Accuracy of energy measurement and reversible operation of a microcanonical Szilard engine

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In a recent paper [Vaikuntanathan and Jarzynski, Phys. Rev. E 83, 061120 (2011)], a model was introduced whereby work could be extracted from a thermal bath by measuring the energy of a particle that was thermalized by the bath and manipulating the potential of the particle in the appropriate way, depending on the measurement outcome. If the extracted work is \( W_{\text{extracted}} \) and the work \( W_{\text{erasure}} \) needed to be dissipated in order to erase the measured information in accordance with Landauer’s principle, it was shown that \( W_{\text{extracted}} \leq W_{\text{erasure}} \) in accordance with the second law of thermodynamics. Here we extend this work in two directions: First, we discuss how accurately the energy should be measured. By increasing the accuracy one can extract more work, but at the same time one obtains more information that has to be deleted. We discuss what are the appropriate ways of optimizing the balance between the two and find optimal solutions. Second, whenever \( W_{\text{extracted}} \) is strictly less than \( W_{\text{erasure}} \) it means that an irreversible step has been performed. We identify the irreversible step and propose a protocol that will achieve the same transition in a reversible way, increasing \( W_{\text{extracted}} \) so that \( W_{\text{extracted}} = W_{\text{erasure}} \).

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I. INTRODUCTION

One of the various statements of the second law of thermodynamics is the Kelvin-Planck formulation: No process is possible whose sole result is the conversion of thermal energy to mechanical work. One consequence of this is the following. Take a system that is initially in thermal equilibrium, but then isolated from the environment. There is no way to reduce on average the energy of the system by any cyclic variation of external parameters. If this were possible, one could then reconnect the system with the thermal bath and return it to the initial state with the only result that some of the initial thermal energy was extracted as mechanical work in contradiction with the second law. In fact, since the system is to be isolated during the time when energy is to be extracted, this is a statement about the possible time evolution of a dynamical (Hamiltonian) system with a certain distribution of initial conditions. Indeed, it can be directly proven from properties of Hamiltonian systems [1–3] and is true not only for the canonical distribution of initial states but for any distribution function that is a monotonically decreasing function of energy [1,2].

Recently, the violation of this statement in the case of a microcanonical ensemble of systems was discussed in several papers [1,4–6]. That is, if you know the initial energy of the system (but not the precise initial state), you can find a cyclic variation of external parameters such that the average energy of the system is reduced, and therefore work on average extracted. But a canonical ensemble can be “converted” to a microcanonical one by a measurement of the energy. This idea was explored in Ref. [4], where a model is constructed that consists of a single particle in a one-dimensional potential \( U(q) \) (where \( q \) is the position of the particle). It is then shown that if you measure the energy you can find a cyclic variation of the potential, which reduces the energy of the system as close to zero as you wish. The initial energy of the particle is then delivered as work \( W_{\text{extracted}} \) to the agent operating the potential.

An explicit protocol is given for the evolution of the potential in the case where the initial potential is quartic, \( U(q) \sim q^4 \), but the procedure is easily extended to other potentials by adding a step that transforms the initial potential adiabatically to a quartic one, whereby the ordering of the different energy states is kept similarly to what is exploited in Ref. [4] and then performing the cyclic operation they presented. They consider the following sequence of steps:

1. The system is brought into contact and allowed to equilibrate with a thermal bath at temperature \( T \). The bath is then removed.
2. The energy of the now-isolated system is measured.
3. The system is subjected to a cyclic protocol that reduces its kinetic energy close to zero, extracting on average the work \( W_{\text{extracted}} \).

This sequence can be repeated indefinitely, and thereby one has constructed a device that converts thermal energy into mechanical work, in seeming contradiction to the second law. The resolution if the inconsistency is found in Landauer’s principle [7,8], which states that the erasure of information by necessity results in dissipation of heat. That is, to erase the information obtained when measuring the energy of the system, and restore the measuring device to the initial state, one needs on average an amount of work \( W_{\text{erasure}} \) that is converted to thermal energy. In Ref. [4] it is explicitly shown that we have

\[ W_{\text{erasure}} \geq TH \geq W_{\text{extracted}}, \tag{1} \]

where \( H \) is the average Shannon information obtained in a single measurement. This means that to erase the information one needs at least as much energy as one extracted from the thermal bath by the operation of the device. In this article we will investigate the second inequality of Eq. (1) and see under what conditions equality can be obtained.

