AUTOEQUIVALENCES OF THE TENSOR CATEGORY OF $U_qg$-MODULES

SERGEY NESHVEYEV AND LARS TUSET

Abstract. We prove that for $q \in \mathbb{C}^*$ not a nontrivial root of unity the cohomology group defined by invariant 2-cocycles in a completion of $U_qg$ is isomorphic to $H^2(P/Q; \mathbb{T})$, where $P$ and $Q$ are the weight and root lattices of $g$. This implies that the group of autoequivalences of the tensor category of $U_qg$-modules is the semidirect product of $H^2(P/Q; \mathbb{T})$ and the automorphism group of the based root datum of $g$. For $q = 1$ we also obtain similar results for all compact connected separable groups.

For a tensor category $\mathcal{C}$ a natural object to study is its group of symmetries, i.e., the group $\text{Aut}^\otimes(\mathcal{C})$ of monoidal autoequivalences of $\mathcal{C}$ identified up to monoidal natural isomorphisms. A more refined version of this group is the tensor category of autoequivalences of $\mathcal{C}$. It is, for example, used to define what is meant by an action of a group on $\mathcal{C}$, which in turn leads to such constructions as equivariantization and crossed products, see e.g. [8] for applications. At the same time there are not many examples for which the group $\text{Aut}^\otimes(\mathcal{C})$ is explicitly computed. The aim of this note is to calculate it for the category of representations of the $q$-deformation $G_q$ of a simply connected semisimple compact Lie group $G$. Part of the information about the group of autoequivalences in this case is contained in the work of McMullen [3], who showed that that the group of automorphisms of the fusion ring of $G$ is isomorphic to $\text{Out}(G)$, that is, to the automorphism group of the based root datum of $g$. The remaining part is determined by the possible tensor structures one can have on the identity functor, and these are described by the cohomology group defined by invariant 2-cocycles on the dual $\hat{G}_q$ of the quantum group $G_q$. Another motivation for computing this cohomology group is the problem of classifying Drinfeld twists that do not necessarily respect braiding; the ones that do respect braiding have been classified in [5].

In a previous paper [7] we showed that if $G$ is a compact connected group then the cohomology group defined by invariant unitary 2-cocycles on $\hat{G}$ is isomorphic to $H^2(\hat{Z}(G); \mathbb{T})$ and we conjectured that for semisimple Lie groups a similar result holds for the $q$-deformation of $G$. We will prove that this is indeed the case using techniques from our earlier paper [5], where we considered symmetric cocycles and were inspired by the proof of Kazhdan and Lusztig of the equivalence of the Drinfeld category and the category of $U_qg$-modules [2]. For $q = 1$ this gives an alternative proof of the main results in [7, Section 2] and allows us, at least in the separable case, to extend those results to non-unitary cocycles relying neither on ergodic actions nor on reconstruction theorems. At the same time this proof is less transparent than that in [7] and, as opposed to [7], relies heavily on the structure and representation theory of compact Lie groups.

We will follow the notation and conventions in [5]. Let $G$ be a simply connected semisimple compact Lie group, $g$ its complexified Lie algebra, $q \in \mathbb{C}^*$ not a nontrivial root of unity. Fix a Cartan subalgebra of $g$ and a system $\{\alpha_1, \ldots, \alpha_r\}$ of simple roots. The weight and root lattices are denoted by $P$ and $Q$, respectively. For weight $\lambda \in P$ denote by $\lambda(i)$ the coefficients of $\lambda$ in the basis consisting of fundamental weights. Take the ad-invariant symmetric form on $g$ such that for every short root in every simple component of $g$, and put $d_i = (\alpha_i, \alpha_i)/2$ and $q_i = q^{d_i}$. For $q \neq 1$

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*Our main result, Theorem 1, is valid for any ad-invariant symmetric form on $g$ such that its restriction to the real Lie algebra of $G$ is negative definite, under the assumption that either $q = 1$ (in which case the choice of a form does not matter) or that $q_i$ is not a root of unity for all $i$. 

