Bloch-sphere approach to correlated noise in coupled qubits
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Abstract
In order to demonstrate a generalized Bloch-sphere approach for the treatment of noise in coupled qubits, we perform a case study of the decoherence of a system composed of two interacting qubits in a noisy environment. In particular, we investigate the effects of correlations in the noise acting on distinct qubits. Our treatment of the two-qubit system by use of the generalized Bloch vector leads to tractable analytic equations for the dynamics of the four-level Bloch vector and allows for the application of geometrical concepts from the well-known two-level Bloch sphere. We find that in the presence of correlated or anticorrelated noise, the rate of decoherence is very sensitive to the initial two-qubit state, as well as to the symmetry of the Hamiltonian. In the absence of symmetry in the Hamiltonian, correlations only weakly impact the decoherence rate.

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(Some figures may appear in colour only in the online journal)

1. Introduction

Loss of coherence, or quantum decoherence, remains the most important obstacle to overcome in order to build a quantum computer. Decoherence due to noise or to entanglement with uncontrollable degrees of freedom is responsible for the decay of coherent superpositions of qubit states. The result is the irreversible loss of the quantum information required for the operation of the device.

Leading candidates for the design of a quantum computer are superconductor-based qubits involving Josephson junctions [1], qubits based on nitrogen vacancies in diamond [2, 3], optically trapped ions [4] and quantum devices including semiconductor quantum
dots [5–8]. It is commonly assumed that the decoherence mainly originates from low-frequency noise with spectrum of $1/f$-type created by dynamic defects tunneling between two metastable states—the so-called two-level systems (TLSs). In particular, in Josephson charge qubits the TLSs are formed by traps in the amorphous material of the insulating barrier of the Josephson junction, and in the substrate required to fabricate the qubit.

While the effects of fluctuating TLSs on isolated qubits were extensively studied, see, e.g., references [9–15], or [16] for a review, it is clear that each of the tunneling charges present in the insulating substrate might influence several qubits fabricated on the same chip. As a result, the noise acting on different qubits may be correlated, which can manifest itself in modifying the decoherence comparing to the case of many uncorrelated sources of the same total intensity. Specifically in charge qubits, we might have a situation where trapped fluctuators are located in the space between the Cooper-pair-boxes belonging to two different qubits that may lead to correlated or anti-correlated noise acting on the two-qubit system.

Quantum oscillations in superconducting two-qubit systems were demonstrated already in 2003 [17], and have recently been extended to three qubit systems [18, 19]. The effect of correlated or partially correlated noise acting on two-qubit systems have been studied theoretically in [20–27]. However, due to the complexity of the Hilbert space of coupled qubits, the efforts have primarily resulted in numerical surveying of various situations. In addition, a general theory for passive error protection in decoherence-free subspaces (DFSs), subspaces of the multi-qubit Hilbert space which are immune to correlated noise, has been developed ([28–32] or see [33] for a review). The work in this paper does not attempt to extend the general work on DFSs, but rather provide an efficient way of analyzing the decoherence for initial states that are not necessarily in the DFSs.

In this work, we present a Bloch-vector treatment of the two-qubit decoherence problem, generalizing the well-known formulation for a single qubit. The 15-dimensional Bloch sphere of the two-qubit system, and the subspace of available positive definite density matrices, has a much richer mathematical structure than the corresponding construction for a single qubit [34–36]. Still the formalism is similar to the single-qubit problem, and we can take advantage of the familiar geometrical concepts developed for the treatment of two levels.

In order to illustrate the four-level Bloch-vector formulation, we consider a model problem of two coupled qubits, $Q_1$ and $Q_2$, subject to noise from two channels: $\xi_1(t)$ acting on $Q_1$ and $\xi_2(t)$ acting on $Q_2$. We are able to derive simple analytical formulas for the decoherence rate of two qubits as a function of the degree of correlation $\mathcal{S}_c$ of the noise sources. Our results clearly demonstrate a strong impact of correlations on the decoherence rate that is very different in distinct subspaces of the two-qubit Hilbert space. This effect is sensitive to the symmetry of the two-qubit Hamiltonian and in the absence of symmetry, correlations in the noise are not important for the decoherence rate.

The rest of the paper is structured as follows. In section 2, we define our model of two coupled qubits in a generalized environment, and discuss the qualitative aspects of our model using a simplified picture. The details of the initial dephasing of our system, assuming stationary fluctuators with unknown distribution of initial states, is studied in section 3. The long-time decoherence in the weak coupling limit is studied in section 4. Finally, our results are discussed in section 5.

2. Model

In order to demonstrate the generalized Bloch-sphere approach, we apply it to a model of two coupled qubits in a noisy environment, previously studied by Averin and Rabenstein [22]. Following [22], the Hamiltonian describing two coupled qubits (e.g., by electrostatic
interaction through finite coupling capacitance for charge qubits, or magnetic interaction for flux qubits, each at zero bias, can be written as \( \hat{H} = \hat{H}_0 + \hat{H}_1(t) \) where

\[
\hat{H}_0 = \sum_{j=1,2} \Delta_j \sigma^z_j + \nu \sigma^x_j \sigma^z_j. \tag{1}
\]

\[
\hat{H}_1(t) = \sum_{j=1,2} \xi_j(t) \sigma_j^z. \tag{2}
\]

Here \( \xi_j(t), j \in \{1, 2\} \), are random forces acting on the qubits, \( Q_1 \) and \( Q_2 \), respectively. The random forces might originate from the same source, from different sources or a combination, resulting in a different degree of correlation in the signal. The degree of correlation is given by the expression

\[
S_c = \frac{\langle (\xi_1(t) - \bar{\xi}_1)(\xi_2(t) - \bar{\xi}_2) \rangle}{\sqrt{\langle (\xi_1(t) - \bar{\xi}_1)^2 \rangle \langle (\xi_2(t) - \bar{\xi}_2)^2 \rangle}}, \tag{3}
\]

where the brackets indicate ensemble averaging over initial conditions and realizations of the noise process, and \( \bar{\xi}_i \equiv \langle \xi_i(t) \rangle \). In the rest of this paper, we assume that the noise processes \( \xi_1(t) \) and \( \xi_2(t) \) have vanishing first moments and finite second moments.

The eigenstates and eigenvalues of the Hamiltonian (1) are as follows \[22\].