The analysis presented in Ref. [4] shows how the second law is not violated by such a device, but it leaves several puzzles. In order to efficiently extract work from the system it is necessary to know the energy accurately. But an accurate energy measurement means a large amount of information. It
seems that \( W_{\text{erasure}} \to \infty \) in the limit of very precise energy measurements. At the same time, \( W_{\text{extracted}} \) is bounded by the average energy of the system at the time of the measurement. Is there some optimal accuracy with which the energy should be measured in order to extract as large a fraction of the energy as possible while still not having to pay too much in deleting the information? And since the probability of a certain energy depends on the energy, are there some regions in energy where it is more important to make accurate measurements? It is also interesting to understand why in Eq. (1) we sometimes have an inequality, rather than strict equality. In other words, where in the process is there an irreversible step that leads to a net increase of the entropy? In this paper we will address these questions.

The paper is organized as follows: In Sec. II we present the model and find the extracted work and measured information. Different ways of optimizing the extracted work are discussed in Sec. III B, and a protocol for reversibly completing the whole operation in Sec. IV. A short summary and discussion is given in Sec. V.

II. MODEL

We use the model of Vaikuntanathan and Jarzynski [4]. The system consists of a single particle with coordinate \( q \) and momentum \( p \) in a potential \( U(q) \). In Ref. [4] they choose \( U(q) \sim q^2 \). The particle is in thermal equilibrium with a reservoir at a certain temperature \( T \). We assume that we can measure the energy of the particle to a certain accuracy. More precisely, we define a number of energies \( 0 < E_1 < \cdots < E_n < E_{\text{max}} \) between 0 and some maximal energy \( E_{\text{max}} \) and we assume that we can measure in which interval \( X_i = [E_{i-1}, E_i] \) the energy lies. Depending on the outcome of the measurement, we choose an appropriate manipulation protocol. The manipulation consists in adiabatically modifying the shape of the potential through a closed path in the space of potentials, returning it in the end to the initial one. The key to understanding the process is the fact that the action calculated for one full period of the motion, \( I = \int p \, dq \), which is the same as the phase space volume enclosed by an equal energy curve is an adiabatic invariant. Since the action is a monotonous function of the energy, it means that the energy ordering of the trajectories is not changed by an adiabatic change in the potential. This applies as long as the potential has a single minimum, but it is no longer true if the potential has a local maximum with a corresponding phase space separatrix. This is utilized in Ref. [4] to devise a protocol that exchanges the energies of a specified set of initial states; the exact protocol is given in Ref. [4]. The end result is that the states which initially were in the interval \( X_i \) are shifted to the lowest energies; see Fig. 1.

Moreover, the ordering of the states inside the interval \( X_i \) is kept, so that a state with a lower initial energy will also have a lower final energy. This means that we can find the final energy \( H^f_j(E) \) of a particle with initial energy \( E \) when the protocol appropriate for an energy in the interval \( X_i \) is executed. If \( g(E) \) is the density of states in the potential \( U(q) \) we have

\[
\int_{E_{i-1}}^{E_i} dE \, g(E) = \int_{0}^{H^f_j(E)} dE \, g(E)
\]

and \( H^f_j(E) \) is found by solving this equation. In the case of a one-dimensional system in a quadratic potential \( U(q) \sim q^2 \) the density of states is a constant, \( g(E) = g_0 \), and we obtain a particularly simple equation, which gives

\[
H^f_j(E_j) = E_j - E_{j-1}.
\]

In the following we will derive all general equations for an arbitrary potential, but we will only find solutions in this special simple case.