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consider the quantized universal enveloping algebra $U_q\mathfrak{g}$ with generators $E_i$, $F_i$ and $K_i$, $1 \leq i \leq r$, so that we in particular have

$$K_i E_j K_i^{-1} = q_i^{\alpha_{ij}} E_j, \quad K_i F_j K_i^{-1} = q_i^{-\alpha_{ij}} F_j, \quad E_i F_j - F_j E_i = \delta_{ij}(K_i - K_i^{-1})/(q_i - q_i^{-1}).$$

Recall that a $U_q\mathfrak{g}$-module $V$ is called admissible if $V = \bigoplus_{\lambda \in \mathcal{P}} V(\lambda)$, where $V(\lambda)$ consists of vectors $v \in V$ such that $K_i v = q_i^{\lambda_i} v$ for all $i$. Denote by $C_q(\mathfrak{g})$ the tensor category of admissible finite dimensional $U_q\mathfrak{g}$-modules. For $q = 1$ denote by $C(\mathfrak{g}) = C_1(\mathfrak{g})$ the usual tensor category of finite dimensional $U\mathfrak{g}$-modules. Let $\mathcal{U}(G_q)$ be the endomorphism ring of the forgetful functor $C_q(\mathfrak{g}) \to \text{Vec}$. If for every dominant integral weight $\mu \in P_+$ we fix an irreducible $U_q\mathfrak{g}$-module $V_\mu$ with highest weight $\mu$, then the ring $\mathcal{U}(G_q)$ can be identified with $\prod_{\mu \in P_+} \text{End}(V_\mu)$. The comultiplication on $U_q\mathfrak{g}$ extends to a homomorphism $\hat{\Delta}_q: \mathcal{U}(G_q) \to \mathcal{U}(G_q \times G_q) = \prod_{\mu,\eta \in P_+} \text{End}(V_\mu \otimes V_\eta)$.

An invertible element $\mathcal{E} \in \mathcal{U}(G_q \times G_q)$ is called a 2-cocycle on $\hat{G}_q$ if

$$(\mathcal{E} \otimes 1)(\hat{\Delta}_q \otimes \iota)(\mathcal{E}) = (1 \otimes \mathcal{E})(\iota \otimes \hat{\Delta}_q)(\mathcal{E}).$$

A cocycle is called invariant if it commutes with elements in the image of $\hat{\Delta}_q$. The set of invariant 2-cocycles forms a group under multiplication, which we denote by $Z^2_{G_q}(\hat{G}_q; \mathbb{C}^*)$. Cocycles of the form $(a \otimes a)\hat{\Delta}_q(a)^{-1}$, where $a$ is an invertible element in the center of $\mathcal{U}(G_q)$, form a subgroup of the center of $Z^2_{G_q}(\hat{G}_q; \mathbb{C}^*)$. The quotient of $Z^2_{G_q}(\hat{G}_q; \mathbb{C}^*)$ by this subgroup is denoted by $H^2_{G_q}(\hat{G}_q; \mathbb{C}^*)$.

The set of $\mathcal{U}(G_q) = \prod_{\mu \in P_+} \text{End}(V_\mu)$ is identified with the algebra of functions on the set $P_+$ of dominant integral weights. By [5, Proposition 4.5] a function on $P_+$ is a group-like element of $\mathcal{U}(G_q)$ if and only if it is defined by a character of $P/Q$. Therefore the Hopf algebra of functions on $P/Q$ embeds into the center of $\mathcal{U}(G_q)$. Hence every 2-cocycle $c$ on $P/Q$ can be considered as an invariant 2-cocycle $\mathcal{E}_c$ on $\hat{G}_q$. Explicitly, $\mathcal{E}_c$ acts on $V_\mu \otimes V_\eta$ as multiplication by $c(\mu, \eta)$. We can now formulate our main result.

**Theorem 1.** The homomorphism $c \mapsto \mathcal{E}_c$ induces an isomorphism

$$H^2(P/Q; \mathbb{T}) \cong H^2_{G_q}(\hat{G}_q; \mathbb{C}^*).$$

In particular, if $\mathfrak{g}$ is simple and $\mathfrak{g} \not\cong \mathfrak{so}_{4n}(\mathbb{C})$ then $H^2_{G_q}(\hat{G}_q; \mathbb{C}^*)$ is trivial, and if $\mathfrak{g} \cong \mathfrak{so}_{4n}(\mathbb{C})$ then $H^2_{G_q}(\hat{G}_q; \mathbb{C}^*) \cong \mathbb{Z}/2\mathbb{Z}$.

The last statement follows from the fact that for simple Lie algebras the group $P/Q$ is cyclic unless $\mathfrak{g} \cong \mathfrak{so}_{4n}(\mathbb{C})$, in which case $P/Q \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, see e.g. Table IV on page 516 in [1].

Note that for $q > 0$ the same result holds for unitary cocycles. This easily follows by polar decomposition, see [5, Lemma 1.1].

In the proof of the theorem we will assume that $q \neq 1$, the case $q = 1$ is similar and for unitary cocycles is also proved by a different method in [7].