- \( |\psi_1\rangle = (\eta_+ + \eta_-)|0011\rangle_+ + (\eta_+ - \eta_-)|0110\rangle_+ \)
- \( |\psi_2\rangle = (\eta_+ - \eta_-)|0011\rangle_+ + (\eta_+ + \eta_-)|0110\rangle_+ \)
- \( |\psi_3\rangle = (\gamma_+ + \gamma_-)|0011\rangle_- + (\gamma_+ - \gamma_-)|0110\rangle_+ \)
- \( |\psi_4\rangle = (\gamma_+ - \gamma_-)|0011\rangle_- + (\gamma_+ + \gamma_-)|0110\rangle_+ \)

where

\[
|kl\rangle_\pm = (|kl\rangle \pm |jl\rangle)/\sqrt{2}, \quad \Delta_\pm = \Delta_1 \pm \Delta_2, \quad \Omega_\pm = \sqrt{\Delta^2_\pm + \nu^2},
\]

\[
\gamma_\pm = \sqrt{(1 \pm \Delta_-/\Omega_-)/2}, \quad \eta_\pm = \sqrt{(1 \pm \Delta_+/\Omega_+)/2}.
\]

The states \( |kl\rangle \), \( k, l \in \{0, 1\} \) are the eigenstates of the operator \( \sigma^l_z \otimes \sigma^z_j \). Having defined our model and its parameters, we present our procedure for the study of four-level systems by use of the Bloch-vector construction.

### 2.1. Coordinate transformations

Noise-induced decoherence leads to deviation from coherent oscillations of the states due to the Hamiltonian \( \hat{H}_0 \). In order to study the effect of noise, it is useful to first transform our system to the reference frame where the density matrix is stationary in the absence of noise. This procedure allows us to separate decoherence from rapid quantum oscillations. The transformation is just the standard transform to the interaction picture:

\[
\rho'(t) = U(t) \hat{\rho}(t) U(t), \quad V'(t) = U(t) \hat{H}_1(t) U(t), \tag{5}
\]

where \( U(t) = e^{-i\hat{H}_0 t/\hbar} \), and \( \hat{\rho}(t) \) is the density matrix of the two-qubit system in the Schrödinger picture. The explicit expression for \( V'(t) \) is given by equation (A.1) in the appendix.

We proceed by specifying the initial state, \( \rho_0' \), of our two-qubit system. Next, we perform a similarity transform

\[
\rho = S^{-1} \rho' S, \quad V(t) = S^{-1} V'(t) S, \tag{6}
\]
where $S$ is the eigenvector matrix of the initial state $\rho_0'$. The motivation behind the similarity transform is the following treatment in terms of coherence vectors on the generalized Bloch ball.

For a two-qubit system, the set $\mathbb{E}_4$ of positive definite density matrices of trace one is a convex subset of the four-level Bloch sphere in $\mathbb{R}^{15}$, contrary to the two-dimensional case where the set of density matrices are equivalent to the set defined by the Bloch sphere. Specifically, the set of density matrices corresponding to pure states is only a six-dimensional subset of the surface of the four-level Bloch sphere. For more in-depth analysis of the geometry of Bloch vectors in the two-qubit system see [37]. The set $\mathbb{E}_4$ can be parametrized by use of the generators $\lambda_i$ of $SU_4$, which are tabulated in [37]. Using this parametrization, we write

$$\rho = \frac{1}{4} + \frac{1}{2} \sum_{i=1}^{15} m_i(t) \lambda_i, \quad V(t) = \frac{1}{2} \sum_{i=1}^{15} \beta_i(t) \lambda_i.$$  

(7)

The motivation behind the similarity transform, (6), is to obtain the special initial state:

$$\rho_0 = S^{-1} \rho_0' S = \frac{1}{4} - \frac{1}{2} \sqrt{\frac{3}{2}} \lambda_{15} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$  

(8)

where all the components, $m_i$, of the Bloch-vector expansion vanish, except for the component $m_{15} = -\sqrt{3}/2$. The factor $\sqrt{3}/2$ originates from the normalization condition for this particular parametrization of the density matrix $\rho$, given by $\sum_{i=1}^{15} |m_i|^2 = 3/2$. The transform to the coordinate frame given by (8) greatly simplifies the further analysis of the equations of motion for $\rho$. This is the geometric equivalent to the conventional choice of the coordinate system for the four-level Bloch sphere where the initial state is parallel to the $z$-axis. Note that the transformation to the form (8) is only possible if the initial state is a pure state.

2.2. Equations of motion on the Bloch sphere

The time evolution of the density matrix in the interaction picture is given by the standard von Neumann equation:

$$\frac{d\rho(t)}{dt} = -i[V(t), \rho(t)].$$  

(9)

By use of the Bloch-vector parametrization of $\rho(t)$ and $V(t)$, given by (7), we are now able to calculate the time evolution of $\rho(t)$ in a way that simplifies the further analysis of the decoherence. The time evolution of the coefficients $m(t)$ of (7) can be derived from the commutator relation

$$[\lambda_i, \lambda_j] = \sum_{k=1}^{15} 2i f_{ijk} \lambda_k,$$  

(10)

where $f_{ijk}$ are the structure factors of $SU_4$ (a table can be found in [38]). We obtain the following equations for the components:

$$\dot{m}_i(t) = \sum_{j,k=1}^{15} f_{ijk} \beta_j(t) m_k(t).$$  

(11)

Due to the similarity transform by $S$, the initial state is always specified by $m_i = 0$ for $i < 15$ and $m_{15} = -\sqrt{3}/2$. With this choice, we can simplify (11) by writing $m_i(t) = m_i(0) + \alpha_i(t)$ and

$$\dot{m}_i(t) = \sum_{j=1}^{15} f_{15j} \beta_j(t) [m_{15}(0) + \alpha_{15}(t)] + \sum_{j=1}^{14} \sum_{k=1}^{14} f_{ijk} \beta_j(t) \alpha_k(t).$$

4
for $k \neq 15$. To the first order in the parameter $\xi(t)$, we obtain the approximation

$$\dot{\alpha}_i(t) \approx \sum_{j=1}^{15} f_{ij} \beta_j(t) m_{15}(0).$$

(12)

This approximation is valid for sufficiently short times, such that all components of the Bloch-vector tangent to the Bloch sphere due to noise are small compared to the component along the initial state.