When we know \( H^f_j(E) \) we can find the energy that on average can be extracted with a given \( n \) and \( E_{\text{max}} \):

\[
W_{\text{extracted}} = \frac{1}{Z} \sum_j \int_{E_{j-1}}^{E_j} dE \, g(E) e^{-\beta E} \left[ E - H^f_j(E) \right],
\]

where

\[
Z = \int_0^\infty dE \, g(E) e^{-\beta E}
\]

is the partition function. If \( P_i \) is the probability that the energy is in interval \( X_i \), we have

\[
P_i = \frac{1}{Z} \int_{E_{i-1}}^{E_i} dE \, g(E) e^{-\beta E}.
\]

Here \( i = 1 \cdots n + 2 \) where we identify \( E_{n+1} = E_{\text{max}} \) and \( E_{n+2} = \infty \). That is, \( P_{n+2} \) is the probability that \( E > E_{\text{max}} \) and we assume that the device will not operate in this case. The information obtained during a measurement is on average

\[
H = - \sum_{i=1}^{n+2} P_i \ln P_i.
\]

If the information is to be erased at a heat bath of temperature \( T_E \), the corresponding work of erasure is \( W_{\text{erasure}} \geq T_E H \), and Eq. (1) is valid when \( T_E = T \).

III. HOW EFFICIENTLY CAN WE EXTRACT WORK?

Let us assume that the maximal energy \( E_{\text{max}} \) is fixed and represents the upper limit of what our device can operate on.

![Fig. 1](image_url)
If the energy is found to be above this value we cannot extract it. If the density of states does not grow too quickly, the probability of this happening decreases quickly with increasing $E_{\text{max}}$. The free parameters of the model are then the number $n$ of energy intervals and the positions $E_i$ of the interval boundaries. There are several ways one can consider to optimize these. The simplest is to find the maximal amount of energy $W_{\text{extracted}}$ that can be extracted in a single run of the cycle that is presented in Sec. I and analyzed in Sec. III A. This means that we are disregarding the work of erasure, $W_{\text{erasure}}$. We can also define the useful work $W = W_{\text{extracted}} - W_{\text{erasure}}$, which according to Eq. (1) is negative when $T_E = T$. In Sec. III B we discuss how to maximize the useful work or minimize the information needed at a specified extracted work $W_{\text{extracted}}$. Finally, we can consider erasing the information at a temperature $T_E < T$, in which case the device will operate as a heat engine between the two thermal baths. One can then define the efficiency $\eta = W / W_{\text{extracted}}$ in the ordinary way as the ratio of the useful work to the energy extracted from the thermal bath. In Sec. III C we will show that one can get the efficiency arbitrarily close to one but only in the limit where $W_{\text{extracted}} = W_{\text{erasure}} = 0$, that is by doing nothing, and we will find the optimal efficiency at a given $W_{\text{extracted}}$.

### A. Maximal $W_i$

How much energy can on average be extracted with a given $n$ and $E_{\text{max}}$? To find this we have to maximize the work $W_{\text{extracted}}$ given by Eq. (4) as a function of the energies $E_i$ marking the boundaries of the energy intervals. In Appendix A it is shown that if we consider the simplest case of $U(q) \sim q^2$ the energy intervals $u_i = \beta(E_i - E_{i-1})$ have to satisfy the equation

$$u_i = 1 - e^{-u_{i+1}}.$$  \hfill (5)

This equation can be solved numerically, but in Appendix A it is also shown how to derive an approximate solution in the limit of large $n$. The result is that for $\beta E_{\text{max}} \gg 1$ we have

$$E_i \approx -2T \ln \left(1 - \frac{i}{n}\right).$$  \hfill (6)

Using this we find

$$W_{\text{extracted}} \approx T \left(1 - \frac{2}{n}\right)$$  \hfill (7)

$$H \approx \ln n - \ln 2 + \frac{1}{2},$$  \hfill (8)

which shows that in the limit of large $n$ we get $W_{\text{extracted}}$ close to the average internal energy of $T$ as given by the equipartition theorem, and that the minimal $W_{\text{erasure}}$ will grow logarithmically with $n$.

