Introductory Dynamic Macroeconomics

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10 January 2005
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Preface

This note is written for the course ECON 3410 / 4410 International macroeconomics and finance. It reviews the key concepts and models needed to start addressing economic dynamics in a systematic way. The level of mathematics used does not go beyond simple algebra. References to the textbooks by Burda and Wyplosz (B&W hereafter) and Rødseth’s Open Economy Macroeconomics (OEM hereafter) are integrated several places in the text.
Chapter 1
Dynamic models in macroeconomics

1.1 Introduction

In many areas of economics, time plays an important role: firms and households do not react instantly to changes in for example taxes, wages and business prospects but take their time to make decisions and adjust behaviour. Moreover, because of information and processing lags, time often goes by before changes in economic conditions are fully recognized. There are also institutional arrangements, social and legal agreements and norms that hinder continuous adjustments of economic variables. Annual (or even biannual) wage bargaining rounds is one important example. The manufacturing of goods is usually not instantaneous but takes time, often several years in the case of projects with huge capital investments. Dynamic behaviour is also induced by the fact that many economic decisions are heavily influenced by what firms, households and the government anticipate about the future. Often expectation formation will attribute a large weight to past developments, since anticipations usually build on past experience.

In macroeconomic models, dynamic behaviour often takes the form of relationships between flow and stock variables. Examples of flow variables are GDP (and the expenditure components), exports and imports, hours worked and inflation. Flow variables are measured in for example million kroner, or thousand hours worked, per year (or quarter, or month). Inflation is measured as a rate, or percentage per time period and is another example of a flow variable. Values of stock variables, in contrast, refer to particular point in time. For example, statistics of national debt often refer to the end of the year (thought it can also be given as an annual average of end-of-quarter or end-of-month observations). Prices indices are also examples of stock variables. They represent the cost of buying a basket of goods with reference to a particular time period. The annual consumption price index is a stock variable which is obtained as the average of the 12 monthly indices (each being a stock variable).

As noted, economic dynamics often arise from the combination of flow and stock variables, and in order to substantiate that a little more we consider a few examples.

First consider a nation’s net foreign debt. In principle, the dynamic behaviour of debt (a stock) is linked to the value of the current account (flow) in the following way

\[ \text{debt} = - \text{current account} + \text{last periods debt} + \text{corrections}. \]

Hence if the current account is zero over the period, there are no corrections of the debt value (typically due to financial transactions), this periods net foreign debt will be equal to last period’s debt. However, and ignoring corrections for simplicity, if there is a primary account surplus for some time, this will lead to a gradual reduction of debt—or an increase in the nation’s net wealth. Conversely, a consistent current account deficit raises a nation’s debt.

Figure 1.1 shows the development of the Norwegian current account and of Norwegian foreign debt. At the start of the period, Norway’s net debt (a stock variable) was hovering at around 100 billion, despite a current account surplus (albeit small) in the early 1980s. Evidently, there was a substantial debt stemming from the 1970s which the nation’s net financial savings (a flow) had not yet been able to wipe out. In most of the quarters in the years 1986-1989, the current account deficit was changed to a deficit, and, as one would expect from the “debt equation” above, debt is seen to increase again until it peaked in the first quarters of 1990. Later in the 1990s, the current account surplus returned, and therefore the net debt was gradually reduced.
Later, at the end of the millennium, surpluses grew to unprecedented magnitudes, up to 75 billion per quarter, resulting in a sharp build up of net financial wealth, accumulating to 75 billion kroner at the end of the period.

The theory of economic growth provides another example of the importance of stock and flow dynamics. For example, the level of production (a flow variable) in the economy depends on the size of the labour stock (literary speaking), and the capital stock. Due to the phenomenon of capital depreciation, some of today’s production needs to be saved just in order to keep the capital stock intact in the first period of the future. Moreover, due to population growth (and a declining marginal product of labour), next period’s capital stock will have to be larger than it is today if we want to avoid that output per capita declines in the next period. Hence, economic growth in terms of GDP per head necessitates that the flow of net investments (gross investments minus capital depreciation) is positive. In line with this, the dynamic equation of the capital stock is written as

\[ K_t = K_{t-1} + J_t - D_t \]

where \( K_t \) denotes the capital stock at the end of period \( t \), \( J_t \) is the flow of gross investments during period \( t \), and \( D_t \) is capital depreciation (also a flow). A much used assumption is that \( D_t \) is proportional to the pre-existing capital stock, i.e., \( D_t = \delta K_{t-1} \) where the rate of capital depreciation \( \delta \) is positive number which is less than one. This gives a well known expression for the development of the capital stock

\[ K_t = (1 - \delta) K_{t-1} + J_t \]

(1.1)

which plays an important role, not only in growth theory, but in real business cycle theory and intertemporal macroeconomics in general.

A third example is the dynamic process between private consumption, savings and wealth, which is illustrated in Figure 1.2. \( C_1 \), near the centre of the picture represents private consumption “today”. Private savings today, \( S_1 \) in the picture, is defined as current income minus consumption, hence today’s consumption affects \( S_1 \) as indicated by the line between \( C_1 \) and \( S_1 \). Suppose, despite households’ best attempt to balance income and consumption, today’s savings \( S_1 \) turned out to be too low. Rational consumers will then seek to correct this in the next period, by cutting back on consumption. This is an example of a dynamic relationship between two flow variables: today’s saving is causing tomorrow’s consumption, as indicated by the line between \( S_1 \) and \( C_2 \), representing tomorrow’s consumption. But the dynamic effect of today’s saving doesn’t necessarily end there. Household (financial) wealth is a stock variable which is given by last periods wealth plus current savings, as illustrated by the line from \( S_1 \) to \( W_1 \).

Experience tells us that wealth may have separate effect on future consumption, and this possibility is captured by the line from \( W_1 \) to \( C_2 \) in the picture. This again illustrates dynamics between flow (\( S_1 \) and \( C_2 \)) and stock (\( W_1 \)) variables. Of course, tomorrow’s eventual private consumption depends of a host of other factors, for example period 2 income which is denoted \( INC_2 \) in the picture. Moreover, \( C_1 \) and \( INC_2 \) together determine tomorrow’s savings \( S_2 \). By now the causal chain is closed: \( S_2 \) is going to affect \( C_3 \) either directly or through \( W_2 \) and so on.

Yet another case of stock-flow dynamics is the relationship between wages and unemployment, which we will discuss in detail in chapter 2. The rate of unemployment is a stock variable which influences wage growth (a flow variable). At the same time, the rate of unemployment depends on accumulated wage growth which determines the real wage level. Similar linkages exist between nominal and real exchange rates, and provide one of the key dynamic mechanisms in the macroeconomic models of the national economy that we encounter later in the course.

Because dynamics are a fundamental feature of the macroeconomy, all serious policy analysis is based on a dynamic approach. Hence, the officials responsible for fiscal and monetary policy use dynamic models as an aid in their decision process. In recent years, monetary policy had taken a more prominent and important role in activity regulation, and as we will explain later in the course, central banks in many countries have defined the rate of inflation as the target variable of economic policy. The instrument of monetary policy nowadays is the central banks sight deposit rate, i.e., the interest rate on banks’ deposits in the central bank. However, no central bank hopes for an immediate and strong effect on the rate of inflation after a change in the interest rate. Rather, because of the many dynamic effects triggered by a change in the interest rate, central bank governors prepare themselves to wait a substantial amount of time before the full effect of the interest rate change hits the target variable. The following statement from the web pages of Norges Bank [The Norwegian Central Bank] is typical of many central banks’ view:

\[ K_t = K_{t-1} + J_t - D_t \]
1.2. STATIC AND DYNAMIC MODELS, AN EXAMPLE

As students of economics you will be familiar with model analysis, both graphical and algebraic. Presumably, most of the models you have used have been static, and time has not played an essential role in the model formulation or in the analysis.

We therefore start by contrasting static models with models that have a dynamic formulation.
In the log linear model (1.4), since both real income and real consumption are transformed by applying the natural logarithm to each variable, it is common to say that each variable have been “log-transformed”. By taking the differential of the log-linear consumption function we obtain (see appendix A for a short reference on logarithms):

\[
\frac{dC_t}{C_t} = \beta_1 \frac{d\ln INC_t}{INC_t} \quad \text{or} \quad \frac{dC_t}{INC_t} \frac{\ln C_t}{C_t} = \beta_1.
\]

Hence, in equation (1.4), \( \beta_1 \) is interpreted as the elasticity of consumption with respect to income: \( \beta_1 \) represents the (approximate) percentage increase in real consumption due to a 1% increase in income.

Note that the log-linear specification (1.4) implies that the marginal propensity to consume is itself a function of income (see exercise 1). In that sense, the log-linear model is more flexible of the two functional forms and this is part of the reason for its popularity. To gain familiarity with the log-linear specification, we choose that functional form in the rest of this section, and in the next, but later we will also use the linear functional form, when that choice makes for easier exposition.

We include a stochastic term \( \epsilon_t \) in the two specified consumption functions. You can think of \( \epsilon_t \) as a completely random variable, with mean value of zero, but which cannot be predicted from \( C_t \) or \( Y_t \) (or from the history of these two variables). In a way, the inclusion of \( \epsilon_t \) in the models is a concession to reality, since economic theory (\( f(INC_t) \) in this case) cannot hope to capture all the vagaries of \( C_t \), which instead is represented by the stochastic term \( \epsilon_t \). Put differently, even if our theory is “correct”, it is only true on average. Hence, even if we know \( \beta_0 \), \( \beta_1 \) and \( INC_t \) with certainty, the predicted consumption expenditure in period \( t \) will only be equal to \( \hat{C}_t \) on average, due to the random disturbance term \( \epsilon_t \).

There is another reason for including a disturbance term in the consumption function, which has to do with how we confront our theory with the time series evidence. So let us consider real data corresponding to \( C_t \) and \( Y_t \), and assume that we have a good way of quantifying the intercept \( \beta_0 \) and the elasticity of consumption with respect to income, \( \beta_1 \). You will learn about so called least-squares estimation in courses in econometrics, but intuitively, least-squares estimation is a way of finding the numbers for \( \beta_0 \) and \( \beta_1 \) that give the on average best prediction of \( C_t \) for a given value of \( Y_t \). Using quarterly data for Norway, for the period 1967(1)-2002(4)—the number in brackets denotes the quarter—we obtain by using the least squares method:

\[
\ln \hat{C}_t = 0.02 + 0.99 \ln INC_t \quad (1.5)
\]

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\[
\ln \hat{C}_t = 0.02 + 0.99 \ln INC_t \quad (1.5)
\]
1.2. Static and Dynamic Models, An Example

The literature refers to this type of model as an autoregressive distributed lag model, ADL model for short. “Autoregressive” refers to the inclusion of \( \ln C_t - 1 \) on the right hand side of the equation. “Distributed lag” refers to the presence of current and lagged income in the model. Arguably, a more precise formulation would be “presence of current and/or lagged income in the model”, since even with \( \beta_1 = 0 \) or \( \beta_2 = 0 \) we would still refer to the equation as an ADL model. Only when \( \beta_1 = \beta_2 = 0 \) does the model reduce to a purer autoregressive model (AR for short).

With reference to Figure 1.2 above, note that equation (1.7) is compatible with the flow dynamics between consumption and income. In order to accommodate the effects that operate via wealth accumulation, we would of course need a more complicated model.

How can we evaluate the claim that the ADL model in equation (1.7) gives a better description of the data than the static model? A full answer to that question would take us into the realm of econometrics, but intuitively, one indication would be if the empirical counterpart to the disturbance of (1.7) are smaller and less systematic than the errors of equation (1.4). To test this, we obtain the residual \( \hat{\varepsilon}_t \) again using the method of least squares to find the best fit of \( \ln C_t \) according to the dynamic model:

\[
\ln C_t = 0.04 + 0.13 \ln INC_t + 0.08 \ln INC_{t-1} + 0.79 \ln C_{t-1}
\]

Figure 1.4 shows the two residual series \( \hat{\varepsilon}_t \) (static model) and \( \hat{\varepsilon}_t \) (ADL model), and it is immediately clear that the dynamic model in (1.8) is a much better description of the behaviour of private consumption than the static model (1.5). As already stated, this is a typical finding with macroeconomic data.

Judging from the estimated coefficients in (1.8), one main reason for the improved fit of the dynamic model is the lag of consumption itself, i.e., the autoregressive aspect of the equation. That the lagged value of the endogenous variable is an important explanatory variable is also a typical finding, and just goes to show that dynamic models represent essential tools for empirical macroeconomics. The rather low values of the income elasticities (0.130 and 0.081) may reflect that households find that a single quarterly change in income is “too little to build on” in their expenditure decisions. As we will see in the next section the results in (1.8) imply a much higher impact of a permanent change in income than of a temporary rise.

1.3 Dynamic Multipliers

The quotation from Noroses Bank’s web pages on monetary policy shows that the Central Bank has formulated a view about the dynamic effects of a change in the interest rate on inflation. In the quotation, the Central Bank states that the effect will take place within two years, i.e., 8 quarters in a quarterly model of the relationship between the rate of inflation and the rate of interest. That statement may be taken to mean that the effect is building up gradually over 8 quarters and then dies away quite quickly, but other interpretations are also possible. In order to inform the public more fully about its view on the monetary policy transmission mechanism (see topic 5 in our course), the Bank would have to give a more detailed picture of the dynamic effects of a change in the interest rate. Similar issues arise whenever it is of interest to study how fast and how strongly an exogenous perturbation or a policy change affect the economy, for example how private consumption is likely to be affected by a certain amount of tax-cut.

The key concept that we need to introduce in detail is called the dynamic multiplier. In order to explain dynamic multipliers, we first show what our estimated consumption function implies about the dynamic effect of a change in income, see section 1.3.1. We then derive dynamic multipliers using a general notation for autoregressive distributed lag models, in section 1.4.

1.3.1 Dynamic Effects of Increased Income on Consumption

We want to consider what the estimated model in (1.8) implies about the dynamic relationship between income and consumption. For this purpose there is no point to distinguish between fitted and actual values of consumption, so we drop the above \( C_t \).

Assume that income rises by 1% in period \( t \), so instead of \( INC_t \) we have \( INC_t = INC_t (1 + 0.01) \). Since income increases, consumption also has to rise. Using (1.8) we have

\[
\ln(C_t(1 + \delta_{0.01})) = 0.04 + 0.13 \ln(INC_t(1 + 0.01)) + 0.08 \ln INC_{t-1} + 0.79 \ln C_{t-1}
\]

Figure 1.4: Residuals of the two estimated consumptions functions (1.5), and (1.8).
1.3. Dynamic Multipliers

where $\delta_{c,t}$ denotes the relative increase in consumption in period $t$, the first period of the income increase. Using the approximation $\ln(1+\delta_{c,t}) \approx \delta_{c,t}$ when $-1 < \delta_{c,t} < 1$, and noting that

$$\ln C_t - 0.04 - 0.13 \ln INC_t - 0.08 \ln INC_{t-1} - 0.79 \ln C_{t-1} = 0,$$

we obtain $\delta_{c,t} = 0.0013$ as the relative increase in $C_t$. In other words, the immediate effect of a one percent increase in $INC$ is a 0.13% rise in consumption.