For each individual realization of the stochastic process $\xi_j(t)$, the two-qubit system evolves as a pure state on the surface of its Bloch sphere. Since the set of density matrices corresponding to pure states is only a six-dimensional subset of the 14-dimensional four-level Bloch sphere, there can to first order only be six nonvanishing components in the equations of motion, (12). Averaged over the noise process $\xi_j(t)$, we obtain the diffusive dynamics of the Bloch vector in this six-dimensional tangent plane. The probability distribution for the Bloch vector will grow in width in each of the six directions, and in general the distribution width might grow faster in some directions. Note that to first order the components normal to the surface of the Bloch sphere are conserved.

The diffusive dynamics of the Bloch vector obtained after averaging over the noise process $\xi_j(t)$ leads to a decay of the purity of the two-qubit system. As long as we consider times sufficiently short so that the uncertainty of the Bloch vector due to noise is small relative to the size of the Bloch sphere, we can calculate the length of the Bloch vector given by the component $m_{15}(t) = -\sqrt{3}/2 + \alpha_{15}(t)$ along the $\lambda_{15}$-axis, by use of the normalization condition

$$\alpha_{15}(t) = \sqrt{\frac{3}{2} - \frac{3}{2} - \sum_{i=1}^{14} \alpha_i^2(t)} \approx \sqrt{\frac{1}{6} \sum_{i=1}^{14} \alpha_i^2(t)}.$$  

(13)

The length of the Bloch vector is a measure of the purity of the density matrix, see e.g., [39].

Averaged over initial conditions and realizations of the noise process, we obtain

$$\langle \alpha_{15}(t) \rangle \approx \sqrt{\frac{1}{6} \sum_{i=1}^{14} \langle \alpha_i^2(t) \rangle},$$

where the mean square of the components $\alpha_i$ is defined by

$$\langle \alpha_i^2(t) \rangle = \int_0^t \int_0^t dt_1 dt_2 \dot{\alpha}_i(t_1) \dot{\alpha}_i(t_2).$$

(15)

2.3. Simple picture of decoherence of a two-qubit system by a single TLS

Before we move on to the calculations, it may be instructive to look at a simple model that still contains essential features regarding the sensitivity of our two-qubit system to external noise. Assume two (coupled) qubits, $Q_1$ and $Q_2$, both again coupled to a single degree of freedom $S_E$, e.g., a TLS or an impurity level in their common environment. The qubits, $Q_1$ and $Q_2$, couple to this environmental system $S_E$ through two channels $\xi_1$ and $\xi_2$ as illustrated in figures 1(a) and (b) and described by the interaction Hamiltonian

$$H_I = (\xi_1 \sigma_1^1 + \xi_2 \sigma_2^1) \hat{E} = \hat{X} \hat{E},$$

(16)

where $\hat{E}$ is an operator acting in the Hilbert space of $S_E$.

The decoherence of the qubits, $Q_1$ and $Q_2$, is given by the transfer of information about the qubit state to the environment [40]. Assume that the two-qubit system is prepared in a
be accounted for in the design process of many qubit circuits. The nature of the eventual correlation in the noise source is extremely important and needs to be accounted for in the design process of many qubit circuits.

Figure 1. (a) Two qubits, $Q_1$ and $Q_2$, coupled to a TLS in the substrate between the two qubits, where the charge can tunnel between two sites. (b) The qubits are coupled to a charged impurity through its image charge on the metallic gate, the charge can tunnel between the gate and the impurity. The coupling strengths are given by $\xi_1$ and $\xi_2$. The configuration demonstrated in (a) gives origin to anticorrelated noise, while configuration (b) gives origin to correlated noise.

The qubits are coupled to a charged impurity $\xi$ and the metallic gate gives origin to a correlated noise on the two qubits. If each interaction can be illustrated schematically as

$$\xi Q_1 \text{ couple with the same strength to the gate, }$$

the matrix element $\langle \hat{X} \rangle$ vanishes for states within the subspace spanned by the states $|\hat{X}\rangle$.

The configuration demonstrated in (a) gives origin to anticorrelated noise, while configuration (b) gives origin to correlated noise. The same is true for states of the form $|\alpha\rangle = |0\rangle$, since the interaction 16 vanishes for states within the subspace spanned by the states $|\hat{X}\rangle$.

Neglecting possible internal couplings in the two-qubit system, the dynamics due to the interaction can be illustrated schematically as

$$|\psi_{Q_1+Q_2}\rangle = \sum_i c(i)|i\rangle,$$

where $|\psi_{Q_1+Q_2}\rangle$ is the state of $S_E$ conditioned upon that the two-qubit system was initially in the state $|\phi_E(t)\rangle$. The dynamics of the state of $S_E$, $|\phi_E(t)\rangle$, will then in general depend on the state of the two-qubit system due to the interaction 16. Neglecting possible internal couplings in the two-qubit system, the dynamics due to the interaction can be illustrated schematically as

$$|\psi_{Q_1+Q_2}\rangle |\phi_E(0)\rangle = \sum_i c(i)|i\rangle |\phi_E(0)\rangle \rightarrow \sum_i c(i)|i\rangle |\phi_E(t)\rangle,$$

where $|\phi_E(t)\rangle$ is the state of $S_E$ conditioned upon that the two-qubit system was initially in the state $|i\rangle$. The decay of decoherence in the two-qubit system is given by the overlap matrix elements $|\langle \phi_E(t) | \phi_E(t) \rangle |$, obtained by tracing over the degrees of freedom of the environment $S_E$.

In the configuration described in figure 1(b), tunneling of electric charges between the impurity and the metallic gate gives origin to a correlated noise on the two qubits. If each qubit couple with the same strength to the gate, $\xi_1 = \xi_2$, and the two qubits are prepared in the state $|0\rangle = |3\rangle$, then the expectation value of the operator $\hat{X}$ vanishes, $\langle 3|\hat{X}|3\rangle = 0$. The same is true for states of the form $\alpha|0\rangle + \beta|01\rangle$ for all values of $\alpha$ and $\beta$. Since the interaction 16 vanishes for states within the subspace spanned by the states $|3\rangle$ and $|0\rangle = |01\rangle$, the matrix element $|\langle \phi_E(t) | \phi_E(t) \rangle |$ is conserved in time and no decay of coherence takes place.

Therefore, in this model, the states $|0\rangle$ and $|10\rangle$ span a DFS in the presence of correlated noise [28, 29, 41]. However, this requires that the coupling parameters are exactly identical $\xi_1 = \xi_2$ and that there is no internal dynamics in the two qubit system. We will, in the following sections, investigate the decoherence of the two qubits system if these requirements are relaxed.