### B. Maximal $W$ for a given $W_1$

In the previous section we found the position of the $E_i$ such that the extracted work, $W_{\text{extracted}}$, was maximized for a given $E_{\text{max}}$ and $n$. However, to extract this energy we have to obtain a certain amount of information, which later has to be deleted. It is therefore possible that by extracting an energy $W_{\text{extracted}} < W_{\text{max}}$ a bit less than the maximal found above, we can reduce the information and thereby the work of erasure, $W_{\text{erasure}}$, so that the amount of useful work $W = W_{\text{extracted}} - W_{\text{erasure}}$ could be increased. This means that we should look for other values of $E_i$ which minimizes the information for a given $W_{\text{extracted}}$. Introducing the Lagrange multiplier $\lambda$ we define the function

$$I = H + \lambda W_{\text{extracted}}.$$  \hfill (9)

We have to minimize this function subject to the constraint Eq. (4) with a specified $W_{\text{extracted}}$. In Appendix B it is shown that this leads to the equations

$$\lambda(u_i + e^{-u_{i+1}} - 1) = \ln \left(\frac{1 - e^{-u_{i+1}}}{u_i - 1}\right), \quad i = 1 \ldots n$$  \hfill (10)

and the constraint

$$\sum_{i=1}^{n} v_i(e^{-v_i} - e^{-v_{i+1}}) = w_1,$$  \hfill (11)

where $v_i = \sum_{j=1}^{i} u_j$ and $w_1 = \beta W_{\text{extracted}}$. We have solved these equations numerically, using Newton’s iterative method. Numerically solving these equations is complicated by the fact that there are in general several solutions. By choosing a large number of initial guesses for the solution we can by reasonable security find the solutions with the smallest necessary information, $H$. To make the structure clearer, it is instructive to use the not-too-large value $\beta E_{\text{max}} = 3$ for the maximal energy. It is also helpful to subtract the expected information, $H_0$, according to Eq. (8) as explained in Eq. (B1). The result is shown in Fig. 2.

Figure 2 shows $H = H_0$ for the minimal $H$ as a function of $W_{\text{extracted}}$ for different $n$. The dots mark the maximal $W_{\text{extracted}}$ and corresponding $H$ for each $n$. We can observe the following: Starting from one of the dots of maximal $W_{\text{extracted}}$ for a given $n$ we can observe a smooth decrease in $H$ as $W_{\text{extracted}}$ is decreased, so that the minimal $H$ is encountered for some $W_{\text{extracted}}$. This minimal $H$ is denoted by $H_{\text{opt}}$ and it is shown that $H_{\text{opt}}$ decreases logarithmically as $n$ increases.

FIG. 2. The minimal information $S$ as a function of the extracted work $w_1$ for different $n$ and with $\beta E_{\text{max}} = 3$. 

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n, we can see that the reduction in $H$ that can be achieved by increasing $n$ at the same $W_{\text{extracted}}$ remains close to constant, at least for the range of $n$ studied. Also, following each curve from the dot, we see that as it crosses the curves for larger $n$, those curves makes a jump. For example, the $n = 2$ curve crosses the $n = 3$ curve around $W_{\text{extracted}} = 0.43$, and the $n = 3$ curve jumps at that point. This is because as $W_{\text{extracted}}$ is reduced, the $n = 3$ minimal solution has one $u_i$, which vanishes at that point. For smaller $W_{\text{extracted}}$ this solution is not found (it is at the boundary of the domain, and not at an interior point), while the $n = 2$ solution represents the true minimum.

C. Maximal $W$ at given temperatures of the baths of energy and erasure

Using the same data we can also demonstrate that there does not exist any optimal efficiency except the trivial solution of doing nothing, as discussed above. The efficiency is

$$\eta = \frac{W}{W_{\text{extracted}}} = 1 - \frac{T_E}{T} \frac{H}{w_1},$$

which means that we can make the efficiency higher by making the ratio $H/w_1$ smaller. In Fig. 3 we plot $H/w_1$ as a function of $w_1$.

As we see, $H/w_1$ grows with $w_1$, which means that we can always increase the efficiency by reducing $W_{\text{extracted}}$. For a given $n$ this continues until one of the intervals $u_i$ (the one at the upper limit of the energy range) collapses to zero and it becomes favorable to decrease $n$ by one. Then the process continues with the new $n$ until all $u_i$ are zero except $u_1$, which then cover the whole range $[0, u_{\text{opt}}]$.