The effect on consumption in the second period depends on whether the rise in income is permanent or only temporary. It is convenient to first consider the effects of a permanent shock to income. Note first that equation (1.8) holds also for period $t + 1$, hence we have

$$\ln(C_{t+1}) = 0.04 + 0.13 \ln INC_{t+1} + 0.08 \ln INC_t + 0.79 \ln C_t$$

before the shock, and

$$\ln(C_{t+1}(1 + \delta_{c,t})) = 0.04 + 0.13 \ln(INC_{t+1}(1 + 0.01)) + 0.08 \ln INC_t(1 + 0.01) + 0.79 \ln C_t,$$

after the shock. Remember that in period $t + 1$, not only $INC_{t+1}$ has changed, but also $INC_t$ and period $t$ consumption (by $\delta_{c,t}$). Using (again) the approximation that $\ln(1+\delta_{c,t}) \approx \delta_{c,t}$ ($t = 0, 1$), the second multiplier is found to be:

$$\delta_{c,t} = -[\ln(C_{t+1}) - 0.04 - 0.13 \ln INC_{t+1} - 0.08 \ln INC_t - 0.79 \ln C_t]$$

$$+0.0013 + 0.0008 + 0.79 \times 0.0013 = 0.003125,$$

or 0.3% (remember that the expression inside the brackets is zero!). By the same way of reasoning, we find that the percentage increase in consumption in period $t + 2$ is 0.46% (formally $\delta_{c,t+2} = 100$).

Since $\delta_{c,t}$ measures the direct effect of a change in $INC$, it is usually called the impact multiplier, and can be defined directly by taking the partial derivative $\partial \ln C_t / \partial \ln INC_t$ in equation (1.8) (more on the relationship between derivative and multipliers in section 1.4 below). The dynamic multipliers $\delta_{c,1}, \delta_{c,2}, \ldots \delta_{c,\infty}$ are in turn linked by exactly the same dynamics as in equation (1.8), namely

$$\delta_{c,j} = 0.135 \delta_{c,j-1} + 0.086 \delta_{c,j-1} + 0.79 \delta_{c,j-1}, \quad \text{for } j = 1, 2, \ldots, \infty. \tag{1.9}$$

For example, for $j = 3$, and setting $\delta_{c,0,3} = \delta_{c,0,2} = 0.01$ since we consider a permanent rise in income, we obtain

$$\delta_{c,3} = 0.0013 + 0.0008 + 0.79 \times 0.0014 = 0.005734$$

or 0.57% in percentage terms. In this example, the multipliers increase from period to period, but the positive increment ($\delta_{c,j} - \delta_{c,j-1}$) is becoming smaller and smaller. From equation (1.9) this is seen to be due to the multiplication of the multiplier $\delta_{c,j-1}$ by the coefficient of the autoregressive term, which is less than 1. Eventually, the sequence of multipliers converges to zero.

The dynamic multipliers for a one percent rise in income are shown for convenience in the first column of Table 1.1. If we instead consider a temporary rise in income (by 0.01), equation (1.8) implies a different sequence of multipliers: The impact multiplier is again 0.0013, but the second multiplier becomes 0.13 $\times 0.018 \times 0.01 + 0.79 \times 0.0013 = 0.0018$, and the third is found to be 0.79 $\times 0.0018 = 0.0014$, so these multipliers are rapidly approaching zero, which becomes the long-run multiplier in this case.

Table 1.1: Dynamic multipliers of the estimated consumption function in (1.8), percentage change in consumption after a 1 percent rise in income.

<table>
<thead>
<tr>
<th>Period</th>
<th>Impact</th>
<th>Temporary</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. period after shock</td>
<td>0.31</td>
<td>0.18</td>
</tr>
<tr>
<td>2. period after shock</td>
<td>0.46</td>
<td>0.15</td>
</tr>
<tr>
<td>3. period after shock</td>
<td>0.57</td>
<td>0.11</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>long-run multiplier</td>
<td>1.00</td>
<td>0.00</td>
</tr>
</tbody>
</table>

Note that the second multiplier of the permanent change is equal to the sum of the two first multipliers of the transitory shock (0.13 + 0.18 = 0.31). This is due an underlying algebraic relationship between the two types of multipliers: multiplier $j$ of a permanent shock is the cumulated sum of the first multipliers of a temporary shock.

Specifically, if we supplement the multipliers in the column to the right with a few more periods and then sum the whole sequence, you will find that the sum is close to 1, which is the long-run multiplier of the permanent change. A relationship like this always holds, no matter what the long-run effect of the permanent change may be. Heuristically, this is because the effect of a permanent change in an explanatory
1.3. DYNAMIC MULTIPLIERS

Figure 1.5 shows graphically the two classes of dynamic multipliers for our consumption function example. Panel a) shows the temporary change in income, and below it, in panel c), you find the graph of the consumption multipliers. Correspondingly, panel b) and d) show the graphs with permanent shift in income and the corresponding (cumulated) dynamic multipliers.²

²In this book we often use figures with multiple graphs, such as Figure 1.5. In the text, we refer to the panels as a), b) etc., according to the rule

\[
\begin{array}{c}
\text{Panel a) } \\
\text{Panel b) } \\
\text{Panel c) } \\
\text{Panel d) }
\end{array}
\]

for a 2 x 2 figure. In the case of a 3 x 3 figure for example, c) denotes the third panel of the first row, while d) is the third panel of the second row.

¹These graphs were constructed using PcGive and GiveWin, but it is of course possible to use Excel or other programs.

1.4 General notation of the ADL model

As noted in the consumption function example, the impact multiplier is identical to the partial derivative of \( C_t \) with respect to \( INC_t \). We now establish more formally that also the second, third and higher order multipliers can be interpreted as derivatives. At this stage it is convenient to introduce a more general notation for the autoregressive distributed lag model. In (1.10), \( y_t \) is the endogenous variable while \( x_t \) and \( x_{t-1} \) make up the distributed lag part of the model:

\[
y_t = \beta_0 + \beta_1 x_t + \beta_2 x_{t-1} + \alpha y_{t-1} + \epsilon_t. \tag{1.10}
\]

In the same way as before, \( \epsilon_t \) symbolizes the small and random part of \( y_t \) which is unexplained by \( x_t, x_{t-1} \) and \( y_{t-1} \).

In many applications, as in the consumption function example, \( y \) and \( x \) are in logarithmic scale, due to the specification of a log-linear functional form. However, in other applications, different units of measurement are the natural ones to use. Thus, depending of which variables we are modelling, \( y \) and \( x \) can be measured in million kroner, or in thousand persons or in percentage points. Mixtures of measurement are also often used in practice: for example in studies of labour demand, \( y_t \) may denote the number of hours worked in the economy (or by an individual) while \( x_t \) denotes real wage costs per hour. The measurement scale does not affect the derivation of the multipliers, but care must be taken when interpreting and presenting the results. Specifically, only when both \( y \) and \( x \) are in logs, are the multipliers directly interpretable as percentage changes in \( y \) following a 1% increase in \( x \), i.e., they are (dynamic) elasticities.

To establish the connection between dynamic multipliers and the derivatives of \( y_t, y_{t+1}, y_{t+2}, \ldots \) it is convenient to think of \( x_t, x_{t+1}, x_{t+2}, \ldots \) as functions of a continuous variable \( h \). When \( h \) changes permanently, starting in period \( t \), we have \( \partial x_t / \partial h > 0 \), while there is no change in \( x_{t-1} \) and \( y_{t-1} \) since those two variables are predetermined in the period when the shock kicks in. Since \( x_t \) is a function of \( h \), so is \( y_t \), and the effect of \( y_t \) of the change in \( h \) is found as

\[
\frac{\partial y_t}{\partial h} = \beta_1 \frac{\partial x_t}{\partial h}. \tag{1.11}
\]

In the outset \( \partial x_t / \partial h \) can be any number, and in the consumption function example we used 0.01 to denote a small change income, but in general it is convenient to evaluate the multipliers for the case of \( \partial x_t / \partial h = 1 \) (it is customary to refer to this as a “unit change”). Hence the first multiplier is simply

\[
\frac{\partial y_t}{\partial h} = \beta_1. \tag{1.11}
\]

The second multiplier is found by considering the equation for period \( t + 1 \), i.e.,

\[
y_{t+1} = \beta_0 + \beta_1 x_{t+1} + \beta_2 x_t + \alpha y_t + \epsilon_{t+1}.
\]
and calculating the derivative $\partial y_{t+1}/\partial h$. Note that, due to the change in $h$ occurring already in period $t$, both $x_{t+1}$ and $x_t$ have changed, i.e., $\partial x_{t+1}/\partial h > 0$ and $\partial x_t/\partial h > 0$. Finally, we need to keep in mind that also $y_t$ is a function of $h$, hence:

$$\frac{\partial y_{t+1}}{\partial h} = \beta_1 \frac{\partial x_{t+1}}{\partial h} + \beta_2 \frac{\partial x_t}{\partial h} + \alpha \frac{\partial y_t}{\partial h} \quad (1.12)$$

Again, considering a unit-change, $\partial x_t/\partial h = \partial x_{t+1}/\partial h = 1$, and using (1.11), the second multiplier can be written as

$$\frac{\partial y_{t+1}}{\partial h} = \beta_1 + \beta_2 + \alpha \delta_1 = \beta_1(1 + \alpha) + \beta_2 \quad (1.13)$$

To find the third derivative, consider

$$y_{t+2} = \beta_0 + \beta_1 x_{t+2} + \beta_2 x_{t+1} + \alpha y_{t+1} + \epsilon_{t+2}$$

Using the same logic as above, we obtain

$$\frac{\partial y_{t+2}}{\partial h} = \beta_1 \frac{\partial x_{t+2}}{\partial h} + \beta_2 \frac{\partial x_t}{\partial h} + \alpha \frac{\partial y_{t+1}}{\partial h} \quad (1.14)$$

$$= \beta_1 \frac{\partial x_{t+2}}{\partial h} + \beta_2 \frac{\partial x_t}{\partial h} + \alpha \delta_1$$

$$= \beta_1 (1 + \alpha + \alpha^2) + \beta_2 (1 + \alpha)$$

where the conventional unit-change, $\partial x_t/\partial h = \partial x_{t+1}/\partial h = 1$, is used in the second line, and the third line is the result of substituting $\partial y_{t+1}/\partial h$ by the right hand side of (1.13). The comparison of equation (1.12) with the first line of (1.14) shows that there is a clear pattern: The third and second multipliers are linked by exactly the same form of dynamics that the govern the $y$ variable itself. This also holds for higher order multipliers, and means that the multipliers can be computed recursively. Once we have found the second multiplier, the third can be found easily by using the second line of (1.14).

Table 1.2 contains a summary of the results we have obtained. In the table, we use the notation $\delta_j$ ($j = 0, 1, 2, \ldots$) for the multipliers. For example, $\delta_0$ is identical to $\partial y_0/\partial h$, and $\delta_2$ is identical to the third multiplier, $\partial y_{t+1}/\partial h$ above. In general, because the multipliers are linked recursively, multiplier number $j + 1$ is given as

$$\delta_j = \beta_1 + \beta_2 + \alpha \delta_{j-1}, \text{ for } j = 1, 2, 3, \ldots \quad (1.15)$$

In the consumption function example, we saw that as long as the autoregressive parameter is less than one, the sequence of multipliers is converging towards a long run multiplier. In this more general case, the condition needed for the existence of a long-run multiplier is that $\alpha$ is less than one in absolute value, formally $-1 < \alpha < 1$. In later sections, namely 1.7 and 1.9 specifically, this condition is explained in more detail. For the present purpose, we simply assume that the condition holds, and define the long run multiplier as $\delta_j = \delta_{j-1} = \delta_{\text{long-run}}$. Using (1.15), the expression for $\delta_{\text{long-run}}$ is found to be

$$\delta_{\text{long-run}} = \frac{\beta_1 + \beta_2}{1 - \alpha} \text{ if } -1 < \alpha < 1. \quad (1.16)$$

Clearly, if $\alpha = 1$, the expression does not make sense mathematically, since the denominator is zero. Economically, it doesn’t make sense either since the long run effect of a permanent unit change in $x$ is an infinitely large increase in $y$ (if $\beta_1 + \beta_2 > 0$). The case of $\alpha = -1$, may at first sight seem to be acceptable since the denominator is 2, not zero. However, as explained in section 1.9 below, the dynamics is unstable also in this case, meaning that the long run multiplier is not well defined for the case of $\alpha = -1$.

Hence, while we can use the table to calculate dynamic multipliers also for the cases where the absolute value of $\alpha$ is equal to or larger than unity, the long-run multiplier of a constant change in $x$ does not exist in this case. Correspondingly, the multipliers of a temporary change do not converge to zero in the case where $-1 < \alpha < 1$ does not hold.

Notice that, unlike $\alpha = 1$, the case of $\alpha = 0$ is unproblematic. This restriction, which excludes the autoregressive term $y_{t-1}$ from the model, only serves to simplify the multiplier analysis. All of the dynamic response in of $g$ then due to the distributed lag in the explanatory variable, and it is referred to by economists as the distributed lag model, denoted DL-model.
1.5 A TYPOLOGY OF LINEAR MODELS

Table 1.3: A model typology.

<table>
<thead>
<tr>
<th>Type</th>
<th>Equation</th>
<th>Restrictions</th>
</tr>
</thead>
<tbody>
<tr>
<td>ADL</td>
<td>( y_t = \beta_0 + \beta_1 x_t + \beta_2 z_{t-1} + \alpha y_{t-1} + \epsilon_t )</td>
<td>None</td>
</tr>
<tr>
<td>Static</td>
<td>( y_t = \beta_0 + \beta_1 x_t + \epsilon_t )</td>
<td>( \beta_2 = \alpha = 0 )</td>
</tr>
<tr>
<td>DL</td>
<td>( y_t = \beta_0 + \beta_1 x_t + \beta_2 z_{t-1} + \epsilon_t )</td>
<td>( \alpha = 0 )</td>
</tr>
<tr>
<td>Difference data</td>
<td>( \Delta y_t = \beta_0 + \beta_1 \Delta x_t + \epsilon_t )</td>
<td>( \beta_2 = -\beta_1, \alpha = 1 )</td>
</tr>
<tr>
<td>ECM</td>
<td>( \Delta y_t = \beta_0 + \beta_1 \Delta x_t + (\beta_1 + \beta_2) x_{t-1} + \epsilon_t )</td>
<td>None</td>
</tr>
<tr>
<td>Homogenous ECM</td>
<td>( \Delta y_t = \beta_0 + \beta_1 \Delta x_t + (\alpha - 1) y_{t-1} + \epsilon_t )</td>
<td>( \beta_1 + \beta_2 = -(\alpha - 1) )</td>
</tr>
</tbody>
</table>

\( \Delta \) is the difference operator, defined as \( \Delta z_t \equiv z_t - z_{t-1} \).

1.5 A typology of linear models

The discussion at the end of the last section suggests that if the coefficient \( \alpha \) in the ADL model is restricted to for example 1 or to 0, quite different dynamic behaviour of \( y_t \) is implied. In fact the resulting models are special cases of the unrestricted ADL model. For reference, this section gives a typology of models that are encompassed by the ADL model. Some of these model we have already mentioned, while others will appear later in the book.

Table 1.3 contains three columns, for model Type, defining Equation and Restrictions, where we give the coefficient restrictions that must be true for each models to be a valid simplifications of the ADL. Hence for example, for the Static model to be valid, both \( \beta_2 = 0 \) and \( \alpha = 0 \) must be true (in practice this amounts to estimation of the ADL and then testing the joint hypothesis with a F-test). The danger of using a static model when the restrictions do not hold is that we get a misleading impression of the adjustment lags (the dynamic multipliers). Specifically, the response of \( y_t \) to a change in \( x_t \) is represented as immediate when it is in fact distributed over several periods.