In the opposite case of anti-correlated noise, where $\xi_1 = -\xi_2 = 1$, which could in fact be realized by a charge trap placed in the substrate between two charge qubits, e.g. in a configuration shown in figure 1(a), the expectation value of the operator coupling to the environment $\hat{X}$ is $\langle 10|\hat{X}|10\rangle = -2$, while $\langle 01|\hat{X}|01\rangle = 2$. However, we easily see that $\langle 00|\hat{X}|00\rangle = \langle 11|\hat{X}|11\rangle = 0$. The subspace spanned by the states $|00\rangle$ and $|11\rangle$ is therefore the DFS in this configuration, while we expect the subspace spanned by $|01\rangle$ and $|10\rangle$, which was a DFS in the presence of correlated noise, to be very susceptible to anti-correlated noise. Thus, the nature of the eventual correlation in the noise source is extremely important and needs to be accounted for in the design process of many qubit circuits.
3. Initial decoherence in the stationary path approximation

In this section, we will study the initial decoherence of our two-qubit system focusing on the sensitivity of the decoherence to correlations in the noise. Previously it has been found that the sensitivity to noise in two-qubit systems is reduced if the noise is correlated, and the two-qubit system is prepared to be in a state within the SWAP subspace, spanned by the states $|10\rangle$, $|01\rangle$ [22]. In order to check this, we calculate the decoherence rate from four different initial states, two of which belong to the SWAP subspace, and two belonging to the subspace spanned by $|00\rangle$ and $|11\rangle$.

As an example to illustrate the method, we will calculate the initial decoherence rate when the two-qubit system is initially prepared in the singlet state $|\psi_0\rangle = |10\rangle - |01\rangle$, where we have omitted the normalization factor. Proceeding as described in section 2, we transform to the coordinate frame where the initial Bloch vector of the two-qubit system lies on the south pole of the Bloch sphere, $m_i = -\sqrt{3}/2$ for $i = 15$ and 0 otherwise. The transformation matrix is for this initial state given by

$$S = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & \sqrt{2} & 0 \end{pmatrix}, \quad S^{-1} = S^{T}. \quad (18)$$

We can then compute the noise matrix

$$V = S^{T}V'S = \frac{1}{2} \sum_{i=1}^{15} \beta_i \lambda_i$$

and use (12) to find the differential equations for the components $\alpha_i(t)$ parameterizing the density matrix $\rho(t)$. For the initial state $|\psi_0\rangle = |10\rangle - |01\rangle$ and the Hamiltonian given by (2), the differential equations for $\alpha_i$ are obtained by tedious, but straightforward algebra and are given by equation (A.2). We note that there are only six nonvanishing components $\alpha_i$.

Since the general formulas given by (A.2) are rather complicated, it is instructive to investigate the simplified case where the two-qubit system is at the co-resonant point $\Delta_1 = \Delta_2 = \Delta$. The problem simplifies further if we assume that the inter-qubit coupling is much less than the single qubit tunneling element, $\nu \ll \Delta$. With this symmetric Hamiltonian, we obtain a simplified set of parameters $\gamma_s = 1/\sqrt{2}$, $\eta_+ = 1$ and $\eta_- = 0$.

Furthermore, if we are interested only in the initial decoherence rate, we may assume the stationary path approximation, where the noise sources are stationary and decoherence is solely due to the different realizations of initial values of the noise sources at the start of each experiment (e.g., a statistical distribution of the gate charge at the start of each run of the measurement protocol [25]). After averaging over many experiments with different initial conditions for the noise sources, we obtain an uncertainty in the precise value of the Bloch vector of our two-qubit system. Inserting the expressions for the derivatives of the component $\alpha_i$ given in (A.2) into (15), we obtain the following expressions for the mean square deviations along each component due to noise:

$$\begin{align*}
\{\alpha^2_{13}(t)\} &= \left\{\frac{1}{8}\right\}(\delta \xi)^2 [c_{-}(t) + c_{+}(t)]^2, \\
\{\alpha^2_{10}(t)\} &= \left\{\frac{1}{8}\right\}(\delta \xi)^2 [s_{-}(t) - s_{+}(t)]^2, \\
\{\alpha^2_{11}(t)\} &= \left\{\frac{1}{4}\right\}(\delta \xi)^2 [c_{-}(t) - c_{+}(t)]^2, \\
\{\alpha^2_{12}(t)\} &= \left\{\frac{1}{4}\right\}(\delta \xi)^2 [s_{-}(t) + s_{+}(t)]^2.
\end{align*} \quad (19)$$
fluctuations along the six components, (19), we obtain the following simple formula
where \( \langle \delta \xi \rangle_\pm \equiv \xi_1(0) \mp \xi_2(0) \), \( \omega_\pm = \Omega_+ \pm \Omega_- \), \( c_\pm(t) = \omega_\pm^{-1}(1 - \cos \omega_\pm t) \), \( s_\pm(t) = \omega_\pm^{-1} \sin \omega_\pm t \). The expression is valid for times shorter than the correlation time of the noise sources, \( t \ll \tau_c \).

From the normalization condition, (14), and the expressions for the mean square fluctuations along the six components, (19), we obtain the following simple formula determining the initial decoherence of the two-qubit system:

\[
\langle \alpha_{15}(t) \rangle = \frac{\langle \delta \xi \rangle_\pm^2}{\sqrt{6}} f(t), \quad f(t) = \frac{c_-(t)}{\omega_-} + \frac{c_+(t)}{\omega_+}.
\]

As expected, we find that at the symmetric co-resonance point the coherence of the initial singlet state \(|10\rangle - |01\rangle\) is very sensitive to the degree of correlation in the noise sources \( \xi_1 \) and \( \xi_2 \). If the noise sources are completely correlated there is no decoherence.

The expressions for \( \langle \alpha_{15}(t) \rangle \) for different initial states are summarized in the table 1, where

\[
g(t) = (\omega_+ \omega_-)^{-1}(1 - \cos \omega_- t - \cos \omega_+ t + \cos 2\Omega_+ t).
\]

In the absence of correlation,

\[
\langle (\delta \xi)^2 \rangle = \langle (\delta \xi)^2 \rangle_+ = \langle (\xi_1(0))^2 \rangle + \langle (\xi_2(0))^2 \rangle,
\]

for the fully correlated noise \( \langle (\delta \xi)^2 \rangle_\pm = 0 \) while for the anti-correlated noise \( \langle (\delta \xi)^2 \rangle_\pm = 0 \).