Using the minimal $H$ as a function of $W_{\text{extracted}}$, we can also find the maximal work $W$, which can be performed at a given temperature $T$ of the system and when the information is erased at a lower temperature $T_E$, and the optimal number $n_{\text{opt}}$ of energy intervals that one needs to achieve this maximal work.

Figure 4 (left) shows $n_{\text{opt}}$ as a function of $T_E/T$. Two graphs are shown, one which uses the optimal solution for producing the least entropy as found in Sec. III B. The other uses the solution giving the maximal heat transfer $W_{\text{extracted}}$ from the heat bath with the given $n$ as found in Sec. III A. The corresponding useful work $W/T$ is shown in Fig. 4 (right). As we can see, the optimal number of intervals $n_{\text{opt}}$ grows as $T_E/T$ decreases. This is natural, since the cost of deleting information becomes less in this case. The optimal solution has a larger $n_{\text{opt}}$ as we expect from Fig. 2, since for a given $W_{\text{extracted}}$ we can reduce the information in measurement by increasing the number of intervals $n$. The useful work $W$ is also larger, but the gain in $W$ is not large and becomes smaller as $T_E/T$ decreases.

IV. RESTORING REVERSIBILITY: UTILIZING ALL THE INFORMATION

The process discussed leads to a curious situation. We have a system in thermal equilibrium. Then we decouple it from the environment and make a measurement on it. That is, we gain information about the system (the energy interval which it is in), thereby increasing our knowledge and reducing the entropy of the system accordingly (note that we are here discussing the statistical entropy, which depends on our knowledge of the system state). In principle, if the measurement process is dissipation-free, this process is reversible, and the amount of information gained is equal to the reduction in entropy. Then we manipulate the system in a deterministic way, extracting energy. During this process it is assumed that the system remains isolated from the environment. Therefore, the entropy is constant by Liouville’s theorem. The information is then erased, and this is also a reversible process in the sense that the energy that needs to be dissipated as heat increases the entropy of the environment by exactly the same amount as the information that is deleted. The whole process is then completely reversible. Yet, if the information is deleted at the same temperature as the system had initially, we have seen that there will be a net conversion of energy from mechanical energy to thermal energy: $W_{\text{erasure}} > W_{\text{extracted}}$. How can this be?

The answer is that the system initially was in thermodynamic equilibrium, but after the process it is not. This means that we have not utilized all the information that we gained during the measurement. We have only extracted as much as possible of the energy that was stored in the system at the moment it was decoupled from the reservoir. At the end, we
are left with a system that has lower entropy than when it started. It means that it is a resource for extracting energy from a thermal reservoir, if it can be reconnected to one. In order to fully exploit the information that we gained during measurement, the system has to be returned reversibly to its initial thermal state before we delete the information. If we do not do this but rather delete the information and reconnect the system to the bath directly, this will be an irreversible process, and this is where entropy is generated. Returning to the steps in the process as described in Sec. I we see that it is in going from step (3) and back to step (1) that the irreversible process takes place. We now describe how to add two further steps to the process, so that the whole cycle becomes reversible and equality $\Delta E = W_{\text{extracted}}$ of the minimal work of erasure and the total extracted work is restored.

(1) In the initial state, the system is in thermal equilibrium with a bath at temperature $T$. The potential is $U(q)$ and the average energy is $E_1$ and the entropy $S_1$.

(2) We decouple the system from the bath and measure in which interval $X_i$ the energy lies. The average energy (which is both thermal average and average over measurement results) is not changed, $E_2 = E_1$, as it has to be since we have only measured and not changed the energy. The average entropy is $S_2 = S_1 - H$, where $H$ is the average information gained by the measurement.

(3) We manipulate the potential in the way described in Sec. II, bringing the interval $X_i$ to the bottom of the potential and returning the potential in the end to $U(q)$. The average energy is $E_3$ and the entropy still $S_2$ since it cannot change in an adiabatic process in an isolated system. During this operation the work $W_{\text{extracted}} = E_2 - E_3 > 0$ is extracted as considered in previous sections. It is the maximal work that can be extracted, keeping the system isolated.

