Motivic complexes from the stable rank filtration

John Rognes

June 8th 2010

Contents

1 Introduction

We consider the stable rank filtration

$$\mathbf{S}[BF^\times] \simeq F_1\mathbf{K}(F) \to F_2\mathbf{K}(F) \to \cdots \to \mathbf{K}(F)$$

of the algebraic K-theory spectrum of a field F, and the associated homological spectral sequence. The k-th filtration quotient is given by the homotopy orbits of an explicit simplicial complex with $GL_k(F)$-action, called the common basis complex $D'(F^k)$:

$$F_k\mathbf{K}(F)/F_{k-1}\mathbf{K}(F) \simeq \mathbf{\Sigma}^\infty \Sigma D'(F^k)//GL_k(F)$$

We conjecture that $\tilde{H}_*(D'(F^k))$ is concentrated in a single degree, namely $* = 2k-3$, and prove this for $k \le 3$. Letting $\Delta_k(F) = \tilde{H}_{2k-3}(D'(F^k))$ and

$$\Gamma_{\mathrm{rk}}(t, F)^{t-s} = H_{t-s}(GL_{s+1}(F); \Delta_{s+1}(F))$$

we then get a cochain complex $\Gamma_{\mathrm{rk}}(t, F)$, concentrated in degrees $0 \le * \le t$, which is a candidate for a rational motivic complex in the sense of Beilinson and Lichtenbaum. In particular, its cohomology

$$H^{t-s}_{\mathrm{rk}}(F; \mathbb{Z}(t)) = H^{t-s}(\Gamma_{\mathrm{rk}}(t, F))$$

gives the E^2-term of a spectral sequence converging to $H_{s+t}\mathbf{K}(F)$, which is rationally isomorphic to $K_{s+t}(F)$. We conjecture that this spectral sequence is rationally isomorphic to the motivic spectral sequence, and prove this on the vertical axis. In combination, these conjectures would prove the Beilinson–Soulé conjecture on the vanishing of motivic cohomology in negative degrees.

2 The Beilinson–Soulé vanishing conjecture

2.1 Eigenspaces of algebraic K-theory

Let F be any field. The rational algebraic K-group $K_n(F)_{\mathbb{Q}} = K_n(F) \otimes_{\mathbb{Z}} \mathbb{Q}$ decomposes into eigenspaces for the Adams operations, with $\psi^k(x) = k^t x$ for

x in the weight t eigenspace, for all integers k. For $n \geq 1$, it is known that only the eigenspaces with $1 \leq t \leq n$ can be nontrivial [20]. According to the Beilinson–Soulé vanishing conjecture, only the eigenspaces with $[n/2] < t \leq n$ can be nontrivial [1]. (A weak form asserts nontriviality only for $[n/2] \leq t \leq n$.) This is one of the major unsolved conjectures about algebraic K-theory.

2.2 Motivic cohomology

The conjecture can be reformulated in terms of motivic cohomology. Bloch's cycle complex $z^t(F, *)$ is generated in degree n by certain codimension t varieties in affine n-space over F, and the motivic cohomology group

$$H^{t-s}_{mot}(F; \mathbb{Z}(t)) = CH^t(F, s+t) = H_{s+t}(z^t(F, *))$$

is given by the homology of this complex in degree $(s+t)$ [3]. It is clear that $H^{t-s}_{mot}(F; \mathbb{Z}(t)) = 0$ for $s < 0$ or $t < 0$, since in these cases there are no codimension t varieties in affine $(s+t)$-space.

2.3 The motivic spectral sequence

The motivic spectral sequence

$$E^2_{s,t}(\mathrm{mot}) = H^{t-s}_{mot}(F; \mathbb{Z}(t)) \Longrightarrow K_{s+t}(F)\,,$$

is an algebra spectral sequence concentrated in the first quadrant [7], [23]. Here is the expected picture, with the origin in the lower left hand corner.

$K_4^M(F)$	$H^3_{mot}(F; \mathbb{Z}(4))$	$H^2_{mot}(F; \mathbb{Z}(4))$	$H^1_{mot}(F; \mathbb{Z}(4))$
$K_3^M(F)$	$H^2_{mot}(F; \mathbb{Z}(3))$	$H^1_{mot}(F; \mathbb{Z}(3))$	(0?)
$K_2^M(F)$	$H^1_{mot}(F; \mathbb{Z}(2))$	(0?)	(0?)
$F^\times$	0	0	0
$\mathbb{Z}$	0	0	0

After rationalization, this spectral sequence is known to collapse at $E^2 = E^\infty$, with the weight t eigenspace of $K_{s+t}(F)_{\mathbb{Q}}$ represented by $H^{t-s}_{mot}(F; \mathbb{Q}(t))$ in the t-th row [8]. For $t \geq 1$, the Beilinson–Soulé conjecture amounts to the assertion that the motivic cohomology group $H^{t-s}_{mot}(F; \mathbb{Q}(t))$ vanishes for $s \geq t$, which is far from obvious from its definition in terms of Bloch's cycle complex. (The weak form asserts vanishing for $s > t$.)

2.4 Rational vs. integral vanishing

In view of the proven Milnor and Bloch–Kato conjectures, the groups

$$H^{t-s}_{\mathrm{mot}}(F;\mathbb{Z}/\ell(t)) \cong H^{t-s}_{\mathrm{et}}(F;\mu_\ell^{\otimes t})$$

are zero for $s \geq t$, for all primes ℓ (not equal to the characteristic of F). It follows that the Beilinson–Soulé conjecture is equivalent to the vanishing of the integral motivic cohomology groups $H^{t-s}_{\mathrm{mot}}(F;\mathbb{Z}(t))$ for $s \geq t$. We will focus on rational algebraic K-theory and motivic cohomology.

2.5 Motivic complexes

Beilinson [2] and Lichtenbaum [10] conjectured that for each $t \geq 1$ there should exist motivic complexes $\Gamma(t,F)$ concentrated in degrees $1 \leq * \leq t$, with cohomology $H^{t-s}(\Gamma(t,F))$ equal to the motivic cohomology groups $H^{t-s}_{\mathrm{mot}}(F,\mathbb{Z}(t))$. For $t=0$ we let $\Gamma(0,F)$ be $\mathbb{Z}$ concentrated in degree 0. For $t=1$, $\Gamma(1,F)$ is $F^\times$ concentrated in degree 1, and Lichtenbaum [11] has constructed a candidate for the motivic complex $\Gamma(2,F)$. Goncharov [9] has defined polylogarithmic complexes $\Gamma_{\mathrm{pol}}(t,F)$ for all $t \geq 0$, but it is not known that their cohomology is rationally isomorphic to motivic cohomology. In this paper we define rank complexes $\Gamma_{\mathrm{rk}}(t,F)$ for all $t \geq 0$, whose cohomology groups give the E^2-term of a spectral sequence converging to the spectrum homology $H_*\mathbf{K}(F)$, which is rationally isomorphic to $K_*(F)$.

3 Tits buildings

3.1 The first delooping of algebraic K-theory

Let $\mathscr{V} = \mathscr{V}(F)$ be the exact category of finite-dimensional F-vector spaces. By definition, $K_n(F) = \pi_{n+1}K(F)_1$, where the space $K(F)_1$ can be defined as the classifying space $|Q\mathscr{V}|$ of Quillen's Q-construction [14], or, equivalently, as the classifying space $|iS_\bullet\mathscr{V}|$ of Waldhausen's $S_\bullet$-construction [25].

3.2 Quillen's rank filtration

The objects of $Q\mathscr{V}$ are the same as those of $\mathscr{V}$, i.e., finite-dimensional F-vector spaces. For each $k \geq 0$ let $F_kK(F)_1$ be the subspace of $K(F)_1$ given by the classifying space $|F_kQ\mathscr{V}|$ of the full subcategory of $Q\mathscr{V}$ generated by the vector spaces of dimension $\leq k$. Then $F_0K(F)_1 \simeq *$, and there is a homotopy equivalence

$$F_kK(F)_1/F_{k-1}K(F)_1 \simeq \Sigma^2 B(F^k)//GL_k(F)$$

for each $k \geq 1$ [15]. Here $B(F^k)$ is the Tits building, which we recall in the next subsection. We write $X//G$ for the homotopy orbit space $EG_+ \wedge_G X$ of a based G-space X. Note that there is a homotopy orbit spectral sequence

$$E^2_{p,q} = H_p(G;\tilde{H}_q(X)) \Longrightarrow \tilde{H}_{p+q}(X//G)\,.$$

3.3 Tits buildings

The *Tits building* $B(F^k)$ of F^k is a simplicial complex, with vertices the set of proper, nontrivial subspaces V of F^k. A set of $(p+1)$ vertices $\{V_0, \dots, V_p\}$ spans a p-simplex if and only if these subspaces form a flag, i.e., when indexed in order of increasing dimension they form a strictly increasing sequence

$$0 \subset V_0 \subset \cdots \subset V_p \subset F^k \, .$$

This is the flag complex associated to the partially ordered set of proper, nontrivial subspaces of F^k.

