ signifies (right/left), and g' and g are coupling constants, not related to any metric. Notice that here we have choices, considering left or right handed systems. From the above one can easily deduce the:

Electromagnetic Sector, as I have explained above, in section (1.9), see also [18]. *The new thing for the Electroweak case is that one considers representations on the Dirac Spinors, of the type,*

$$\mathbf{D}_\mu = \gamma^\mu D_\mu.$$

The corresponding energy equation is therefore,

$$(Q - E^2)\psi = 0,$$

where,

$$Q = \sum_{\mu,\nu} g^{\mu,\nu} \mathbf{D}_\mu \mathbf{D}_\nu.$$

Assume we have a trivial metric, and consider the equation,,

$$(\sum_p \gamma_p D_p - E)(\sum_q \gamma_q D_q + E) = \sum_{p,q} \gamma_p D_p \gamma_q D_q - E^2,$$

where we either assume that $[\gamma, D_p] = 0$, so that $\sum_{p,q} \gamma_p D_p \gamma_q D_q = \sum_p D_p D_p$or accept that the difference is without interest, we find a solution of the energy equation is the same as a solution of the Dirac type equation,

$$\sum_p (\gamma_p D_p \pm E)(\psi) = 0.$$

The consequence is that we should look for $B_\mu, W_\mu^k \in \tilde{H}$, and ψ, such that,

$$\overline{\psi}(\sum_p \gamma_p D_p)\psi,$$

is constant. This is exactly the Lagrangian in the electroweak sector,

$$\mathbf{L}_{EW} = \sum_\mu \psi^\dagger \gamma^\mu (\imath\delta_\mu - g'1/2Y_W B_\mu - g1/2\overline{\tau}_L \overline{W}_\mu)\psi$$

The Quantum Chronodynamic Sector, is the theory that takes care of quarks, and fermions. Here we are interested in the representations ρ of $\tilde{H}(\sigma)(\mathfrak{g}_1)$ on $\tilde{\Delta}_\mathbf{C} := \mathbf{C} \otimes_\mathbf{R} \tilde{\Delta}$. We know that they are of the form,

$$\rho(dt_p) = D_p := \xi_p + \nabla_p, \nabla_p \in U(\mathbf{C} \otimes \mathfrak{su}(3)),$$

treating ∇_p as a potential. In QCD the corresponding representations are taken as,

$$D_\mu = \delta_\mu - \imath g_s G_\mu^i T^i,$$

where $\{T^i\}$ is a basis for $\mathfrak{su}(3)$. The Lagrangian is then guessed to be,

$$\mathbf{L}_{QCD} = \imath U^\dagger(\delta_\mu - \imath g_s G_\mu^i T^i)\gamma^\mu U + \imath D^\dagger(\delta_\mu - \imath g_s G_\mu^i T^i)\gamma^\mu D$$

where U are Dirac spinors associated with up-quarks, and D are Dirac spinors associated with down-quarks, and g_s is the strong coupling constant.

The last provisos of the equation, seems very peculiar. However, if we understand that the model for this Lagrangian is not the usual energy equation, but the first order Dirac equation and moreover, that the operators γ^μ are just defined on $B_o \oplus B_p$, and the up-quark is not an element in $B_o \oplus B_p$, then it becomes reasonable. Recall that all elements in $\mathfrak{g}$ kill the up quark d_3, but do not kill the two down-quarks d_1, d_2. However, there should be a difference between left and right handedness, here as above.

The Higgs Sector, is the last case I shall mention. The Higgs Lagrangian is, before Symmetry Breaking,

$$\mathbf{L}_H = \phi^\dagger(\delta^\mu - \imath/2(g'Y_W B^\mu + g\tau\overline{W}^\mu))(\delta_\mu + \imath/2(g'Y_W B_\mu + g\tau\overline{W}_\mu))\phi - \lambda^2/4(\phi^\dagger\phi - v^2)^2$$

where $\phi = (\phi^+, \phi^0)$ (not the coordinate in $\tilde{H}$!) is a Spinor, with the electric charges, Q, as indicated, both components having weak isospin, $Y_W = 1$.

The classical interpretation seems strange, and I propose that this has to do with the restriction to the 4-dimensional Minkowski/Einstein-space. Working in our 6-dimensional moduli-space, we have much more room to explain, as we have done, the difference between left and right-handedness, and to treat mass, total and kinetic energy on the same level, as we have seen above. The rest, I assume, is physics.

10. End Words

This note is an extension of several papers, see [18], [20], [19], *and has been the subject of quite a number of talks, at Paris/ENS, Stockholm/Mittag Leffler, Marseille/Luminy, Oslo, Angers, Trieste, Belfast, Mulhouse, Nice, Praha, Lahore, Tallinn, Casablanca, Rabat, and Strasbourg, during the last 20 years or so. It is based on the ideas and the general philosophy presented first in* [16], *and then in* [18], *in which the main new idea is TIME, considered as a metric on the moduli space of the phenomena we choose to study.*

The underlying mathematical tool is non-commutative deformation theory, see [12], *and the non-commutative algebraic geometry, based on this deformation theory, see* [13].