(4) To reversibly return the system to the initial state we first modify the potential adiabatically in such a way that the distribution function is thermal at the right temperature $T$. This means that we have to find a potential $U_4(q)$ with a corresponding density of states $g_4(E)$ such that after the process the particle is left with a distribution function of the energy $P_4(E)$ such that

$$P_4(E) = \frac{g_4(E)}{Z_4} e^{-\beta E} \quad Z_4 = \int_0^\infty dE \ g_4(E)e^{-\beta E}.$$ 

Note that the potential $U_4(q)$ will in general depend on the interval $X_i$ where the system energy was found to be. The form of the potential can in principle be found, but we do not need it. It is sufficient to know that it exists, which seems clear at least for simple potentials with a single minimum. The average energy is $E_4$ and the entropy is still $S_2$. This process requires a work $W_{\text{compression}} = E_3 - E_4 < 0$. It is negative since the energy of the system has to increase since we know that initially it is close to the bottom of the potential. We have to use external work to achieve this, but it prepares the system for the last step where a larger amount of work is extracted from a thermal reservoir.

(5) Finally, we can now safely reconnect the system to the bath, which is a reversible process and does not change anything on average, since the system already is prepared in a thermal state. We can then adiabatically return the potential to the initial $U(q)$. This gives the same average energy $E_1$ and entropy $S_1$ as in the initial state. The process produces the work

$$W_{\text{expansion}} = T \Delta S - \Delta U = T(S_1 - S_2) - (E_1 - E_4) > 0.$$ 

The total work obtained in the full cycle is

$$W_{\text{total}} = W_{\text{extracted}} + W_{\text{compression}} + W_{\text{expansion}} = TH,$$

which according to Landauer’s principle is exactly the minimal energy that must be dissipated to erase the information obtained in the measurement. This statement is true for any number of energy intervals in the measurement scheme and any set if interval boundaries $E_i$. The only requirement is that all processes are adiabatic, which means that they have to be performed infinitely slowly. The bound on the total work is also in accord with the general theory of optimal processes in information processing [9].

V. SUMMARY

We have discussed the model of Vaikuntanathan and Jarzynski [4] for extracting work from a thermal bath by measuring the energy of a particle that was thermalized with the bath and manipulating the potential of this particle in the appropriate way, depending on the measurement outcome. We have addressed the question of how accurately the energy should be measured. This is formalized in the same way as in Ref. [4] by dividing the energy axis in subintervals $X_i$ and assuming that the measurement tells with perfect accuracy in which interval the energy is. We have optimized the boundaries $E_i$ of the intervals according to different criteria: for extracting the maximal energy, for minimizing the entropy production, and for maximizing the efficiency of a heat engine at a given power.

We have identified the irreversible step in the protocol of Ref. [4] as the one where the system is known to be close to the lowest energy state and is reconnected with a thermal bath. In this process the available phase space of the particle suddenly increases, and the process is irreversible and there is a net increase in entropy. This is in principle the same situation as in the paradigmatic example of free expansion of an ideal gas following a sudden increase in the accessible volume. In the context of information-driven heat engines (Maxwell’s demons), similar situations have been recently discussed. In Ref. [10] an overdamped particle in a potential was considered and the potential was manipulated in order to extract energy following the measurement of position. It was found that to get the maximal work possible by the measured information one had to strongly confine the particle initially close to the measured position and then gradually make the potential less steep while extracting energy. In the context of single-electron devices [11,12], it was found that when opening the barrier between two possible states for a particle, this has to be done in an optimized way so that at no point will the available phase space suddenly increase. Similarly, in this paper we have described a protocol whereby the irreversible step in Ref. [4] can be reversibly performed, thereby increasing the extracted work up to the maximal achievable by the measured information, so that the extracted work is exactly the same as what is needed in order to erase the information in accordance with Landauer’s principle.
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APPENDIX A: MAXIMAL $W_1$ FOR A GIVEN $n$