Our first goal will be to construct a homomorphism $H^2_{G_q}(\hat{G}_q; \mathbb{C}^*) \to H^2(P/Q; \mathbb{T})$. For every $\mu \in P_+$ fix a highest weight vector $\xi_\mu \in V_\mu$. Recall [5, Section 2] that for $\mu, \eta \in P_+$ there exists a unique morphism

$$T_{\mu,\eta}: V_{\mu+\eta} \to V_\mu \otimes V_\eta$$

such that $\xi_{\mu+\eta} \mapsto \xi_\mu \otimes \xi_\eta$. The image of $T_{\mu,\eta}$ is the isotypic component of $V_\mu \otimes V_\eta$ with highest weight $\mu + \eta$. Hence if $\mathcal{E}$ is an invariant 2-cocycle then it acts on this image as multiplication by a nonzero scalar $c_{\mathcal{E}}(\mu, \eta)$. As in the proof of [5, Lemma 2.2], the identity $(T_{\mu,\eta} \otimes \iota)T_{\mu,\eta,\nu} = (\iota \otimes T_{\eta,\nu})T_{\mu,\eta+\nu}$ immediately implies that $c_{\mathcal{E}}$ is a 2-cocycle on $P_+$. Furthermore, the cohomology class $[c_{\mathcal{E}}]$ of $c_{\mathcal{E}}$ in $H^2(P_+; \mathbb{C}^*)$ depends only on the class of $\mathcal{E}$ in $H^2_{G_q}(\hat{G}_q; \mathbb{C}^*)$, since if $a \in \mathcal{U}(G_q)$ is a central element acting on $V_\mu$ as multiplication by a scalar $a(\mu)$ then the action of $(a \otimes a)\hat{\Delta}_q(a)^{-1}$ on the image of $T_{\mu,\eta}$ is multiplication by $a(\mu) a(\eta) a(\mu + \eta)^{-1}$. Thus the map $\mathcal{E} \mapsto c_{\mathcal{E}}$ defines a homomorphism $H^2_{G_q}(\hat{G}_q; \mathbb{C}^*) \to H^2(P_+; \mathbb{C}^*)$. 


Given a cocycle on \( P/Q \), we can consider it as a cocycle on \( P \) and then get a cocycle on \( P_+ \) by restriction. Thus we have a homomorphism \( H^2(P/Q; \mathbb{T}) \to H^2(P_+; \mathbb{C}^*) \). It is injective since the quotient map \( P_+ \to P/Q \) is surjective and a cocycle on \( P/Q \) is a coboundary if it is symmetric.

**Lemma 2.** For every invariant 2-cocycle \( \mathcal{E} \) on \( \hat{G}_q \) the class of \( c_{\mathcal{E}} \) in \( H^2(P_+; \mathbb{C}^*) \) is contained in the image of \( H^2(P/Q; \mathbb{T}) \).

**Proof.** Consider the skew-symmetric bi-quasicharacter \( b: P_+ \times P_+ \to \mathbb{C}^* \) defined by

\[
b(\mu, \eta) = c_{\mathcal{E}}(\mu, \eta)c_{\mathcal{E}}(\eta, \mu)^{-1}.
\]

It extends uniquely to a skew-symmetric bi-quasicharacter on \( P_+ \). To prove the lemma it suffices to show that the root lattice \( Q \) is contained in the kernel of this extension. Indeed, since \( H^2(P/Q; \mathbb{T}) \) is isomorphic to the group of skew-symmetric bi-characters on \( P/Q \), then follows that there exists a cocycle \( c \) on \( P/Q \) such that the cocycle \( c_{\mathcal{E}}c^{-1} \) on \( P_+ \) is symmetric. Then by [6, Lemma 4.2] the cocycle \( c_{\mathcal{E}}c^{-1} \) is a coboundary, so \( c_{\mathcal{E}} \) and the restriction of \( c \) to \( P_+ \) are cohomologous.

To prove that \( Q \) is contained in the kernel of \( b \), recall [5, Section 2] that for every simple root \( \alpha_i \) and weights \( \mu, \eta \in P_+ \) with \( \mu(i), \eta(i) \geq 1 \) we can define a morphism

\[
\tau_{\mu, \eta}: V_{\mu + \eta - \alpha_i} \to V_{\mu} \otimes V_{\eta} \text{ such that } \xi_{\mu + \eta - \alpha_i} \mapsto [\mu(i)]_q \xi_\mu \otimes F_i \xi_\eta - q^{\mu(i)}[\eta(i)]_q F_i \xi_\mu \otimes \xi_\eta.
\]