We find that the effect of correlated noise is strongly dependent on the initial state of the qubit. The singlet state \(|\psi\rangle = |01\rangle - |10\rangle\) is a DFS in the presence of completely correlated noise, while the state \(|01\rangle + |10\rangle\) is only partly protected from correlated noise. The difference is due to the fact that the singlet state is an eigenstate of \( H_0 \), while the initial state \(|01\rangle + |10\rangle\) is not an eigenstate of \( H_0 \) and will therefore due to the time evolution obtain components in subspaces of the Hilbert space that are sensitive to correlated noise. For a general initial state, \( |\psi_0\rangle \), the component in the subspace of the Hilbert space spanned by the singlet state, \( \langle (|01\rangle - |10\rangle)|\psi_0\rangle \), will not decay in time. For this state, we will find an initial decay of the coherence, and then the persisting coherence of amplitude given by the overlap element. The initial states \(|\psi_0\rangle = |00\rangle - |11\rangle\) and \(|\psi_0\rangle = |00\rangle + |11\rangle\) are decoherence-free and weakly protected, respectively, to anticoherent noise. It is important to note that we assumed co-resonance. If the Hamiltonian, (1), is not perfectly symmetric, the symmetric states will no longer be eigenstates of \( H_0 \) and will therefore be less sensitive to correlated/anticoherent noise.

To compare with the above result, if the two-qubit Hamiltonian commutes with the operator representing coupling to the noise source, i.e. for the conventional Hamiltonian \([41]\)

\[
H = \sum_{j=1,2} \Delta \sigma_j^z + v \sigma_z^i \sigma_z^j + \sum_{j=1,2} \xi_j(t) \sigma_j^i,
\]

then the subspace spanned by the set of states \(|01\rangle, |10\rangle\) is a two-dimensional DFS with respect to a completely correlated noise, but vulnerable to an anticoherent noise, and vice versa for the set of states spanned by \(|00\rangle, |11\rangle\).
4. Decoherence for intermediate times in the weak coupling limit

In the previous section, we studied the initial dynamics of the two-qubit system, due to noise from the channels $\xi_1(t)$ and $\xi_2(t)$, by use of the stationary path approximation. In this section, we will consider the decoherence for times longer than the correlation time $\tau_c$ of the noise sources.

We proceed by use of the four-level Bloch-vector method applied in the same manner as presented earlier, by (11), (14) and (15). The decoherence is restricted to a six-dimensional subspace of the 14-dimensional Bloch sphere as long as we observe the qubits for times $t/|\langle \delta \xi \rangle_{\pm}| \ll 1$. We are required to solve the integrals, of the form given by (15), determining the mean square component in each of the six directions spanning the subspace.

To compare with the results of [22, 25], we start with the initial state $|\psi_0\rangle = |01\rangle$. After transforming to the coordinate frame where $\rho_0 = 1/4 - \sqrt{3/8} \lambda_{15}$, we obtain the following equations of motion:

\[
\begin{align*}
\dot{\alpha}_9(t) &= \xi_1(t) (\xi_1 \sin \omega \tau + \xi_2 \sin \omega \tau),
\dot{\alpha}_{10}(t) &= \xi_1(t) (\mu_{1+} + \mu_{2-}) (\cos \omega \tau - \cos \omega \tau),
\dot{\alpha}_{11}(t) &= [\xi_1(t) - \xi_2(t)] [-(\mu_{2+} + \mu_{1+}) \sin \omega \tau + (\mu_{2+} - \mu_{1+}) \sin \omega \tau],
\dot{\alpha}_{12}(t) &= 0,
\dot{\alpha}_{13}(t) &= \xi_2(t) (-\xi_1 \sin \omega \tau - \xi_1 \sin \omega \tau),
\dot{\alpha}_{14}(t) &= \xi_2(t) (\mu_{1-} - \mu_{2-}) (-\cos \omega \tau + \cos \omega \tau),
\end{align*}
\]

where we have introduced the notation

\[
\begin{align*}
\xi_1 &= \eta_1^2 \gamma_1^2 - \eta_2^2 \gamma_1^2, & \xi_2 &= \eta_1^2 \gamma_2^2 - \eta_2^2 \gamma_2^2,
\mu_{i\pm} &= \eta_{i+} \eta_{i-} (y_1^2 \pm y_2^2), & \mu_{2\pm} &= \eta_{2+} \eta_{2-} (y_2^2 \pm y_2^2).
\end{align*}
\]

When computing the mean square of each component $\langle \alpha_i^2(t) \rangle$, by use of (15), we assume time translation invariance, which allows us to make the standard transformation [42]

\[
\langle \alpha_i^2(t) \rangle = \int_0^t dt \int_{-\infty}^\infty d\tau \dot{\alpha}_i \left(-\frac{\tau}{2} + T\right) \dot{\alpha}_i \left(\frac{T}{2} + T\right),
\]

where $\tau = t_2 - t_1$ and $T = (t_1 + t_2)/2$. We have also assumed observation times to be longer than the correlation time of the noise sources, $t \gg \tau_c$, which allows us to extend the integral over $\tau$ to infinity. In addition, we have to assume that the coupling to the environment is sufficiently weak to ensure that the six components of the initial ‘hypersurface’ where dynamics take place, $\langle \alpha_i^2(t) \rangle$ for $i = 9 - 14$, are small compared to $\langle m_{15}(t) \rangle$.

Inserting (22) into (23), we obtain terms in the integrand proportional to $\cos(\Omega_+ \pm \Omega_-) \tau$, $\cos 2(\Omega_+ \pm \Omega_-) T$ and $\cos(\pm \Omega_+ \tau \pm 2 \Omega_- T)$. If we assume that $t \gg 1/\omega_{1+}$, the terms that oscillate as a function of $T$ will be negligible compared to the slowly varying ones. By this approximation, the expressions for the mean square of the six components, from the initial state $|\psi_0\rangle = |01\rangle$, reduce to the following:

\[
\langle \alpha_9^2(t) \rangle \approx \int_0^T dt \int_{-\infty}^\infty d\tau \langle \xi_1(\tau) \xi_1(0) \rangle (\xi_1^2 \cos \omega \tau + \xi_2^2 \cos \omega \tau).
\]