The general linear group $GL_k(F)$ acts on $B(F^k)$ by its usual action on subspaces, with $g \in GL_k(F)$ taking $V \subset F^k$ to $g(V) \subset F^k$. The maximal flags, with $\dim(V_a) = a+1$ for each $0 \le a \le p$, define $(k-2)$-simplices that cover $B(F^k)$. Hence $B(F^k)$ is covered by the $(k-2)$-simplex

$$0 \subset F^1 \subset \cdots \subset F^{k-1} \subset F^k \, ,$$

and all of its $GL_k(F)$-translates.

3.4 Steinberg representations

The Tits building $B(F^k)$ has the homotopy type of a wedge sum of $(k-2)$-dimensional spheres [15]. Hence the reduced homology of $\Sigma^2 B(F^k)$ is concentrated in degree k. The natural $GL_k(F)$-action on $B(F^k)$ induces a $GL_k(F)$-action on this homology group, which defines the *Steinberg representation*

$$\mathrm{St}_k(F) = \tilde{H}_k(\Sigma^2 B(F^k)) \, .$$

The homotopy orbit spectral sequence collapses to isomorphisms

$$\tilde{H}_{p+k}(\Sigma^2 B(F^k)//GL_k(F)) \cong H_p(GL_k(F); \mathrm{St}_k(F)) \, .$$

3.5 Finite generation

It follows from the above that $F_k K(F)_1 \to K(F)_1$ is k-connected. When F is a number field and $\mathscr{O} = \mathscr{O}_F$ its ring of integers, Quillen used the fact that $H_*(GL_k(\mathscr{O}); \mathrm{St}_k(F))$ is of finite type [16] to deduce that $H_*(K(\mathscr{O})_1)$ is of finite type, and hence that $\pi_*(K(\mathscr{O})_1)$ and $K_*(\mathscr{O})$ are of finite type. To get more precise information, we shall pass from the first to the higher deloopings of algebraic K-theory.

4 Stable buildings

4.1 The K-theory spectrum

The algebraic K-groups of F are also realized as the homotopy groups of a symmetric spectrum $\mathbf{K}(F)$, with level n space given by the n-fold iterated $S_\bullet$-construction

$$K(F)_n = |iS^{(n)}_\bullet \mathscr{V}| \, .$$

Rationally, the algebraic K-groups can be recovered from the homology groups of this spectrum, since the Hurewicz homomorphism

$$K_n(F) = \pi_n \mathbf{K}(F) \to H_n \mathbf{K}(F)$$

is a rational isomorphism.

4.2 The stable rank filtration

The objects of $iS_\bullet^{(n)}\mathscr{V}$ in simplicial degree p are certain diagrams $X\colon \mathrm{Ar}[p]^n \to \mathscr{V}$, where $\mathrm{Ar}[p]$ is the arrow category of $[p] = \{0 < 1 < \cdots < p\}$. For each $k \geq 0$ let $F_k\mathbf{K}(F)$ be the subspectrum of $\mathbf{K}(F)$ with level n space

$$F_k K(F)_n = |F_k iS_\bullet^{(n)}\mathscr{V}|$$

given by the full subcategory of the n-fold $S_\bullet$-construction generated by the diagrams X taking values in vector spaces of dimension $\leq k$. The sequence of spectra

$$* \simeq F_0\mathbf{K}(F) \to F_1\mathbf{K}(F) \to \cdots \to F_k\mathbf{K}(F) \to \cdots \to \mathbf{K}(F)$$

is called the *stable rank filtration* [17].

4.3 Stable buildings

There is a spectrum with $GL_k(F)$-action $\mathbf{D}(F^k)$, called the *stable building* (or dwelling, den or demesne), and a stable equivalence

$$F_k\mathbf{K}(F)/F_{k-1}\mathbf{K}(F) \simeq \mathbf{D}(F^k)//GL_k(F)$$

for each $k \geq 1$ [17, 3.8]. We shall give an explicit description of the stable building, as a suspension spectrum, in the next subsection. The right hand side is the homotopy orbit spectrum for the $GL_k(F)$-action on $\mathbf{D}(F^k)$. In the case $k = 1$, $\mathbf{D}(F^1) = \mathbf{S}$ is the sphere spectrum with the trivial $F^\times = GL_1(F)$-action, so

$$F_1\mathbf{K}(F) \simeq \mathbf{S}[BF^\times] = \mathbf{\Sigma}^\infty(BF^\times_+)\,.$$

The tensor product of F-vector spaces makes the stable rank filtration a diagram of $\mathbf{S}[BF^\times]$-module spectra, and $\mathbf{K}(F)$ is a commutative $\mathbf{S}[BF^\times]$-algebra spectrum.

4.4 Common basis complexes

Definition 4.4.1 ([17, 14.5])**.** The *common basis complex* $D'(F^k)$ is a simplicial complex, with vertices the set of proper, nontrivial subspaces V of F^k. A set of $(p+1)$ vertices $\{V_0, \dots, V_p\}$ spans a p-simplex if and only if these subspaces admit a common basis, i.e., if there is a basis $\mathscr{B} = \{b_1, \dots, b_k\}$ for F^k such that each V_a, for $0 \leq a \leq p$, is spanned by a subset of $\mathscr{B}$.

The group $GL_k(F)$ acts on $D'(F^k)$ by its usual action on subspaces. For each basis $\mathscr{B}$ as above, the set of subspaces $\langle b_i \mid i \in S\rangle$ generated by the $(2^k - 2)$ proper, nonempty subsets of $\mathscr{B}$ defines a $(2^k - 3)$-simplex in $D'(F^k)$. As $\mathscr{B}$ varies, these simplices cover $D'(F^k)$. As a special case, let $\mathscr{E} = \{e_1, \dots, e_k\}$

denote the standard basis for F^k. Then $D'(F^k)$ is covered by the (2^k-3)-simplex consisting of all *axial subspaces* $F^S = \langle e_i \mid i \in S\rangle$, and all of its $GL_k(F)$-translates.

Proposition 4.4.2 ([17, 14.6])**.** *There is a stable equivalence*

$$\mathbf{D}(F^k) \simeq \mathbf{\Sigma}^\infty \Sigma D'(F^k)$$

of spectra with $GL_k(F)$-action. Hence there is a stable equivalence

$$F_k\mathbf{K}(F)/F_{k-1}\mathbf{K}(F) \simeq \mathbf{\Sigma}^\infty \Sigma D'(F^k)//GL_k(F)$$

for each $k \geq 1$.

4.5 The component filtration

Each flag $V_0 \subset \cdots \subset V_p$ admits a common basis, so there is an inclusion $B(F^k) \subseteq D'(F^k)$. It is part of a finite filtration

$$B(F^k) \simeq F_1 D'(F^k) \subset \cdots \subset F_k D'(F^k) = D'(F^k)$$

where the subquotient $F_c D'(F^k)/F_{c-1}D'(F^k)$ has homology concentrated in degree $(k+c-3)$, for each $2 \leq c \leq k$. It follows that the homology of $\mathbf{D}(F^k)$ is concentrated in degrees $k-1 \leq * \leq 2k-2$, see Corollary 9.2.4.

Theorem 4.5.1. *(a) $H_*(\mathbf{D}(F^1)) = \mathbb{Z}$ is concentrated in degree* 0.

(b) $H_{k-1}(\mathbf{D}(F^k)) = 0$ for all $k \geq 2$, so $H_*(\mathbf{D}(F^2))$ is concentrated in degree 2.

(c) $H_3(\mathbf{D}(F^3)) = 0$, so $H_*(\mathbf{D}(F^3))$ is concentrated in degree 4.

Part (a) is clear. We prove parts (b) and (c) in Lemmas 9.3.1 and 9.4.1 below.