The image of \( \tau_{\mu, \eta} \) is the isotypic component of \( V_\mu \otimes V_\eta \) with highest weight \( \mu + \eta - \alpha_i \). Since the element \( \mathcal{E} \) is invariant, it acts on this image as multiplication by a nonzero scalar \( c_i(\mu, \eta) \). As in the proof of [5, Lemma 2.3], consider now another weight \( \nu \) with \( \nu(i) \geq 1 \). The isotypic component of \( V_\mu \otimes V_\eta \otimes V_\nu \) with highest weight \( \mu + \eta + \nu - \alpha_i \) has multiplicity two, and is spanned by the images of \( (\iota \otimes \tau_{\eta, \nu}) \tau_{\mu, \eta + \nu} \) and \( (\iota \otimes \tau_{\iota, \nu}) \tau_{\mu, \eta + \nu - \alpha_i} \), as well as by the images of \( (T_{\mu, \eta} \otimes \iota) \tau_{\iota, \mu + \nu} \) and \( (\tau_{\iota, \mu} \otimes \iota) T_{\mu + \nu, \eta} \). We have

\[
[\eta(i)]_q (T_{\mu, \eta} \otimes \iota) \tau_{\iota, \mu + \nu} - [\nu(i)]_q (\tau_{\iota, \mu} \otimes \iota) T_{\mu + \nu, \eta} = [\mu(i) + \eta(i)]_q (\iota \otimes \tau_{\iota, \nu}) T_{\mu, \eta + \nu - \alpha_i}.
\]

Apply the element

\[
\Omega := (\mathcal{E} \otimes 1)(\hat{\Delta}_q \otimes \iota)(\mathcal{E}) = (1 \otimes \mathcal{E})(\iota \otimes \hat{\Delta}_q)(\mathcal{E})
\]

to this identity. The morphisms \( (T_{\mu, \eta} \otimes \iota) \tau_{\iota, \mu + \eta, \nu} \) and \( (\tau_{\iota, \mu} \otimes \iota) T_{\mu + \nu, \eta} \) are eigenvectors of the operator of multiplication by \( \Omega \) on the left with eigenvalues \( c_{\mathcal{E}}(\mu, \eta)c_i(\mu + \eta + \nu, \nu) \) and \( c_i(\mu, \eta)c_{\mathcal{E}}(\mu + \eta + \nu - \alpha_i, \nu) \) respectively. Since the morphisms \( (T_{\mu, \eta} \otimes \iota) \tau_{\iota, \mu + \nu} \) and \( (\tau_{\iota, \mu} \otimes \iota) T_{\mu + \nu, \eta} \) are linearly independent, by applying \( \Omega \) to (1) we conclude that these three eigenvalues coincide. In particular,

\[
c_i(\mu, \eta)c_{\mathcal{E}}(\mu + \eta - \alpha_i, \nu) = c_i(\eta, \nu)c_{\mathcal{E}}(\mu, \eta + \nu - \alpha_i).
\]

Applying this to \( \eta = \nu = \mu \) we get

\[
b(2\mu - \alpha_i, \mu) = 1.
\]

Since \( b \) is skew-symmetric, this gives \( b(\alpha_i, \mu) = 1 \). The latter identity holds for all \( \mu \in P_+ \) with \( \mu(i) \geq 1 \). Since every element in \( P \) can be written as a difference of two such elements \( \mu \), it follows that \( \alpha_i \) is contained in the kernel of \( b \).

Therefore the map \( \mathcal{E} \mapsto c_{\mathcal{E}} \) induces a homomorphism \( H^2(G_\mathcal{E}_q; \mathbb{C}^*) \to H^2(P/Q; \mathbb{T}) \). Clearly, it is a left inverse of the homomorphism \( H^2(P/Q; \mathbb{T}) \to H^2(G_\mathcal{E}_q; \mathbb{C}^*) \), \( [c] \mapsto [c_{\mathcal{E}}] \), constructed earlier. Thus it remains to prove that the homomorphism \( H^2(G_\mathcal{E}_q; \mathbb{C}^*) \to H^2(P/Q; \mathbb{T}) \) is injective.

Assume that \( \mathcal{E} \) is an invariant 2-cocycle such that the cocycle \( c_{\mathcal{E}} \) on \( P_+ \) is a coboundary. Our goal is to show that \( \mathcal{E} \) is the coboundary of a central element in \( \mathcal{U}(G_q) \). We will follow the strategy in [5], where this was shown under the additional assumption that \( \mathcal{E} \) is symmetric, that is, \( R_\mathcal{E} \mathcal{E} = \mathcal{E} \mathcal{E} R_\mathcal{E} \) for an \( R \)-matrix \( R_\mathcal{E} \in \mathcal{U}(G_q \times G_q) \), which depends on the choice of a number \( h \in \mathbb{C} \) such that \( q = e^{\pi i h} \).