We have to maximize Eq. (4) with respect to $E_i$:

$$\frac{\partial W_{\text{extracted}}}{\partial E_i} = \frac{1}{Z} \sum_j g(E_j) e^{-\beta E_j} \left[ E_i - H_j(E_i) \right] (\delta_{j,i} - \delta_{j-1,i})$$

$$- \frac{1}{Z} \sum_j \int_{E_{i-1}}^{E_i} dE \ g(E) e^{-\beta E} \frac{\partial H_j(E)}{\partial E_i}.$$  

Differentiating Eq. (2) we get

$$\frac{\partial H_j(E)}{\partial E_i} = - \frac{g(E_j)}{g[H_j(E_j)]} \delta_{j-1,i}.$$  

The equations $\frac{\partial W_{\text{extracted}}}{\partial E_i} = 0$ then becomes

$$H_j(E_i) = e^{\beta E_i} \int_{E_j}^{E_{i+1}} dE e^{-\beta E} \frac{g(E)}{g[H_j(E_j)]} = - \frac{1}{\beta} \left[ e^{-\beta E_{i+1}} - e^{-\beta E_i} \right].$$

Eq. (A1) becomes Eq. (5).

We can find an approximate solution to this equation for large $n$ when all $u_i \ll 1$ and we can expand the exponential

$$u_{i-1} = u_i - \frac{1}{2} u_i^2 + \cdots.$$  

Treating $i$ as a continuous variable, we get the differential equation

$$\frac{du}{dt} = \frac{1}{2} u^2,$$

which is integrated to give

$$u_i = \frac{1}{A - i/2}. \quad \text{(A2)}$$

Here $A$ is a constant of integration, which has to be found from the boundary condition $\sum_i u_i = u_m = \beta E_{\text{max}}$. We have

$$\int_0^n di u_i = -2 \ln \left| \frac{n - 2A}{-2A} \right| = u_m,$$

which gives

$$A = \frac{n/2}{1 - e^{-u_m/2}}. \quad \text{(A3)}$$

We can now find

$$E_i = \frac{1}{\beta} \sum_{j<i} u_j \approx \frac{1}{\beta} \int_0^i di \frac{A - i/2}{A - i/2} = - \frac{2}{\beta} \ln \left( 1 - \frac{iB}{n} \right). \quad \text{(A4)}$$

We can now calculate the extracted work and information. First we find

$$Z = \int_0^{\infty} dE \ g_0 e^{-\beta E} = \frac{g_0}{\beta}$$

and

$$P_i = \frac{1}{Z} \int_{E_{i-1}}^{E_i} dE \ g_0 e^{-\beta E} = e^{-\beta E_{i+1}} - e^{-\beta E_i}$$

$$= \frac{B}{n} \left[ 2 - \frac{(2i-1)B}{n} \right]. \quad \text{(A5)}$$

The probability to find $E > E_{\text{max}}$ is $P_{n+2} = 1 - e^{-\beta E_{\text{max}}}$ and the information

$$H = - \sum_{i=1}^{n+2} P_i \ln P_i.$$  

We replace the sum by an integral:

$$\int di \frac{B}{n} \left[ 2 - \frac{(2i-1)B}{n} \right] \ln \left[ \frac{B}{n} \left( 2 - \frac{(2i-1)B}{n} \right) \right]$$

$$= \left[ B(2-B) + \frac{B^2}{n} \right] \ln \frac{B}{n} - \frac{1}{4} \left[ 2 - \frac{(2n-1)B}{n} \right]^2$$

$$\times \left[ \ln \left( 1 - \frac{2n-1}{n} \right) - \frac{1}{2} \right] + \frac{1}{4} \left( 2 + \frac{B}{n} \right)^2$$

$$\times \left[ \ln \left( 2 + \frac{B}{n} \right) - \frac{1}{2} \right].$$

When $u_m \gg 1$, we have $B \to 1$ and $P_{n+2} \to 0$. We then get Eq. (8). Combining Eqs. (4) and (3), the extracted work is

$$W_{\text{extracted}} = \sum_i E_i P_{i+1}. \quad \text{(A6)}$$

Using Eqs. (6) and (A5) we get

$$W_{\text{extracted}} = - \frac{2B}{\beta n} \int_0^n di \ ln \left| 1 - \frac{1-B}{n} \right| \left[ 2 - \frac{B}{n} (2i+1) \right]$$

$$= T(1-B)^2 \left[ 2 \ln(1-B) - 1 - \frac{2TB(1-B)}{n} \right]$$

$$\times \left[ \ln(1-B) - 1 \right] + T \left( 1 - \frac{2B}{n} \right). \quad \text{(A7)}$$