4.6 The connectivity conjecture

Conjecture 4.6.1 (Connectivity conjecture [17, 12.3])**.** *The homology of* $\mathbf{D}(F^k)$ *is concentrated in degree $(2k-2)$, for each field F and integer $k \geq 1$.*

The conjecture is known to hold for $1 \leq k \leq 3$. It can be reformulated as stating that $\mathbf{D}(F^k)$ is $(2k-3)$-connected, or that $\tilde{H}_*(D'(F^k))$ is concentrated in degree $(2k-3)$.

Definition 4.6.2. Let

$$\Delta_k(F) = H_{2k-2}(\mathbf{D}(F^k)) = \tilde{H}_{2k-3}(D'(F^k))$$

be the "top" homology of $\mathbf{D}(F^k)$, viewed as a $GL_k(F)$-representation.

The following addendum is known for $k=2$, when $H_0(GL_2(F);\Delta_2(F)) \cong \mathbb{Z}/2$, and apparently $H_0(GL_3(F);\Delta_3(F)) \cong \mathbb{Z}/3$ [18, 3.4].

Conjecture 4.6.3 (Strong connectivity conjecture)**.** *The group of coinvariants $H_0(GL_k(F);\Delta_k(F))$ is torsion, for each field F and integer $k \geq 2$.*

5 Rank cohomology

5.1 The rank spectral sequence

Placing $F_k\mathbf{K}(F)$ in filtration $s = k-1$ and applying homology, we get the *rank spectral sequence* [18, §4]

$$\begin{aligned} E^1_{s,t}(\mathrm{rk}) &= H_{s+t}(\mathbf{D}(F^{s+1})//GL_{s+1}(F)) \\ &\Longrightarrow H_{s+t}\mathbf{K}(F)\,. \end{aligned}$$

It is concentrated in the first quadrant, and is a module spectral sequence over its 0-th column

$$H_*(\mathbf{S}[BF^\times]) \cong H_*(F^\times)$$

(group homology).

5.2 Expected appearance

If the connectivity conjecture 4.6.1 holds for $k = s+1$, the homotopy orbit spectral sequence collapses to an isomorphism

$$E^1_{s,t}(\mathrm{rk}) \cong H_{t-s}(GL_{s+1}(F); \Delta_{s+1}(F))\,.$$

This leads to the following picture in the first quadrant, with the origin at the lower left hand corner. We write $GL_k = GL_k(F)$ and $\Delta_k = \Delta_k(F)$, for brevity.

$$\begin{array}{ccccccc}
H_4(F^\times) & \longleftarrow & H_3(GL_2;\Delta_2) & \longleftarrow & H_2(GL_3;\Delta_3) & \longleftarrow & H_1(GL_4;\Delta_4) \\
H_3(F^\times) & \longleftarrow & H_2(GL_2;\Delta_2) & \longleftarrow & H_1(GL_3;\Delta_3) & \longleftarrow & H_0(GL_4;\Delta_4) \\
H_2(F^\times) & \longleftarrow & H_1(GL_2;\Delta_2) & \longleftarrow & H_0(GL_3;\Delta_3) & & (0?) \\
F^\times & \longleftarrow & H_0(GL_2;\Delta_2) & & 0 & & (0?) \\
\mathbb{Z} & & 0 & & 0 & & 0
\end{array}$$

If the strong connectivity conjecture 4.6.3 holds, the groups $H_0(GL_k; \Delta_k)$ on the diagonal are rationally trivial for $k \geq 2$.

5.3 The stable rank conjecture

We now have two spectral sequences, the motivic spectral sequence

$$E^2_{s,t}(\mathrm{mot}) = H^{t-s}_{\mathrm{mot}}(F; \mathbb{Z}(t)) \Longrightarrow K_{s+t}(F)$$

and the rank spectral sequence

$$E^1_{s,t}(\mathrm{rk}) = H_{s+t}(\mathbf{D}(F^{s+1})//GL_{s+1}(F)) \Longrightarrow H_{s+t}\mathbf{K}(F)\,.$$

They appear to be closely related, so we formulate the following hypothesis:

Conjecture 5.3.1 (Stable rank conjecture)**.** *The motivic spectral sequence and the rank spectral sequence are rationally isomorphic, from the E^2-term and onwards.*

In particular, this conjecture contains the assertion that for each $t \geq 0$, the t-th row $(E^1_{*,t}(\mathrm{rk}), d^1)$ of the rank spectral sequence is a rational motivic complex of type $\Gamma(t, F)$, in the sense that its homology is rationally isomorphic to motivic cohomology:

$$E^2_{s,t}(\mathrm{rk}) \cong_{\mathbb{Q}} H^{t-s}_{\mathrm{mot}}(F; \mathbb{Z}(t)) .$$

Furthermore, the conjecture contains the assertion that after rationalization the rank spectral sequence collapses at the E^2-term.

An unstable rank conjecture, asserting that the weight filtration on $K_*(F)_{\mathbb{Q}} \subset H_*(GL_\infty(F))_{\mathbb{Q}}$ is complementary to the filtration by the images of the homomorphisms $H_*(GL_k(F))_{\mathbb{Q}} \to H_*(GL_\infty(F))_{\mathbb{Q}}$, is apparently due to Suslin [6]. It was proved to be correct for number fields by Borel and Yang [5].

5.4 Rank cohomology

Definition 5.4.1. For each $t \geq 0$ we define the *rank complex* $\Gamma_{\mathrm{rk}}(t, F)$ to be given by the t-th row in the rank spectral sequence, with cohomological indexing:

$$\Gamma_{\mathrm{rk}}(t, F)^{t-s} = E^1_{s,t}(\mathrm{rk}) = H_{s+t}(\mathbf{D}(F^{s+1})//GL_{s+1}(F)) .$$

If the connectivity conjecture 4.6.1 holds for $k = s+1$, we have the algebraic description

$$\Gamma_{\mathrm{rk}}(t, F)^{t-s} = H_{t-s}(GL_{s+1}(F); \Delta_{s+1}(F)) .$$

In particular, the rank complex is then concentrated in degrees $0 \leq * \leq t$. If the strong connectivity conjecture 4.6.3 holds, we can (brutally) truncate the rank complex to degrees $1 \leq * \leq t$ for $t \geq 1$, with no change in the rational cohomology.

Definition 5.4.2. The coboundary map in the rank complex is given by the d^1-differentials in the rank spectral sequence. Its cohomology

$$H^{t-s}_{\mathrm{rk}}(F; \mathbb{Z}(t)) = H^{t-s}(\Gamma_{\mathrm{rk}}(t, F))$$

will be called the *rank cohomology* of F.

With this notation, the stable rank conjecture 5.3.1 asserts that rank cohomology is rationally isomorphic to motivic cohomology. Independently of that conjecture, rank cohomology is well related to rational algebraic K-theory, in view of the rank spectral sequence

$$E^2_{s,t}(\mathrm{rk}) = H^{t-s}_{\mathrm{rk}}(F; \mathbb{Z}(t)) \Longrightarrow H_{s+t}\mathbf{K}(F)$$

and the rational isomorphism $K_{s+t}(F) \cong_{\mathbb{Q}} H_{s+t}\mathbf{K}(F)$.

5.5 The vanishing conjecture

If the connectivity and stable rank conjectures hold, we deduce that

$$H^{t-s}_{\mathrm{mot}}(F;\mathbb{Z}(t)) \cong_{\mathbb{Q}} E^2_{s,t}(\mathrm{rk}) = 0$$

for all $s > t$. This gives the weak form of the Beilinson–Soulé vanishing conjecture. If furthermore each group $H_0(GL_k(F);\Delta_k(F))$ is rationally trivial for $k \geq 2$, then the full vanishing conjecture (for $s \geq t$) follows.

6 Rings and schemes

[[Can extend the definition of the rank filtration to the case of rings with the invariant rank property, including all commutative rings R. Use the exact category $\mathscr{F}(R)$ of finitely generated free R-modules to get a stable rank filtration

$$* \to \mathbf{S}[BR^\times] \to F_2\mathbf{K}(R) \to \cdots \to F_k\mathbf{K}(R) \to \cdots \to \mathbf{K}^f(R)$$

of free K-theory, and stable equivalences $F_k\mathbf{K}(R)/F_{k-1}\mathbf{K}(R) \simeq \mathbf{D}(R^k)//GL_k(R)$ and $\mathbf{D}(R^k) \simeq \mathbf{\Sigma}^\infty \Sigma D'(R^k)$. Define presheaves of rank complexes by

$$\Gamma_{\mathrm{rk}}(t, \mathrm{Spec}(R))^{t-s} = H_{s+t}(\mathbf{D}(R^{s+1})//GL_{s+1}(R))$$

and sheafify in Zariski or étale topology. Is the result homotopy invariant for regular R?]]