The first step in [5], see the discussion following Lemma 2.2 in [5], was to show that \( \mathcal{E} \) is cohomologous to a cocycle such that

\[
\mathcal{E} T_{\mu, \eta} = T_{\mu, \eta} \quad \text{and} \quad \mathcal{E} \tau_{\iota, \mu, \omega} = \tau_{\iota, \mu, \omega}
\]

(2)
for all \( \mu, \eta \in P_+ \), \( 1 \leq i \leq r \) and \( \nu, \omega \in P_+ \) such that \( \nu(i), \omega(i) \geq 1 \). This part goes through in the non-symmetric case without any changes, as the symmetry requirement was needed only to conclude that \( c_E \) is a coboundary.

Therefore to prove the injectivity of \( H^2_{G_q}(\hat{G}_q; C^*) \to H^2(P/Q; T) \) it suffices to establish the following result, which extends [5, Corollary 4.4].

**Proposition 3.** If \( \mathcal{E} \) is an invariant 2-cocycle on \( \hat{G}_q \) with property (2) then \( \mathcal{E} = 1 \).

The proof of this statement in [5] for symmetric cocycles is based on considering the action of \( \mathcal{E} \) on a comonoid representing the forgetful functor on \( C_{\mu, \eta}^4 \). Recall briefly how this comonoid, essentially constructed by Kazhdan and Lusztig, is defined. For every weight \( \mu \in P_+ \), fix an irreducible \( U_q \mathfrak{g} \)-module \( V_\mu \) with lowest weight \(-\mu\) and a lowest weight vector \( \xi_\mu \). For \( \lambda \in P \) and \( \mu, \eta \in P_+ \) such that \( \lambda + \mu \in P_+ \), there exists a unique morphism

\[
\text{tr}_{\mu, \lambda}^\eta : V_\mu \otimes V_{\lambda+\mu+\eta} \to V_\mu \otimes V_{\lambda+\mu} \quad \text{such that} \quad \xi_\mu \otimes \xi_{\lambda+\mu+\eta} \mapsto \xi_\mu \otimes \xi_{\lambda+\mu}.
\]

Using these morphisms define an inverse limit \( U_q \mathfrak{g} \)-module

\[
M_\lambda = \lim_\mu \overline{V}_\mu \otimes V_{\lambda+\mu}.
\]

Denote by \( \text{tr}_{\mu, \lambda}^\mu \) the canonical map \( M_\lambda \to V_\mu \otimes V_{\lambda+\mu} \). The module \( M_\lambda \) is considered as a topological \( U_q \mathfrak{g} \)-module with a base of neighborhoods of zero formed by the kernels of the maps \( \text{tr}_{\mu, \lambda}^\mu \), while all modules in our category \( C_{\mu, \eta}^4 \) are considered with discrete topology. Then \( \text{Hom}_{U_q \mathfrak{g}}(M_\lambda, V) \) is the inductive limit of the spaces \( \text{Hom}_{U_q \mathfrak{g}}(\overline{V}_\mu \otimes V_{\lambda+\mu}, V) \). The vectors \( \xi_\mu \otimes \xi_{\lambda+\mu} \) define a topologically cyclic vector \( \Omega_\lambda \in M_\lambda \). For any finite dimensional admissible \( U_q \mathfrak{g} \)-module \( V \) the map

\[
\eta_V : \text{Hom}_{U_q \mathfrak{g}}(\oplus_\lambda M_\lambda, V) \to V, \quad \eta_V(f) = \sum_\lambda f(\Omega_\lambda),
\]

is an isomorphism, so the topological \( U_q \mathfrak{g} \)-module \( M = \oplus_\lambda M_\lambda \) represents the forgetful functor. Furthermore, the representation of \( U_q \mathfrak{g} \) in the endomorphism ring of the forgetful functor is implemented by the antihomomorphism \( \pi : U_q \mathfrak{g} \to \text{End}_{U_q \mathfrak{g}}(M) \) defined by \( \pi(E_i) \Omega_\lambda = E_i \Omega_{\lambda-\alpha_i}, \pi(F_i) \Omega_i = F_i \Omega_{\lambda+\alpha_i} \), and \( \pi(K_i) \Omega_\lambda = q_i^{\lambda(i)} \Omega_\lambda \). In other words, \( M \) is a \( U_q \mathfrak{g} \)-bimodule.