For the DL model to be a valid simplification of the ADL, only one coefficient restriction needs to be true, namely \( \alpha = 0 \). That said, the DL model is also a quite restrictive model of the dynamic response to a shock, since the whole adjustment of

\[ y_t \text{ to a change in } x_t \text{ is completed in the course of only two periods.} \]

The fourth model in Table 1.3, called the Difference data model is included since it is popular in modern macroeconomics. A technical motivation for the model in applied work is that, by taking the difference of \( y_t \) and \( x_t \) prior to estimation, the econometric problem of residual autocorrelation is reduced. However, unless the two restrictions that define the model type are empirically acceptable, choosing the Difference data model may create more problems than it solves. For example, the regression coefficient of \( \Delta x_t \) will not be a correct estimate of the impact multiplier \( \beta_1 \), nor of the long-run multiplier \( \Delta_{long-run} \). The economic interpretation of the Difference data model (‘growth rate model’ if \( y_t \) and \( x_t \) are in logs) is also problematic, since there appears to be no cost of being out of equilibrium in terms of the levels variables.

The two last models do not have the shortcomings of the Static, DL or Difference data models. Hence, the error-correction model (ECM) is consistent with a long run relationship between \( y_t \) and \( x_t \), and it also describes the behaviour of \( y_t \) outside equilibrium. As explained in section 1.7 below, the ECM is actually a reformulation of the ADL, hence there are no restrictions involved. The Homogenous ECM has the same advantages as the ECM in terms of interpretation, but the long-run multiplier is restricted to unity. One example, which is relevant for our course, is when \( y_t \) is the log of the domestic price level, and \( x_t \) is the log of an index of foreign prices, denoted in domestic currency. If the hypothesis of purchasing power parity applies, that would motivate the Homogenous ECM with \( \beta = 1 \).

1.6 Extensions and examples

In this subsection we briefly point to some important extensions of the ADL model. Second, to help solidify the understanding of the ADL framework, we provide additional economic examples.

1.6.1 Extensions

The ADL model in equation (1.10) is general enough to serve as an introduction to most aspects of dynamic analysis in economics. However, the ADL model, and the dynamic multiplier analysis, can be extended in several directions to provide additional flexibility in applications. The most important extensions are:

1. Several explanatory variables
2. Longer lags
3. Systems of ADL equations

In economics, more than one explanatory variable is usually needed to provide a satisfactory explanation of the behaviour of a variable \( y_t \). The ADL model (1.10) can be generalized to any number of explanatory variables. However, nothing is lost
1.6. EXTENSIONS AND EXAMPLES

in terms of understanding by only considering the case of two exogenous variables, $x_{1,t}$ and $x_{2,t}$. The extension of equation (1.10) to this case is

$$y_t = \beta_0 + \beta_{11}x_{1,t} + \beta_{21}x_{1,t-1} + \beta_{31}x_{2,t} + \beta_{22}x_{2,t-1} + \alpha_{0} + \epsilon_{t},$$  

(1.17)

where $\beta_{k}$ is the coefficient of the $k^{th}$ lag of the explanatory variable $x$. The dynamic multipliers of $y_t$ can be with respect to either $x_{1,t}$ or $x_{2,t}$, the derivation being exactly the same as above. Formally, we can think of each set of multipliers as corresponding to partial derivatives. In applications, the dynamic multipliers of the different explanatory variables are often found to be markedly different. For example, if $y_t$ is the log of the hourly wage, while $x_{1,t}$ and $x_{2,t}$ are the rate of unemployment and productivity respectively, dynamic multipliers with respect to unemployment is usually much smaller in magnitude than the multipliers with respect to productivity, see section 2 below.

Longer lags in either the $x$-es or in the autoregressive part of the model makes for more flexible dynamics. Even though the dynamics of such models can be quite complicated, the dynamic multipliers always exist, and can be computed by following the logic of section 1.3.1 above. However, as such calculation quickly become tiring, practitioners use software when they work with such models (good econometric software packages includes options for dynamic multiplier analysis). In this course, we do not consider the formal analysis of higher order autoregressive dynamics.

Often, an ADL equation is joined up with other equations to form dynamic system of equations. For example, in (1.17) while $x_{2,t}$ is truly exogenous, there is separate equation for $x_{1,t}$ with $y_t$ on the right hand side. Hence, $x_{1,t}$ and $y_t$ are determined in a simultaneous equations system. Another, very common case, is that the equation for $x_{1,t}$ contains the lag $y_{t-1}$, not the contemporaneous $y_t$. Also in this case is $x_{2,t}$ and $y_t$ jointly determined, but in what is referred to as a recursive system of equations. In either case the true multipliers of $y_t$ with respect to the exogenous variable $x_{2,t}$, cannot be derived from equation (1.17) alone, because that would make us miss the feed-back that a change in $x_{2,t}$ has on $y_t$ via the endogenous variable $x_{1,t}$. To obtain the correct dynamic multipliers of $y$ with respect to $x_1$ we must use the two equation system. In this course we will not pursue the formal analysis of dynamic system is general. However, specific economic examples of dynamic system and how they are analyzed will crop up several times as we proceed. For example, section 1.10 takes the dynamic version of the Keynesian multiplier models as one specific example. Other applications of dynamic systems are found in the sections on wage-price dynamics and on economic policy analysis.

1.6.2 A few more examples

There are more examples of applications of the ADL model than one can mention, so this section focuses on a few of the examples that we encounter later in the course.

1.6.2.1 The dynamic consumption function (again)

This has been the main example so far, and in section 1.3 we discussed the dynamics of a log-linear specification in detail. Of course exactly the same analysis applies to a linear functional form of the consumption function, except that the multipliers will be in units of million kroner (at fixed prices) rather than percentages. In section 1.10 the linear consumption function is combined with the general budget equation to form a dynamic system.

In modern econometric work on the consumption function, more variables are usually included than just income. Hence, there are other multipliers to consider than the ones with respect to $INC_t$. The most commonly found additional explanatory variables are wealth, the real interest rate and indicators of demographic developments.

1.6.2.2 The Phillips curve

In Chapter 2, and several times later in the course, we will consider the so called expectations augmented Phillips curve. An example of such a relationship is

$$\pi_t = \beta_0 + \beta_{11}u_t + \beta_{21}u_{t-1} + \beta_{31}\pi_{t-1} + \varepsilon_t.$$  

(1.18)

$\pi_t$ denotes the rate of inflation, $\pi_t = \ln(P_t/P_{t-1})$, where $P_t$ is an index of the price level of the economy. $u_t$ is the rate of unemployment—or its log. Whether $u_t$ is a rate (or percentage) or the log of a rate (percentage) is another of example of a choice between different functional forms (as in the consumption function example at the beginning of this chapter). If $u_t$ is a rate (or percentage) the Phillips curve is linear, and the effect of a small change in $u_t$ on $\pi_t$ is independent of the initial rate of unemployment.

If $u_t = \ln$ the rate of unemployment), the Phillips curve is non-linear: starting from a low level, a rise in the rate of unemployment leads to a larger reduction in $\pi_t$ than if the initial rate of unemployment is high. The graph of the Phillips curve, with the rate of unemployment on the horizontal axis, is thus curved towards the origin (a convex function). In many countries, a convex Phillips curve is known to give a better data fit than the linear function, and it is important to keep that in mind when we frequently in this book, find it convenient to interpret $u_t$ as the rate of unemployment.

In equation (1.18), the distributed lag in the rate of unemployment captures several interesting economic hypotheses. For example, if the rise in inflation following a fall in unemployment is first weak but then gets stronger, we might have that both $\beta_{11} < 0$ and $\beta_{12} < 0$. On the other hand, some economist have argued the opposite.
1.6. EXTENSIONS AND EXAMPLES

that the inflationary effects of changes in unemployment are likely to be strongest in the first periods after shock to unemployment, in which case we might expect to find that $\beta_1 < 0$ while $\beta_2 > 0$.

Finally, in equation (1.18), $\pi_{t+1}$ denotes the expected rate of inflation one period ahead, and in the same manner as earlier in this chapter, $\pi_t$ denotes a random disturbance term. In sum, (1.18) includes two explanatory variables: the rate of unemployment, and the expectation of the future value of the endogenous variable. The rate of unemployment is observable, but expectations are usually not. In order to make progress from (1.18) it is therefore necessary to specify a hypothesis of expectations. The simplest hypothesis is that expectations build on the last observed rate of inflation, hence

$$\pi_{t+1} = \pi_{t-1},$$

(1.19)

where the parameter $\tau$ is typically positive, but not larger than 1, i.e., $0 < \tau \leq 1$.

Substitution of $\pi_{t+1}$ by (1.19), the Phillips curve in (1.18) becomes

$$\pi_t = \beta_0 + \beta_1 u_t + \beta_2 \pi_{t-1} + \alpha \pi_{t-1} + \varepsilon_t,$$

(1.20)

which is an ADL equation.

As explained, the hypothesis in equation (1.19) is just one out of many possible formulations about expectations formation. Alternative specifications give rise to other dynamic models of the rate of inflation. Consider for example a situation where the government have introduced an explicit inflation target, which we denote $\bar{\pi}$. In this case, it is reasonable to believe that firms and households will base their expectations of future inflation on the attainment of the inflation target. In the simplest case we may then set

$$\pi_{t+1} = \bar{\pi},$$

(1.21)

in place of (1.19). As an exercise, you should convince yourself that equations (1.18) and (1.21) imply an equation for inflation which is an example of a distributed lag model (DL model). More generally, firms and households take into consideration the possibility that future inflation is not exactly on target. Hence they may adopt a more robust forecasting rule, for example

$$\pi_{t+1} = (1-\tau)\bar{\pi} + \tau \pi_{t-1}, \quad 0 < \tau \leq 1.$$  

(1.22)

In this case, the derived dynamic equation for inflation again takes the form of an ADL model.

1.6.2.3 Exchange rate dynamics

Later in the course we will discuss the market for foreign exchange. As we will learn, in the case of a floating exchange rate regime, the following relationship can

be derived for the nominal exchange rate (kroner/USD), $E_t$:

$$\ln E_t = \beta_0 + \beta_1 (i_t - i_t^*) + \varepsilon_t, \quad \beta_1 < 0$$

(1.23)

where $i^*$ is the foreign interest rate and $\varepsilon$ denotes the expected rate of depreciation. $\pi_t - i_t - \pi_e^*$ is known as the risk premium. In passing, you may note that the parameter $\beta_1$ is called a semi-elasticity. It measures the relative change in $E_t$ due to a unit increase in the risk premium. If expectations are regressive, meaning that

$$\varepsilon = -\tau (E_{t-1}, \tau > 0),$$

(see Rødseth (2000, Ch 3.1)) the equation for $\ln E_t$ can be written as

$$\ln E_t = \beta_0 + \beta_1 (i_t - i_t^*) + \alpha E_{t-1} + \varepsilon_t$$

(1.24)

with $\beta_1 < 0$ and $\alpha = \beta_1 \tau < 0$, which we recognize as another example of an ADL model. Note that unlike the earlier examples, the coefficient of the lagged endogenous variable is negative in this example.

1.6.3 Multipliers in the text books to this course

As already noted in the introduction, the distinction between short and long-run multipliers permeates modern macroeconomics, and so is not special to the consumption function example above! The reader is invited to be on the look-out for expressions like short and long-run effects/multipliers/elasticities in the textbooks by Burda and Wyplosz (2001). One early example in the book is found in Chapter 8, on money demand, Table 8.4. Note the striking difference between the short-run and long-run multipliers for all countries, a direct parallel to the consumption function example we have worked through in this section. Hence, the precise interpretation of (log) linearized money demand function in Burda’s Chapter 8.4 is as a so called steady state relationship, thus the parameter $\mu$ is a long-run elasticity with respect to income. In the next section of this note, the relationship between long-run multipliers and steady state relationships in explained. The money demand function also plays an important role in Chapter 9 and 10, and in later chapters in B&W.

In the book by Burda, the distinction between short and long-run is a main issue in Chapter 12, where short and long-run supply curves are derived. For example, the slope of the short-run curves in figure 12.6 correspond to the impact multipliers of the respective models, while the vertical long-run curve suggest that the long-run multipliers are infinite. In section 2 below, we go even deeper into wage-price dynamics, using the concepts that we have introduced here.

Later on in the course we will encounter models of the joint determination of inflation, output and the exchange rates which are also dynamic in nature, so a good understanding the logic of dynamic multipliers will prove very useful. Interest rate setting with the aim of controlling inflation is one specific example. However, we will not always need (or be able to) give a full account of the multipliers of these more complex model. Often we will concentrate on the impact and long-run multipliers.
1.7 The error correction model

If we compare the long-run multiplier of a permanent shock to the estimated regression coefficient (or elasticity) of a static model, there is often a close correspondence. This is the case in our consumption function example where the multiplier is 1.00 and the estimated coefficient in equation (1.5) is 0.99. This is not a coincidence, since the dynamic formulation in fact accommodates a so-called steady-state relationship in the form of a static equation. In this sense, a static model is therefore embedded in a dynamic model.

To look closer at the correspondence between the static model formulation and the steady state properties of the dynamic model, we consider again the ADL model:

\[ y_t = \beta_0 + \beta_1 x_t + \beta_2 x_{t-1} + \alpha y_{t-1} + \varepsilon_t. \]  

(1.25)

Above, we have emphasized two properties of this model. First it usually explains the behaviour of the dependent variable much better than a simple static relationship, which imposes on the data that all adjustments of \( y \) to changes in \( x \) takes place without delay. Second, it allows us to calculate the very useful dynamic multipliers. But what does (1.25) imply about the long-run relationship between \( y \) and \( x \)? The sort of relationship that we expect to hold when both \( y_t \) and \( x_t \) are growing with constant rates, as so-called steady state situation? To answer this question it is useful to rewrite equation (1.25), so that the relationship between levels and growth becomes clear. The reason we do this is to establish that changes in \( y_t \) are not only caused by changes in \( x_t \), but also by past period's deviation between \( y_t \) and the steady state equilibrium value of \( y_t \) which we denote \( y^* \). Thus, the period to period changes in \( y_t \) are correcting past deviations from equilibrium, as well as responding to (new) changes in the explanatory variable. The version of the model which shows this most clearly is known as the error correction (or equilibrium correction) model, ECM for short, see Table 1.3 above.

To establish the ECM transformation of the ADL, we need to make two algebraic steps, and to establish a little more notation (related to the concept of the steady state). In terms of algebra, we first subtract \( y_{t-1} \) from either side of equation (1.25), and then subtract and add \( \beta_1 x_{t-1} \) on the right hand side. This gives

\[ \Delta y_t = \beta_0 + \beta_1 x_t + \beta_2 x_{t-1} + (\alpha - 1)y_{t-1} + \varepsilon_t \]  

(1.26)

where \( \Delta \) is known as the difference operator, defined as \( \Delta z_t = z_t - z_{t-1} \) for a time series variable \( z_t \). If \( y_t \) and \( x_t \) are measured in logarithms (like consumption and income in our consumption function example) \( \Delta y_t \) and \( \Delta x_t \) are their respective growth rates. Hence, for example, in the consumption function example of section 1.3.1:

\[ \Delta \ln C_t = \ln(C_t/C_{t-1}) = \ln(1 + \frac{C_t - C_{t-1}}{C_{t-1}}) \approx \frac{C_t - C_{t-1}}{C_{t-1}}. \]

Thus, \( \Delta \ln C_t \) is explained by two factors: first the change in the explanatory variable, \( \Delta x_t \), and second, the correction of the last period's disequilibrium, the
1.7. THE ERROR CORRECTION MODEL

deviation between $y_{-1}$ and the last periods equilibrium level $y^*$. With reference to Table 1.3, we note that the Homogenous ECM has exactly the same properties and interpretation, in that case, the long run multiplier $\gamma$ is unity.