Here we omitted the explicit expressions for the other five components. We note that the integral grows linearly in time.
By use of the normalization condition, (14), we can make the following approximation for \( t |(\delta \xi)_\pm | \ll 1 \),

\[
\langle a_1(0) \rangle \approx (1/\sqrt{6}) \sum_{i=1}^{15} \langle a_i^2(0) \rangle \approx \Gamma \sqrt{6},
\]

where we introduced the decoherence rate \( \Gamma \) characterizing the decay of the Bloch vector given by \( \langle a_1(0) \rangle \) to lowest order in \( t \). With the initial condition \( |\psi_0 \rangle = |01 \rangle \), we obtain the following expression for \( \Gamma \):

\[
\Gamma = \frac{1}{\sqrt{6}} \sum_{\pm} \{(S_{12}(0) + S_{22}(0))a_\pm - 2S_{12}(0)(\mu_2 \pm \mu_1)^2\},
\]

\[
a_+ = \xi_1^2 + (\mu_2 + \mu_1)^2 + (\mu_2 - \mu_1)^2,
\]

\[
a_- = \xi_2^2 + (\mu_2 - \mu_1)^2 + (\mu_2 - \mu_1)^2.
\]

(26)

Here we have introduced the spectral density

\[
S_{ij}(\omega) = \int_{-\infty}^{\infty} dt \langle \xi_i(\tau) \xi_j(0) \rangle \cos(\omega \tau)
\]

(27)

for the Fourier transform of the (cross) correlation spectra from channels \( i, j \) at frequency \( \omega \). The expression given by (26) is valid for max\{1/\( \omega_+ \), \( \tau_\pm \)\} \( \ll t \ll 1/|(\delta \xi)_\pm | \). We note that the noise at the two frequencies, \( \Omega_+ \pm \Omega_- \), is the only relevant component of the noise in the weak coupling limit. This result agrees with the findings of [22]. It is, however, important to stress that the above result is to first order in the coupling strength \( \xi_i \). For a single qubit, with energy splitting \( \Delta \), subject to transverse noise, it is known [43, 44] that we will obtain a contribution to the dephasing from the noise at zero frequency:

\[
\langle a_j^2(t) \rangle \propto \xi^2 S(\omega) + \xi^4 S(0) + \cdots,
\]

where \( \omega = \Delta \) is the qubit frequency at zero bias, and the zero frequency contribution is due to the net shift in the precession rate due to the noise, \( \omega = \sqrt{\Delta^2 + \xi^2} = \Delta + \xi^2/2\Delta^2 \). This effect will also be present in the multi-qubit problem.

In the work by Averin and Rabenstein [22], it was also reported that the decoherence rate was more sensitive to correlation than the two-qubit system was prepared in the state \( |\psi_0 \rangle = |01 \rangle \) than in the state \( |\psi_0 \rangle = |00 \rangle \), see figures 2 and 3 in that paper. Our analysis for intermediate times shows, however, that these two states are initially equally sensitive to correlations. If the noise sources have the same magnitude, \( |S_{11}(\omega)| = |S_{22}(\omega)| \), we can express the decoherence rate as a function of the correlation parameter \( S_C \) given by (3). The decoherence rate \( \Gamma \) as a function of the correlation \( S_C \) is plotted in figure 2, where we find that the coherence time of the state \( |\psi_0 \rangle = |01 \rangle \) increases as a function of the degree of correlation \( S_C \), while the coherence time of the initial state \( |\psi_0 \rangle = |00 \rangle \) decreases as a function of \( S_C \). The sensitivity to correlations \( d\Gamma/dS_C \) has the same magnitude, but opposite sign for these two initial states. Further analysis shows that our results are fully consistent with those of Averin, and the 'seeming' discrepancy has the following explanation. In [22], the oscillations of the probability \( p_{1,2}(t) \) to find \( Q_1 \) and \( Q_2 \) in the excited state at time \( t \) are plotted. Our results for the decoherence rate agree at short times (this has been checked numerically in the regime \( t |(\delta \xi)_\pm | \ll 1 \) where our approximation is valid), but thereafter in [22] slowly damped oscillations of amplitude \( \sim 0.7 \) for the initial state \( |01 \rangle \) were found. This behavior is due to the fact that the state \( |01 \rangle \) is not an eigenstate of \( H_0 \). For the Hamiltonian (1) at the co-resonance point, the eigenstates are the usual singlet and triplet states. The singlet state is protected against correlated noise, while the triplet is not. Decomposed in these two subspaces,
Figure 2. Sensitivity of the decoherence rate $\Gamma$ to correlations between $\xi_1(t)$ and $\xi_2(t)$ and to symmetry in the Hamiltonian $H_0$. We plot the total decoherence rate $\Gamma$ against the correlation coefficient $S_C$ for four different initial states, $|\psi_0\rangle = |01\rangle - |10\rangle$ (red), $|\psi_0\rangle = |01\rangle$ (blue), $|\psi_0\rangle = |00\rangle$ (black) and $|\psi_0\rangle = |00\rangle - |11\rangle$ (green). In the left figure, we are at the co-resonance point, where $\Delta_1 = \Delta_2$, while in the right figure $\Delta_1 = 0.7\Delta_2$. The coupling parameter is $\nu = 0.1$.

the initial state $|01\rangle$ has one component in each subspace and both components have the same magnitude $\sim 1/\sqrt{2} \approx 0.7$. In the presence of a correlated noise, the component in the triplet will therefore decay rapidly, while we will find persisting oscillations of amplitude $\sim 0.7$ due to the component in the singlet subspace. The Hamiltonian considered in [22] is not fully symmetric, but the picture is still valid. One component decays at a fast rate, while the component in the approximate ‘singlet’-subspace decays much slower. Interestingly, it was reported in [45], that concurrence oscillations from the initial state $|01\rangle$ lasts even longer than expected from its components in the DFS and its compliment, due to an interference effect between the components that decreases the decoherence. From a practical point of view, however, we can focus on the initial decoherence rate. If the purity of the state has decayed more than a few per cent, then quantum computation by error correction is practically impossible. Furthermore, our results show that the initial state is even more important than it was previously reported [22, 25]. In the presence of completely correlated noise, $S_C = 1$, the singlet state $|\psi_0\rangle = |01\rangle - |10\rangle$ is decoherence-free. However, it is important to mention that this is the only DFS. There is no two-dimensional DFS for our Hamiltonian $H_0$. The state $|\psi_0\rangle = |01\rangle - |10\rangle$ is decoherence-free if and only if we are at the co-resonance point, as shown in figure 2. If we move away from this point, the sensitivity to correlations is reduced. This behavior is controlled by the symmetry of the Hamiltonian. For our Hamiltonian, (1), the relevant parameters are $\kappa_1 = \Delta_-/\Delta_+$ and $\kappa_2 = \Delta_+/\nu$. For either $\kappa_1 \ll 1$ or $\kappa_2 \ll 1$, the Hamiltonian has a high degree of symmetry and there exist states that are protected against correlated or anticorrelated noise. If the Hamiltonian has low degree of symmetry, then initial states that are initially symmetric or antisymmetric with respect to the noise source, i.e. the state $|\psi_0\rangle = |01\rangle - |10\rangle$, will decompose in the non-symmetric eigenstates of $H_0$ and quickly lose its symmetry by the time evolution induced by $H_0$. In this case, the sensitivity to correlations in the noise source will be very weak.