It was shown in [5, Section 4], see the arguments up to (but not including) Lemma 4.3 there, that condition (2) is exactly what is needed to define an action of any invariant cocycle \( \mathcal{E} \) satisfying (2) on the \( U_q \mathfrak{g} \)-bimodule \( M \). More precisely, we showed that there exist a character \( \chi \) of \( P/Q \), an invertible morphism \( \mathcal{E}_0 \) of \( M = \oplus_\lambda M_\lambda \) onto itself preserving the direct sum decomposition, and an invertible element \( c \) in the center of \( \mathcal{U}(G_q) \) such that

\[
\text{tr}_{\mu, \lambda}^\mu \mathcal{E}_0 = \chi(\mu)^{-1} \mathcal{E} \text{tr}_{\mu, \lambda}^\mu \quad \text{and} \quad \eta_V(f \mathcal{E}_0) = c \eta_V(f)
\]

for all \( \mu \in P_+, \lambda \in P \) such that \( \lambda + \mu \in P_+ \), and for all finite dimensional admissible \( U_q \mathfrak{g} \)-modules \( V \) and \( f \in \text{Hom}_{U_q \mathfrak{g}}(M_\lambda, V) \). We will show now that this is already enough to conclude that \( \mathcal{E} \) is, in fact, symmetric.

**Proof of Proposition 3.** We want to show that \( \mathcal{R}_h \mathcal{E} = \mathcal{E}_{21} \mathcal{R}_h \) for some \( h \) such that \( q = e^{\pi i h} \). We will prove a stronger statement: \( \sigma \mathcal{E} = \tilde{\sigma} \mathcal{E} \) for any braiding \( \sigma \) on \( C_{\mu, \eta}^4 \).

By (3), since \( \text{tr}_{\mu, \lambda}^\mu(\Omega_\lambda) = \xi_\mu \otimes \xi_{\lambda+\mu} \) for any \( \mu, \eta, \nu \in P_+ \) and \( f \in \text{Hom}_{U_q \mathfrak{g}}(\overline{V}_\mu \otimes V_{\nu}, V_{\eta}) \) we have

\[
\chi(\mu)^{-1} f \mathcal{E}(\xi_\mu \otimes \xi_{\eta}) = c(\nu) f(\xi_\mu \otimes \xi_{\eta}).
\]

As the vector \( \xi_\mu \otimes \xi_{\eta} \) is cyclic, this means that \( f \mathcal{E} = \chi(\mu)c(\nu) f \). Since this is true for all \( f \), we conclude that \( \mathcal{E} \) acts on the isotypic component of \( V_\mu \otimes V_{\eta} \) with highest weight \( \nu \) as multiplication by \( \chi(\mu)c(\nu) \). In other words, \( \mathcal{E} \) acts on the isotypic component of \( V_\mu \otimes V_{\eta} \) with highest weight \( \nu \) as multiplication by \( \chi(\mu)c(\nu) \). It follows that

\[
\mathcal{E} = \chi(\mu-\eta) \mathcal{E} \sigma \quad \text{on} \quad V_\mu \otimes V_{\eta},
\]
But by assumption (2) the element $E$ is the identity on the isotypic component of $V_\mu \otimes V_\eta$ with highest weight $\mu + \eta$, so by considering the above identity on this isotypic component we conclude that $\chi(\mu - \eta) = 1$. Thus $\chi$ is the trivial character and $\sigma E = E \sigma$. By [5, Corollary 4.4] we then get that $E = 1$. This completes the proof of Proposition 3 and hence of Theorem 1.

As our first application we will classify Drinfeld twists, relating the coproducts on $U_q\mathfrak{g}$ and $U\mathfrak{g}$, that do not necessarily respect braiding.

**Corollary 4.** Let $\varphi : \mathcal{U}(G_q) \to \mathcal{U}(G)$ be an isomorphism extending the canonical identifications of the centers of these algebras with the algebra of functions on $P_+$, and let $h$ be such that $q = e^{i\pi h}$. Suppose $F \in \mathcal{U}(G \times G)$ is an invertible element such that

(i) $(\varphi \otimes \varphi) \Delta_q = F \Delta \varphi(\cdot) F^{-1};$

(ii) the element $(t \otimes \hat{\Delta})(F^{-1})(1 \otimes F^{-1})(\hat{\Delta} \otimes t)(F)$ coincides with Drinfeld’s KZ-associator $\Phi_{KZ}(ht_{12}, ht_{23})$, where $t \in \mathfrak{g} \otimes \mathfrak{g}$ is the element defined by our fixed ad-invariant form on $\mathfrak{g}$.