7 Rank one

7.1 Milnor K-theory

Let $\Lambda^* F^\times$ be the free graded commutative (= exterior) ring generated by the abelian group $F^\times$ in degree 1. The Milnor K-theory ring $K^M_*(F)$ is its quotient by the ideal $I \subset \Lambda^* F^\times$ generated by the products $x \wedge y$ in degree 2, for $x, y \in F^\times$ with $x + y = 1$. There is an isomorphism $K^M_t(F) \cong H^t_{\mathrm{mot}}(F;\mathbb{Z}(t))$ for all $t \geq 0$ [13], [24], compatible with the standard ring homomorphism $K^M_*(F) \to K_*(F)$ and the edge homomorphism $E^2_{0,*}(\mathrm{mot}) \twoheadrightarrow E^\infty_{0,*}(\mathrm{mot}) \rightarrowtail K_*(F)$ in the motivic spectral sequence.

7.2 A comparison result

Theorem 7.2.1. *The motivic spectral sequence and the rank spectral sequence are rationally isomorphic on the vertical axis, from the E^2-term and onwards.*

More precisely, the canonical homomorphism $\Lambda^* F^\times \to H_*(F^\times) = E^1_{0,*}(\mathrm{rk})$ induces a rational isomorphism $K^M_*(F) = E^2_{0,*}(\mathrm{mot}) \cong_{\mathbb{Q}} E^2_{0,*}(\mathrm{rk})$. All later differentials that map to the vertical axis are rationally trivial, in both spectral sequences.

Proof. Consider the commutative diagram of solid arrows

$$\begin{array}{ccccc}
\Lambda^* F^\times & \twoheadrightarrow & K^M_*(F) & \longrightarrow & K_*(F) \\
\downarrow {\scriptstyle \cong_{\mathbb{Q}}} & & \downarrow & & \downarrow {\scriptstyle \cong_{\mathbb{Q}}} \\
H_*(F^\times) & \twoheadrightarrow & E^2_{0,*}(\mathrm{rk}) & \longrightarrow & H_*\mathbf{K}(F)\,.
\end{array}$$

The image of the differential $d^1 \colon E^1_{1,*}(\mathrm{rk}) \to E^1_{0,*}(\mathrm{rk})$ is an ideal $J \subset H_*(F^\times)$, since the stable rank filtration is a diagram of $\mathbf{S}[BF^\times]$-module spectra, so this is a diagram of graded commutative rings. The left and right hand vertical maps are rational isomorphisms. [[Reference for $\Lambda^* F^\times \cong_{\mathbb{Q}} H_*(F^\times)$?]] We know that $F_2\mathbf{K}(F) \to \mathbf{K}(F)$ is 3-connected, by Theorem 4.5.1, so $E^2_{0,2}(\mathrm{rk}) = E^\infty_{0,2}(\mathrm{rk})$ maps injectively to $H_2\mathbf{K}(F)$. Hence $x \wedge y$ in $\Lambda^* F^\times$ maps to zero in $E^2_{0,*}(\mathrm{rk})$, for all $x, y \in F^\times$ with $x + y = 1$. Therefore the ideal $I \subset \Lambda^* F^\times$ maps into the ideal $J \subset H_*(F^\times)$, and there is a unique vertical homomorphism in the middle making the whole diagram commute.

Looking at the left hand square, it follows that $K^M_*(F) \to E^2_{0,*}(\mathrm{rk})$ is rationally surjective. By the rational collapse at E^2 of the motivic spectral sequence, we know that $K^M_*(F) \to K_*(F)$ is rationally injective. Looking at the right hand square, it follows that the middle vertical homomorphism is a rational isomorphism, and that the edge homomorphism $E^2_{0,*}(\mathrm{rk}) \to H_*\mathbf{K}(F)$ is rationally injective. This implies that $E^2_{0,*}(\mathrm{rk}) \twoheadrightarrow E^\infty_{0,*}(\mathrm{rk})$ is a rational isomorphism, also for the rank spectral sequence. □

8 Rank two

8.1 Connectivity

Let $k = 2$. The common basis complex $D'(F^2)$ is the 1-dimensional simplicial complex with vertices the lines $L \subset F^2$, or equivalently, the points $x \in P^1(F)$ on the projective line over F, and a 1-simplex $\{x_0, x_1\}$ for each pair of distinct points $x_0, x_1 \in P^1(F)$. As usual, we think of $x \in F$ as the point in $P^1(F)$ given by the line through $(1, x) \in F^2$, and write ∞ for the point given by the line through $(0, 1) \in F^2$.

The group $GL_2(F)$ acts transitively on the 0-simplices, with 0 stabilized by the subgroup P_1 of upper-triangular matrices $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$. It also acts transitively on the ordered 1-simplices, with $(0, \infty)$ stabilized by the subgroup T_2 of diagonal matrices $\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$. Hence the augmented, oriented chain complex $\bar{D}'_*(F^2)$ associated to $D'(F^2)$ [21, §4.1] is

$$0 \leftarrow \mathbb{Z} \xleftarrow{d_0} \mathbb{Z}[GL_2(F)/P_1] \xleftarrow{d_1} \mathbb{Z}[GL_2(F)/T_2] \otimes_{\Sigma_2} \mathbb{Z}_{\mathrm{sgn}} \leftarrow 0\,,$$

where $\mathbb{Z}_{\mathrm{sgn}}$ denotes the sign representation. It is concentrated in degrees $-1 \le * \le 1$. Writing $x_0 \wedge x_1$ for the oriented chain generated by the ordered simplex (x_0, x_1) (so that $x_1 \wedge x_0 = -(x_0 \wedge x_1)$), we have $d_1(x_0 \wedge x_1) = x_1 - x_0$ and $d_0(x) = 1$. The complex is clearly exact at $\mathbb{Z}$ and $\mathbb{Z}[GL_2(F)/P_1]$, with homology

$$\Delta_2(F) = \ker(d_1) = \tilde{H}_1(D'(F^2)) = H_2(\mathbf{D}(F^2))$$

in degree 1. In particular, the connectivity conjecture holds for $k = 2$. In this case, the Tits building $B(F^2) \subset D'(F^2)$ appears as the 0-skeleton of the common basis complex, and there is a short exact sequence

$$0 \leftarrow \mathrm{St}_2(F) \xleftarrow{d_1} \mathbb{Z}[GL_2(F)/T_2] \otimes_{\Sigma_2} \mathbb{Z}_{\mathrm{sgn}} \leftarrow \Delta_2(F) \leftarrow 0$$

of $GL_2(F)$-representations.

8.2 A resolution

The common basis complex $D'(F^2)$ equals the 1-skeleton of the complete simplex $E(F^2)$ spanned by all points $x \in P^1(F)$, with one p-simplex for each $(p+1)$-tuple of pairwise distinct points $\{x_0, x_1, \ldots, x_p\}$. The augmented, oriented chain complex $\bar{E}_*(F^2)$ is exact, hence there is an exact sequence

$$0 \leftarrow \Delta_2(F) \xleftarrow{d_2} \bar{E}_2(F^2) \xleftarrow{d_3} \bar{E}_3(F^2) \xleftarrow{d_4} \bar{E}_4(F^2) \leftarrow \ldots$$

providing a $GL_2(F)$-resolution of $\Delta_2(F)$.

The analysis of this resolution, presented in this and the following two subsections, is the oriented version of that given for the ordered chain complex in [22, §2]. See also Bloch's 1978 lectures [4].

The group $GL_2(F)$ acts transitively on the ordered 2-simplices, with $(0, \infty, 1)$ stabilized by the subgroup $F^\times$ of scalar matrices, i.e., the center of $GL_2(F)$. Hence

$$\bar{E}_2(F^2) = \mathbb{Z}[GL_2(F)/F^\times] \otimes_{\Sigma_3} \mathbb{Z}_{\text{sgn}} \,.$$

More generally, for $p \geq 2$ each orbit for the $GL_2(F)$-action on the ordered p-simplices contains a unique element $[x_3, \ldots, x_p] := (0, \infty, 1, x_3, \ldots, x_p)$, where the $x_3, \ldots, x_p$ are pairwise distinct elements in $P^1(F) \setminus \{0, \infty, 1\} = F \setminus \{0, 1\}$. The stabilizer is $F^\times$ in each case.