When $u_m \gg 1$, we have $B \to 1$ and we find Eq. (7).

To show the accuracy of the approximate solution we compare it with the exact result found by numerical solution of Eq. (5). Figure 5 (left) shows $H$ as function of $n$ together with Eq. (8), while Fig. 5 (right) shows $(1 - W_{\text{extracted}}/T)^{-1}$ as function of $n$ together with Eq. (7), both for $u_m = 10$. We conclude that the approximate solution works well even for $n$ not much larger than $u_m$, which means that the $u_i$ need not be much smaller than 1.

APPENDIX B: MAXIMAL $W$ FOR A GIVEN $W_1$

To minimize $I$ in Eq. (9) we have to solve $\frac{\partial I}{\partial E_i} = 0$ together with the constraint Eq. (4). We have

$$\frac{\partial S}{\partial E_i} = - \sum_j (\ln P_j + 1) \frac{\partial P_j}{\partial E_i} = - \frac{1}{Z} g(E_i) e^{-\beta E_i} \ln P_{i+1},$$

$$\frac{\partial E_i}{\partial E_i} = - \frac{1}{\beta} \sum_{j<i} u_j \approx \frac{1}{\beta} \int_0^i di \frac{A - i/2}{A - i/2} = - \frac{2}{\beta} \ln \left( 1 - \frac{iB}{n} \right).$$
where we use
\[ \frac{\partial P_j}{\partial E_i} = \frac{1}{Z} g(E_i) e^{-\beta E_i} (\delta_{j,i} - \delta_{j-1,i}), \]
and this gives
\[ \frac{1}{\lambda} \ln \frac{P_{i+1}}{P_i} = H_i^{i+1}(E_i) - e^{\beta E_i} \int_{E_i}^{E_{i+1}} dE e^{-\beta E} g(E) g[H_j^{i+1}(E_i)]. \]
For constant \( g(E) = g_0 \), we get similar to Eq. (5)
\[ \frac{1}{\lambda} \ln \frac{P_{i+1}}{P_i} = u_i - 1 + e^{-u_{i+1}}, \]
and from Eq. (A5) we have
\[ P_i = e^{-\beta E_{i+1}} - e^{-\beta E_i}, \]
which gives
\[ \frac{P_{i+1}}{P_i} = \frac{e^{-\beta E_i} - e^{-\beta E_{i+1}}}{e^{-\beta E_{i-1}} - e^{-\beta E_i}} = \frac{1 - e^{-u_i}}{e^{-u_i} - 1}. \]
The constraint is in this case given by Eq. (A6), which gives Eqs. (10) and (11).
To show the results it is instructive to subtract the expected entropy \( S_0 \) according to Eq. (8). For this, let us apply Eq. (A7), which we rewrite as
\[ W_{\text{extracted}} = C - \frac{D}{n}, \]
with
\[ C = T(1 - B)^2 [2 \ln(1 - B) - 1] + T, \]
\[ D = 2BT + 2BT(1 - B)[\ln(1 - B) - 1] \]
and Eq. (8) (it is sufficient to keep the approximate expression for \( H \), but not for \( W_{\text{extracted}} \) when \( E_{\text{max}} \) is not large). Eliminating \( n \) we get the relation
\[ H_0 = \ln \frac{D/2}{C - W_{\text{extracted}}} + \frac{1}{2} \]
between the entropy \( H_0 \) and the extracted work. Note that this relation is only approximate since it is based on the approximate solution of Eq. (5), and that Eq. (5) applies to the maximal extracted work for a given \( n \).