Consider next a theoretical steady state situation in which growth rates are constants, $\Delta x_t = \gamma g$, $\Delta y_t = g_y$, and the disturbance term is equal to its mean, $\varepsilon_t = 0$. Imposing this in (1.31), and noting that $\{y - y^*\}_{-1} = 0$ by definition of a steady state, gives

$$g_y = \beta_0 + \beta_1 g_x - (1 - \alpha)k,$$

meaning that the intercept in the long-term relationship (1.28) can be expressed as

$$k = \frac{-\beta_0 + \beta_1 g_x}{1 - \alpha},$$

(1.32)

again subject to a condition $-1 < \alpha < 1$. Often we only consider a static steady states, with no growth, so $g_y = g_x = 0$. In that case $k$ is simply $\beta_0/(1 - \alpha)$.

In sum, there is an important correspondence between the dynamic model and a static relationship like (1.28) motivated by economic theory:

1. A theoretical linear relationship $y^* = k + \gamma z$ can be retrieved as the steady state solution of the dynamic model (1.25). This generalizes to theory models with more than one explanatory variable (e.g., $y^* = k + \gamma_1 x_1 + \gamma_2 x_2$) as long as both $x_1$ and $x_2$ (and/or their lags) are included in the dynamic model. In section 2 we will discuss some details of this extension in the context of models of wage and price setting (inflation).

2. The theoretical slope coefficient $\gamma$ is identical to the corresponding long-run multiplier (of a permanent increase in the respective explanatory variables).

3. Conversely, if we are only interested in quantifying a long-run multiplier (rather than the whole sequence of dynamic multipliers), it can be found by using the identity in (1.29).

A reasonable objection to #3 is that, if we are only interested in the theoretical long-run slope coefficient, why don’t we simply estimate $\gamma$ from a static model, rather than bother with a dynamic model? After all, in the consumption function example, the direct estimate (1.5) is practically identical to the long-run multiplier which we derived from the estimated dynamic model? The short answer is that, as a rule, static models yields poor estimates of long-run multipliers. To understand why takes us into time series econometrics, but intuitively the direct estimate (from the static model) is only reliable when the theoretical relationship has a really strong presence in the data. This seems to be the case in our consumption function example, but in a majority of applications, the theoretical relationship, though valid, is obscured by slow adjustment and influences from other factors. Therefore, it generally pays off to first formulate and estimating the dynamic model.

Returning to the beginning of this section, we note that the transformation of the ADL model into level and differences is known as the error correction transformation.
1.9. SOLUTION AND SIMULATION OF DYNAMIC MODELS

The second rationale for (interpretation of) static models is exactly the interpretation that we have highlighted most in this chapter. It applies to asymptotically stable dynamic models, where the steady state relationships represents an equilibrium situation.

Ideally, the two interpretations should not be mixed! Nevertheless authors do exactly that, perhaps because they need to make short-cuts in order to complete their models and to be able to “tell a story”. Consider for example Ch 12.3 in B&W where equations which reasonably can be interpreted as long-run tendencies (interpretation 2 above), namely equation (12.6) and (12.7) are turned into a seemingly dynamic model by using a algebraic trick (see section 12.3.4 in B&W) and an ad hoc theory of dynamics in the mark-up. In section 2 we present two other models of wage-and-price formation where the short-run is reconciled with the long-run in a consistent manner.

1.9 Solution and simulation of dynamic models

The reader will have noted that the existence of a finite long-run multiplier, and thereby the validity of the correspondence between the ADL model and long-run relationships, depends on the autoregressive parameter α in (1.25) being different from unity. In section 1.9.1 we show that the parameter α is also crucial for the nature and type of solution of equation (1.25). Several of the insights won by considering the solution of the ADL equation (1.25) in some detail, carry over to more complicated equations as well as to systems of equations. For such systems the practical way to find the solution is by a computer based technique called simulation, which we outline in section 1.9.2.

1.9.1 Solution of ADL equations

In order to discuss the solution of ADL models, it is useful to first consider the simpler case of a deterministic autoregressive model, where we abstract from the distributed lag part \((x_t \text{ and } x_{t-1})\) of the model, as well as from the disturbance term \((\varepsilon_t)\). One way to achieve this simplification of (1.25) is to assume that both \(x_t\) and \(\varepsilon_t\) are fixed at their respective constant means:

\[ x_t = m_x \text{ for } t = 0, 1, \ldots \text{ and } \varepsilon_t = m_\varepsilon \text{ for } t = 0, 1, \ldots \]

Hence we follow convention an assume that each \(\varepsilon_t\), representing a small and random influence on \(y\), has a common mean of zero. For the explanatory variable \(x_t\) the mean is denoted \(m_x\). We write the simplified model as

\[ y_t = \beta_0 + Bm_x + \alpha y_{t-1}, \text{ where } B = \beta_1 + \beta_2 \]  

(1.33)

In the following we proceed as if the the coefficients \(\beta_0, \beta_1, \beta_2\) and \(\alpha\) are known numbers. This is a simplifying assumption which allows us to abstract from estimation issues, which in any case belong to a course in time series econometrics.

We assume that equation (1.33) holds for \(t = 0, 1, 2, \ldots\). It is usual to refer to \(t = 0\) as the initial period or the initial condition. The assumption we make about the initial period is crucial for the existence and uniqueness of a solution. A standard result is the following: If \(y_0\) is a fixed and known number, then there is a unique sequence of numbers \(y_0, y_1, y_2, \ldots\) which is the solution of (1.33). This can be seen by induction: Consider first \(t = 0\). From the assumption of known initial conditions, \(y_0\) then follows. Next, set \(t = 1 \in (1.33)\), and \(y_1\) is seen to determined uniquely since we already know \(y_0\). And so on; having established \(y_{t-1}\) we find \(y_t\) be solving (1.33) one period forward. This procedure is also known as solving the equation recursively from known initial conditions.

You may note that unlike other “conditions for solution” occurring in other areas of mathematical economics, the requirement of fixed and known initial conditions is almost trivial since \(y_0\) is simply given “from history”.11

The algebraic properties of the solution of (1.33) carries over to other, less stylized, cases so they are worth considering in more detail. Using the suggested recursive procedure, we obtain for the three first periods:

\[ y_1 = \beta_0 + Bm_x + \alpha y_0 \]
\[ y_2 = \beta_0 + Bm_x + \alpha y_1 = \beta_0(1 + \alpha) + Bm_x(1 + \alpha) + \alpha^2 y_0 + \beta_1(1 + \alpha) + \alpha^2 y_0 \]
\[ y_3 = \beta_0 + Bm_x + \alpha y_2 = \beta_0 + Bm_x(1 + \alpha + \alpha^2) + \alpha^3 y_0 \]

Evidently there is as pattern, allowing us to write \(y_t\) as

\[ y_t = \beta_0 + Bm_x \sum_{k=0}^{t-1} \alpha^k + \alpha^3 y_0, \quad t = 1, 2, \ldots \]  

(1.34)

which is the general solution of (1.33) for the initial condition of known \(y_0\). Equation (1.34) is a useful reference for discussion of the three types of solution of ADL models, namely the stable, unstable and explosive solutions. We start with the stable solution, and then consider the two others.

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11 If we open for the possibility that the initial conditions are not determined by history, but that they can “jump” at any point in time, there are other solutions to consider. Such solutions play a large role in macroeconomics, but they belong to more advanced courses.
Stability of solution  

The condition

\[-1 < \alpha < 1\]  \hspace{1cm} (1.35)

is the necessary and sufficient condition for the existence of a (globally asymptotically) stable solution. The stable solution has the characteristic that asymptotically there is no trace left of the initial condition \( y_0 \). From (1.34) we see that as the distance in time between \( y_t \) and the initial condition increases, \( y_t \) has less and less influence on the solution. When \( t \) becomes large (approaches infinity), the influence of the initial condition becomes negligible. Since \( \sum_{s=0}^{t-1} \alpha^s \to \frac{1}{1-\alpha} \) as \( t \to \infty \), we have asymptotically:

\[ y^* = \frac{\beta_0 + Bm_s}{1-\alpha} \]  \hspace{1cm} (1.36)

where \( y^* \) denotes the equilibrium of \( y_t \). As stated, \( y^* \) is independent of \( y_0 \).

Assume that there is permanent change in \( x_t \). Such a change can be implemented as a shift in the mean \( m_x \). Keeping in mind that \( B = \beta_1 + \beta_2 \), the derivative of \( y^* \) with respect to a permanent change in \( x \) is

\[ \partial y^*/\partial m_x = (\beta_1 + \beta_2)/(1-\alpha), \]
Nevertheless, the solution is perfectly valid mathematically speaking: given an initial condition, there is one and only one sequence of numbers \( y_1, y_2, \ldots, y_T \) which satisfy the model. The instability is however apparent when we consider a sequence of solutions. Assume that we first find a solution conditional on \( y_0 \), and denote the solution \( \{y_1^0, y_2^0, \ldots, y_T^0\} \). This is actually a forecast for the period \( t = 1, \ldots, T \) conditional on \( y_0 \). After one period, we will usually want to recalculate the solution because something we did not anticipate occurred in period 1, making \( y_1^0 \neq y_1 \) and we will want to condition the forecast on the observed \( y_1 \). The updated solution is \( \{y_1^1, y_2^1, \ldots, y_T^1\} \) since we now condition on \( y_1 \). From (1.36) we see that as long as \( y_1^0 \neq y_1 \) (the same as saying that \( c_1 \neq 0 \)) we will have \( y_1^1 - y_1^0 \neq 0, y_2^1 - y_1^0 \neq 0, \ldots, y_T^1 - y_T^0 \neq 0 \). Moreover, when the time arrives to condition on \( y_T \), the same phenomenon is going to be observed again. The solution is indeed unstable in the sense that any (small) change in initial conditions have a permanent effect on the solution. Economists like to refer to this phenomenon as hysteresis, see Burda and Wyplosz (2001, p 538). In the literature on wage-price setting, the point has been made that failure of wages to respond properly to shocks to unemployment (in fact the long-run multiplier of wages with respect to is zero) may lead to hysteresis in the rate of unemployment. The case of \( \alpha = -1 \) is more curious, but it is in any case useful to check the solution and dynamics implied by (1.34) also in this case.

Panel c) of Figure 1.6 illustrates the unstable case by setting \( \beta_0 + B_0 \nu = 0.025 \) and \( \alpha = 1 \) in (1.33), corresponding to for example 2.5% annual growth if \( \gamma \) is a variable in logs. Actually there are two solutions. One takes \( y_0 = 3.57 \) as the initial period, and the other is conditioned by \( y_0 = 3.59 \). Although there is a relatively small difference between the two initial conditions in this example, the lasting influence on the different initial values on the solutions is visible in the graph.

**Explosive solution** When \( \alpha \) is greater than unity in absolute value the solution is called explosive, for reasons that should be obvious when you consult (1.34). In panel d) of Figure 1.6 the explosive solution obtained when setting \( \beta_0 + B_0 \nu = 0.025 \) and \( \alpha = 1.05 \) in (1.33) is shown. We plot the explosive solution over a longer period than the others, in order to make the explosive nature of the solution become clearly visible to the eye.

At this point it is worth recalling that everything we have established about the existence and properties of solutions, has been based on making equation (1.25):

\[
y_t = \beta_0 + \beta_1 x_t + \beta_2 x_{t-1} + \alpha y_{t-1} + \epsilon_t
\]

subject to the two simplifying assumptions: \( \epsilon_t = 0 \) for \( t = 0, 1, \ldots, \), and \( x_t = m_t \) for \( t = 0, 1, \ldots \). Luckily, the qualitative results (stable, unstable or explosive solution) are quite general and independent of which assumptions we make about the disturbance and the \( x \) variable. However, any particular numerical solution that we obtain for (1.25) is conditioned by these assumptions. For example, in the stable case with \( 0 < \alpha < 1 \), if we instead of \( x_t = m_t \) for \( t = 0, 1, \ldots \) set \( x_t = 2m_t, t = 0, 1, \ldots \) then both

![Figure 1.7: Solution of the consumption function in equation (1.8) for the period 1990(1)-2001(4). The actual values of log(Ct) are also shown, for comparison.](image-url)
by a formidable 50% over the period, and the graph shows that this development is well explained by the solution of the consumption function. That the solution is conditioned by the true values of income, must be taken into account when assessing the close capture of the trend-growth in consumption.

Also apparent from the graph is a marked seasonal pattern: consumption is highest in the 3rd and 4th quarter of each year. Seasonality is a feature of many quarterly or monthly time series. How to represent seasonality in a model is question in its own right and cannot be answered in any detail within the scope of this book. However the interested reader might note that in order to capture seasonality as well as we do in the figure, we have included so called seasonal dummies, but for simplicity, the estimated coefficient have been suppressed in equation (1.8).\textsuperscript{12}

1.9.2 Simulation of dynamic models

In applied macroeconomics it is not common to refer to the solution of an ADL model. The common term is instead simulation. Specifically, economists use dynamic simulation to denote the case where the solution for period 1 is used to calculate the solution for period 2, and the solution for period 2 is in its turn used to find the solution for period 3, and so on. From what we have said above, in the case of a single ADL equation like

\[ y_t = \beta_0 + \beta_1 x_t + \beta_2 x_{t-1} + \alpha y_{t-1} + \epsilon_t, \]

dynamic simulation amounts to finding a solution of the model. Concretely, the correspondence between solution and dynamic simulation also holds for complex equations (with several lags and/or several explanatory variables), as well as for the systems of dynamic equations which are used in practice. However, while the mathematical description of the solution becomes more involved for such systems, simulation offers a practical way of finding the solution that always works as long as the initial condition is fixed. Reflecting the usefulness of simulation in macroeconomics is the fact that modern econometric software packages contain a wide range of built-in possibilities for simulation analysis, as well as rich possibilities for reporting of the results (in numerical or graphical form). Three main types of simulation is track-analysis, forecasting and multiplier analysis.

When we do a track-analysis, simulation is used to find how fell the solution of an estimated macroeconomic model tracks the actual values of the endogenous variable over the sample period. Figure 1.7 provides an example of this kind of track analysis, where we in fact have used the built in simulation possibilities in PcGive. Dynamic simulation is also the practical way produce forecasts from dynamic models. However, a main complication that distinguishes a historical dynamic simulation from a dynamic forecast is that, since observations of the \( x \) variable are only available for the historical sample period (\( t = 1, 2, ..., T \)) they need to be forecasted themselves before we can simulate \( y \) for \( H \) periods ahead (\( H \) is often referred to as the forecast horizon). Having fixed a set of values \( x_{T+1}, x_{T+2}, ..., x_{T+H} \), the simulated values \( y_{T+1}, y_{T+2}, ..., y_{T+H} \) correspond to the solution based on the initial values \( x_T, x_{T-1}, y_{T} \). Finally, dynamic simulation is used to produce dynamic multipliers. In that case we do two dynamic simulations: first, a baseline simulation using a baseline or reference set of values of the \( x \) variable (often these are the historical values, or a baseline forecast), and second, an alternative simulation based on an alternative (or “shocked”) set of \( x \) values. As always, the initial condition has to be specified before the solution can be found. Since the focus is on the effect of a change in \( x \), the same initial condition is used for both the baseline and the alternative simulation. If we denote the baseline solution \( y^B \) and the alternative \( y^A \), the difference \( y^B - y^A \) are the dynamic multipliers. Referring back to section 1.3.1, you will now appreciate that what we did there was nothing but doing dynamic simulation “by hand”, in order to obtain the multipliers of consumption with respect to (permanent and temporary changes) in income. Figure 1.5, in contrast is produced “automatically” by the built in multiplier option in PcGive, and in exercise 8 you are asked to replicate that graph.