For example, the orbit of (x_0, x_1, x_2, x_3) contains $(0, \infty, 1, z)$ where

$$z = \frac{(x_0 - x_3)}{(x_0 - x_2)} \frac{(x_1 - x_2)}{(x_1 - x_3)}$$

is the classical *cross-ratio*. Hence

$$\bar{E}_3(F^2) = \mathbb{Z}[GL_2(F)/F^\times]\{[z]\} \otimes_{\Sigma_4} \mathbb{Z}_{\text{sgn}}$$

where z ranges over the set $F \setminus \{0, 1\}$. Here $(12) \in \Sigma_4$ maps z to z^{-1} and (1234) maps z to $1 - z$. Likewise,

$$\bar{E}_4(F^2) = \mathbb{Z}[GL_2(F)/F^\times]\{[x, y]\} \otimes_{\Sigma_5} \mathbb{Z}_{\text{sgn}} \,,$$

where (x, y) ranges over the distinct pairs of elements in $F \setminus \{0, 1\}$.

8.3 Group homology

The resolution above leads to a hyperhomology spectral sequence

$$E^1_{p,q} = H_q(GL_2(F); \bar{E}_p(F^2)) \Longrightarrow H_{p+q-2}(GL_2(F); \Delta_2(F))$$

where $p \geq 2$ and $q \geq 0$. We can rewrite the first page as

$$0 \leftarrow H_*(\Sigma_3 \times F^\times; \mathbb{Z}_{\text{sgn}}) \xleftarrow{d^1_{3,*}} H_*(\Sigma_4 \times F^\times; \mathbb{Z}_{\text{sgn}}\{[z]\})$$
$$\xleftarrow{d^1_{4,*}} H_*(\Sigma_5 \times F^\times; \mathbb{Z}_{\text{sgn}}\{[x, y]\}) \leftarrow \ldots$$

with $z, x \neq y$ as above. The 0-th row is

$$0 \leftarrow \mathbb{Z}/2 \longleftarrow \mathbb{Z}\{[z]\} \otimes_{\Sigma_4} \mathbb{Z}_{\text{sgn}} \longleftarrow \mathbb{Z}\{[x, y]\} \otimes_{\Sigma_5} \mathbb{Z}_{\text{sgn}} \leftarrow \ldots$$

where $[z]$ maps to 0 and $[x, y] = (0, \infty, 1, x, y)$ maps to the alternating sum of the cross ratios of $(\infty, 1, x, y)$, $(0, 1, x, y)$, $(0, \infty, x, y)$, $(0, \infty, 1, y)$ and $(0, \infty, 1, x)$. This leads to the following definitions.

Definition 8.3.1. Let the *pre-Bloch group* $\mathscr{P}(F)$ be the abelian group generated by the set of symbols $[z]$, where $z \in F \setminus \{0,1\}$, subject to the relations

$$[x] - [y] + [\frac{y}{x}] - [\frac{1-x^{-1}}{1-y^{-1}}] + [\frac{1-x}{1-y}] = 0$$

for all $x \neq y \in F \setminus \{0,1\}$. Let its quotient $\bar{\mathscr{P}}(F)$ be subject to the additional relations

$$-[z] = [z^{-1}] = [1-z]$$

for all $z \in F \setminus \{0,1\}$. The kernel of the homomorphism $\mathscr{P}(F) \twoheadrightarrow \bar{\mathscr{P}}(F)$ is 6-torsion [22, 1.2, 1.4], hence this is a rational isomorphism.

Proposition 8.3.2. *(a)* $E^{\infty}_{2,0} = H_0(GL_2(F); \Delta_2(F)) \cong \mathbb{Z}/2$.

(b) $E^{\infty}_{3,0} \cong \bar{\mathscr{P}}(F)$ *and* $E^{\infty}_{2,1}$ *is* 6*-torsion, so*

$$H_1(GL_2(F); \Delta_2(F)) \twoheadrightarrow \bar{\mathscr{P}}(F)$$

is a rational isomorphism.

[[Is $H_1(GL_2(F); \Delta_2(F)) \cong \mathscr{P}(F)$?]]

8.4 The Bloch group

Definition 8.4.1. The *Bloch group* $\mathscr{B}(F)$ is the kernel of the homomorphism

$$\varphi \colon \mathscr{P}(F) \longrightarrow \Lambda^2 F^{\times}$$

taking $[z]$ to $z \wedge (1-z)$.

Presumably the rank spectral sequence differential

$$d^1_{1,2} \colon H_1(GL_2(F); \Delta_2(F)) \longrightarrow H_2(F^{\times})$$

is compatible, under the rational isomorphisms $H_1(GL_2(F); \Delta_2(F)) \cong_{\mathbb{Q}} \mathscr{P}(F)$ and $H_2(F^{\times}) \cong_{\mathbb{Q}} \Lambda^2 F^{\times}$, with the homomorphism φ. We know from Theorem 7.2.1 that the cokernel of either map is rationally isomorphic to $K_2(F)$.

If, furthermore, $H_0(GL_3(F); \Delta_3(F))$ is torsion and $\mathbf{D}(F^4)$ is at least 4-connected, as predicted by the connectivity conjecture, then we get rational isomorphisms

$$K_3(F)_{\mathrm{ind}} \cong_{\mathbb{Q}} H^1_{\mathrm{rk}}(F; \mathbb{Z}(2)) \cong_{\mathbb{Q}} \mathscr{B}(F)\,.$$

Here $K_3(F)_{\mathrm{ind}} = \mathrm{cok}(K^M_3(F) \to K_3(F))$ is the indecomposable part of algebraic K-theory [12]. In the case of motivic cohomology (in place of rank cohomology), the integral isomorphism $K_3(F)_{\mathrm{ind}} \cong H^1_{\mathrm{mot}}(F; \mathbb{Z}(2))$ is immediate from the motivic spectral sequence.

[[Extend to a formula for $H_2(GL_2(F); \Delta_2(F))$ and $H^2_{\mathrm{rk}}(F; \mathbb{Z}(3))$. Discuss $K_4(F)_{\mathrm{ind}}$ rationally.]]

8.5 The dilogarithm

The classical dilogarithm is defined on the unit disc in $\mathbb{C}$ by the absolutely convergent series

$$\mathrm{Li}_2(z) = \sum_{n\geq 1} \frac{z^n}{n^2}$$

(similar to the series $\sum_{n\geq 1} z^n/n$ for $-\log(1-z)$), and admits a multivalued analytic continuation over $P^1(\mathbb{C}) \setminus \{0, \infty, 1\}$. The Bloch–Wigner function

$$\mathscr{L}_2(z) = \mathrm{Im}\big(\mathrm{Li}_2(z) + \log(1-z)\log|z|\big)$$

is single-valued real analytic on $P^1(\mathbb{C}) \setminus \{0, \infty, 1\}$, and continuous with value 0 at 0, 1 and ∞. It satisfies the Spence(–Abel) functional equation

$$\mathscr{L}_2(x) - \mathscr{L}_2(y) + \mathscr{L}_2(\frac{y}{x}) - \mathscr{L}_2(\frac{1-x^{-1}}{1-y^{-1}}) + \mathscr{L}_2(\frac{1-x}{1-y}) = 0$$

and the relations

$$-\mathscr{L}_2(z) = \mathscr{L}_2(z^{-1}) = \mathscr{L}_2(1-z)$$

for $x \neq y, z \in P^1(\mathbb{C}) \setminus \{0, \infty, 1\}$.

Proposition 8.5.1. *Let $F \subseteq \mathbb{C}$. The rule $[z] \mapsto \mathscr{L}_2(z)$ defines a well-defined function $\mathscr{L}_2 \colon \bar{\mathscr{P}}(F) \to \mathbb{R}$. The composite*

$$K_3(F) \cong_{\mathbb{Q}} H_3\mathbf{K}(F) \to H_1(GL_2(F); \Delta_2(F)) \cong_{\mathbb{Q}} \bar{\mathscr{P}}(F) \xrightarrow{\mathscr{L}_2} \mathbb{R}$$

agrees with the Borel regulator [[up to an explicit unit]].