5. Discussion

Our Bloch-vector treatment of the two-qubit decoherence problems shows that the two qubit problem can be separated in two regimes. In the symmetric regime, the symmetry of the initial state $|\psi_0\rangle$ is conserved by the Hamiltonian in the absence of noise. In this regime, the system is very sensitive to correlations or anticorrelations in the noise sources $\xi_1(t)$ and $\xi_2(t)$
acting on qubits $Q_1$ and $Q_2$, respectively. This sensitivity, which is consistent with the general DFS picture, is dependent on the initial state and on the parameters of the problem in a more complex way than it was reported in [22, 25], where it was assumed that states in the SWAP subspace spanned by the states $|01\rangle$ and $|10\rangle$ should be less influenced by a correlated noise. At the same time, the decoherence of states outside this subspace should only be weakly influenced by correlations. This thought is based on the concept of DFSs defined as the states where the dissipative part of the Markovian master equation is zero [32], but does not fully take into account the intrinsic dynamics of the system. A general theory for dynamically stable DFSs was developed in [32]. We find that it is both the symmetry of the initial state and how much this state overlaps with an eigenstate of the Hamiltonian in the absence of noise that determines the rate of decoherence. For the Hamiltonian given by (2), at the co-resonance point, all the three initial states $|01\rangle$, $|10\rangle$ and $|01\rangle - |10\rangle$ look similar initially for an external noise source coupled diagonally in the basis-forming states, see figure 1. However, only the last state is an eigenstate of $H_0$ and therefore fully protected during the time evolution, i.e. a true DFS. Qualitatively, we expect that the same analysis is valid also for $n$-qubit systems.

Consider, e.g., a three-qubit system where each qubit couples with $\sigma_z$ coupled to a common noise source; this is an extension of the previously given example (16) to three qubits. The three-qubit system is prepared in, e.g., the state $|001\rangle + |010\rangle$ (or any superposition of states with one spin up and two down). The noise source cannot tell the difference between these two states; thus, it is in a DFS with regards to correlated noise. However, if this is not an eigenstate of the Hamiltonian, then the state will (rapidly) evolve to a state which is more susceptible to correlated noise. If $|001\rangle + |010\rangle$ has a large overlap with an eigenstate of the Hamiltonian, then the decoherence of this state will be very sensitive to correlations in the noise, while if the overlap is small it will not. This is exactly the same picture as in the two-qubit case.

While previous works, except for the general classifications of DFSs, have mainly focused on a correlated noise in the SWAP subspace, we find that this is not the only interesting subspace in this regard. Considering two qubits at the co-resonance point, we find that the subspace spanned by the states $|00\rangle$, $|11\rangle$ is protected against an antisymmetric noise. For this system, an antisymmetric noise also leads to increase of the initial decoherence rate $\Gamma_1$ for states in the SWAP subspace, and an increase in $\Gamma_1$ by factor 2 for the singlet state.

For the case of noise due to charged traps in the amorphous substrate, a trap located between two qubits, giving rise to anticorrelated noise, seems just as plausible as the opposite situation of strict correlations. In an experimental setup, one might take advantage of this situation by first looking for correlated or anticorrelated noise, and thereafter deciding the encoding of the two qubits in the subspace that is less sensitive to the noise present in the relevant setup.

Our treatment, by use of the four-level Bloch-vector construction, gives a tractable set of equations compared to alternative methods. The familiar geometrical concepts used when treating two-level systems by the same method can be carried over to higher dimensions. In our treatment, different rates of decoherence along different directions on the Bloch sphere, as well as their sensitivity to correlations in the noise, follow naturally from the formalism. We have shown by explicit calculations how this method is applied in the case of a two-qubit (four-level) system, and it is interesting to consider how to extend this to systems with more levels.

For a system with an $n$-dimensional Hilbert space, the set of density matrices (which are Hermitian with trace 1) is $n^2 - 1$ dimensional. One can show (see e.g. [36]) that the pure states in all cases are on the surface of a sphere in this space. That is, all pure states are lying in a space of dimension $n^2 - 2$. The mixed states, being convex combinations of pure states, necessarily
lie inside this sphere, but only for \( n = 2 \) do all points inside the sphere correspond to positive density matrices and thus to allowed quantum states. In general, the allowed density matrices constitutes a subset of the interior of the Bloch sphere. The set of pure states is determined by \( n \) complex numbers (the components along each of the basis vectors of the Hilbert space), but the number of independent parameters is two less because of normalization and the global phase, so that the space of pure states has \( 2n - 2 \) dimensions. That is, the pure states are a \((2n - 2)\)-dimensional subset of the \( n^2 - 2 \) dimensional surface of the Bloch sphere. One of the features of our method is that this means that the number of independent equations of the form (12) is also \( 2n - 2 \), that is, it scales only linearly with \( n \), rather than quadratic as the dimensionality of the Bloch sphere. However, one still has to diagonalize the full Hamiltonian and the initial density matrix, and this quickly becomes difficult analytically as \( n \) increases. This means that this step probably has to be performed numerically in all but the simplest cases where some symmetry allows for analytic solutions. This means that the coefficients \( \beta_j \) in (12) are found numerically. The conceptual framework of our method should still be helpful in analyzing these equations and give geometric insight into the decoherence processes.

In conclusion, we have demonstrated a generalized Bloch-sphere method by treating the problem of two coupled qubits coupled to statistically-independent or correlated noise sources. By use of this method, we find expressions for the dephasing of the two-qubit system as a function of the degree of correlation of the noise sources. At the co-resonance point, the decoherence rate is very sensitive to the initial two-qubit state. In the absence of symmetry of the Hamiltonian, correlation in the noise seems to have a very weak impact on the decoherence rate.