We end this section by noting that care must be taken to distinguish between static simulation, which is also part of the vocabulary used by macroeconomists, and dynamic simulation which we have discussed so far. In a static simulation, the actual (not the simulated) values for \( y \) in period \( t-1 \) is used to calculate \( y_t \). Thus the sequence of \( y \)'s obtained from a static simulation does not correspond to the solution of the ADL model. Static and dynamic simulations give an identical result only for the first period of the simulation period. Since, in practice, we use estimated values of the model’s coefficient when we simulate a model, the resulting \( y \)-values from a static simulation over the sample period are identical to the equation’s fitted values.\textsuperscript{13}

1.10 Dynamic systems

In macroeconomics, the effect of a shock or policy change is usually dependent on system properties. As a rule it is not enough to consider only one equation in order to obtain the correct dynamic multipliers. Consider for example the consumption function of section 1.2 and 1.3, where we implicitly assumed that income (\( \text{INC} \)) was an exogenous variable. This exogeneity assumption is only tenable given some further assumptions about the rest of the economy: for example if there is a general equilibrium with flexible prices (see for example Burda and Wyplosz (2001, ch 10.5)) and the supply of labour is fixed, then output and income may be regarded as independent of \( C_t \). However, with sticky prices and idle resources, i.e., the Keynesian...
1.10. DYNAMIC SYSTEMS

case, INC must be treated as endogenous, and to use the multipliers that we derived in section 1.3 are in fact misleading.

Does this mean that all that we have said so far about multipliers and stability of a dynamic equation is useless (apart from a few special cases)? Fortunately things are not that bad. First, it is often quite easy to bring the system on a form with two reduced form dynamic equations that are of the same form that we have considered above. After this step, we can derive the full solution of each endogenous variable of the system (if we do want). Second, there are ways of discussing stability and the dynamic properties of systems, without first deriving the full solution. One such procedure is the so called phase-diagram which however goes beyond the scope of this course. Third, in many cases the dynamic system is after all rather intuitive and transparent, so it is possible to give a good account of the dynamic behaviour, simply based on our understanding of the economics of the problem under consideration.

Fourth, as hinted above, simulation represents a practical way to find the solution of systems of dynamic equations with quantified coefficient values and known initial conditions, i.e., the sort of system used in practice for forecasting and policy analysis.

In this section, we give a simple example of the first approach (finding the solution) based on the consumption function again. However, it is convenient to use a linear specification:

\[ C_t = \beta_0 + \beta_1 INC_t + \alpha C_{t-1} + \varepsilon_t \]  

(1.39)

which together with the stylized product market equilibrium condition

\[ INC_t = C_t + J_t \]  

(1.40)

where \( J_t \) denotes autonomous expenditure, and INC is now interpreted as the gross domestic product, GDP. We assume that there are idle resources (unemployment) and that prices are sticky. The 2-equation dynamic system has two endogenous variables \( C_t \) and \( INC_t \), while \( J_t \) and \( \varepsilon_t \) are exogenous.

To find the solution for consumption, simply substitute INC from (1.40), and obtain

\[ C_t = \tilde{\beta}_0 + \tilde{\alpha} C_{t-1} + \tilde{\beta}_2 J_t + \tilde{\varepsilon}_t \]  

(1.41)

where \( \tilde{\beta}_0 \) and \( \tilde{\alpha} \) are the original coefficients divided by \((1-\beta_1)\), and \( \tilde{\beta}_2 = \beta_1/(1-\beta) \), (what about \( \tilde{\varepsilon}_t \)?)

Equation (1.41) is yet another example of an ADL model, so the theory of the previous sections applies. For a given initial condition \( C_0 \) and known values for the two exogenous variables (e.g., \( J_1, J_2, \ldots, J_T \)) there is a unique solution. If \(-1 < \tilde{\alpha} < 1\) the solution is asymptotically stable. Moreover, if there is a stable solution for \( C_t \), there is also a stable solution for \( INC_t \), so we don’t have to derive a separate equation for \( INC_t \) in order to check stability of income.

The impact multiplier of consumption with respect to autonomous expenditure is \( \tilde{\beta}_2 \), while in the stable case, the long-run multiplier is \( \tilde{\beta}_2/(1-\tilde{\alpha}) \).

Equation (1.41) is called the final equation for \( C_t \). The defining characteristic of a final equation is that (apart from exogenous variables) the right hand side only contains lagged values of the left hand side variable. It is often feasible to derive a final equation for more complex system than the one we have studied here. The conditions for stability is then expressed in terms of the so called characteristic roots of the final equation. The relationship between \( \tilde{\alpha} \) and \( \tilde{\beta} \) and the characteristic roots goes beyond the scope of this course, but we mention that sufficient condition for stability of a second order difference equation is that both \( \tilde{\alpha} \) and \( \tilde{\beta} \) are less than one.

**Exercises**

1. Show that, after setting \( \varepsilon_t = 0 \) (for convenience), \( MFC \equiv \partial C_t/\partial INC_t = k \cdot \beta_1 INC_t^{\alpha_2-1} \), where \( k = \exp(\beta_0) \).

2. Use the numbers from the estimated consumption function and check that by using the formulae of Table 1.2, you obtain the same numerical results as in section 1.3.1.

3. Check that you understand and are able to derive the results in the column for a temporary change in \( x_t \) in Table 1.2. Confirm that in the case of a distributed lag model, the two first multipliers of a temporary change are equal the coefficients of \( x_t \) and \( x_{t-1} \) respectively, while \( \delta_j = 0 \) for \( j = 2, 3, \ldots \). Show also that in the case of a permanent change, \( \delta_{long-run} \) is equal to \( \delta_2 \).

4. In a dynamic system with two endogenous variables, explain why we only need to derive one final equation in order to check the stability of the system.

5. In our example, derive the final equation for \( INC_t \). What is the relationship between the demand multipliers that you know from Keynesian models, and the impact and long-run multipliers of INC with respect to a one unit change in autonomous expenditure?

6. Let \( \sigma \) denote the real exchange rate (defined in the same way as in B&B) in period t. Assume that the time period is annual, and that we are given the following ADL model which explains \( \sigma \):

\[ \sigma_t = \beta_0 + \beta_1 x_t + \beta_2 x_{t-1} + \alpha \sigma_{t-1} \]  

(1.42)

\( x_t \) represents an exogenous variable. (For simplicity, the random disturbance \( \varepsilon_t \) has been omitted). Economists have suggested that after a shock, the real exchange rate can “overshoot” its (new) long-run level.

(a) With reference to (1.42), formulate a precise definition of overshooting.

(b) Illustrate the concept by drawing graphs (by hand or computer) of dynamic multipliers which are consistent with both overshooting with no-overshooting.
1.10. DYNAMIC SYSTEMS

(c) Give one example of specific parameter values which gives rise to overshooting, and of another specific set which does not imply overshooting. Formulate a definition of overshooting that you find meaningful in the light of (1.42)

7. Figure 1.7 in the text shows a solution for ln(C_t) which trends upwards. This may indicate an unstable solution. Assume that you did not have access to the value of α in the consumption function, only to Figure below 1.8. Why does this figure indicate that the solution is in fact stable?

![Figure 1.8: 5 solutions of the consumption function in 1.8, corresponding to different initial periods: 1989(4), 1990(4), 1992(4), 1995(4) and 1998(4).](image)

8. Download the file norcons.zip from the web page. Use the instruction provided in the included .txt file to replicate the dynamic multipliers in Figure 1.5.
Chapter 2

Wage-price dynamics

2.1 Introduction

In economics the term ‘inflation’ generally describes the prevailing annual rate at which the prices of goods and services are increasing. It is commonplace that all prices tend to rise at broadly the same rate, because, when prices of domestic goods are rising fast this will generally be true also of wages, and of the price of imported goods. This is because inflation in one sector of the economy permeates rapidly into other sectors. The phrase, a ‘high rate of inflation’ therefore usually describes a situation in which the money values of all goods in an economy are rising at a fast rate.

History shows many examples of countries which have been hit by extremely high rates of inflation, so called hyperinflation. Any such episode is a result of crisis and disorganization in the economic and political system of a country, and of a breakdown in the public trust in the country’s monetary system. The typical case is however that inflation can be moderately high for long periods of time without doing serious damage to the stability of monetary institutions. Nevertheless, even if we aside the phenomenon of hyperinflation, it is generally agreed that a high and volatile inflation rate is a cause of concern, and towards the end of last century, the governments of the “Western world” invested heavily in curbing inflation. Institutional changes took place (crushing of labour unions in the UK, revitalization of incomes policies in Norway, a new orientation of monetary policy, the European Union’s stability pact, etc.), and governments also allowed unemployment to rise to level not seen since the 2WW. Was all this necessary, or was there other ways to curb inflation that would have meant a lower toll in terms of welfare loss? Finally, given that inflation has now become a prime target of economic policy, what is the inflation outlook and how can inflation be controlled using the policy instruments that are recognized as legitimate in liberalized economies? Answers to any of these important issues are necessarily model dependent. By model dependency we mean that, before the answer is given, a view has been formulated, either explicitly or implicitly, about the major determinants of inflation and about which instruments are available for controlling inflation, and so forth. In this chapter we therefore discuss inflation models.

It is a familiar fact that prices influences each other: wages follow the cost of living index, which is based on prices on tradable and non-tradable goods, which in their turn (and to varying extent) reflect labour costs that are determined by wage settlements of the past. Hence the ‘inflation spiral’ has a dynamic structure that results from the interaction of several markets, where the markets for labour and goods are perhaps the two most important. Models of the dynamic structure of wage and price setting are therefore of the greatest interest in a course like ours, and they also represent one area where the modelling concepts of Chapter 1 are indispensable.

In this chapter, we present two important models of wage and price setting behaviour that are relevant to the inflation process of small open economies: the Norwegian model and the Phillips curve model. The Norwegian model, or main-course model, is an important example of a macroeconomic theory that is based on realistic assumptions about wage and price setting behaviour. In Norway, it has been the framework both for analysis and policy decisions for decades. The Phillips curve is of course covered by every textbook in macroeconomics, and in section 1.6.2 above we have already introduced a Phillips curve relationship, as an example of how ADL models are applied in macroeconomics. In this chapter, we attempt to give a fresh lick to the Phillips curve by comparing it with the main-course model of inflation, which in turn is seen to be compatible with modern bargaining models of wage setting. In fact we show that the Phillips curve can be seen as a special case of the richer dynamics implied by the Norwegian model. At the end of the chapter we give a some empirical evidence from the Norwegian economy: Which model is supported by the evidence? The Phillips curve or the bargaining model?

2.2 The main-course model of open economy inflation

The Norwegian model of inflation was formulated in the 1960s1. It soon became the framework for both medium term forecasting and normative judgements about "sustainable" centrally negotiated wage growth in Norway, as explained in the seminal paper by Aukrust (1977).2 However, it is not the historical importance of the

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1 In fact there were two models, a short-term multisector model, and the long-term two sector model that we re-construct using modern terminology in this chapter. The models were formulated in 1966 in two reports by a group of economists who were called upon by the Norwegian government to provide background material for that year’s round of negotiations on wages and agricultural prices. The group (Aukrust, Holte and Stolz) produced two reports. The second (dated October 26 1966, see Aukrust (1977)) contained the long-term model that we refer to as the main-course model. Later, there was similar development in e.g., Sweden, see Edgren et al. (1969) and the Netherlands, see Driehuis and de Wolf (1976).

2 In later usage the distinction between the short and long-term models seems to have become blurred, in what is often referred to as the Scandinavian model of inflation. We acknowledge Aukrust’s clear exposition and distinction in his 1977 paper, and use the name Norwegian main-course model for the long-term version of his theoretical framework.

3 On the role of the main-course model in Norwegian economic planning, see Hjerkholt (1998).
Norwegian model that is our concern here, but instead the rather remarkable fact that the model, when presented in modernized form, is fully consistent with recently formulated and currently popular models of wage formation, and of the modern view on the role of wage setting in the inflation process of small open economies. Specifically, we show that the model can be formulated as a set of propositions about long-run relationships and equilibrating dynamics, using the models and concepts of Chapter 1.

As will be explained below, another name for the model is the main-course model, since a joint trend made up of productivity and foreign prices define the scope for wage growth in an open economy (the main-course for wage development over time). We use the name main-course model in the following, since it also discourages the misunderstanding that the model only applies to the Norwegian economy.

As mentioned, part of our agenda is to show that the (reconstructed) Norwegian model serves as a reference point for, and in many respects also as a corrective to, the modern models of wage formation and inflation in open economies. Macroeconomic textbooks and papers are known to state that models of inflation have evolved progressively, from the basic (wage) Phillips curve, via the expectations augmented Phillips curve and towards the present day post-Keynesian Phillips curve. It emerges from our discussion that the theoretical development is better characterized by a branching-out of new specialized models. For example, the expectations augmented wage Phillips curve is a special case of the Norwegian model, and is thus not an evolution of it. The latest branch of theory building, post Keynesian Phillips curve, adds a phenomenon which is not covered by the main-course model, namely forward looking expectations. However, post-Keynesians achieve this at the cost of making unrealistic assumptions about (or simply ignoring) about other parts of the inflation process. Arguably, a better way to explore the potential role of forward looking rational expectations in a main-course model of the modern and extended form that we presented below. However, that project takes us far beyond the scope of this book.

2.2.1 A framework for long-run wage and price setting

Central to the model is the distinction between a tradables sector where firms act as price takers, either because they sell most of their produce on the world market, or because they encounter strong foreign competition on their domestic sales markets, and a nontradables sector where firms set prices as mark-ups on wage costs. The two sectors are dubbed the exposed (e) and sheltered (s) sectors of the economy.

The model’s main propositions are, first, that exposed sector wage growth will follow a long run tendency defined by the exogenous price and productivity trends which characterize that sector. The joint productivity and price trend is called the main-course of e-sector wage development. Second, it is assumed that the relative wage between the two sectors are constant in the long term. For consistency, the

\[ W_e = \frac{Q_e}{Q_0} A_e \]

where \( W_e \) denotes the long-run equilibrium wage level consistent with the twin assumptions of exogenous price and productivity trends and a constant normal wage share. Using lower case letters to denote natural logarithms of a variable, i.e., for example \( \ln w_e = \ln(W_e) \) for the wage rate, we obtain the following, equivalent formulation, which we will use in the following:

\[ \ln H_{1\text{MC}}: \quad \ln w_e^* = q_e + a_e + \ln mc_e \]
productivity. If this is not the case, then $w^e$, cannot be the equilibrium to which the actual wage converges in a steady-state. The joint trend made up of productivity and foreign prices, traces out a central tendency for wage growth, it represents a long-run sustainable scope for wage growth. Aukrust (1977) aptly refers to this joint trend as the main-course for wage determination in the exposed industries. For reference we therefore define the main-course variable (in logs) as

$$mc = a_e + q_e.$$  

(2.3)

It is easy to show, by way of citation, that Aukrust meant exactly what we have asserted, namely that equation (2.2) is a long-run relationship corresponding to a steady state situation. Consider for example the following citation:

The relationship between the “profitability of E industries” and the “wage level of E industries” that the model postulates, therefore, is a certainly not a relation that holds on a year-to-year basis. At best it is valid as a long-term tendency and even so only with considerable slack. It is equally obvious, however, that the wage level in the E industries is not completely free to assume any value irrespective of what happens to profits in these industries. Indeed, if the actual profits in the E industries deviate much from normal profits, it must be expected that sooner or later forces will be set in motion that will close the gap. (Aukrust, 1977, p 114-115).

Aukrust coined the term ‘wage corridor’ to represent the development of wages through time and used a graph similar to figure 2.1 to illustrate his ideas. The main-course defined by equation (2.3) is drawn as a straight line since the wage is measured in logarithmic scale. The two dotted lines represent what Aukrust called the “elastic borders of the wage corridor”.