[[Problem: edge homomorphism $H_3\mathbf{K}(F) \to H^1_{\mathrm{rk}}(F; \mathbb{Z}(2))$ might not map to $\Gamma_{\mathrm{rk}}(2, F)^1 = H_1(GL_2; \Delta_2)$, due to differential from $H_0(GL_3; \Delta_3)$ (or $E^2_{3,1}(\mathrm{rk})$).]]

[[For a number field F with r_2 pairs of complex embeddings, the collected homomorphism $K_3(F) \to \mathbb{R}^{r_2}$ is rationally injective.]]

9 Higher ranks

9.1 Finite topologies

The common basis complex $D'(F^k)$ is a $(2^k - 3)$-dimensional simplicial complex with $GL_k(F)$-action. To analyze its homology, it is convenient to filter its simplices by their stabilizer type. These turn out to correspond to homeomorphism classes of topologies on the set $\{1, \dots, k\}$.

Recall that a p-simplex in $D'(F^k)$ is a set of $(p+1)$ proper, nontrivial subspaces $\{V_0, \dots, V_p\}$ of F^k that admit a common basis $\mathscr{B} = \{b_1, \dots, b_k\}$. Choosing such a common basis, each subspace V_a for $0 \leq a \leq p$ can be written in the form $\langle b_i \mid i \in S_a \rangle$ for a unique proper, nonempty subset S_a of $\{1, \dots, k\}$. The collection of subsets $\sigma = \{S_0, \dots, S_p\}$ can be viewed as a *subbasis* for a *topology* τ on $\{1, \dots, k\}$, where τ is the closure of σ with respect to all unions and intersections. There is a one-to-one correspondence between such finite topologies τ and *preorders* ω on $\{1, \dots, k\}$, where $i \leq j$ in ω if and only if $j \in T$ implies $i \in T$, for all $T \in \tau$.

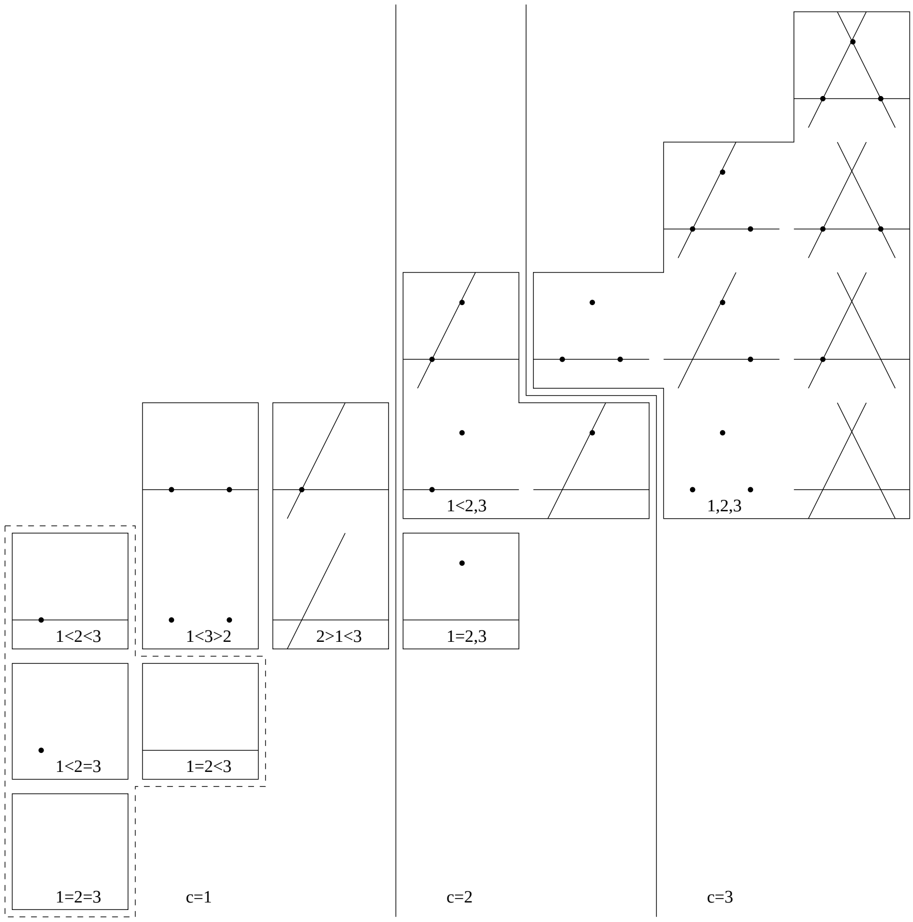


Figure 1: Topologies on $\{1,2,3\}$

A different choice of common basis for the subspaces $\{V_0, \dots, V_p\}$ leads to an equivalent subbasis, up to a permutation of the elements of $\{1, \dots, k\}$. The associated topologies (resp. preorders) are therefore homeomorphic (resp. isomorphic). Let us write $[\tau]$ for the homeomorphism class of τ.

The family of all topologies on $\{1, \dots, k\}$ is partially ordered by inclusion, with $\tau' \le \tau$ if $T \in \tau'$ implies $T \in \tau$. The trivial (= indiscrete) topology is initial and the discrete topology is final in this partial ordering. There is an induced partial ordering on the family of isomorphism classes of topologies on $\{1, \dots, k\}$, with $[\tau'] \le [\tau]$ if τ' is homeomorphic to some τ^* with $\tau^* \le \tau$.

In the case $k = 3$ there are 20 equivalence classes of subbases, corresponding to 9 homeomorphism classes of topologies on $\{1,2,3\}$, as shown in Figure 1. In the next case, there are 33 homeomorphism classes of topologies on $\{1,2,3,4\}$.

9.2 The component filtration

For each $1 \le c \le k$, let $F_c D'(F^k) \subseteq D'(F^k)$ be the simplicial subcomplex consisting of simplices $\{V_0, \dots, V_p\}$ whose associated topology $[\tau]$ has c or fewer connected components. Deleting one or more of the V_a leads to a coarser topology, which cannot increase the number of connected components, so this condition defines a subcomplex. We call

$$B(F^k) \simeq F_1 D'(F^k) \subset \cdots \subset F_k D'(F^k) = D'(F^k)$$

the *component filtration* of $D'(F^k)$ [19, §7].

Definition 9.2.1. Let the *Lie representation* Lie_c be the free abelian group generated by the iterated Lie brackets on c symbols $x_1, \dots, x_c$, where each symbol occurs exactly once. It is free abelian of rank $(c-1)!$. The symmetric group Σ_c acts on Lie_c by permuting the generators. Let $\mathrm{Lie}_c^* = \mathrm{Hom}(\mathrm{Lie}_c, \mathbb{Z})$ be the dual (= contragredient) Σ_c-representation.

These representations were denoted XL_c and W_c, respectively, in [17, 13.6, 11.10].) For example, $\mathrm{Lie}_1^* = \mathbb{Z}$ is trivial and $\mathrm{Lie}_2^* = \mathbb{Z}_{\mathrm{sgn}}$ is the sign representation. There is a basis for $\mathrm{Lie}_3^* = \mathbb{Z}\{w_1, w_2\}$ such that (12) acts by $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and (123) acts by $\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$.

Definition 9.2.2. For each partition of k as a sum of c natural numbers $\vec{k} = (k_1, \dots, k_c)$, with $k_1 \geq \cdots \geq k_c$, there is a direct sum decomposition

$$F^k = F^{k_1} \oplus \cdots \oplus F^{k_c} .$$

As an ordered sum it is stabilized by the product group

$$GL_{\vec{k}}(F) = GL_{k_1}(F) \times \cdots \times GL_{k_c}(F) \subseteq GL_k(F) ,$$

while as an unordered sum it is stabilized by the semidirect product

$$\Sigma_{\vec{k}} \ltimes GL_{\vec{k}}(F) \subseteq GL_k(F) ,$$

where $\Sigma_{\vec{k}} \subseteq \Sigma_c$ is the stabilizer of $(k_1, \dots, k_c) \in \mathbb{N}^c$, under the permutation action.