This non-commutative algebraic geometry is quite different from the more well known non-commutative geometry, proposed by Alain Connes, see e.g. [28], *and its application to modern physics is equally different from his. In particular, it seems to me that the set up of Connes, assuming that existence of a C^*-algebra of operators on some Hilbert space, and a convenient spectral triple, is lacking the kind of explanatory power that I needed to be able to work with these ideas from physics.*

My purpose has been to construct a model for the study of most natural phenomena, given a reasonable mathematical model for the primary observation. The Toy Model, $\underline{\tilde{H}}$ studied above, is one simple choice, and it would be a miracle if it turned out to be more than an intermediate step on the long road towards understanding (some nontrivial part of) nature.

However, it has a series of interesting properties. The choice of Minkowski, and of Einstein, of the metric $ds^2 = dt^2 - (dx_1^2 + dx_2^2 + dx_3^2)$, considering time as a 4th dimension, works fine in explaining the maximality of the velocity of light, but it also contains the mathematical model of a tachyon!, which seems to have been forgotten, regretted or denied.

In the Toy Model $\underline{\tilde{H}}$, the mathematical notion of velocity is different, and the set of velocities turn out to be a compact space, so there exist an absolute maximal speed.

Treating Einstein's proper time as one of the 0-velocities in the Toy Model, we see that the Minkowski metric turns into the trivial metric of a 4-dimensional Riemannian space, $dt^2 = dx_1^2 + dx_2^2 + dx_3^2 + ds^2$, *and the Schwartzchild metric turns into a metric very close to ours, see section (2). Deforming these metrics into metrics on the corresponding blow-up of the s-line, makes it part of the restriction of the Toy Model* $\underline{\tilde{H}}$, *to* $M(B)$, *but the interpretation of this space as a moduli space of an observer observing an observed in 3-space, goes lost. For this we need more dimensions, at least 6.*

This interpretation is basic; as we shall identify tangent vectors at a point of $\underline{\tilde{H}}$ *as (glimpses of) elementary particles at that point. This provides models for 3 charge-and mass-less, and 3 charged and massy, elementary states, generating all the other state-spaces of particles in our natural fauna.*

The moduli spaces of the mathematical models of the phenomena we are interested in, and their dynamical superstructure, create our algebras of observables. Their representations, or their measurements, are the main objects of our study. Therefore the basic elements of the theory are the representations of the algebras of observables. The theory of non-commutative algebraic geometry furnishes a framework for this theory, by making clear the role of the symmetries, the gauge groups, in the choice of representations to be studied.

The evolution equations defined by the universal Dirac derivation, and a choice of Dynamical Structure, give us an analytic continuation of a properly prepared representation. We have not, in this paper, ventured into concrete calculations of solutions to the equations that pop up. This will be the subject of later works. The main theoretical problem that is not yet covered is, however, related to the physical notion of Interaction, see above. We have included a short sketch of a mathematical model for interactions, for the general model, including a bold proposal for a mathematical explanation of the electro-week force. But this is not entirely satisfactory. What happens when two elementary particles "comes close" to each other? How should we interpret this in terms of the representations representing the particles? Here the non-commutative algebraic geometry and the theory of non-commutative deformations of families of representations will, certainly, be essential, but the detailed treatment of this has to wait, as will the many other questions that pop up when one wants to use the model in "real life". In relation to the short Subsection (4.4), on Cosmology, it is certainly tempting to make comparison with the very speculative literature treating notions like wormholes, firewalls, complementarity, entanglement, and the likes. The exceptional fibers, the black holes $E(\underline{\lambda})$, *in our Toy Model, could be used to "make wormholes," and certainly to give examples of hologram-like phenomena, and the solution of the Furniture equation in Subsection (2.4) might look like a mathematical model of a firewall at the Horizon. The fact, that we in this model see what "happened before" in cosmological time, where the gravitational mass-density was bigger, could be used as an argument for the non-existence of both Black Energy, and Black Matter.*

This may all be interesting, but my goal for this work has been more modest. I wanted a "simple" but mathematically sound model for treating the kind of physics I never understood as a student, with the hope of, through this, eventually to understand a little more of the "nature" we seem to live in.

References

[1] St. Augustin *Les Confessions de Saint Augustin* par Paul Janet. Charpentier, Libraire-Éditeur, Paris (1861).

[2] A.A. Beilinson, *Coherent sheaves on*P^n *and problems of linear algebras,* Func. Anal. Appl. 12 (1978), pp. 214-216.

[3] Harald Bjar and O. A. Laudal (1990) *Deformations of Lie algebras, and Lie algebras of Deformations* Comp. Math.75:69-111,1990. .

[4] E.Eriksen, O.A.Laudal, A.Siqveland Noncommutative Deformation Theory Book in preparation.

[5] Brian Greene The Fabric of the Cosmos Allen Lane 2004

[6] Kitty Ferguson (2010) *Pythagoras. His lives and the legacy of a rational universe* Icon Books Ltd.