Figure 2.1: The ‘Wage Corridor’ in the Norwegian model of inflation.

In modern theories of wage formation, labour unions, either unitarily or as a result of bargaining with firms, are assumed to achieve an expected real wage which is a mark-up on average productivity, see e.g., Burda and Wyplosz (2001, Ch. 12). If “expected real wage” is defined as $w^e - q_e$, Aukrust’s model is perfectly compatible with today’s consensus theory of wage formation. The main difference is that Aukrust’s formulation is more precise about the time horizon—his model being a theory of long term tendencies. Moreover, just as the mark-up is assumed to be a function of labour market pressure in modern bargaining models, the normal rate of profit in the main-course model can be seen as not being completely invariant, but instead as conditioned by economic, social and institutional factors. Hence, one plausible generalization of $H_{mc}$ is represented by

$$H_{mc} \quad w^e = m_{e,0} + mc + \gamma_{e,1}u_i,$$

where $u$ is the log of the rate of unemployment, and $m_{e,0}$ is the wage-share after the effect of unemployment has been netted out. In the following we will take $H_{mc}$ as the long-run relationship implied by the main-course model.

Other candidate variables for inclusion in an extended main-course hypothesis include the ratio between unemployment insurance payments and earnings (the so called replacement ratio, see for example Burda and Wyplosz (2001, p 89-90, and Table 4.8)), and variables that represent unemployment composition effects (unemployment duration, the share of labour market programmes in total unemployment), see Nickell (1987), Calmfors and Forslund (1991). However, in the following we will keep things as simple as possible, and only include the rate of unemployment in the...
extended main-course proposition.

As mentioned above, there are two other long-run propositions which complete Aukrust’s theory. First, an assumption about a constant relative wage between the sectors, and, second, the assumed existence of a normal sustainable wage share also in the sheltered sector of the economy. We dub these two additional propositions H2mc and H3mc respectively:

\[ H2mc \quad w_s^* - w_e^* = m_{es}, \]
\[ H3mc \quad w_s^* - q_1^* - a_s = m_s. \]

1 is the log of the long run ratio between e-sector and s-sector wages. \( a_s \) is the exogenous productivity trend in the sheltered sector, and \( m_s \) is the (log) of the equilibrium wage rate in the sheltered sector.

Note that, if the long run wage in the exposed sector is determined by the exogenous main course, then \( H2mc \) determines the long run wage in the sheltered sector, and \( H3mc \), in turn determines the long run price level of the sheltered sector. Hence, re-arranging \( H3mc \), gives

\[ q_1^* = w_s^* - a_s + m_s \]

which is similar to modern theories of so called normal cost pricing: the price is set as a mark-up on average labour costs.

The other two long-run propositions (\( H2mc \) and \( H3mc \)) in Aukrust’s model have not received nearly as much attention as \( H1mc \) in empirical research, but exceptions include Roseth and Holden (1990) and Nymoen (1991). In part, this is due to lack of high quality data wage and productivity data and for the private service and retail trade sectors. Another reason is that both economists and policy makers in the industrialized countries place most emphasis on understanding and evaluating wage setting in manufacturing, because of its continuing importance for the overall economic performance. In the next paragraph we will focus custom and focus on wage setting in the e-sector of the economy. However, we return to the two other proposition at the end of the section, where we draw the implications of the main course model for domestic price level and inflation.

2.2.2 Dynamic adjustment

As we have seen, Aukrust was clear about two things. First, the main-course relationship for e-sector wages should be interpreted as a long-run tendency, not as a relationship that governs wage development on a year to year basis. Hence, the theory makes us anticipate that actual observations of e-sector wages will fluctuate around the theoretical main-course. Second, if e-sector wages deviate too much from the long-run tendency, we expect that forces will begin to act on wage setting so that adjustments are made in the direction of the main-course. For example, profitability below the main-course level will tend to lower wage growth, either directly or after a period of higher unemployment. In Aukrust’s words:
thus excluding negative values of the autoregressive coefficient. Hence, subject to the stability condition in (2.6), (2.5) can be written as

\[
\Delta w_{e,t} = \beta_0 + \beta_{11} \Delta mc_t + \beta_{21} \Delta u_t - (1 - \alpha) \left[ w_{e,t-1} - \frac{\beta_{11} + \beta_{22}}{1 - \alpha} mc_{t-1} - \frac{\beta_{12}}{1 - \alpha} u_{t-1} \right] + \epsilon_t.
\]

(2.7)

To reconcile this with the hypothesized long run relationship \(m_{mc} \), we make use of

\[
\gamma_{e,1} = \frac{\beta_{11} + \beta_{22}}{1 - \alpha},
\]

the long-run multiplier of wage with respect to \( u \), and then impose the following restriction on the coefficient of \( mc_{t-1} \):

\[
\beta_{11} + \beta_{22} = (1 - \alpha)
\]

(2.8)

since \( m_{mc} \) implies that the long-run multiplier with respect to \( mc \) is unity. Then (2.7) becomes

\[
\Delta w_{e,t} = \beta'_0 + \beta_{11}' \Delta mc_t + \beta_{21}' \Delta u_t - (1 - \alpha) \left[ w_{e,t-1} - \gamma_{e,1} u_{t-1} \right] + \epsilon_t
\]

(2.9)

which is an example of the homogenous ECM of section 1.5. The short-run multiplier with respect to the main-course is of course \( \beta_{11} \), which can be considerably smaller than unity without violating the main-course hypothesis \( m_{mc} \).

The formulation in (2.9) is an equilibrium correction model, ECM, since the term in brackets captures that wage growth in period \( t \) partly corrects last period’s deviation from the long-run equilibrium wage level. We can write it as

\[
\Delta w_{e,t} = \beta'_0 + \beta_{11}' \Delta mc_t + \beta_{21}' \Delta u_t - (1 - \alpha) \left[ w_{e,t-1} - w^*_e \right]_{t-1} + \epsilon_t
\]

where \( w^*_e \) is given by the left hand side of the extended main-course relationship \( H_{1mc} \).\footnote{The interested reader will have noted that, for consistency, the intercept \( \beta'_0 \) is given as \( \beta'_0 = \beta_0 + (1 - \alpha) \mu_{mc} \).} Subject to the condition \( 0 < \alpha < 1 \) stated above, wage growth is seen to bring the wage level in the direction of the main-course. For example, assume that the sum of price and productivity growth is constant, \( \Delta mc_t = \gamma_{mc} \) and \( \Delta u_t = 0 \) (constant rate of unemployment). If there is disequilibrium in period \( t - 1 \), for example \( \{w_e - w^*_e\}_{t-1} > 0 \), wage growth from \( t - 1 \) to \( t \) will be reduced, and this leads to \( \{w_e - w^*_e\}_t < \{w_e - w^*_e\}_{t-1} \) in the next period.

Figure 2.2 illustrates the dynamics following an exogenous and permanent change in the rate of unemployment: We consider a hypothetical steady state with an initially constant rate of unemployment (see upper panel) and wages growing along the main-course. In period \( t_0 \) the steady state level of unemployment increases permanently. Wages are now out of equilibrium, since the steady state path is shifted down in period \( t_0 \), but because of the corrective dynamics, the wage level adjusts gradually towards the new steady state growth path. Two possible paths are indicated by the two thinner line. In each case the wage is affected by \( \beta_{21} < 0 \) in period \( t_0 \). Line \( a \) corresponds to the case where the short-run multiplier is smaller in absolute value than the long-run multiplier, (i.e., \( -\beta_{21} < \gamma_{e,1} \)). A different situation, is shown in adjustment path \( b \), where the short-run effect of an increase in unemployment is larger than the long-run multiplier.
2.2.3 A simulation model of the main-course

We can use computer simulation to confirm our conclusions about the dynamic behaviour of the main-course model. The following three equations make up a representative main-course model of wage-setting in the exposed sector:

\[
\begin{align*}
    w_{t,t} &= 0.1mc_{t} + 0.3mc_{t-1} - 0.06\ln U_{t-1} + 0.6w_{t,t-1} + \varepsilon_{w, t} \quad (2.10) \\
    mc_{t} &= 0.03 + mc_{t} + \varepsilon_{mc, t} \quad (2.11) \\
    U_{t} &= 0.005 + 0.01 \cdot S1989 + 0.8U_{t-1} + \varepsilon_{U, t} \quad (2.12)
\end{align*}
\]

Equation (2.10) is a simplified version of (2.4), where we omit the constant term and the current value of the rate of unemployment. Note that the equation is written with an explicit log transformation of the rate of unemployment, \( U_t \). The disturbance \( \varepsilon_{w, t} \) is normally distributed with mean zero and standard error 0.01 (which is representative of the residual standard error estimated wage equations on annual data). Equation (2.11) defines the main-course variable as a so-called stochastic trend. The average growth rate of the main-course is 0.03, which again seems OK since we have annual data in mind for our calibration exercise.\(^6\) The standard error of \( \varepsilon_{mc, t} \) is also set to 0.01. The third equation (2.12), is the equation for the rate of unemployment. It specifies \( U_t \) as an exogenous variable (hence there is no presence of \( w_{t,t} \) or its lags in the \( U \)-equation). The equation has a steady state solution for \( U_t \) since the autoregressive coefficient is set to 0.8.

Next, we want to add a little more colour to the calibrated model, namely a structural change in the unemployment period, and this explains why the term 0.01 \( \cdot \) S1989, is included the model. \( S1989 \) is a so called ‘step-dummy’, which is zero until 1988 (\( t < 1989 \)) and one for later periods. You can check that this calibrates the steady state rate of unemployment to 0.025 before 1989, and to 0.05 after. Hence there is a regime shift in the equilibrium rate of unemployment, taking place in 1989.\(^7\)

In Figure 2.3, the dotted line shows the solution for \( w_{t,t} \) over the period 2003-2010, using the initial values \( mc_{2002} \) and \( U_{2002} \), and the values for the three disturbances, drawn randomly by the computer programme for the solution period 2003 – 2010. The line closest to the line for the solution is (the extended) main-course, and the boundaries of the wage-corridor is given as \( \pm 2 \) standard errors of the main-course. Clearly, all the properties of the theoretically motivated Figure 2.1 are also found in the simulation results in Figure 2.3.

Figure 2.4 emulates Figure 2.2 of a regime shift in the rate of unemployment.

---

\(^1\)Due to the disturbance term, the period by period growth rates vary, hence the rate of change, and thus the trend of \( mc_{t} \), is stochastic as the name suggests.

\(^2\)For reference, the standard error of \( \varepsilon_{U, t} \) is set to 0.003.

\(^6\)For simplicity, all disturbances are set to their mean value of zero in the simulations underlying this figure.
2.2. The lower panel of the graph shows that the wage level is gradually reduced compared to what we would have observed if the regime shift had not kicked in during 1989 (indicated by the dotted line).

Although this section has used simulation of a calibrated model to illustrate wage and unemployment dynamics, the model remains wholly theoretical. However, in section 2.5.2 we will show an estimated model, using real world data, which has very similar properties.

2.2.4 The main-course and the price level

So far, we have looked at only e-sector wage formation. To sketch the theory’s implication for the overall price level, we introduce a simplified definition equation for the log of the price level, \( p \):

\[
p_t = \phi g_{e,t} + (1 - \phi) q_{e,t}, \quad 0 < \phi < 1.
\]

(2.13)

\( \phi \) is a coefficient that reflects the weight of non-traded goods in private consumption.\(^6\)

Starting from

\[
p_t = \phi (g_{e,t} - q_{e,t}^*) + q_{e,t}^* + (1 - \phi) q_{e,t},
\]

and using H\(_{pue}\), H\(_{pec}\) and H\(_{3mc}\), we have

\[
p_t = \phi (g_{e,t} - q_{e,t}^*) + q_{e,t}^* + \phi (a_{e,t} - a_{s,t}) + \phi_1 e_{t} u_t
\]

+ terms with \( m_a, \) \( m_s \) and \( m_{es} \)

saying that, the change from \( p_t \) to \( p_{t+1} \), and hence inflation will depend on

1. Disequilibrium in the e-sector price level,
2. development of foreign prices,
3. development of productivity (differential),
4. unemployment

We will not specify the dynamics here, but note that if a steady state exist, with \( \Delta g_{e,t} = g_{e} \) and constant unemployment, then \( q_{e,t} - q_{e,t}^* = 0 \) and

\[
\Delta p_t = g_s + \phi (g_s - g_e).
\]

(2.14)

Note the somewhat surprising implication that increased productivity in the exposed sector of the economy increases the steady state level of inflation, see also Rødseth (2000, Ch. 7.6).

\(^{6}\)For reference, due to the log-form, \( \phi = x_s/(1 - x_s) \) where \( x_s \) is the share of non-traded good in consumption.

2.2.5 The main-course model and the Scandinavian model of inflation

In this chapter we have taken care to explain how the assumption about a constant long-term wage share in the e-sector can be made consistent with an ADL equation for the e-sector nominal wage level. The distinction between long-term steady-state propositions and dynamics has not always been made clear, though. This is true for the Scandinavian model, which the main-course model is often confounded with.

The Scandinavian model specifies the same three underlying assumptions as the main-course model: H\(_{1mc}\) (we do not need the extended version of the hypothesis for the point we wish to make here), H\(_{2mc}\) and H\(_{3mc}\). But the distinction between long-run and dynamics is blurred in the Scandinavian model. Hence, for example, the dynamic equation for e-sector wages in the Scandinavian model is usually written as:

\[
\Delta w_{e,t} = \beta_0 + \Delta m_{e,t},
\]

(without a disturbance term for simplicity) which is seen to place the rather unrealistic restriction of \( \alpha = 1 \) on e-sector wage dynamics. Hence, the Scandinavian model specification can only be expected to be reasonable if the time period subscript \( t \) refers to a span of several years, so that \( \Delta w_{e,t} \) and \( \Delta m_{e,t} \) refer to for example 10 years averages of the two growth rates.\(^{10}\)

2.2.6 The main-course and the Battle of the Markups

Chapter 12.3 in the book by B&W presents a modern framework for thinking about inflation. The basic idea is that in modern economies firms typically attempt to mark-up on prices on unit labour costs, while workers and unions on their part strive to make their real wage reflect the profitability of the firms, thus their real wage claim is a mark-up on productivity. Hence there is a conflict between workers and firms, both are interested in controlling the real wage, but they have imperfect control: Workers influence the nominal wage, while the nominal price is determined by firms.

As already hinted the Norwegian model of inflation fits into this modern framework. Hence, using H\(_{1mc}\)–H\(_{3mc}\) above we have that

\[
\begin{align*}
w^* &= m_e + q_e + a_s, \\
q_e^* &= -m_s + w^* - a_s,
\end{align*}
\]

saying that the desired wage level is a mark up on prices and productivity, exactly as in equation (12.5) in B&W, albeit in the exposed sector of the economy, while the desired (e-sector) price level is a mark-up on unit-labour costs (as in B&W’s equation (12.4)).

\(^{10}\)For an exposition and appraisal of the Scandinavian model, in terms of contemporary, see Rødseth (2000, Chapter 7.6)
2.3. THE MAIN-COURSE AND THE PHILLIPS CURVE

BkW present the battle of mark-ups model in a static setting. The implication is that (if only workers’ price expectations are correct in each period) actual wages and prices are determined by the static model. This of course runs against our main message, namely that actual wage and prices are better described by a dynamic system. Above, we have seen how the Norwegian model represents a consistent theory about both the long-run, and of the dynamics of wage setting.