Recall the Steinberg representation $\mathrm{St}_k(F) = \tilde{H}_{k-1}(\Sigma B(F^k))$ of $GL_k(F)$. The tensor product action of $GL_{\vec{k}}(F)$ on $\mathrm{St}_{k_1}(F) \otimes \cdots \otimes \mathrm{St}_{k_c}(F)$ extends to an action of $\Sigma_{\vec{k}} \ltimes GL_{\vec{k}}(F)$ on $\mathrm{Lie}_c^* \otimes \mathrm{St}_{k_1}(F) \otimes \cdots \otimes \mathrm{St}_{k_c}(F)$, where $\Sigma_{\vec{k}} \subseteq \Sigma_c$ acts on Lie_c^* as defined above, and by permuting the Steinberg representations. See [19, 7.7] for the topological precursor of the following result.

Proposition 9.2.3 ([19, 7.8])**.** *For each $2 \leq c \leq k$ the relative homology $H_*(F_c D'(F^k), F_{c-1} D'(F^k))$ is concentrated in degree $(k + c - 3)$, where it is isomorphic as a $GL_k(F)$-module to the direct sum*

$$Z'_{k+c-3} = \bigoplus_{\vec{k}} \mathbb{Z}[GL_k(F)] \underset{\Sigma_{\vec{k}} \ltimes GL_{\vec{k}}(F)}{\otimes} \mathrm{Lie}_c^* \otimes \mathrm{St}_{k_1}(F) \otimes \cdots \otimes \mathrm{St}_{k_c}(F) .$$

The sum runs over the partitions $\vec{k} = (k_1, \dots, k_c)$ of k as a sum of c natural numbers, with $k_1 \geq \cdots \geq k_c$.

Corollary 9.2.4. *$\tilde{H}_*(D'(F^k))$ is isomorphic to the homology of a chain complex*

$$0 \leftarrow \mathrm{St}_k(F) = Z'_{k-2} \leftarrow Z'_{k-1} \leftarrow \cdots \leftarrow Z'_{2k-3} = \mathbb{Z}[GL_k(F)/T_k] \otimes_{\Sigma_k} \mathrm{Lie}^*_k \leftarrow 0$$

where $T_k \subset GL_k(F)$ is the subgroup of diagonal matrices.

The connectivity conjecture asks that the homology of this complex is concentrated at the top end, in degree $(2k-3)$. In this case,

$$\Delta_k(F) = \tilde{H}_{2k-3}(D'(F^k)) = \ker(Z'_{2k-3} \to Z'_{2k-4}) .$$

[[Expect Z'_{2k-3} is generated by relative $(2k-3)$-cycles given by $(2k-2)$-tuples $\{L_2, \dots, L_k, H_2, \dots, H_k\}$, where $L_1 \oplus \cdots \oplus L_k = F^k$ and H_i is the hyperplane spanned by the L_j with $i \neq j$. Make Z'_{2k-4} and the boundary map d_{2k-3} explicit.]]

9.3 Weight zero

Lemma 9.3.1. *For $k \geq 2$ the boundary map $d_{k-1}\colon Z'_{k-1} \to Z'_{k-2}$ is surjective, so $\tilde{H}_{k-2}(D'(F^k)) = 0$ and $E^1_{k-1,0}(\mathrm{rk}) = \Gamma_{\mathrm{rk}}(0, F)^{1-k} = 0$ for $* < 0$.*

Proof. We discussed $k = 2$ earlier. For $k \geq 3$, d_{k-1} is a homomorphism

$$\bigoplus_{\vec{k}=(k_1,k_2)} \mathbb{Z}[GL_k(F)] \underset{GL_{\vec{k}}(F)}{\otimes} \mathrm{St}_{k_1}(F) \otimes \mathrm{St}_{k_2}(F) \longrightarrow \mathrm{St}_k(F) .$$

By the proof of the Solomon–Tits theorem in [15] [[Explain the details?]], there is a homotopy equivalence

$$\bigvee_L \Sigma B(F^{k-1}) \xrightarrow{\simeq} B(F^k) ,$$

where L ranges over the lines in F^k that are transverse to the hyperplane F^{k-1}. Furthermore, the induced isomorphism

$$\bigoplus_L \mathrm{St}_{k-1}(F) \xrightarrow{\cong} \mathrm{St}_k(F)$$

factors through d_{k-1} on the summand $\vec{k} = (k-1, 1)$, hence d_{k-1} is surjective. □

9.4 Weight one and rank three

Let $k = 3$. The chain complex Z'_* associated to the component filtration

$$B(F^3) \subset F_2 D'(F^3) \subset D'(F^3)$$

is

$$0 \leftarrow \mathrm{St}_3(F) \xleftarrow{d_2} \mathbb{Z}[GL_3(F)] \underset{GL_{(2,1)}(F)}{\otimes} \mathrm{St}_2(F) \xleftarrow{d_3} \mathbb{Z}[GL_3(F)/T_3] \otimes_{\Sigma_3} \mathrm{Lie}^*_3 \leftarrow 0 ,$$

concentrated in degrees $1 \leq * \leq 3$.

Lemma 9.4.1. *This complex is exact at Z'_2, so $\tilde{H}_2(D'(F^3)) = 0$ and $E^1_{2,1}(\mathrm{rk}) = \Gamma_{rk}(1, F)^{-1} = 0$.*

Proof. The Steinberg module $Z'_1 = \mathrm{St}_3(F)$ is a submodule of the free abelian group $\mathbb{Z}\{(L \subset P)\}$ generated by the maximal flags $L \subset P$ in F^3, where each line L and each plane P occurs algebraically zero times. These are 1-cycles in $B(F^3) \simeq F_1 D'(F^3)$ given by pairs $\{L, P\}$, with associated preorder $(1 < 2 < 3)$.

The module Z'_2 is the submodule of the free abelian group $\mathbb{Z}\{(L \subset P, L')\}$ generated by the maximal flags $L \subset P$ and direct sum decompositions $P \oplus L' = F^3$, where for each plane P and complementary line L' there are algebraically zero lines L in P. These are relative 2-cycles in $(F_2 D'(F^3), F_1 D'(F^3))$ given by triples $\{L, L', P\}$ with associated preorder $(1 < 2, 3)$.

The module Z'_3 is generated by symbols (L_1, L_2, L_3), where $L_1 \oplus L_2 \oplus L_3 = F^3$ is a sum decomposition into three lines. These are relative 3-cycles in $(D'(F^3), F_2 D'(F^3))$ given by quadruples $\{L_2, L_3, P_{12}, P_{13}\}$, with $P_{12} = L_1 \oplus L_2$ and $P_{13} = L_1 \oplus L_3$, with associated preorder $(1, 2, 3)$.

In these terms, the boundary maps are given by

$$d_2(L \subset P, L') = (L \subset P) - (L \subset L \oplus L') + (L' \subset L \oplus L')$$

and

$$d_3(L_1, L_2, L_3) = (L_1 \subset P_{12}, L_3) - (L_2 \subset P_{12}, L_3) + (L_1 \subset P_{13}, L_2) - (L_3 \subset P_{13}, L_2)\,.$$

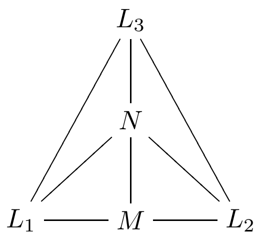
Note that Z'_2 is generated by the differences

$$[L_1, L_2, L_3] := (L_1 \subset L_1 \oplus L_2, L_3) - (L_2 \subset L_1 \oplus L_2, L_3)$$

where L_1, L_2, L_3 range over the triples of lines with $L_1 \oplus L_2 \oplus L_3 = F^3$. Clearly $[L_2, L_1, L_3] = -[L_1, L_2, L_3]$, and

$$[L_1, L_2, L_3] \equiv [L_3, L_1, L_2]$$

modulo the image of d_3, so in this sense $[L_1, L_2, L_3]$ is an alternating function in the triple of lines.