Assume $F' \in \mathcal{U}(G \times G)$ is another element with the same properties. Then there exist a $\mathbb{T}$-valued 2-cocycle $c$ on $P/Q$ and an invertible central element $a \in \mathcal{U}(G)$ such that $F' = E_c F(a \otimes a) \Delta(a)^{-1}$.

**Proof.** The proof is similar to that of [5, Theorem 5.2]. Define $E = (\varphi^{-1} \otimes \varphi^{-1})(F') F^{-1} \in \mathcal{U}(G_q \times G_q)$. It is easy to check that $E$ is an invariant 2-cocycle on $\hat{G}_q$. By Theorem 1, $E = E_c (b \otimes b) \Delta_q(b)^{-1}$ for a 2-cocycle $c$ on $P/Q$ and a central element $b \in \mathcal{U}(G_q)$. Letting $a = \varphi(b)$, we obtain $F' = E_c (a \otimes a) (\varphi \otimes \varphi)(\Delta_q(b)^{-1}) F = E_c F(a \otimes a) \Delta(a)^{-1}$. □

Note that this corollary implies that the Dirac operator defined as in [4] is the same (for fixed $\varphi$) for any choice of a unitary element $F$ satisfying properties (i) and (ii). This extends [5, Theorem 6.1].

We now turn to our main application, the computation of the group of $\mathbb{C}$-linear monoidal autoequivalences of $\mathcal{C}_q(\mathfrak{g})$ identified up to monoidal natural isomorphisms.

Any automorphism $\alpha$ of the based root datum $\Psi_\mathfrak{g}$ of $\mathfrak{g}$ defines an automorphism of the Hopf algebra $U_q\mathfrak{g}$, hence an autoequivalence $\tilde{\alpha}$ of $\mathcal{C}_q(\mathfrak{g})$. On the other hand, for any 2-cocycle $c$ on $P/Q$ we can define an autoequivalence $\beta_c$ which acts trivially on objects and morphisms, while the tensor structure is given by the action of $E_c^{-1}$. It turns out that any autoequivalence of $\mathcal{C}_q(\mathfrak{g})$ is monoidally naturally isomorphic to a composition of two autoequivalences defined either by an automorphism of $\Psi_\mathfrak{g}$ or by a cocycle on $P/Q$.

**Theorem 5.** The group of $\mathbb{C}$-linear monoidal autoequivalences of the tensor category $\mathcal{C}_q(\mathfrak{g})$ is canonically isomorphic to $H^2(P/Q; \mathbb{T}) \rtimes \text{Aut}(\Psi_\mathfrak{g})$.

**Proof.** The proof is essentially identical to [7, Theorem 2.5]. Briefly, by a result of McMullen [3] any automorphism of the fusion ring of $\mathcal{C}_q(\mathfrak{g})$, mapping irreducibles into irreducibles, is implemented by an automorphism of $\Psi_\mathfrak{g}$. Hence, for any autoequivalence $\gamma$ of $\mathcal{C}_q(\mathfrak{g})$ there exists a unique automorphism $\alpha$ of $\Psi_\mathfrak{g}$ such that $\tilde{\alpha} \circ \gamma$ maps every object into an isomorphic one; that is, ignoring the tensor structure, $\tilde{\alpha} \circ \gamma$ is naturally isomorphic to the identity functor. Possible tensor structures on the identity functor are, in turn, described by invariant 2-cocycles on $\hat{G}_q$. □

We next consider $q = 1$ and extend the above results to compact connected groups.

The group $P/Q$ is canonically identified with the dual of the center $Z(G)$ of the group $G$, and so, for $q = 1$, Theorem 1 can be formulated as $H^2_G(\hat{G}; \mathbb{C}^*) \cong H^2(Z(\bar{G}); \mathbb{C}^*)$.

**Theorem 6.** For any compact connected separable group $G$ we have a canonical isomorphism $H^2_G(\hat{G}; \mathbb{C}^*) \cong H^2(Z(\bar{G}); \mathbb{C}^*)$.

**Proof.** For Lie groups the proof is essentially the same as above, with $P$ replaced by the weight lattice of a maximal torus of $G$. In the general case we have a homomorphism $H^2(Z(\bar{G}); \mathbb{C}^*) \to H^2_G(\hat{G}; \mathbb{C}^*)$...
obtained by considering \( \mathcal{U}(Z(G)) \) as a subring of \( \mathcal{U}(G) \). To construct the inverse homomorphism, for every quotient \( H \) of \( G \) which is a Lie group consider the composition

\[
H^2_c(\hat{G}; \mathbb{C}^*) \to H^2_H(\hat{H}; \mathbb{C}^*) \to H^2(\hat{Z}(H); \mathbb{C}^*),
\]

where the first homomorphism is defined using the quotient map \( \mathcal{U}(G) \to \mathcal{U}(H) \). The map \( Z(G) \to Z(H) \) is surjective (since this is true for Lie groups), so \( Z(G) \) is the inverse limit of the groups \( Z(H) \).