[7] O. Gravir Imenes (2011) *The concept of charge in a model based on non-commutative algebraic geometry* Dissertation presented for the degree of Philosophiae Doctor. Faculty of Mathematics and Natural Sciences, University of Oslo, September 2011.

[8] Stephen Hawking (1988) *A Brief History of Time* Bantam Books, Transworld Publishers, London. (1988)

[9] O. A. Laudal (1965) *Sur la thorie des limites projectives et inductives* Annales Sci. de l'Ecole Normale Sup. 82(1965) pp. 241-296..

[10] O. A. Laudal (1979) *Formal moduli of algebraic structures* Lecture Notes in Math.754, Springer Verlag, (1979).

[11] O. A. Laudal (1986) *Matric Massey products and formal moduli I* in (Roos, J.E.ed.) Algebra, Algebraic Topology and their interactions Lecture Notes in Mathematics, Springer Verlag, vol 1183, (1986) pp. 218–240.

[12] O. A. Laudal (2002) *Noncommutative deformations of modules* Special Issue in Honor of Jan-Erik Roos, Homology, Homotopy, and Applications, Ed. Hvedri Inassaridze. International Press, (2002).See also: Homology, Homotopy, Appl. 4(2002), pp. 357-396,

[13] O. A. Laudal (2000) *Noncommutative Algebraic Geometry* Max-Planck-Institut für Mathematik, Bonn, 2000 (115).

[14] O. A. Laudal (2001) *Noncommutative algebraic geometry* Proceedings of the International Conference in honor of Prof. Jos Luis Vicente Cordoba, Sevilla 2001. Revista Matematica Iberoamericana.19 (2003), 1-72.

[15] O. A. Laudal (2003) *The structure of* $Simp_n(A)$ (Preprint, Institut Mittag-Leffler, 2003-04.) Proceedings of NATO Advanced Research Workshop, Computational Commutative and Non-Commutative Algebraic Geometry. Chisinau, Moldova, June 2004.

[16] O. A. Laudal (2005) *Time-space and Space-times* Conference on Noncommutative Geometry and Representatioon Theory in Mathematical Physics. Karlstad, 5-10 July 2004. Ed. Jürgen Fuchs, et al. American Mathematical Society, Contemporary Mathematics, Vol. 391, 2005. ISSN: 0271-4132.

[17] O. A. Laudal (2007) *Phase Spaces and Deformation Theory* Preprint, Institut Mittag-Leffler, 2006-07. See also the part of the paper published in: Acta Applicanda Mathematicae, 25 January 2008.

[18] O. A. Laudal (2011) *Geometry of Time Spaces* World Scientific, (2011).

[19] O. A. Laudal (2013) *Mathematics in the 21st Century* Springer Proceedings in Mathematics and Statistics, ed. P.Cartier et al. Springer Basel 2014.

[20] O. A. Laudal (2014) *The structure of Ph*, generalised de Rham, and entropy* Proceedings from the QQQ Conference 2012, Journal of Physics: Conference Series 532(2014)012013.

[21] O. A. Laudal and G. Pfister (1988) *Local moduli and singularities* Lecture Notes in Mathematics, Springer Verlag, vol. 1310, (1988)

[22] J.Madore, S. Schraml, P. Schupp, J. Wess (2000) *Gauge Theory on Noncommutative Spaces* arXiv:hep-th/0001203v1 28 Jan 2000.

[23] F.Mandl and G. Shaw (1984) *Quantum field theory* A Wiley-Interscience publication. John Wiley and Sons Ltd. (1984)

[24] R. Penrose *What came before the Big Bang? Cycles of Time* Vintage Books (2011)

[25] C. Procesi (1967) *Non-commutative affine rings* Atti Accad. Naz. Lincei Rend.Cl. Sci. Fis. Mat. Natur. (8)(1967), 239-255

[26] C. Procesi (1973): *Rings with polynomial identities* Marcel Dekker, Inc. New York, (1973).

[27] R.K.Sachs and H.Wu (1977) *General Relativity for Mathematicians* Springer Verlag, (1977)

[28] T. Schücker (2002) *Forces from Connes' geometry* arXiv:hep-th/0111236v2, 7 June 2002.

[29] N. Seiberg and E.Witten (1999) *String Theory and Noncommutative Geometry* JHEP 9909,032(1999),hep-th/9908142.

[30] S. Weinberg (1995) *The Quantum Theory of Fields. Vol I, II, III* Cambridge University Press, (1995).

[31] S. Itaka. *Symmetric forms and Weierstrass cycles* Proc. Japan Acad. **54** (1978), Ser. A.

[32] D. Laksov and A. Thorup. *These are the differentials of order n* Trans. Amer. Math. Soc. **351** (1999), 1293-1353.

[33] O. A. Laudal. *Phase Spaces and Deformation Theory* Acta Appl Math **101** (2008), 191–204.

[34] M. Schlessinger. *Functors of Artin rings* Trans. Amer. Math. Soc. **130** (1968), .208–222.

[35] I. Reiten. *An introduction to representation theory of Artin algebras* Bull. London Math. Soc. **17** (1985), 209–233.

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