2.2.7 Role of exchange rate regime

In his 1977 paper, Aukrust argues that the validity of his model hinges on the assumption of a controlled (“fixed”) exchange rate. If the rate of foreign exchange is floating, one cannot be sure that \( q_f \), which is the log of the world price in USD + the Kroner/USD exchange rate, is a pure causal factor of the domestic wage level. It may itself reflect deviations from the main-course, thus

\[
mc - w > 0 \quad \text{and} \quad \Delta mc < 0,
\]

The framework we use is an ECM system, with a wage Phillips curve as one of the equations. Without loss of generality we concentrate on the wage Phillips curve, and recall that according to Aukrust’s theory it is assumed that

\[
\begin{align*}
\Delta w_t &= \beta_{w0} + \beta_{w1} \Delta mc_t + \beta_{w2} u_{t-1} + \varepsilon_{w,t} \quad \beta_{w2} < 0, \\
u_t &= \beta_{u0} + \alpha_u u_{t-1} + \beta_{u1} (w - mc)_{t-1} + \varepsilon_{u,t} \quad \beta_{u1} > 0, \\
0 &< \alpha_u < 1,
\end{align*}
\]

where we have simplified the notation somewhat by dropping the “e” sector subscript.\(^{12}\) Compared to equation (2.5) above, we have simplified by assuming that only the current unemployment rate affects wage growth. On the other hand, since we are considering a dynamic system, we have added a \( w \) in the subscript of the coefficients. Note that compared to (2.5) the autoregressive coefficient \( \alpha_u \) is set to unity in (2.15)—this is of course not a simplification but a defining characteristic of the Phillips curve. Equation (2.16) represents the idea that low profitability causes

\(^{11}\)See Aukrust (1977, p. 114).

\(^{12}\)Alternatively, given \( H_{w,c} \), \( \Delta w \) represents the average wage growth of the two sectors.
unemployment. Hence if the wage share is too high relative to the main-course, unemployment will increase, i.e., $\beta_{u1} > 0$.

(2.5) and (2.15) make up a dynamic system. Building on what we have learnt about stability in section 1.9 and 1.10, we know that for a given set of initial values $(w_0, u_0, mc_0)$, the systems determines a solution $(w_0, u_0), (w_1, u_1), ..., (w_T, u_T)$ for the period $t = 1, 2, ..., T$. We also know that the solution depends on the values of $mc_t, c_{at}, c_{e1}$ over that period.

Without further restrictions on the coefficients, it becomes too complicated to derive the final equation of for example $w_1$. This is frequently the case for dynamic systems. However, we are usually able to characterize the steady state, assuming that it exists (i.e., the solution is stable). Usually we can also discuss the dynamics in qualitative terms. We follow this approach in the following.

As a first step we derive the steady state solution of the system, assuming stable dynamics, so that a solution exists. We choose the solution based on $c_{a1} = c_{e1} = 0$ and $\Delta mc_t = g_{mc}$ (i.e., the constant growth rate of the main course), which gives rise to the following steady state:

\[
\Delta w_1 = g_{mc},
\]

\[
u_t = u_{t-1} = w^{phil},
\]

the equilibrium rate of unemployment.

since wages cannot reach a steady state unless also unemployment attains its steady state value, which we dub the equilibrium rate of unemployment, $w^{phil}$. Substitution into (2.16) and (2.15) gives the following long run system:

\[
g_{mc} = \beta_{w0} + \beta_{u1} g_{mc} + \beta_{u2} w^{phil}
\]

\[
w^{phil} = \beta_{a0} + \alpha_{u} w^{phil} + \beta_{a1}(w - mc)
\]

The first equation gives the solution for $w^{phil}$

\[
w^{phil} = \left( \frac{\beta_{w0}}{-\beta_{u2}} + \frac{\beta_{a1} - 1}{-\beta_{u2}} g_{mc} \right), (2.17)
\]

which is the natural rate of unemployment as implied by the main-course Phillips curve. We can also call it the “main-course rate of unemployment”, since it is the rate of unemployment required to keep wage growth on the main-course.

Next, let us consider the dynamics of the system. Consider the dynamic solution based on $\Delta mc_t = g_{mc}$, and $c_{e1} = c_{a1} = 0$ in all periods, starting from any historically determined initial condition. In this case, (2.5) becomes:

\[
\Delta w_t - g_{mc} = \beta_{w0} + (\beta_{u1} - 1) g_{mc} + \beta_{u2} w_t,
\]

or, using (2.17):

\[
\Delta w_t - g_{mc} = \beta_{u2}(u_t - w^{phil}). \quad (2.18)
\]

Because of the assumption that $\beta_{u2} < 0$, wage growth is higher than the main-course growth as long as unemployment is below the natural rate. Moreover, from the second equation of the system, (2.16):

\[
u_t = \beta_{a0} + \alpha_{u} u_{t-1} + \beta_{a1}(w - mc)_{t-1}, \beta_{a1} > 0
\]

it is seen that a higher value of $w - mc$ contributes to higher unemployment in the next period. This analysis suggests that from any starting point on the Phillips curve, stable dynamics leads to the steady state solution. This indicates that the twin assumption of $\beta_{u2} < 0$ and $\beta_{a1} > 0$ is important for stability. And that for example $\beta_{u2} = 0$ harms stability.

It is important to note that the Phillips curve needs to be supplemented by an equilibrating mechanism in the form of an equation for the rate of unemployment. Without such an equation in place, the system is incomplete, and there is a missing term.

The question about the dynamic stability of the natural rate cannot be addressed in the single equation Phillips curve “system”.

The dynamics of the Phillips curve case is illustrated in Figure 2.5. Assume that the economy is initially running at a low level of unemployment, i.e., $u_0$ in the figure. The short-run Phillips curve (2.15) determines the rate of wage inflation $\Delta w_0$. Consistent with equation (2.18), the figure shows that $\Delta w_0$ is higher than the growth in the main-course $g_{mc}$. Given this initial situation, a process starts where the wage share is increasing from period to period. From equation (2.16) is seen that the consequence must be a gradually increased rate of unemployment, away from $u_0$ and towards the natural rate $w^{phil}$. Hence we can imagine that the dynamic stabilization process takes place “along” the Phillips curve in Figure 2.5. When the rate of unemployment reaches $w^{phil}$ the dynamic process stops, because the impetus of the rising wage share has dried out.

The steep Phillips curve in the figure is defined for the case of $\Delta w_0 = \Delta mc_0$. The slope of this curve is given by $-\beta_{w0}/(1 - \beta_{u1})$, and it has been dubbed the
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long run Phillips curve in the literature. The issue about the slope of the long-run Phillips curve is seen to hinge on the coefficient $\beta_{e1}$, the elasticity of wage growth with respect to the growth in the main-course. In the figure, the long-run curve is downward sloping, corresponding to $\beta_{e1} < 1$ which is conventionally referred to as dynamic inhomogeneity in wage setting. The converse, referred to as dynamic homogeneity in the literature, implies $\beta_{e1} = 1$, and the long run Phillips curve is then vertical. Subject to dynamic homogeneity, the equilibrium rate $w^{stab}$ is independent of world inflation and productivity growth $g_{mc}$.

The slope of the long-run Phillips curve represented one of the most debated issues in macroeconomics in the 1970 and 1980s. One arguments in favour of a vertical long-run Phillips curve is that workers are able to obtain full compensation for price increases. Therefore, in the context of our model, $\beta_{e1} = 1$ is perhaps the only reasonable parameter value. The downward sloping long-run Phillips curve has also been denounced on the grounds that it gives a too optimistic picture of the powers of economic policy: namely that the government can permanently reduce the level of unemployment below the natural rate by “fixing” a suitably high level of inflation, see e.g., Romer (1996, Ch 5.5). In the context of an open economy this discussion appears as somewhat exaggerated, since a long-run trade-off between inflation and unemployment in any case does not follow from the premise of a downward-sloping long-run curve. Instead, as shown in figure 2.5, the steady state level of unemployment is determined by the rate of imported inflation and productivity growth as represented by $g_{mc}$. Neither of these are instruments of economic policy.13

Both the wage Phillips curve of this section, and the model for wages in section 2.2 have been kept deliberately simple. In the real economy, cost-of-living considerations play a significant role in wage setting. Thus, in applied econometric work, one usually includes current and lagged consumer price inflation in the wage equation. Section 2.5 shows an extended empirical example. The above formal framework above can extended to accommodate this, without changing the conclusion about the steady state solution.

Another important factor which we have omitted so far from the formal analysis, is expectations. For example, instead of (2.15) we might have

$$\Delta w_1 = \beta_{e0} + \beta_{e1} \Delta w_3 + \beta_{e2} u_0 + \varepsilon_{w1},$$

or $w_{t+1}$ for that matter. However, as long as expectations are influenced by the main-course, we get the same conclusion as above. For example

$$\Delta w_1 = \varphi \Delta g_{mc} + (1 - \varphi) \Delta u_{t-1}, \quad 0 < \varphi \leq 1$$

In steady state there are no expectations errors, so

$$\Delta w^*_1 = \Delta u_{t-1} = g_{mc}$$

13To affect $w^{stab}$, policy needs to incur a higher or lower permanent rate of currency depreciation.

as before.

In the previous section, we drew the implication of the main-course model for the steady state of the domestic rate of inflation. To establish the corresponding result for the case of the Phillips curve, we start by repeating the definition equation for the consumer price index:

$$p_t = \phi_{q,t} + (1 - \phi_{q,t}), \quad 0 < \phi < 1,$$

or, in differences

$$\Delta p_t = \phi \Delta q_{t-1} + (1 - \phi) \Delta q_t . \quad (2.19)$$

If we make the assumption that sheltered sector price growth is always on the long-run equilibrium path implied by the main-course theory, we obtain

$$\Delta q_{t-1} = \Delta q_t - \Delta q_{t-1},$$

which can be combined with (2.15) to give a price Phillips curve for the sheltered sector, and, by using (2.19), also a Phillips curve with $\Delta p_t$ on the left hand side of the equation. The steady state rate of inflation can then be found by inserting the long-run relationships $\Delta q_{t-1} = \Delta g_{mc} - \Delta u_0$ and $\Delta q_{t} = \Delta g$: $\Delta p_t = \phi (\Delta g_{mc} - \Delta u_0) + (1 - \phi) g_t = g_0 + \phi (g_0 - g_0)$

$$(2.2)$$

since $g_{mc} = g_0 + g_u$. The is the same expression as in equation (2.14) above, showing that even though the ECM model for wage growth has been replaced by a Phillips curve, the implication for the steady state rate of inflation is unchanged. The reason is that both model versions align long run wage growth with the main course. The difference is the mechanism: In section 2.2.2 wage growth error corrects deviations from the main course directly. In the Phillips curve case, the stabilization is indirect and works through the rate of unemployment: In the case of too large wage increases, unemployment is a “disciplining device” which forces wage claims back on to the main course.

2.4 Summing up the main-course model

In this chapter we have presented the main-course theory of long run stability of the wage shares of the exposed and sheltered sector of a small open economy. It is interesting to note that the long run relationships for wage and price levels predicted by the main course are compatible with modern battle of mark-up models of wage and price setting. We then developed two dynamic models of wage setting which are both consistent with the long run properties of the main-course theory. The main findings are summarized in 5 points:

1. The first model is an ADL model for wage level: The distributed lag part was made up of main-course variable $(mc_q)$ and the rate of unemployment $(u_t)$. 

2.4. SUMMING UP THE MAIN-COURSE MODEL

(a) If we assume a stable solution of the ADL for wages (i.e., $0 < \alpha < 1$), it can be transformed to an ECM for wage growth.

(b) For a given (exogenously determined) level of $w_t$, wage growth error-corrects deviations from the main-course.

(c) Therefore, there is a steady state level for the wage share $w_t = mc_q$.

(d) Stability of wage growth and inflation also follows.

2. The second model postulates a wage Phillips curve, PCM, consistent with setting $\alpha = 1$ in the wage ADL model. The following results were found to hold for the PCM specification of wage dynamics.

(a) The PCM (by itself, viewed isolated from the rest of the model economy) gives an unstable solution for the wage-share $w_t - mc_q$.

(b) If the PCM is linked up with a second equation, which explains $q_t$ as an increasing function of the wage share, the two equation system implies a steady state level for the wage share $w_t - mc_q$.

(c) The PCM implies a natural rate of unemployment $(\bar{w}^{Phil})$ which corresponds to the steady state level of unemployment implied by the 2-equation system.

3. Hence both models (ECM and PCM) stabilizes the wage share, and the rates of wage and price inflation.

4. The difference between the models lies in the mechanism that secures stability of the wage share

(a) In the ECM case: There is an amount of collective rationality in wage setting institutions. For example: Unions adjust their wage claims to the last years profitability. Inflation is stabilized at any given rate of unemployment, also low ones that prevailed in Europe until 1980 and in Scandinavia until the end of last century.

(b) In the PCM case: There is less collective rationality. Instead unemployment serves as a disciplining device: There is only one level of unemployment at which the rate of inflation is stable, i.e., the natural rate of unemployment.

5. There seems to be some important implications for policy: For example:

(a) If PCM is the true model, then self-defeating policy to try to target “full unemployment” below the natural rate.

(b) If ECM is the true model then it is not only possible to target full unemployment, it may also be advisable in order to maintain collective rationality (avoid breakdown in the bargaining system/institutions).

2.5 Norwegian evidence on the main-course model

In this section we illustrate how well wage equations corresponding to the Phillips curve (PCM), and the extended main-course model, fit the data for Norwegian manufacturing. Readers with no familiarity with econometrics should read on: we abstract from all technical detail and try instead to explain to the economic interpretation of the findings, and how they relate to the two theoretical models presented in this chapter.

2.5.1 A Phillips curve model

We use an annual data set for the period 1965-1998. In the choice of explanatory variables and of data transformations, we build on existing studies of the Phillips curve in Norway, cf. Stølen (1990,1993). The variables are in log scale (unless otherwise stated) and are defined as follows:

- $w_{t-1}$: hourly wage cost in manufacturing;
- $q_t$: index of producer prices (value added deflator);
- $p_t$: the official consumer price index (CPI index);
- $\bar{q}_t$: average labour productivity;
- $t_{wu}$: rate of total unemployment (i.e., unemployment includes participants in active labour market programmes);
- $rpr_t$: the replacement ratio;
- $h$: the length of the ”normal” working day in manufacturing;
- $t1$: the manufacturing industry payroll tax-rate (not log).

Note, with reference to the previous sections, that the main-course variable would be

$$mc_q = q_t + \bar{q}_t.$$

The estimated Phillips curve, using ordinary least squares, OLS, is shown in equation (2.21). The term of the left hand side, $\Delta w_{t-1}$, is the fitted value of the growth rate of hourly wage costs (hence the OLS residual would be $\Delta w_{t-1} - \hat{\Delta w}_{t-1}$). On the left hand side, we have the variables that are found to be significant.\footnote{We have followed a strategy of first estimating a quite general ADL, where both lagged wage growth (the AR term) and productivity growth (current and lagged), and the lagged rate of unemployment is included. A procedure called general-to-specific modelling has been used to derive the final specification in equation (2.21).} The numbers in brackets below the coefficients are the estimated coefficient standard errors, which are used to judge the statistical significance of the coefficients. A rule-of-thumb
satisfies that if the ratio of the coefficient and its standard error is larger than 2 (in absolute value for negatively signed coefficients), the underlying true parameter (the $\beta$) is almost certainly different from zero. Applying this rule, we find that all the coefficients of the Norwegian Phillips curve are significant.

$$\Delta w_{t} = -0.0683 + 0.26 \Delta q_{t-1} + 0.20 \Delta q_{t} + 0.29 \Delta q_{t-1}$$

$$-0.0316 t_{u} - 0.07 IP_{t}$$

(0.01) (0.09) (0.06) (0.01)

The estimated equation is recognizable as an augmented Phillips curve, although there are some noteworthy departures from the ‘pure’ wage Phillips curve of section 2.3. First, there is the rate of change in the CPI-index, $\Delta q_{t-1}$, which shows that Ankerst’s basic model is too stylized to fit the data. Wage setters in the (exposed) manufacturing sectors obviously care about the evolution of cost-of living, not only about the wage-scope. For reference, it might be noted that before the start of each bargaining round in Norway, representatives of unions and of organizations on the firm side, aided by a team of experts, work out a consensus view on the outlook for CPI price increases. It is possible that the significance of the lagged rate of inflation in equation (2.21) is due to these institutionalized forecasts on the actual wage outcome.