**Appendix. Detailed expressions**

**A.1. The noise matrix \( V'(t) \)**

The explicit expression for the matrix elements of the interaction term of the Hamiltonian in the interaction picture \( V'(t) = U^{-1}H_1(t)U \) is given by the following.

\[
\begin{align*}
V_{11}' &= \xi_1[(\eta_-\gamma_+ + \eta_+\gamma_-)^2 \cos \omega_1t + (\eta_-\gamma_+ - \eta_+\gamma_-)^2 \cos \omega_2t] \\
&\quad + \xi_2[(\eta_-\gamma_+ + \eta_+\gamma_-)^2 \cos \omega_1t + (\eta_-\gamma_+ - \eta_+\gamma_-)^2 \cos \omega_2t], \\
V_{12}' &= \xi_2[(-\mu_1 + \mu_2) \cos \omega_1t + (\mu_1 - \mu_2) \cos \omega_1t - i\xi_2 \sin \omega_1t - i\xi_1 \sin \omega_2t], \\
V_{13}' &= \xi_2[(\mu_1 + \mu_2) \cos \omega_1t - (\mu_1 - \mu_2) \cos \omega_1t - i\xi_1 \sin \omega_1t - i\xi_2 \sin \omega_2t], \\
V_{14}' &= (\xi_1 + \xi_2)[-i(\mu_2 + \mu_1) \sin \omega_1t + i(\mu_2 - \mu_1) \sin \omega_2t], \\
V_{22}' &= \xi_1[(\eta_-\gamma_+ + \eta_+\gamma_-)^2 \cos \omega_1t + (\eta_-\gamma_+ - \eta_+\gamma_-)^2 \cos \omega_2t] \\
&\quad - \xi_2[(\eta_-\gamma_+ + \eta_+\gamma_-)^2 \cos \omega_1t + (\eta_-\gamma_+ - \eta_+\gamma_-)^2 \cos \omega_2t], \\
V_{23}' &= (\xi_1 - \xi_2)i(\mu_2 + \mu_1) \sin \omega_1t - i(\mu_2 - \mu_1) \sin \omega_2t], \\
V_{24}' &= \xi_1(-\cos \omega_1t(\mu_1 - \mu_2) + \cos \omega_1t(\mu_1 + \mu_2) - i \sin \omega_1t \xi_1 - i \sin \omega_2t \xi_2], \\
V_{33}' &= -\xi_1[(\eta_-\gamma_+ + \eta_+\gamma_-)^2 \cos \omega_1t + (\eta_-\gamma_+ - \eta_+\gamma_-)^2 \cos \omega_2t] \\
&\quad + \xi_2[(\eta_-\gamma_+ + \eta_+\gamma_-)^2 \cos \omega_1t + (\eta_-\gamma_+ - \eta_+\gamma_-)^2 \cos \omega_2t], \\
V_{34}' &= \xi_2[(\mu_1 - \mu_2) \cos \omega_1t - (\mu_1 - \mu_2) \cos \omega_1t - i\xi_2 \sin \omega_1t - i\xi_1 \sin \omega_2t], \\
V_{44}' &= -\xi_1[(\eta_-\gamma_+ + \eta_+\gamma_-)^2 \cos \omega_1t + (\eta_-\gamma_+ - \eta_+\gamma_-)^2 \cos \omega_2t] \\
&\quad - \xi_2[(\eta_-\gamma_+ + \eta_+\gamma_-)^2 \cos \omega_1t + (\eta_-\gamma_+ - \eta_+\gamma_-)^2 \cos \omega_2t],
\end{align*}
\]  

(A.1)

where we have only given the upper triangle of the Hermitian matrix \( V' \).
A.2. Differential equations for the density matrix components $\alpha_i$

The differential equations for the components of the density matrix evolving from the initial state $|\psi_0\rangle = |10\rangle - |01\rangle$:

\[
\dot{\alpha}_0 = \frac{\xi_1(t)}{\sqrt{2}} [\xi_1 \sin \omega_1 t + \xi_2 \sin \omega_2 t - \frac{\xi_3(t)}{\sqrt{2}} [\xi_1 \sin \omega_1 t + \xi_2 \sin \omega_2 t],
\]

\[
\dot{\alpha}_{10} = \frac{\xi_1(t)}{\sqrt{2}} [-(\mu_{1+} + \mu_{2-}) \cos \omega_1 t + (\mu_{1-} + \mu_{2+}) \cos \omega_2 t]
\]

\[
+ \frac{\xi_2(t)}{\sqrt{2}} [-(\mu_{1-} - \mu_{2+}) \cos \omega_1 t + (\mu_{1+} - \mu_{2-}) \cos \omega_2 t],
\]

\[
\dot{\alpha}_{11} = (\xi_1(t) - \xi_2(t)) [(\mu_{1+} + \mu_{2+}) \sin \omega_1 t + (\mu_{1-} + \mu_{2+}) \sin \omega_2 t],
\]

\[
\dot{\alpha}_{12} = - \frac{\xi_1(t)}{\sqrt{2}} [(\eta_{1-} - \eta_{1+})^2 \cos \omega_1 t + (\eta_{1+} - \eta_{1-})^2 \cos \omega_2 t]
\]

\[
+ \frac{\xi_2(t)}{\sqrt{2}} [(\eta_{1+} + \eta_{1-})^2 \cos \omega_1 t + (\eta_{1+} - \eta_{1-})^2 \cos \omega_2 t],
\]

\[
\dot{\alpha}_{13} = \frac{\xi_1(t)}{\sqrt{2}} [\xi_1 \sin \omega_1 t + \xi_2 \sin \omega_2 t] - \frac{\xi_2(t)}{\sqrt{2}} [\xi_2 \sin \omega_1 t + \xi_1 \sin \omega_2 t],
\]

\[
\dot{\alpha}_{14} = \frac{\xi_1(t)}{\sqrt{2}} [-(\mu_{1+} + \mu_{2-}) \cos \omega_1 t + (\mu_{1-} + \mu_{2+}) \cos \omega_2 t]
\]

\[
+ \frac{\xi_2(t)}{\sqrt{2}} [-(\mu_{1-} - \mu_{2+}) \cos \omega_1 t + (\mu_{1+} - \mu_{2-}) \cos \omega_2 t].
\]  

(A.2)

References

[33] Lidar D A and B W 2003 Decoherence-free subspaces and subsystems Irreversible Quantum Dynamics (Lecture Notes in Physics vol 622) ed F Benatti and R Floreanini (Berlin: Springer) pp 83–120
[40] Zurek W H 2003 Rev. Mod. Phys. 75 715–75