Let M be a third line in the plane P spanned by L_1 and L_2. Then

$$\begin{aligned}[L_1, L_2, L_3] &= (L_1 \subset P, L_3) - (M \subset P, L_3) + (M \subset P, L_3) + (L_2 \subset P, L_3) \\ &= [L_1, M, L_3] + [M, L_2, L_3]\,.\end{aligned}$$

Let N be a third line in the plane spanned by M and L_3. Then

$$\begin{aligned}[L_1, L_2, L_3] &= [L_1, M, L_3] + [M, L_2, L_3] \\ &\equiv [M, L_3, L_1] + [L_3, M, L_2] \\ &= [M, N, L_1] + [N, L_3, L_1] + [L_3, N, L_2] + [N, M, L_2] \\ &\equiv [L_1, M, N] + [M, L_2, N] - [L_1, L_3, N] + [L_2, L_3, N] \\ &= [L_1, L_2, N] - [L_1, L_3, N] + [L_2, L_3, N]\,.\end{aligned}$$

Now consider any $x \in Z_2'$ with $d_2(x) = 0$. We can write

$$x = \sum_i n_i [L_1^i, L_2^i, L_3^i]$$

as a finite sum with integers coefficients of terms $[L_1, L_2, L_3]$, with the corresponding finite sum of expressions

$$d_2([L_1, L_2, L_3]) = \sum_{\pi \in \Sigma_3} \operatorname{sgn}(\pi)(L_{\pi(1)} \subset L_{\pi(1)} \oplus L_{\pi(2)})$$

equal to zero.

By assuming that F is an infinite field, we can find a line N in F^3 in general position with respect to all the lines and planes appearing in x. For each triple L_1, L_2, L_3, the plane through N and L_3 meets the plane through L_1 and L_2 in a line M, and we are in the situation above. Hence x is congruent, modulo the image of d_3, to a finite sum y of expressions

$$[L_1, L_2, N] - [L_1, L_3, N] + [L_2, L_3, N] = \sum_{\pi \in \Sigma_3} \operatorname{sgn}(\pi)(L_{\pi(1)} \subset L_{\pi(1)} \oplus L_{\pi(2)}, N)\,.$$

This finite sum is obtained from $d_2(x)$ by taking each $(L \subset P)$ to $(L \subset P, N)$, for the fixed N. Since $d_2(x)$ is algebraically zero, so is y. Hence x is in the image of d_3.

The case where F is finite can be handled by extending the definition of $[L_1, L_2, L_3]$ to be 0 if the three lines do not span all of F^3, and checking that all the formulas are still satisfied. □

9.5 Resolutions

[[Discuss extensions $D'(F^k) \subset E(F^k)$ with $E(F^k)$ contractible or highly connected, and use quotient complex $\bar{E}_*(F^k)/\bar{D}'_*(F^k)$ to get a resolution of the $GL_k(F)$-representation $\Delta_k(F)$.]]

[[A candidate for the higher pre-Bloch group $\bar{\mathscr{P}}_k(F)$ is the cokernel of

$$H_0(GL_k(F); \bar{E}_{2k}(F^k)) \to H_0(GL_k(F); \bar{E}_{2k-1}(F^k))$$

and the closely related groups $H_1(GL_k(F); \Delta_k(F)) = \Gamma_{\mathrm{rk}}(k, F)^1$ and $H^1_{\mathrm{rk}}(F; \mathbb{Z}(k))$ are then candidates for higher pre-Bloch and Bloch groups $\mathscr{P}_k(F)$ and $\mathscr{B}_k(F)$.]]

9.6 Polylogarithms

[[Discuss $H_1(GL_3(F); \Delta_3(F))$ and relation to trilogarithm and Borel regulator $K_5(F) \to \mathbb{R}$ for $F \subseteq \mathbb{C}$.]]

References

[1] A. A. Beĭlinson, *Higher regulators and values of L-functions*, Current problems in mathematics, Vol. 24, Itogi Nauki i Tekhniki, Akad. Nauk SSSR Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1984, pp. 181–238 (Russian).

[2] ———, *Height pairing between algebraic cycles*, K-theory, arithmetic and geometry (Moscow, 1984), Lecture Notes in Math., vol. 1289, Springer, Berlin, 1987, pp. 1–25.

[3] Spencer Bloch, *Algebraic cycles and higher K-theory*, Adv. in Math. **61** (1986), no. 3, 267–304.

[4] Spencer J. Bloch, *Higher regulators, algebraic K-theory, and zeta functions of elliptic curves*, CRM Monograph Series, vol. 11, American Mathematical Society, Providence, RI, 2000.

[5] Armand Borel and Jun Yang, *The rank conjecture for number fields*, Math. Res. Lett. **1** (1994), no. 6, 689–699.

[6] Jean-Louis Cathelineau, *Homologie du groupe linéaire et polylogarithmes (d'après A. B. Goncharov et d'autres)*, Astérisque (1993), no. 216, Exp. No. 772, 5, 311–341 (French, with French summary). Séminaire Bourbaki, Vol. 1992/93.

[7] Eric M. Friedlander and Andrei Suslin, *The spectral sequence relating algebraic K-theory to motivic cohomology*, Ann. Sci. École Norm. Sup. (4) **35** (2002), no. 6, 773–875 (English, with English and French summaries).

[8] H. Gillet and C. Soulé, *Filtrations on higher algebraic K-theory*, Algebraic K-theory (Seattle, WA, 1997), Proc. Sympos. Pure Math., vol. 67, Amer. Math. Soc., Providence, RI, 1999, pp. 89–148.

[9] A. B. Goncharov, *Polylogarithms and motivic Galois groups*, Motives (Seattle, WA, 1991), Proc. Sympos. Pure Math., vol. 55, Amer. Math. Soc., Providence, RI, 1994, pp. 43–96.

[10] S. Lichtenbaum, *Values of zeta-functions at nonnegative integers*, Number theory, Noordwijkerhout 1983 (Noordwijkerhout, 1983), Lecture Notes in Math., vol. 1068, Springer, Berlin, 1984, pp. 127–138.

[11] Stephen Lichtenbaum, *The construction of weight-two arithmetic cohomology*, Invent. Math. **88** (1987), no. 1, 183–215.

[12] A. S. Merkur′ev and A. A. Suslin, *The group K_3 for a field*, Izv. Akad. Nauk SSSR Ser. Mat. **54** (1990), no. 3, 522–545 (Russian); English transl., Math. USSR-Izv. **36** (1991), no. 3, 541–565.

[13] Yu. P. Nesterenko and A. A. Suslin, *Homology of the general linear group over a local ring, and Milnor's K-theory*, Izv. Akad. Nauk SSSR Ser. Mat. **53** (1989), no. 1, 121–146 (Russian); English transl., Math. USSR-Izv. **34** (1990), no. 1, 121–145.

[14] Daniel Quillen, *Higher algebraic K-theory. I*, Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972), Springer, Berlin, 1973, pp. 85–147. Lecture Notes in Math., Vol. 341.

[15] ______, *Finite generation of the groups K_i of rings of algebraic integers*, Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972), Springer, Berlin, 1973, pp. 179–198. Lecture Notes in Math., Vol. 341.

[16] M. S. Raghunathan, *A note on quotients of real algebraic groups by arithmetic subgroups*, Invent. Math. **4** (1967/1968), 318–335.

[17] John Rognes, *A spectrum level rank filtration in algebraic K-theory*, Topology **31** (1992), no. 4, 813–845.

[18] ______, *Approximating $K_*(\mathbf{Z})$ through degree five*, K-Theory **7** (1993), no. 2, 175–200.

[19] ______, *$K_4(\mathbf{Z})$ is the trivial group*, Topology **39** (2000), no. 2, 267–281.

[20] Christophe Soulé, *Opérations en K-théorie algébrique*, Canad. J. Math. **37** (1985), no. 3, 488–550 (French).

[21] Edwin H. Spanier, *Algebraic topology*, Springer-Verlag, New York, 1981. Corrected reprint.

[22] A. A. Suslin, *K_3 of a field, and the Bloch group*, Trudy Mat. Inst. Steklov. **183** (1990), 180–199, 229 (Russian). Translated in Proc. Steklov Inst. Math. **1991**, no. 4, 217–239; Galois theory, rings, algebraic groups and their applications (Russian).

[23] A. Suslin, *On the Grayson spectral sequence*, Tr. Mat. Inst. Steklova **241** (2003), no. Teor. Chisel, Algebra i Algebr. Geom., 218–253; English transl., Proc. Steklov Inst. Math. (2003), no. 2 (241), 202–237.

[24] Burt Totaro, *Milnor K-theory is the simplest part of algebraic K-theory*, K-Theory **6** (1992), no. 2, 177–189.

[25] Friedhelm Waldhausen, *Algebraic K-theory of spaces*, Algebraic and geometric topology (New Brunswick, N.J., 1983), Lecture Notes in Math., vol. 1126, Springer, Berlin, 1985, pp. 318–419.