A second departure from the theoretical main-course Phillips curve is the absence of productivity growth in equation (2.21) (estimation shows that they are statistically insignificant in this specification). Hence, the variables that represent the main-course are the current an lagged growth rate of the product price index.

The rate of unemployment (in log form, remember) is a significant explanatory variable in the model, and of course what turns this into an empirical Phillips curve. The last left hand side variable, $IP_{t}$, represents the effects of incomes policies and wage-price freezes, which have been used several times in the estimation period, as part of the wider setting of coordinated and centralized wage bargaining. As discussed above, a key parameter of interest in the Phillips curve model is the main-course natural rate of unemployment, denoted $u_{phil}$ in equation (2.17). Using the coefficient estimates in (2.21), and setting the growth rate of prices ($\dot{q}$) and productivity growth equal to their sample means of 0.06 and 0.027, we obtain an estimated natural rate of 0.0355 (which is as nearly identical to the sample mean of the rate of unemployment (0.0313)).

Figure 2.6 shows the sequence of natural rate estimates over the last part of the sample—together with $\pm 2$ estimated standard errors and with the actual unemployment rate for comparison. The figure shows that the estimated natural rate of unemployment is relatively stable, and that it is appears to be quite well determined. 1990 and 1991 are notable exceptions, when the natural rate apparently increased from to 0.033 and 0.040 from a level 0.028 in 1989. However, compared to confidence interval for 1989, the estimated natural rate has increased significantly in 1991, which represents an internal inconsistency since one of the assumptions of this model is that $u_{phil}$ is a time invariant parameter.

Another point of interest in figure 2.6 is how few times the actual rate of unemployment crosses the line of the estimated natural rate. This suggest very sluggish adjustment of actual unemployment to the purportedly constant equilibrium rate. In order to investigate the dynamics more formally, we have grrafted the Phillips curve equation (2.21) into a system that also contains the rate of unemployment as an endogenous variable, i.e., an empirical counterpart to equation (2.16) in the theory of the main-course Phillips curve. As noted above, the endogeneity of the rate of unemployment is just as much a part of the natural rate framework as the wage Phillips curve itself, since without the “unemployment equation” in place one cannot show that the natural rate of unemployment obtained from the Phillips curve corresponds to a steady state of the system.

We have therefore estimated a version of the dynamic Phillips curve system given by equation (2.15)-(2.16) above. We do not give the detailed estimation results here, but Figure 2.7 offers visual inspection of some of the properties of the estimated model. The first four graphs shows the actual values of $\Delta p_{t}$, $t_{u}$, $\Delta w_{t}$ and the wage share $w_{t} - q_{t} - a_{t}$ together with the results from dynamic simulation. As could be expected, the fits for the two growth rates are quite acceptable. However, the “near instability” property of the system manifests itself in the graphs for the level of the unemployment rate and for the wage share. In both cases there are
In section 2.2.1. A full analysis is documented in Bårdsen et al. (2004, chapter 6).

So-called cointegration techniques have been used to estimate the relationship dubbed H1 in section 2.2.1. A full analysis is documented in Bårdsen et al. (2004, chapter 6).

Figure 2.7: Dynamic simulation of the Phillips curve model. Panel a)-d) Actual and simulated values (dotted line). Panel e)-f): multipliers of a one point increase in the rate of unemployment.

The main motivation here is however to compare this model with the estimated Phillips curve of the previous section. First note that the coefficient captures that the pay-losses that would otherwise have followed from the reductions in working hours have been partially compensated in the negotiated wage settlements.

Several consecutive year of under- or overprediction. The last two displays contain the cumulated dynamic multipliers of tu and the wage share, resulting from a 0.01 point increase in the unemployment rate. The striking feature is that any evidence of dynamic stability is hard to gauge from the two responses. Instead, it is as if the level of unemployment and the wage share “never” return to their initial values. Thus, in this Phillips curve system, equilibrium correction is found to be extremely weak.

As already mentioned, the belief in the empirical relevance a the Phillips curve natural rate of unemployment was damaged by the remorseless rise in European unemployment in the 1980s, and the ensuing discovery of great instability of the estimated natural rates. In that perspective, the variations in the Norwegian natural rate estimates in figure 2.6 are quite modest, and may pass as relatively acceptable as a first order approximation of the attainable level of unemployment. However, the estimated model showed that equilibrium correction is very weak. After a shock to the system, the rate of unemployment is predicted to drift away from the natural rate for a very long period of time. Hence, the natural rate thesis of asymptotically stability is not validated.

There are several responses to this result. First, one might try to patch-up the estimated unemployment equation, and try to find ways to recover a stronger relationship between the real wage and the unemployment rate, i.e., in the empirical counterpart of equation (2.16). In the following we focus instead on the other end of the problem, namely the Phillips curve itself. In fact, it emerges that when the Phillips curve framework is replaced with a wage model that allows equilibrium correction to any given rate of unemployment rather than to only the “natural rate”, all the inconsistencies are resolved.

2.5.2 An error correction model that integrates the main-course

In section 2.2 we discussed the main-course model and its extensions to modern wage-bargaining theory. Equation (2.22) shows an empirical version of an equilibrium correction model for wages, similar to equation (2.9) above:

$$
\Delta w_t = -0.197 - 0.478 \text{ecm}_{ecm,t-1} + 0.413 \Delta p_{t-1} + 0.333 \Delta \pi_t - 0.835 \Delta \pi_t - 0.0582 IP_t
$$

(2.22)

The first explanatory variable is the error correction term ecm_{ecm,t-1} which corresponds to w_{t-1} - w^* in section 2.2.2. The estimated w^* is a function of mc, with the homogeneity restriction (2.8) imposed. The estimated value of γ_{t-1} is −0.01, hence we have:

$$
\text{ecm}_{ecm,t-1} = w_{t-1} - mc_{t-1} - 0.01 tu_{t-1}
$$

The variable Δ\pi_t represents institutional changes in the length of the working week. The estimated coefficient captures that the pay-losses that would otherwise have followed from the reductions in working hours have been partially compensated in the negotiated wage settlements.

The main motivation here is however to compare this model with the estimated Phillips curve of the previous section. First note that the coefficient of ecm_{ecm,t-1} is relatively large, a result which is in direct support of Aukrust’s view that there are wage-stabilizing forces at work even at a constant rate of unemployment. To make further comparisons with the Phillips curve, we have also grafted (2.22) into dynamic system that also contains an equation for the rate of unemployment (the estimated equation for tu_t is almost identical to its counterpart in the Phillips curve system). Hence we have in fact an estimated version of the calibrated simulation model of section 2.2.3.

Some properties of that system is illustrated in figure 2.8. For each of the four endogenous variables shown in the figure, the model solution (i.e. the lines denoted “simulated values”) is closer to the actual value than is the case in Figure 2.7 for the Phillips curve system. The two last panels in the figure show the cumulated dynamic multiplier of an exogenous shock to the rate of unemployment. The difference from figure 2.7, where the steady state was not even “in sight” within the 35 year simulation period, are striking. In figure 2.8, 80% of the long-run effect is
2.5. NORWEGIAN EVIDENCE ON THE MAIN-COURSE MODEL

Figure 2.8: Dynamic simulation of the ECM model Panel a)-d) Actual and simulated values (dotted line). Panel e)-f): multipliers of a one point autonomous increase in the rate of unemployment reached within 4 years, and the system has reached a new steady state by the end of the first 10 years of the solution period. The conclusion is that this system is more convincingly stable than the Phillips curve version of the main-course model. Note also that the estimated model, which uses real data of the Norwegian economy, has the same dynamic properties as the calibrated theory model of section 2.2.3.

Economists are known to state that it is necessary to assume a vertical Phillips curve to ensure dynamic stability of the macroeconomy. In opposition to this view, the evidence presented here shows not only that the wage price system is stable when the Phillips curve is substituted by a wage equation which incorporates direct adjustment also with respect to profitability (consistent with the Norwegian model and with modern bargaining theory). Quite plainly, the system with the alternative (error correction) wage equation has much more convincing stability properties than the Phillips curve system.

Exercises

1. Is $\beta_{22} > 0$ a necessary and/or sufficient condition for path b to occur?
2. What might be the economic interpretation of having $\beta_{21} < 0$, but $\beta_{22} > 0$?

3. Assume that $\beta_{21} + \beta_{22} = 0$. Try to sketch the wage dynamics (in other words the dynamic multipliers) following a rise in unemployment in this case!

4. Above, the expression for the natural rate was found after first establishing the steady state solution for the system. To establish the main-course rate of unemployment more directly, rewrite (2.15) as

$$\Delta w_t = \beta_{w1} \Delta mc_t + \beta_{w2} (u_t - \frac{\beta_{w0}}{\beta_{w2}}) + \epsilon_{w,t}, \quad (2.23)$$

and invoke the steady state situation, i.e., $\Delta w - g_{mc} = 0$, $\epsilon_{w,t} = 0$. Show that (2.23) defines $u_{phil}$ as

$$0 = \beta_{w2} [u_{phil} - \frac{\beta_{w0}}{\beta_{w2}}] + (\beta_{w1} - 1) g_{mc}.$$ 

5. Complete the algebra leading from (2.19) to (2.20).
Appendix A

Variables and relationships in logs

Logarithmic transformations of economic variables are used several times in the text, first in section 1.2. Logarithms possess some properties that aid the formulation of economic relationships (model building), and the visual inspection of model and data properties in graphs. This appendix reviews some key properties of the logarithmic function, and the characteristic of graphs.

Throughout we use logarithms to the base of Euler’s number $e \approx 2.71828 \ldots$, and we use the symbol $\ln$ for these natural logarithms. In general, the natural logarithm of a number or variable $x$ is the power to which $e$ must be raised to yield $a$, i.e., $e^{\ln x} = x$.

For reference, we put down the main rules for operating on the natural logarithmic function (both $x$ and $y$ are positive):

$$\ln(xy) = \ln x + \ln y$$ (A.1)

$$\ln(\frac{x}{y}) = \ln x - \ln y$$ (A.2)

$$\ln(x^a) = a \ln x, \text{ a is any number or variable}$$ (A.3)

From these basic rules, additional ones can be constructed. For example

$$\ln(x^a y^b) = a \ln x + b \ln y$$

which is often called a linear combination of the log transformed variables, with weights $a$ and $b$. In the main text, a prime example of a linear combination is the stylized definitional equation for the log of the consumer price index, see e.g. equation (2.13). In that equation the weights sum to one, corresponding to $b = 1 - a$.

Another example of a linear combination is the unweighted geometric average:

$$\ln((x_1 x_2 \ldots x_n)^{1/n}) = \frac{1}{n} \sum_{i=1}^{n} \ln x_i$$

Assume next that $y$ is a function $f(x)$. The relationship is linear if $f(x) = a + bx$. A much used non-linear specification of $f(x)$ is

$$y = Ax^b, \; x > 0$$ (A.4)

where $A$ and $b$ are constant coefficients. Note first that if we apply the definition of the elasticity

$$E_{x,y} = \frac{f' \cdot x}{y}$$

or, in general, with $n$ different $x$-es:

$$\ln((x_1 x_2 \ldots x_n)^{1/n}) = \frac{1}{n} \sum_{i=1}^{n} \ln x_i$$

to (A.4), we obtain

$$E_{x,y} = b$$ (A.5)

showing that the coefficient $b$ is the elasticity of $y$ with respect to $x$. Second, if we apply the rules for logarithm to (A.4) the following log-linear relationship is obtained:

$$\ln y = \ln A + b \ln x, \; x \text{ is positive.}$$ (A.6)

The equation is linear in the logs of the variables, a property which is well captured by the name log-linear. Conveniently, the elasticity $b$ is the slope coefficient of the relationship. This is confirmed by taking the differential of (A.6):

$$\frac{d \ln y}{d \ln x} = b.$$
The examples considered so far have used computer generated numbers. Figure A.1 shows an example of how well the approximation works. The graph in panel a) shows 200 observations of a time series with a growth rate of 3%. In fact this is the same series as we used to represent the \( x \) variable in Figure A.1. The graph is not completely smooth, since for realism, we have added a random shock to each observation (a stochastic disturbance term). This means that even using the exact computation, each growth rate is likely to be different from 0.03. This is illustrated in panel b) showing the exact periodic growth rates. Evidently, there is a lot of variation around the mean growth rate of 0.03. Panel c), with the graph of \( \ln x \), shows a linear though not completely deterministic evolution through time. The straight line represents the underlying average growth of 0.03.

Figure A.2 shows an example of how well the approximation works. The graph in panel a) shows 200 observations of a time series with a growth rate of 3%. In fact this is the same series as we used to represent the \( x \) variable in Figure A.1. The graph is not completely smooth, since for realism, we have added a random shock to each observation (a stochastic disturbance term). This means that even using the exact computation, each growth rate is likely to be different from 0.03. This is illustrated in panel b) showing the exact periodic growth rates. Evidently, there is a lot of variation around the mean growth rate of 0.03. Panel c), with the graph of \( \ln x \), shows a linear though not completely deterministic evolution through time. The straight line represents the underlying average growth of 0.03.

Finally, panel d) in the figure shows the scatter plot of the exact growth rate against the approximated growth rate based the differenced log transformed series in panel c). To the eye at least, the vast majority to observations lies spot on the drawn least squares regression line, which is evidenced that the approximation works really well. However, there is indication that for more sizable growth rates, for example above 10%, the difference between exact and the approximate computation begin to be of practical interest.

The examples considered so far have used computer generated numbers. Figure
A.3 shows the actual and log transformed series of a credit indicator for Norway. Unlike the stylized properties shown in Figure A.1 and A.2, the real world series in this graph shows a more mixed picture. For example, the upper panel suggests two periods of exponential growth in credit: the first ending in 1989 and the second one beginning in 1994. In between lies the period of falling housing prices and the biggest banking crisis since the 1920s. Despite the smoothing function of the log transform the burst of the bubble is still evident in the bottom panel. Outside the burst period, the linearization works rather well.

Figure A.3: Upper panel: The time graph of the total credit issued in Norway, together with the estimated linear trend. Bottom panel: The time graph of the logarithmic transform of the credit series. Source: Norges Bank, RIMINI database.

Figure A.4 shows annual data of Norwegian real GDP for the period 1865-1999. Panel a) shows the fixed price series in million kroner, 1990 is the base year. The break in series is due to 2WW, when the country was occupied. Over such a long period of time, the exponential growth pattern is visible and the log transform (panel b) therefore shows a much more linear relationship. In the panel c) and d), the exact and approximate growth rates are shown.


References All the formulae in this appendix are standard and can be found in any textbook in mathematical analysis. For example, in Sydsæter (2000), the logarithmic function is presented in chapter 3.10 and rules for derivation in chapter 5.11. Elasticities are found in chapter 5.12. The approximation used in equation (A.7) is discussed on page 253 of Sydsæter (2000). A good reference in English is Sydsæter and Hammond (2002).
Bibliography