Then \( H^2(\hat{Z}(G); \mathbb{C}^*) \) is the inverse limit of the groups \( H^2(\hat{Z}(H); \mathbb{C}^*) \). Therefore the above maps \( H^2_c(\hat{G}; \mathbb{C}^*) \to H^2(\hat{Z}(H); \mathbb{C}^*) \) define a homomorphism \( H^2_c(\hat{G}; \mathbb{C}^*) \to H^2(\hat{Z}(G); \mathbb{C}^*) \). It is clearly a left inverse of the map \( H^2(\hat{Z}(G); \mathbb{C}^*) \to H^2_c(\hat{G}; \mathbb{C}^*) \), so it remains to show that it is injective.

In other words, we have to check that if \( \mathcal{E} \) is an invariant cocycle on \( \hat{G} \) such that its image in \( \mathcal{U}(H \times H) \) is a coboundary for every Lie group quotient \( H \) of \( G \), then \( \mathcal{E} \) itself is a coboundary. If \( \mathcal{E} \) were unitary, this could be easily shown by taking a weak operator limit point of cochains, see the proof of [7, Theorem 2.2], and would not require the separability of \( G \). In the non-unitary case we can argue as follows.

Since \( \hat{G} \) is separable, there exists a decreasing sequence of closed normal subgroups \( N_n \) of \( G \) such that \( \cap_{n \geq 1} N_n = \{ \} \) and the quotients \( H_n = G/N_n \) are Lie groups. Let \( \mathcal{E}_n \) be the image of \( \mathcal{E} \) in \( \mathcal{U}(H_n \times H_n) \). By assumption there exist invertible central elements \( c_n \in \mathcal{U}(H_n) \) such that \( \mathcal{E}_n = (c_n \otimes c_n) \hat{\Delta}(c_n)^{-1} \). For a fixed \( n \) consider the image \( a \) of \( c_{n+1} \) in \( \mathcal{U}(H_n) \). Then \( c_n a^{-1} \) is a central group-like element in \( \mathcal{U}(H_n) \). By [5, Theorem A.1] it is therefore defined by an element of the center of the complexification \( (H_n)_{\mathbb{C}} \) of \( H_n \). Since the homomorphism \( (H_{n+1})_{\mathbb{C}} \to (H_n)_{\mathbb{C}} \) is surjective, we conclude that there exists a central group-like element \( b \) in \( \mathcal{U}(H_{n+1}) \) such that its image in \( \mathcal{U}(H_n) \) is \( c_{n} a^{-1} \). Replacing \( c_{n+1} \) by \( c_n b \) we get an element such that \( \mathcal{E}_{n+1} = (c_n \otimes c_{n+1}) \hat{\Delta}(c_{n+1})^{-1} \) and the image of \( c_{n+1} \) in \( \mathcal{U}(H_n) \) is \( c_n \). Applying this procedure inductively we can therefore assume that the image of \( c_{n+1} \) in \( \mathcal{U}(H_n) \) is \( c_n \) for all \( n \geq 1 \). Then the elements \( c_n \) define a central element \( c \in \mathcal{U}(G) \) such that \( \mathcal{E} = (c \otimes c) \hat{\Delta}(c)^{-1} \).

In [7, Theorem 2.5] we computed the group of autoequivalences of the \( C^* \)-tensor category of finite dimensional unitary representations of \( G \). The above theorem and the same arguments as in the proof of Theorem 5 allow us to get a similar result ignoring the \( C^* \)-structure.

**Theorem 7.** For any compact connected separable group \( G \), the group of \( \mathbb{C} \)-linear monoidal autoequivalences of the category of finite dimensional representations of \( G \) is canonically isomorphic to \( H^2(\hat{Z}(G); \mathbb{C}^*) \rtimes \text{Out}(G) \).

**References**


Department of Mathematics, University of Oslo, P.O. Box 1053 Blindern, NO-0316 Oslo, Norway
E-mail address: sergeyn@math.uio.no

Faculty of Engineering, Oslo University College, P.O. Box 4 St. Olavs plass, NO-0130 Oslo, Norway
E-mail address: Lars.Tuset@iu.hio.no